Fluid Dynamics from Kinetic Equations

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LECTURE 4:

LINEAR AND WEAKLY NONLINEAR FLUID DYNAMICAL LIMITS

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I. INTRODUCTION

Full justification of most of the formal approximations of the past lectures has proven difficult because the basic well-posedness and regularity questions remain open for both the fluid systems and the Boltzmann equation.

The problem is exacerbated by the fact that to bound the error of the asymptotic expansions requires control of successively higher order spatial derivatives of the fluid variables, thereby requiring restrictions to a meager subset of all physically natural initial data and possibly to finite times.

For example, Caflisch used a method based on the Hilbert expansion to justify the compressible Euler system from the Boltzmann equation. His result requires smooth initial data and holds for as long as the limiting solution of the compressible Euler system is smooth.

Because solutions of the compressible Euler system are known to become singular in finite time for a very general class of initial data, such a local result is about the best one can hope for by appealing to such an expansion.

Two approaches to circumventing these difficulties have emerged recently. First, one studies derivations of linear or weakly nonlinear fluid dynamical systems, such as the acoustic system and the incompressible systems, about which more is known. Second, one abandons traditional expansion-based derivations in favor of moment-based formal derivations, which put fewer demands on the well-posedness and regularity theory.

In this lecture we present moment-based formal derivations of the acoustic, incompressible Stokes, incompressible Navier-Stokes, and incompressible Euler systems from the Boltzmann equation.

II. THE GENERAL SETTING

We consider the scaled Boltzmann equation

$$\boxed{\mathsf{St}\,\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathsf{Kn}}\,\mathcal{B}(F, F)\,,}$$

where St is the Strouhal number, Kn is the Knudsen number.

We consider fluid dynamical regimes in which F is close to a spatially homogeneous Maxwellian $M = M(v)$. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{D/2}} \exp(-\tfrac{1}{2}|v|^2)\,.$$

Relative Kinetic Density

It is natural to introduce the relative density, $G = G(v, x, t)$, defined by $F = MG$. The initial-value problem for G is

$$\boxed{\mathrm{St}\, \partial_t G + v \cdot \nabla_x G = \frac{1}{\mathrm{Kn}} \mathcal{Q}(G, G)\,, \qquad G(v, x, 0) = G^{in}(v, x)\,,}$$

where the collision operator is now given by

$$\begin{aligned}\mathcal{Q}(G, G) &\equiv \frac{1}{M} \mathcal{B}(MG, MG) \\ &= \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G'_* G' - G_* G)\, b(\omega, v_* - v,)\, \mathrm{d}\omega\, M_* \mathrm{d}v_*\,.\end{aligned}$$

For simplicity, we consider this problem over the periodic box $\mathbb{T}^D$.

Normalizations

This nondimensionalization has the normalizations

$$\int_{\mathbb{S}^{D-1}} \mathrm{d}\omega = 1\,, \qquad \int_{\mathbb{R}^D} M\,\mathrm{d}v = 1\,, \qquad \int_{\mathbb{T}^D} \mathrm{d}x = 1\,,$$

associated with the domains $\mathbb{S}^{D-1}$, $\mathbb{R}^D$, and $\mathbb{T}^D$, the normalization

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} b(\omega, v_* - v)\,\mathrm{d}\omega\, M_*\mathrm{d}v_*\, M\mathrm{d}v = 1\,.$$

associated with the collision kernel b, and the normalizations

$$\iint_{\mathbb{R}^D\times\mathbb{T}^D} G^{in}\, M\mathrm{d}v\,\mathrm{d}x = 1\,, \quad \iint_{\mathbb{R}^D\times\mathbb{T}^D} v\, G^{in}\, M\mathrm{d}v\,\mathrm{d}x = 0\,,$$

$$\iint_{\mathbb{R}^D\times\mathbb{T}^D} \tfrac{1}{2}|v|^2 G^{in}\, M\mathrm{d}v\,\mathrm{d}x = \frac{D}{2}\,.$$

associated with the initial data G^{in}.

Notation

In this lecture $\langle \xi \rangle$ will denote the average over $\mathbb{R}^D$ of any integrable function $\xi = \xi(v)$ with respect to the positive unit measure $M\mathrm{d}v$:

$$\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v)\, M\mathrm{d}v\,.$$

Because $\mathrm{d}\mu = b(\omega, v_* - v)\,\mathrm{d}\omega\, M_*\mathrm{d}v_*\, M\mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_*, v)$:

$$\langle\!\langle \Xi \rangle\!\rangle = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_*, v)\,\mathrm{d}\mu\,.$$

The measure $d\mu$ is invariant under the coordinate transformations

$$(\omega, v_*, v) \mapsto (\omega, v, v_*)\,, \qquad (\omega, v_*, v) \mapsto (\omega, v'_*, v')\,.$$

These, and compositions of these, are called $\mathrm{d}\mu$-symmetries.

