Mathematical MCQ for international students admitted to École polytechnique

This multiple-choice questionnaire is intended for international students admitted to the first year of the engineering program of Ecole polytechnique. It will also be useful for French students who have not attended preparatory classes for Grandes Ecoles.

The aim of this MCQ is to provide an opportunity for these students to check their knowledge in some mathematical fields used during the engineering polytechnician cycle.

There are 20 exercises which deal with the following topics:

- Functions of one variable. Linear differential equations.
- Integration.
- Linear Algebra.
- Sequences and numerical series. Function sequences and series.

If a student meets some difficulties to answer or does not answer correctly, he should work by itself in order to better understand the corresponding notion.

The online course support (in french) available on this **webpage** (http://www.math.polytechnique.fr/ harinck/RappelsMathematiques.html) summarizes the most important tools concerning the different themes.

Some indications.

- **In Part I**, the questions invoke the themes of integration, sequences, numerical series, and function sequences and series. **There is only one good answer** to each question.
- **In Part II**, the themes evaluated are functions of one variable, function series, linear differential equations, linear algebra and integration. **There can be several good answers (at least one)** to each question. If one answer is omitted, the notion must be reviewed.
- Some exercises involve longer calculations. It is important to calculate quickly and accurately.
- **The answers** are given in this last page. However, we do not explain any one of them. The student have to look for the answers and to justify them by itself.

1 Part I

In this part, each question has only one good answer.

Exercise 1. *1.* Given a and b two real numbers such that $ab < 1$. The quantity $\arctan(a) + \arctan(b)$ is equal to :

- ***a.*** $\arctan\left(\frac{1}{1-ab}\right)$,
- ***b.*** $\arctan\left(\frac{a+b}{1-ab}\right)$,
- ***c.*** $\arctan\left(\frac{ab}{1-ab}\right)$,
- ***d.*** $\arctan\left(\frac{1+ab}{1-ab}\right)$.

2. Let $S = 2\arctan(\frac{1}{4}) + \arctan(\frac{1}{7}) + 2\arctan(\frac{1}{13})$. Then S is equal to

- ***a.*** $\frac{\pi}{3}$,
- ***b.*** $\frac{\pi}{6}$,
- ***c.*** $\frac{\pi}{4}$,
- ***d.*** $\frac{\pi}{5}$.

Exercise 2. Let $f(x) = \operatorname{argch}\big(\frac{1}{2}(x + \frac{1}{x})\big)$, where $x > 0$. We have for all $x > 0$

- ***a.*** $f(x) = \big|\log x\big|$,
- ***b.*** $f(x) = \log\left(x + \frac{1}{x}\right)$,
- ***c.*** $f(x) = \log\left(\sqrt{x + \frac{1}{x}}\right)$,
- ***d.*** $f(x) = \log\left(\sqrt{1 + \frac{1}{x^2}}\right)$.

Exercise 3. Let $a, b \in \mathbb{R}$ such that $a < b$ and f be a continuous function on $I = [a, b]$.

1. if f does not vanish at any points of I, then $\int_a^b f(t)dt \neq 0$.

- ***a.*** True,
- ***b.*** False.

2. Denote by u and v two differentiable functions on $\mathbb{R}$ with values in I. Let

$$\Phi(x) = \int_{u(x)}^{v(x)} f(t)dt.$$

Φ is differentiable and $\Phi'(x) = f(v(x)) - f(u(x))$ for all $x \in \mathbb{R}$.

- ***a.*** True,
- ***b.*** False.

3. Suppose that $I = [-1, 1]$, $v(x) = \cos x$, $u(x) = \sin x$ and $f(t) = \sqrt{1-t^2}$. Then

- ***a.*** $\Phi'(0) = 1$,
- ***b.*** $\Phi'(0) = 2$,
- ***c.*** $\Phi'(0) = 0$,
- ***d.*** $\Phi'(0) = -1$.