Local Conservation Laws

If G solves the scaled Boltzmann equation then G satisfies local conservation laws of mass, momentum, and energy:

$$\boxed{\begin{aligned} \mathrm{St}\,\partial_t \langle G \rangle + \nabla_x \cdot \langle v\, G \rangle &= 0\,, \\ \mathrm{St}\,\partial_t \langle v\, G \rangle + \nabla_x \cdot \langle v \otimes v\, G \rangle &= 0\,, \\ \mathrm{St}\,\partial_t \langle \tfrac{1}{2}|v|^2 G \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 G \rangle &= 0\,. \end{aligned}}$$

Global Conservation Laws

When these are integrated over space and time while recalling the normalizations associated with G^{in}, they yield the global conservation laws of mass, momentum, and energy:

$$\boxed{\begin{aligned}\int_{\mathbb{T}^D} \langle G(t) \rangle \, \mathrm{d}x &= \int_{\mathbb{T}^D} \langle G^{in} \rangle \, \mathrm{d}x = 1 \,,\\ \int_{\mathbb{T}^D} \langle v \, G(t) \rangle \, \mathrm{d}x &= \int_{\mathbb{T}^D} \langle v \, G^{in} \rangle \, \mathrm{d}x = 0 \,,\\ \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G(t) \rangle \, \mathrm{d}x &= \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G^{in} \rangle \, \mathrm{d}x = \frac{D}{2} \,.\end{aligned}}$$

Local Entropy Dissipation Law

If G solves the scaled Boltzmann equation then G satisfies local entropy dissipation law:

$$\boxed{\begin{aligned} \mathsf{St}\,\partial_t\langle(G\log(G)-G+1)\rangle &+ \nabla_x\cdot\langle v\,(G\log(G)-G+1)\rangle \\ &= -\frac{1}{\mathsf{Kn}}\left\langle\!\!\left\langle \tfrac{1}{4}\log\left(\frac{G'_*G'}{G_*G}\right)(G'_*G'-G_*G)\right\rangle\!\!\right\rangle \\ &\leq 0\,. \end{aligned}}$$

Global Entropy Dissipation Law

When this is integrated over space and time, it yields the global entropy equality

$$\boxed{H(G(t)) + \frac{1}{\epsilon}\int_0^t R(G(s))\,\mathrm{d}s = H(G^{in})\,,}$$

where the relative entropy functional H is given by

$$H(G) = \int_{\mathbb{T}^D} \langle (G \log(G) - G + 1) \rangle\,\mathrm{d}x\,,$$

while the entropy dissipation rate functional R is given by

$$R(G) = \int_{\mathbb{T}^D} \tfrac{1}{4} \left\langle\!\!\left\langle \log\left(\frac{G'_* G'}{G_* G}\right)(G'_* G' - G_* G) \right\rangle\!\!\right\rangle \mathrm{d}x\,.$$

Fluctuations

All our derivations will employ a scaling in which the Knudsen number Kn vanishes and the density F is close to the absolute Maxwellian M. We therefore set $\mathrm{Kn} = \epsilon$ and consider families of solutions parametrized by ϵ that have the form

$$G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in} , \qquad G_\epsilon = 1 + \delta_\epsilon g_\epsilon ,$$

where the fluctuations g_ϵ^{in} and g_ϵ are bounded while $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0 .$$

Formal Convergence Assumptions

In these derivations we assume that g_ϵ converges formally to g, where the limiting function is in $L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$, and that all formally small terms vanish.

For example, we express the global conservation laws, which are the same for all of our derivations, in terms of g_ϵ and then formally let $\epsilon \longrightarrow 0$ to obtain

$$\boxed{\int_{\mathbb{T}^D} \langle g(t) \rangle \,\mathrm{d}x = 0\,, \quad \int_{\mathbb{T}^D} \langle v\, g(t) \rangle \,\mathrm{d}x = 0\,, \quad \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 g(t) \rangle \,\mathrm{d}x = 0\,.}$$

Henceforth, the derivations differ.

III. FORMAL DERIVATION OF THE ACOUSTIC SYSTEM

The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units, in this the fluid fluctuations (ρ, u, θ) satisfy

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot u = 0\,, \qquad & \rho(x,0) = \rho^{in}(x)\,,\\
\partial_t u + \nabla_x (\rho + \theta) = 0\,, \qquad & u(x,0) = u^{in}(x)\,,\\
\tfrac{D}{2} \partial_t \theta + \nabla_x \cdot u = 0\,, \qquad & \theta(x,0) = \theta^{in}(x)\,.
\end{aligned}
$$

This is one of the simplest systems of fluid dynamical equations imaginable, being essentially the wave equation.

Acoustic Scaling

It is most natural to derive the acoustic system first because its derivation is simpler and requires no additional assumptions regarding either the scaling or the collision kernel. One sets $\mathtt{St} = 1$ and considers a family of formal solutions G_ϵ to the scaled Boltzmann initial-value problem

$$\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon)\,, \qquad G_\epsilon(v, x, 0) = G_\epsilon^{in}(v, x)\,,$$

with $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ for some δ_ϵ that satisfies

$$\delta_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0\,.$$

The derivation has two steps.

First Step - 1

We first determine the form of the limiting function g. Observe that the fluctuations g_ϵ satisfy

$$\epsilon \left(\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon\right) + \mathcal{L} g_\epsilon = \delta_\epsilon \, \mathcal{Q}(g_\epsilon, g_\epsilon) \,,$$

where the linearized collision operator $\mathcal{L}$ is defined formally by

$$\begin{aligned} \mathcal{L}\tilde{g} &= -2\mathcal{Q}(1, \tilde{g}) \\ &= \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} \left(\tilde{g} + \tilde{g}_* - \tilde{g}' - \tilde{g}'_*\right) b(\omega, v_* - v) \, \mathrm{d}\omega \, M_* \mathrm{d}v_* \,. \end{aligned}$$

We define $\mathcal{L}$ to be the unique nonnegative, self-adjoint extension over $L^2(M\mathrm{d}v)$ of this formal operator. By letting $\epsilon \to 0$ above one finds that $\mathcal{L}g = 0$. Hence, $g(\,\cdot\,, x, t)$ takes values in $\mathrm{Null}(\mathcal{L})$, the null space of $\mathcal{L}$.

First Step - 2

One can show that $\mathrm{Null}(\mathcal{L}) = \mathrm{span}\{1, v_1, \cdots, v_D, |v|^2\}$. Because the limit $g(\,\cdot\,, x, t)$ takes values in $\mathrm{Null}(\mathcal{L})$ and because g is assumed to belong to $L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,dx))$, we conclude that g has the form

$$g = \rho + v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right)\theta\,,$$

for some (ρ, u, θ) in $L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$.

This form is call an *infinitesimal Maxwellian* because

$$\begin{aligned}\mathcal{M}(1 + \delta\rho, \delta u, 1 + \delta\theta) &= \frac{1 + \delta\rho}{(2\pi(1 + \delta\theta))^{D/2}} \exp\left(\frac{|v - \delta u|^2}{2(1 + \delta\theta)}\right) \\ &= M\Big(1 + \delta\, g + O(\delta^2)\Big)\,.\end{aligned}$$

Second Step - 1

Next we show that the evolution of (ρ, u, θ) is governed by the acoustic system. Observe that the fluctuations g_ϵ formally satisfy the local conservation laws

$$
\begin{aligned}
\partial_t \langle g_\epsilon \rangle + \nabla_x \cdot \langle v\, g_\epsilon \rangle &= 0\,, \\
\partial_t \langle v\, g_\epsilon \rangle + \nabla_x \cdot \langle v \otimes v\, g_\epsilon \rangle &= 0\,, \\
\partial_t \langle \tfrac{1}{2}|v|^2 g_\epsilon \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 g_\epsilon \rangle &= 0\,.
\end{aligned}
$$

By letting $\epsilon \to 0$ in these equations and using the infinitesimal Maxwellian form of g, one then finds that (ρ, u, θ) solves the acoustic system.

Second Step - 2

By the formal continuity of the above densities in time, one finds that

$$\left(\rho^{in}, u^{in}, \theta^{in}\right) = \lim_{\epsilon \to 0} \left(\langle g_\epsilon^{in}\rangle, \langle v\, g_\epsilon^{in}\rangle, \langle(\tfrac{1}{D}|v|^2 - 1)\, g_\epsilon^{in}\rangle\right),$$

provided we assume that the limits on the right-hand side exist in the sense of distributions for some $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$.