4. And

- ***a.*** $\Phi(0) = 1$,
- ***b.*** $\Phi(0) = \frac{1}{2}$,
- ***c.*** $\Phi(0) = \frac{\pi}{4}$,
- ***d.*** $\Phi(0) = \frac{1}{4}$.

Exercise 4. Let f be a $\mathcal{C}^\infty$ function on $\mathbb{R}$. We suppose that the asymptotic expansion of f in a neighborhood of 0 is given by

$$f(x) = 3 + 2x + x^2 + o(x^4).$$

1. We have

- ***a.*** $f'(0) = 1$,
- ***b.*** $f''(0) = 1$,
- ***c.*** $f''(0) = 2$,
- ***d.*** $f'(0) = -2$.

2. Let g be a $\mathcal{C}^\infty$ function on $\mathbb{R}$. We suppose that its asymptotic expansion at 0 is given by

$$g(x) = 1 - 2x + x^3 + o(x^4).$$

We have

- ***a.*** $f(x) + g(x) = 4 + x^2 + o(x^4)$,
- ***b.*** $f(g(x)) = 3 - 4x + x^2 + x^3 + o(x^4)$,
- ***c.*** $f(x) - g(x) = 2 + 4x + x^2 - x^3 + o(x^4)$,
- ***d.*** $2f(x) = 6 - 4x + 2x^2 + o(x^4)$.

3. Let F be the primitive of f such that $F(0) = 1$. Then

- ***a.*** $F(x) = 1 + 3x + x^2 + x^3 + o(x^4)$,
- ***b.*** $F'''(0) = 0$,
- ***c.*** $F''(0) = 1$,
- ***d.*** $F(x) = 1 + 3x + x^2 + o(x^2)$.

Exercise 5. ***1.*** The integral $\int_0^1 \frac{2x}{x^2+x+1}dx$ is equal to

a. $\log 3$,

b. $\log 3 - \frac{\pi\sqrt{3}}{9}$,

c. $\log 3 + \frac{\sqrt{3}}{9}$,

d. $\log 3 - \frac{\sqrt{3}}{9}$.

2. The integral $\int_0^1 \frac{dx}{x^4+1}$ is equal to

a. $\log(3+\sqrt{2})$,

b. $\sqrt{2}\log(3+\sqrt{2})$,

c. $\frac{\sqrt{2}}{8}(\log(3+\sqrt{2})+\pi)$,

d. $\frac{\sqrt{2}}{8}\log(3+\sqrt{2})+\pi$.

Exercise 6. For x a real number, we set $\Gamma(x) = \int_0^{+\infty} e^{-t}t^{x-1}dt$. The maximal domain where the function Γ is well defined is

a. $\mathbb{R}$,

b. $]0, +\infty[$,

c. $[0, +\infty[$,

d. $[1, +\infty[$.

Exercise 7. Let $a < b \in \mathbb{R}$, $I \subset \mathbb{R}$ and $f : I \times [a, b] \to \mathbb{R}$ be a continuous function. We set $F(x) = \int_a^b f(x,t)dt$. Without any additional assumption, we claim:

1. F is continuous on I.

a. True,

b. False.

2. If $x \mapsto f(x,t)$ is $\mathcal{C}^1$ on I for all $t \in [a, b]$, then F is also $\mathcal{C}^1$ on I.

a. True,

b. False.

3. Let $I = [0, 1]$. We consider a continuous function $g : [a, b] \longrightarrow \mathbb{R}$ which admits a primitive G such that $G(a) = 0$ and $G(b) = 1$. We set $f(x,t) = \frac{(x-t)^2}{2}g(t)$ and we assume that $F(0) = F'(0) = 0$. Then $F(x) = x^2$.

a. True,

b. False.

Exercise 8. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with values in $\mathbb{R}$.

1. If $\lim\limits_{n\to\infty}(nu_n)$ exists, then $\lim\limits_{n\to\infty} u_n = 0$.

- ***a.*** True,
- ***b.*** False.

2. If $\limsup\limits_{n} u_n = 0$, then $u_n \leq 0$ for all $n \in \mathbb{N}$.

- ***a.*** True,
- ***b.*** False.