Formal Acoustic Limit Theorem

Theorem. 1 *Let G_ϵ be a family of distribution solutions of the scaled Boltzmann initial-value problem with initial data G_ϵ^{in} that satisfy the normalizations. Let $G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in}$ and $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ where $\delta_\epsilon \to 0$ as $\epsilon \to 0$, and the fluctuations g_ϵ^{in} and g_ϵ are bounded in $L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$.*

Moreover:

1. *Assume that in the sense of distributions the family g_ϵ^{in} satisfies*

$$\lim_{\epsilon\to 0}\Big(\langle g_\epsilon^{in}\rangle, \langle v\, g_\epsilon^{in}\rangle, \langle (\tfrac{1}{D}|v|^2 - 1)\, g_\epsilon^{in}\rangle\Big) = \Big(\rho^{in}, u^{in}, \theta^{in}\Big)$$

   for some $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R})$.

2. *Assume that the local conservation laws are also satisfied in the sense of distributions for every* g_ϵ.

3. *Assume that the family* g_ϵ *converges in the sense of distributions as* $\epsilon \to 0$ *to* $g \in L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$*. Assume furthermore that* $\mathcal{L}g_\epsilon \to \mathcal{L}g$*, that the moments*

$$\langle g_\epsilon \rangle\,, \qquad \langle v\, g_\epsilon \rangle\,, \qquad \langle v \otimes v\, g_\epsilon \rangle\,, \qquad \langle v|v|^2 g_\epsilon \rangle\,,$$

converge to the corresponding moments

$$\langle g \rangle\,, \qquad \langle v\, g \rangle\,, \qquad \langle v \otimes v\, g \rangle\,, \qquad \langle v|v|^2 g \rangle\,,$$

and that every formally small term vanishes, all in the sense of distributions as $\epsilon \to 0$.

Then g *is the unique local infinitesimal Maxwellian determined by the solution* (ρ, u, θ) *of the acoustic system with the initial data* $(\rho^{in}, u^{in}, \theta^{in})$.

IV. FORMAL DERIVATION OF INCOMPRESSIBLE SYSTEMS

It is easily seen that any $(\rho, u, \theta) \in L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ such that

$$\nabla_x \cdot u = 0 \,, \qquad \nabla_x (\rho + \theta) = 0 \,,$$

is a stationary solution of the acoustic system which will generally vary in space. On the other hand, it can be shown that absolute Maxwellians are the only stationary solutions of the Boltzman equation.

It is clear that the time scale at which the acoustic system was derived was not long enough to see the evolution of these solutions. By considering the Boltzmann equation over a longer time scale one can give formal moment derivations of three incompressible fluid dynamical systems, depending on the limiting behavior of the ratio $\delta_\epsilon / \epsilon$ as $\epsilon \to 0$. We will show the following:

- When $\delta_\epsilon/\epsilon \to 0$, one considers time scales of order $1/\epsilon$, and an incompressible Stokes system is derived.

- When $\delta_\epsilon/\epsilon \to 1$ (or any other nonzero number), one considers time scales of order $1/\epsilon$, and an incompressible Navier-Stokes system is derived.

- When $\delta_\epsilon/\epsilon \to \infty$, one considers time scales of order $1/\delta_\epsilon$, and an incompressible Euler system is derived.

We now show how these different regimes are identified, and how the fluid dynamical systems are derived.

$\boxed{\textbf{Boussinesq-Balance Incompressible Fluid Systems}}$

The incompressible Stokes, Navier-Stokes, and Euler systems that will be derived all govern the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrim values. By suitable choices of Galilean frame and units, one can assume that these equilibrim values are 1, 0, and 1 respectively. We denote the fluctuations about these values by (ρ, u, θ).

For all three systems these fluctuations satisfy the incompressibility and Boussinesq relations:

$$\nabla_x \cdot u = 0\,, \qquad\qquad \rho + \theta = 0\,.$$

The systems differ however in the equations that govern the dynamics of these fluctuations.

Incompressible Stokes Dynamics

For the Stokes system the dynamical equations are

$$\boxed{\begin{aligned} \partial_t u + \nabla_x p &= \nu \Delta_x u \,, \qquad & u(x,0) &= u^{in}(x)\,, \\ \tfrac{D+2}{2}\partial_t \theta &= \kappa \Delta_x \theta \,, \qquad & \theta(x,0) &= \theta^{in}(x)\,, \end{aligned}}$$

where $\nu > 0$ is the kinematic viscosity and $\kappa > 0$ is the thermal diffusivity. Like the acoustic system, the Stokes system is also one of the simplest systems of fluid dynamical equations imaginable, being essentially a system of linear heat equations.

Incompressible Navier-Stokes Dynamics

For the Navier-Stokes system the dynamical equations are

$$
\boxed{
\begin{aligned}
&\partial_t u + u\cdot\nabla_x u + \nabla_x p = \nu\Delta_x u\,, &\qquad& u(x,0) = u^{in}(x)\,,\\
&\tfrac{D+2}{2}\Big(\partial_t\theta + u\cdot\nabla_x\theta\Big) = \kappa\Delta_x\theta\,, &\qquad& \theta(x,0) = \theta^{in}(x)\,,
\end{aligned}
}
$$

where the kinematic viscosity ν and the thermal diffusivity have the same values as in the Stokes system. Unlike the Stokes system however, the Navier-Stokes system is nonlinear. While this fact does not complicate its formal derivation, it makes the mathematical establishment of its validity much harder.

Incompressible Euler Dynamics

For the Euler system the dynamical equations are

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0\,, \qquad u(x,0) = u^{in}(x)\,,$$
$$\tfrac{D+2}{2}\Big(\partial_t \theta + u \cdot \nabla_x \theta\Big) = 0\,, \qquad \theta(x,0) = \theta^{in}(x)\,,$$

Like the Navier-Stokes system, the Euler system is nonlinear. The full mathematical establishment of its validity is also an open problem.

As was the case for the acoustic system, the Euler system has stationary solutions that vary in space. It is clear that the time scale at which the Euler system was derived was not long enough to see the evolution of these solutions. Even at a formal level it is unclear how this long-time evolution should be governed.

Long Time Scaling

In order to identify how the different regimes arise, we reconsider the Boltzmann initial-value problem on a time scale $1/\tau_\epsilon$, where

$$\tau_\epsilon \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0\,.$$

Upon setting $\mathrm{St} = \tau_\epsilon$, the scaled Boltzmann initial-value problem becomes

$$\tau_\epsilon\,\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon}\,\mathcal{Q}(G_\epsilon, G_\epsilon)\,, \qquad G_\epsilon(v,x,0) = G_\epsilon^{in}(v,x)\,.$$

The idea is to identify possible choices for τ_ϵ by seeking different balances between terms as ϵ tends to zero.

Fluctuations

One considers a family of formal solutions G_ϵ whose fluctuations g_ϵ are given by $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ for some $\delta_\epsilon > 0$ that vanishes as $\epsilon \to 0$. These fluctuations are assumed to satisfy both the Boltzmann equation

$$\boxed{\tau_\epsilon \, \partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon} \mathcal{L} g_\epsilon = \frac{\delta_\epsilon}{\epsilon} \, \mathcal{Q}(g_\epsilon, g_\epsilon) \, ,}$$

and the local conservation laws

$$\boxed{\begin{aligned} \tau_\epsilon \, \partial_t \langle g_\epsilon \rangle + \nabla_x \cdot \langle v \, g_\epsilon \rangle &= 0 \, , \\ \tau_\epsilon \, \partial_t \langle v \, g_\epsilon \rangle + \nabla_x \cdot \langle v \otimes v \, g_\epsilon \rangle &= 0 \, , \\ \tau_\epsilon \, \partial_t \langle \tfrac{1}{2} |v|^2 g_\epsilon \rangle + \nabla_x \cdot \langle v \tfrac{1}{2} |v|^2 g_\epsilon \rangle &= 0 \, . \end{aligned}}$$

First and Second Steps - 1

The first step is to show that the limiting g is an infinitesimal Maxwellian. After multiplying above Boltzmann equation by ϵ and letting $\epsilon \to 0$, we argue as in the first step of the acoustic system derivation to conclude g has the form

$$g = \rho + v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right)\theta\,,$$

for some (ρ, u, θ) in $L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$.

The second step is to show that (ρ, u, θ) satisfy the incompressibility and Boussinesq relations. By letting $\epsilon \to 0$ in the local conservation laws and using the form of g, one finds that

$$\nabla_x \cdot u = 0\,, \qquad \nabla_x(\rho + \theta) = 0\,.$$

The first equation above is the incompressibility relation, while the second says $\rho + \theta$ is a function of time only.

First and Second Steps - 2

By the global energy conservation law one thereby concludes that

$$\rho + \theta = \int_{\mathbb{T}^D} (\rho + \theta)\, \mathrm{d}x = \frac{2}{D} \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 g \rangle \, \mathrm{d}x = 0\,.$$

Hence, (ρ, u, θ) satisfy the incompressibility and Boussinesq relations.

The Boussinesq relation implies g is an infinitesimal Maxwellian of the form

$$g = v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2} \right) \theta\,,$$

for some (u, θ) in $L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R}^D \times \mathbb{R}))$.