3. $\lim\limits_{n\to+\infty}\lim\limits_{m\to+\infty}\left(1-\frac{1}{n}\right)^m = \lim\limits_{m\to+\infty}\lim\limits_{n\to+\infty}\left(1-\frac{1}{n}\right)^m$.

- ***a.*** True,
- ***b.*** False.

4. Suppose that $\lim\limits_{n\to\infty} u_n = \pi$, then $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n} u_n^2 = \pi^2$.

- ***a.*** True,
- ***b.*** False.

5. We define the sequence $(u_n)_{n\in\mathbb{N}}$ by $u_{n+1} = \frac{u_n}{2} + \frac{1}{u_n}$ and $u_0 = 1$. Then, $\sqrt{3}$ is a cluster point of the sequence $(u_n)_{n\in\mathbb{N}}$.

- ***a.*** True,
- ***b.*** False.

6. If $\lim\limits_{n\to\infty}|u_n|$ exists, then (u_n) has at most two cluster points.

- ***a.*** True,
- ***b.*** False.

7. If $u_n > 0$ and $\lim\limits_{n\to+\infty} u_n = 0$ then the series $\sum(-1)^n u_n$ is convergent.

- ***a.*** Always true,
- ***b.*** Always false,
- ***c.*** True with additional assumption.

Exercise 9. *1.* Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions defined on $[0,1]$ for $n \geq 1$ by

$$f_n(x) = \frac{ne^{-x} + x^2}{n+x}.$$

Then the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $[0,1]$.

- ***a.*** true
- ***b.*** false

2. Let $(g_n)_{n\in\mathbb{N}}$ be the sequence of functions defined on $[0,1]$ for $n \geq 1$ by $g_n(x) = \dfrac{n}{1+nx}$. Then the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly convergent on $[0,1]$.

- ***a.*** true
- ***b.*** false

3. Let $(h_n)_{n\in\mathbb{N}}$ be the sequence of functions defined on $[0,+\infty[$ for $n \geq 1$ by

$$h_n(x) = \begin{cases} \frac{x}{n^2} & \text{if } x \in [0,n] \\ \frac{2n-x}{n^2} & \text{if } x \in]n,2n[\\ 0 & \text{otherwise} \end{cases}$$

Then the sequence $(h_n)_{n\in\mathbb{N}}$ is uniformly convergent on $[0,+\infty[$.

- ***a.*** true
- ***b.*** false

4. Taking h_n as above, we have $\displaystyle\lim_{n\to\infty}\int_0^{+\infty} h_n(x)dx = \int_0^{+\infty}\lim_{n\to\infty} h_n(x)dx$.

- ***a.*** true
- ***b.*** false

Exercise 10. *1.* Let $\alpha \in]0,\pi[$. Then, the series $\displaystyle\sum_{n\geq 1}\frac{\sin(nx)}{n}$ is uniformly convergent on $[\alpha, 2\pi-\alpha]$.

- ***a.*** True,
- ***b.*** False.

2. We consider a power series $\sum_n a_n x^n$ with convergence radius R. If the sequence $(\left|\frac{a_n}{a_{n+1}}\right|)_{n\in\mathbb{N}}$ is convergent then $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$.

- ***a.*** True,
- ***b.*** False.

3. The convergence radius of the power series $\displaystyle\sum_n \left(\frac{n}{n+1}\right)^{n^2} x^n$ is

- ***a.*** 1,
- ***b.*** e,
- ***c.*** $\frac{1}{e}$,
- ***d.*** 0.

2 Part II

In this part, the questions can have several good answers (at least one). Give all the good answers.

Exercise 11. *1.* Let F, G and H be subspaces of a vector space E. If $E = F \oplus G$ and $E = F \oplus H$ (direct sums), then $G = H$.

- ***a.*** True,
- ***b.*** False.