First Key Idea - 1

Observe that the momentum and a linear combination of the mass and energy local conservation laws can be expressed as

$$\partial_t \langle v\, g_\epsilon \rangle + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle A\, g_\epsilon \rangle + \frac{1}{\tau_\epsilon} \nabla_x \Big\langle \tfrac{1}{D} |v|^2 g_\epsilon \Big\rangle = 0 \,,$$
$$\partial_t \Big\langle \big(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\big)\, g_\epsilon \Big\rangle + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle B\, g_\epsilon \rangle = 0 \,,$$

where the matrix-valued function A and the vector-valued function B are defined by

$$A(v) = v \otimes v - \tfrac{1}{D}|v|^2 I \,, \qquad B(v) = \tfrac{1}{2}|v|^2 v - \tfrac{D+2}{2} v \,.$$

It is clear that $A \in L^2(M\mathrm{d}v; \mathbb{R}^{D \vee D})$ and $B \in L^2(M\mathrm{d}v; \mathbb{R}^D)$, and that each entry of A and B is in $\mathrm{Null}(\mathcal{L})^\perp$.

First Key Idea - 2

As is common when studying incompressible fluid dynamical limits, the momentum equation will be integrated against divergence-free test functions. The last term in its flux will thereby be eliminated.

One therefore only has to pass to the limit in the flux terms that involve A and B — namely, in the terms

$$\frac{1}{\tau_\epsilon}\langle A\, g_\epsilon\rangle\,, \qquad \frac{1}{\tau_\epsilon}\langle B\, g_\epsilon\rangle\,.$$

There is a chance these terms will have limits because each entry of A and B is in $\mathrm{Null}(\mathcal{L})^\perp$ while g_ϵ is converging to g, which is in $\mathrm{Null}(\mathcal{L})$.

Second Key Idea - 1

In fact, we will use the scaled Boltzmann equation to show that for a proper choice of τ_ϵ every expression of the form

$$\frac{1}{\tau_\epsilon}\langle \mathcal{L}\xi\, g_\epsilon \rangle$$

where $\xi \in \mathrm{Dom}(\mathcal{L})$ has a limit.

We let $\widehat{A} \in L^2(M\mathrm{d}v;\mathbb{R}^{D\vee D})$ and $\widehat{B} \in L^2(M\mathrm{d}v;\mathbb{R}^D)$ be the unique solutions of

$$\begin{aligned} \mathcal{L}\widehat{A} &= A\,, \qquad \widehat{A} \in \mathrm{Null}(\mathcal{L})^\perp \quad \text{entrywise}\,, \\ \mathcal{L}\widehat{B} &= B\,, \qquad \widehat{B} \in \mathrm{Null}(\mathcal{L})^\perp \quad \text{entrywise}\,. \end{aligned}$$

We can then set ξ equal to the entries of $\widehat{A}$ and $\widehat{B}$ to evaluate the limit in the flux terms that involve A and B.

Second Key Idea - 2

First, observe that because $\mathcal{L}$ is formally symmetric, one has

$$\frac{1}{\tau_\epsilon}\langle \mathcal{L}\xi \, g_\epsilon \rangle = \frac{1}{\tau_\epsilon}\langle \xi \, \mathcal{L} g_\epsilon \rangle \, ,$$

for every $\xi \in \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Null}(\mathcal{L})^\perp$, where $\mathrm{Dom}(\mathcal{L}) \subset L^2(M\mathrm{d}v)$. Upon multiplying the Boltzmann equation by ξ and integrating, one obtains

$$\epsilon \, \partial_t \langle \xi \, g_\epsilon \rangle + \frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle v \, \xi \, g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\langle \xi \, \mathcal{L} g_\epsilon \rangle = \frac{\delta_\epsilon}{\tau_\epsilon} \langle \xi \, \mathcal{Q}(g_\epsilon, g_\epsilon) \rangle \, .$$

One therefore has that

$$\frac{1}{\tau_\epsilon}\langle \mathcal{L}\xi \, g_\epsilon \rangle = \frac{\delta_\epsilon}{\tau_\epsilon} \langle \xi \, \mathcal{Q}(g_\epsilon, g_\epsilon) \rangle - \frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle v \, \xi \, g_\epsilon \rangle - \epsilon \, \partial_t \langle \xi \, g_\epsilon \rangle \, .$$

The right-hand side will have a limit as $\epsilon \to 0$ that depends on the limiting behavior of the ratio $\delta_\epsilon / \epsilon$, provided one makes the correct choice of τ_ϵ.

Third Step - 1

- When $\delta_\epsilon/\epsilon \to 0$, one sets $\tau_\epsilon = \epsilon$ and finds that

$$\frac{1}{\tau_\epsilon}\langle \mathcal{L}\xi \, g_\epsilon \rangle \longrightarrow -\nabla_x \cdot \langle v \, \xi \, g \rangle \, .$$

- When $\delta_\epsilon/\epsilon \to 1$, one sets $\tau_\epsilon = \epsilon$ and finds that

$$\frac{1}{\tau_\epsilon}\langle \mathcal{L}\xi \, g_\epsilon \rangle \longrightarrow -\nabla_x \cdot \langle v \, \xi \, g \rangle + \langle \xi \, \mathcal{Q}(g,g) \rangle \, .$$

- When $\delta_\epsilon/\epsilon \to \infty$, one sets $\tau_\epsilon = \delta_\epsilon$ and finds that

$$\frac{1}{\tau_\epsilon}\langle \mathcal{L}\xi \, g_\epsilon \rangle \longrightarrow \langle \xi \, \mathcal{Q}(g,g) \rangle \, .$$

Third Step - 2

We now must evaluate the terms $\nabla_x \cdot \langle v\,\xi\,g\rangle$ and $\langle \xi\,\mathcal{Q}(g,g)\rangle$ that appear on the right-hand sides above. Ones uses the infinitesimal Maxwellian form of g to obtain

$$\begin{aligned} \nabla_x \cdot \langle v\,\xi\,g\rangle &= -\langle \xi\,A\rangle : \nabla_x u - \langle \xi\,B\rangle \cdot \nabla_x \theta\,, \\ \langle \xi\,\mathcal{Q}(g,g)\rangle &= \tfrac{1}{2}\langle \mathcal{L}\xi\,A\rangle : u \otimes u + \langle \mathcal{L}\xi\,B\rangle \cdot u\,\theta + \tfrac{1}{2}\langle \mathcal{L}\xi\,C\rangle\,\theta^2 \end{aligned}$$

where the matrix-valued function A and the vector-valued function B were defined earlier, while the scalar-valued function C is defined by

$$C(v) = \tfrac{1}{4}|v|^4 - \tfrac{D+2}{2}|v|^2 + \tfrac{D(D+2)}{4}\,.$$

It is clear that $C \in L^2(M\mathrm{d}v)$. One also has that $C \in \mathrm{Null}(\mathcal{L})^\perp$ and that C is perpendicular to each entry of A and B.

Third Step - 3

Indeed, because g has the form of an infinitesimal Maxwellian, for sufficiently small δ one has that $\exp(\delta g)$ is in $\mathrm{Dom}(\mathcal{Q})$ and one can show that

$$0 = \mathcal{Q}\Big(\exp(\delta g), \exp(\delta g)\Big) = \delta^2\Big[\mathcal{Q}(g,g) + \mathcal{Q}(1,g^2)\Big] + O(\delta^3)\,.$$

Hence g satifies the identity $\mathcal{Q}(g,g) = \frac{1}{2}\mathcal{L}(g^2)$. Hence,

$$\langle \xi\, \mathcal{Q}(g,g)\rangle = \tfrac{1}{2}\langle \xi\, \mathcal{L}(g^2)\rangle = \tfrac{1}{2}\langle \mathcal{L}\xi\, \mathcal{P}^{\perp}(g^2)\rangle\,,$$

where $\mathcal{P}^{\perp} = \mathcal{I} - \mathcal{P}$ and $\mathcal{P}$ is the orthogonal projection from $L^2(M\mathrm{d}v)$ onto $\mathrm{Null}(\mathcal{L})$, which for every $\tilde{g} \in L^2(M\mathrm{d}v)$ is given by

$$\mathcal{P}\tilde{g} = \langle \tilde{g}\rangle + v\cdot\langle v\,\tilde{g}\rangle + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,\langle(\tfrac{1}{D}|v|^2 - 1)\,\tilde{g}\rangle\,.$$