2. We consider $E = \mathcal{C}^\infty(\mathbb{R})$. Let G be the space of affine functions of E and $H = \mathrm{Span}_{\mathbb{R}}(\cos, \sin)$ be the vector subspace of E generated by the functions $x \mapsto \cos x$ and $x \mapsto \sin x$. We set

$$F = \left\{ f \in E | f(0) = f'(0) = 0 \right\}.$$

We claim

- ***a.*** $F \oplus G = E$,
- ***b.*** $F \oplus H = E$,
- ***c.*** $G = H$,
- ***d.*** For any $f \in E$, if $f \notin F$, then $f \in G$.

Exercise 12. Let

$$S = \left\{ P \in \mathbb{R}[X], P(X^2) = (X^2 + 1)P \right\}.$$

- ***a.*** There exists P in S such that $\deg(P) = 3$,
- ***b.*** $S = \{0\}$,
- ***c.*** $\dim(S) = 1$,
- ***d.*** $\forall d \in \mathbb{N}, \exists P \in S$ such that $\deg(P) = d$.

Exercise 13. *1.* The matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is diagonalizable.

- ***a.*** True,
- ***b.*** False

2. A is nilpotent of index (the minimum such that $A^k = 0$) :

- ***a.*** 1,
- ***b.*** 2,
- ***c.*** 3,
- ***d.*** 4.

Exercise 14. Let $b = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$. We consider the other basis $b' = (e'_1, e'_2, e'_3)$ of $\mathbb{R}^3$ given by

$$\begin{cases} e'_1 = e_1 - e_2 + e_3 \\ e'_2 = e_1 + e_2 - e_3 \\ e'_3 = e_1 - e_3. \end{cases}$$

Recall that the change of basis matrix from b to b' is

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

Let $x \in \mathbb{R}^3$, $x = x_1e_1 + x_2e_2 + x_3e_3 = x'_1e'_1 + x'_2e'_2 + x'_3e'_3$ and denote by X and X' the matrices :

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

We have

a. $X' = PX$,

b. $\det(P) = \det(P^{-1})$,

c.

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 1 & \frac{1}{2} \\ 0 & -1 & -1 \end{pmatrix},$$

d. $X = PX'$.

Exercise 15. ***1.*** We consider the matrix

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}.$$

a. A is diagonalizable,

b. A is not invertible,

c. A has a double eigenvalue,

d. $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ is an eigenvector.

2. We consider the matrices

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 2 & 4 & 2 \\ -1 & 0 & 3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

We have

a. $A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$,

b. $A = PDP^{-1}$ with $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$,

c. $A = P^{-1}DP$ with $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$,

d. $A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

Exercise 16. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a differentiable function such that

$$f'(x) = f(\pi - x). \tag{1}$$

We deduce that

a. $f \in \mathcal{C}^\infty(\mathbb{R})$,

b. f is a solution of $y'' - y = 0$,

c. The set of solutions of (**??**) is exactly $\{f(x) = \lambda\cos(x) + \mu\sin(x) | \lambda, \mu \in \mathbb{R}\}$,

d. The set of solutions of (**??**) is exactly $\{f(x) = \lambda(\cos(x) - \sin(x)) | \lambda \in \mathbb{R}\}$.

Exercise 17. *1.* The function $x \longmapsto 1/x^\alpha$ is integrable on $]0, +\infty[$ for

a. $\alpha < 1$,

b. $\alpha > 1$,

c. none $\alpha \in \mathbb{R}$,

d. every $\alpha \in \mathbb{R}$.

2. The function $x \longmapsto 1/(1+x)^\alpha$ is integrable on $]0, +\infty[$ for

a. $\alpha < 1$,

b. $\alpha > 1$,

c. none $\alpha \in \mathbb{R}$,

d. every $\alpha \in \mathbb{R}$.

3. Let $f :]0, +\infty[\longmapsto \mathbb{R}$ be a continuous function. If f is integrable on $]0, +\infty[$ then $\lim\limits_{x \to +\infty} f(x) = 0$.

a. True,

b. False.