Fourth Step

The fourth step determines the limit of the motion and heat flux terms. Then by letting ξ in the Third Step be the entries of $\widehat{A}$ and $\widehat{B}$ and by using that A and $\widehat{A}$ are even whereas B and $\widehat{B}$ are odd, one finds that

$$\begin{aligned}
\nabla_x \cdot \langle v\,\widehat{A}\,g\rangle &= -\langle \widehat{A} \otimes A\rangle : \nabla_x u && = -\nu\Big(\nabla_x u + (\nabla_x u)^T\Big)\,,\\
\nabla_x \cdot \langle v\,\widehat{B}\,g\rangle &= -\langle \widehat{B} \otimes B\rangle \cdot \nabla_x \theta && = -\kappa \nabla_x \theta\,,\\
\langle \widehat{A}\,\mathcal{Q}(g,g)\rangle &= \tfrac{1}{2}\langle \mathcal{L}\widehat{A} \otimes A\rangle : (u \otimes u) && = u \otimes u - \tfrac{1}{D}|u|^2 I\,,\\
\langle \widehat{B}\,\mathcal{Q}(g,g)\rangle &= \langle \mathcal{L}\widehat{B} \otimes B\rangle \cdot u\,\theta && = \tfrac{D+2}{2} u\,\theta\,,
\end{aligned}$$

where kinematic viscosity ν and thermal conductivity κ are given by

$$\nu = \frac{1}{(D-1)(D+2)}\,\langle \widehat{A} : \mathcal{L}\widehat{A}\rangle\,, \qquad \kappa = \frac{1}{D}\,\langle \widehat{B} \cdot \mathcal{L}\widehat{B}\rangle\,.$$

Fifth Step

The Fifth Step shows how the evolution of (u, θ) is governed by the appropriate motion and heat equations. We pass to the limit using the Fourth Step to evaluate the limiting fluxes.

If we now let Π denote the orthogonal projection from $L^2(\mathrm{d}x; \mathbb{R}^D)$ onto the divergence-free vector fields, then by the formal continuity in time of the densities in the dynamical equations, one sees that

$$\left(u^{in}, \theta^{in}\right) = \lim_{\epsilon \to 0} \left(\Pi \langle v\, g_\epsilon^{in} \rangle, \langle (\tfrac{1}{D+2}|v|^2 - 1)\, g_\epsilon^{in} \rangle\right), \tag{1}$$

provided we assume that the limit on the right-hand side exists in the sense of distributions for some $(u^{in}, \theta^{in}) \in L^2(\mathrm{d}x; \mathbb{R}^D \times \mathbb{R})$.

Formal Incompressible Limit Theorem

Theorem. 2 *Let G_ϵ be a family of distribution solutions of the scaled Boltzmann initial-value problem with initial data G_ϵ^{in} that satisfy the normalizations. Let $G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in}$ and $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ where $\delta_\epsilon \to 0$ as $\epsilon \to 0$, and the fluctuations g_ϵ^{in} and g_ϵ are bounded in $L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$.*

Moreover:

1. *Assume that in the sense of distributions the family g_ϵ^{in} satisfies*

$$\lim_{\epsilon\to 0}\Big(\langle v\, g_\epsilon^{in}\rangle, \langle(\tfrac{1}{D+2}|v|^2 - 1)\, g_\epsilon^{in}\rangle\Big) = \Big(u^{in}, \theta^{in}\Big)$$

   for some $(u^{in}, \theta^{in}) \in L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R})$.

2. *Assume that the local conservation laws are also satisfied in the sense of distributions for every g_ϵ.*

3. *Assume that the family g_ϵ converges in the sense of distributions as $\epsilon \to 0$ to $g \in L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$. Assume furthermore that $\mathcal{L}g_\epsilon \to \mathcal{L}g$, that for every $\xi \in L^2(M\,\mathrm{d}v)$ the moments $\langle \xi\, g_\epsilon \rangle$ converge to $\langle \xi\, g \rangle$, and that every formally small term vanishes, all in the sense of distributions as $\epsilon \to 0$.*

Then g is the unique local infinitesimal Maxwellian determined by the solution (u, θ) of the Stokes system when $\delta_\epsilon/\epsilon \to 0$, the Navier-Stokes system when $\delta_\epsilon/\epsilon \to 1$ (or any finite value), or the Euler system when $\delta_\epsilon/\epsilon \to \infty$ with initial data (u^{in}, θ^{in}).

V. REMARKS

Finally, it should be pointed out that the above systems are not the only incompressible Stokes, Navier-Stokes, and Euler systems that may be derived as fluid dynamical limits of the Boltzmann equation. More refined asymptotic balances lead to incompressible Stokes, Navier-Stokes, and Euler systems that differ from those above in (1) the form of the heat equation and (2) the Boussinesq relation is replaced by $p = \rho + \theta$. These also have moment-based derivations. One should therefore be be careful about referring to “the incompressible Stokes system” (for example) until it is clear to which Stokes system you are referring.

In addition, there are other systems that have moment-based derivations.