4. Let $f :]0,+\infty[\longmapsto \mathbb{R}$ be a real function defined by :

$$f(x) = \begin{cases} n^2 x - n^3 + 1 \text{ si } x \in \left[n - \dfrac{1}{n^2}, n\right] \text{ for } n \geqslant 2 \\ -n^2 x + n^3 + 1 \text{ si } x \in \left[n, n + \dfrac{1}{n^2}\right] \text{ for } n \geq 2 \\ 0 \text{ otherwise.} \end{cases}$$

a. f is continuous,

b. $\lim_{x \to +\infty} f(x) = 0$,

c. f is integrable on $]0, +\infty[$,

d. f is differentiable on $]0, +\infty[$.

Exercise 18. Let I an interval of $\mathbb{R}$ (finite or infinite). Let $\big(u_n(x)\big)_n$ be a sequence of functions defined over I with values in $\mathbb{R}$. Which of the following assertions are true:

a. If $\sum_n u_n(x)$ is uniformly convergent on I then $\sum_n u_n(x)$ is simply convergent on I,

b. If $\sum_n u_n(x)$ is normally convergent on I, then $\sum_n u_n(x)$ is uniformly convergent on I,

c. If $\sum_n u_n(x)$ is absolutely convergent on I then $\sum_n u_n(x)$ is uniformly convergent on I,

d. If $\sum_n u_n(x)$ is uniformly convergent on each segment included in I, then $\sum_n u_n(x)$ is uniformly convergent on I,

e. If $\sum_n u_n(x)$ is uniformly convergent on each segment included in I, then $\sum_n u_n(x)$ is simply convergent on I.

Exercise 19. We consider the following linear differential equation on $\mathbb{R}$

$$y' + y = xe^{-x}. \tag{2}$$

a. There exists a unique solution of (**??**),

b. The solutions of (**??**) are exactly the functions of the form

$$x \mapsto \left(\frac{1}{2}x^2 + K\right) e^{-x},\ K \in \mathbb{R},$$

c. The solutions of (**??**) are exactly the functions of the form

$$x \mapsto (x + K)\, e^{-x},\ K \in \mathbb{R},$$

d. The solutions of (**??**) are exactly the functions of the form

$$x \mapsto \left(\frac{1}{2}x^2 + 3 + K\right) e^{-x},\ K \in \mathbb{R}.$$

Exercise 20. We consider the following linear differential equation on $\mathbb{R}$

$$y'' - 4y' + 3y = (2x+1)e^{-x} \tag{3}$$

and we denote by S the set of solutions of (**??**).

a. $\{f(0), f \in S\} = \mathbb{R}$,

b. The functions in S such that $f(0) = 5/16$ are exactly the functions of the form

$$x \mapsto \left(\frac{x}{4} + 5/16\right) e^{-x} + K(e^x - e^{3x}),\ \ K \in \mathbb{R},$$

c. The functions in S such that $f(0) = 5/16$ are exactly the functions of the form

$$x \mapsto \left(\frac{x}{4} + 5/16\right) e^{-x} + K(e^x + e^{3x}),\ \ K \in \mathbb{R},$$

d. $\dim(S) = 1$.

Answers

Part I		**Part II**
Exercise 1.		**Exercise 11.**
1. b		1. b
2. c		2. a and b
Exercise 2.		**Exercise 12.**
a		c
Exercise 3.		**Exercise 13.**
1. a		1. b
2. b		2. c
3. d		**Exercise 14.**
4. c		d
Exercise 4.		**Exercise 15.**
1. c		1. a and d
2. c		2. b and d
3. d		**Exercise 16.**
Exercise 5.		a and d
1. b		**Exercise 17.**
2. c		1. c
Exercise 6.		2. b
b		3. b
Exercise 7.		4. a and c
1. a		**Exercise 18.**
2. b		a, b and e
3. b		**Exercise 19.**
Exercise 8.		b and d
1. a		**Exercise 20.**
2. b		a and b
3. b		
4. a		
5. b		
6. a		
7. c		
Exercise 9.		
1. a		
2. b		
3. a		
4. b		
Exercise 10.		
1. a		
2. a		
3. b		