Relative trace formula for compact quotient and pseudocoefficients for relative discrete series.

P. Delorme*, P. Harinck

Abstract

We introduce the notion of relative pseudo-coefficient for relative discrete series representations of real spherical homogeneous spaces of reductive groups. We prove that K-finite relative pseudo-coefficient does not exist for semisimple symmetric spaces of type $G_{\mathbb{C}}/G_{\mathbb{R}}$, where K is a maximal compact subgroup of $G_{\mathbb{C}}$, and construct strong relative pseudo-coefficients for some hyperbolic spaces. We establish a toy model for the relative trace formula of H. Jacquet for compact discrete quotient $\Gamma\backslash G$. This allows us to prove that a relative discrete series representation, which admits strong pseudo-coefficients with sufficiently small support, occurs in the spectral decomposition of $L^2(\Gamma\backslash G)$ with a nonzero period.

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1 Introduction

The notion of pseudo-coefficient for a discrete series representation was first introduced in the group case, i.e $X = G \times G/\mathrm{diag}(G) \simeq G$, where G is the group of real points of any connected reductive algebraic group defined over $\mathbb{R}$. It is defined as follows: let μ be the Plancherel measure of G. Given a discrete series representation π of G, a test function $\phi \in C_c^\infty(G)$ is said to be a pseudo-coefficient of π if for μ-almost irreducible tempered representation ρ, then $\mathrm{tr}\ \rho(\phi) \neq 0$ if and only if ρ is equivalent to π. Here $\mathrm{tr}\ \rho(\phi)$ is the trace of the trace operator $\rho(\phi)$. In ([7] Proposition 1), it is shown that every discrete series representation of G admits pseudo-coefficients.

In this article, we consider real spherical homogeneous spaces, that is homogeneous spaces for a real reductive group G, with an open orbit for any minimal parabolic subgroup. On such spaces, we define the notion of relative pseudo-coefficient for relative discrete series representation (see Definition 3.1).

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In the case of semisimple symmetric spaces of type $G_{\mathbb{C}}/G_{\mathbb{R}}$, we show, using properties of orbital integrals (see [6] and [17]), that no relative discrete series representation admits K-finite relative pseudo-coefficient, where K is a maximal compact subgroup of $G_{\mathbb{C}}$ (see Theorem 6.2).

We look also at hyperbolic spaces over $\mathbb{C}$ and over the quaternions $\mathbb{H}$, i.e $\mathbb{X} = G/H$, where $G = U(p, q, \mathbb{K})$ and $H = U(1, \mathbb{K}) \times U(p-1, q, \mathbb{K})$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$.

For the rest of this introduction we fix such a G and H. Using results of J. Faraut, M. Flensted-Jensen and K. Okamoto and results of [10], we show that some relative discrete series representations of these spaces do not have relative pseudo-coefficients (see Proposition 4.1), but there exists a countable family of relative discrete series representations having what we call strong relative pseudo-coefficients i.e. which isolate a single relative discrete series representation among the irreducible unitary representations of the group G having a nonzero H-fixed distribution vector (see Theorem 4.8).

We give an application of the existence of these strong relative pseudo-coefficients, whose support are arbitrary small, to existence of some representations occurring in the spectral decomposition of $L^2(\Gamma\backslash G)$ where Γ is a torsion free cocompact discrete subgroup of G, stable by the involution σ whose fixed point group is H (see Theorem 5.7). Existence of such cocompact discrete subgroups is shown using adelic methods (see Proposition 5.6).

For this application, we establish a toy model for the more sophisticated relative trace formula of H. Jacquet ([18]).

Let ξ_Γ be the distribution vector for the right regular representation of G in $L^2(\Gamma\backslash G)$ given by integration over $\Gamma \cap H\backslash H$, which is compact in our case. We denote by $c_{\xi_\Gamma,\xi_\Gamma}$ the corresponding generalized matrix coefficient.

Then it is easy to give two expressions of $c_{\xi_\Gamma,\xi_\Gamma}(f)$ for $f \in C_c^\infty(\mathbb{X})$. One expression, which is called spectral, involves periods of representations, i.e. H-fixed distribution vectors of irreducible subrepresentations of $L^2(\Gamma\backslash G)$, the other, called geometric, involving relative orbital integrals, i.e. the average of f on orbits of elements of Γ under the action of $H \times H$. We call the equality of these 2 expressions a relative trace formula.

When one plugs a strong relative pseudo-coefficient into the spectral side of the relative trace formula, it singles out the contribution of the relative discrete series representation. On the geometric side, as we can find strong relative pseudo-coefficient with sufficiently small support, we get only the contribution of the neutral element and this contribution is equal to $f(1)$.

Altogether, it shows that the relative discrete series representation with strong relative pseudo-coefficients occurs in $L^2(\Gamma\backslash G)$.

This paper is organized as follows: In section 2 we prove a relative trace formula for $\Gamma\backslash G$ when H is a unimodular closed subgroup of G, Γ is a cocompact discrete subgroup of G such that the volume of $\Gamma \cap H\backslash H$ is finite and the centralizers of elements of Γ in $H \times H$ are unimodular. In section 3, we introduce the notion of relative pseudo-coefficient and prove our application of existence of strong relative pseudo-coefficient when H is the fixed point group of an involution of G. In section 4, we explain our results (existence or non existence of strong relative pseudo-coefficient for relative discrete series representations) for hyperbolic spaces. In section 5, we construct cocompact discrete subgroups of unitary groups on $\mathbb{C}$ and $\mathbb{H}$ satisfying our assumptions and section 6 is devoted to prove that no relative discrete series representation of $G_{\mathbb{C}}/G_{\mathbb{R}}$ admits K-finite relative pseudo-coefficient.

2 A relative trace formula for $\Gamma\backslash G$.

If M is a differentiable manifold, then $C(M)$ and $C^\infty(M)$ will denote the space of continuous functions and smooth functions on M respectively. Let $C_c(M)$ and $C_c^\infty(M)$ be the subspaces of compactly supported functions in $C(M)$ and $C^\infty(M)$ respectively.

Let G be a real reductive group. We consider a discrete cocompact subgroup Γ of G and a closed subgroup H. We assume that

1. H is unimodular,
2. the volume $vol((H\cap\Gamma)\backslash H)$ of $(H\cap\Gamma)\backslash H$ is finite, (2.1)
3. for each $\gamma\in\Gamma$, the subgroup $(H\times H)_\gamma=\{(h,h')\in H\times H; h^{-1}\gamma h'=\gamma\}$ is unimodular.

We set $\Gamma_H := H\cap\Gamma$. We fix Haar measures on these groups, discrete groups being equipped with the counting measure.

If V is a topological vector space then V' will denote its topological dual.
Let (π, V) a continuous representation of G in a Hilbert space V. We denote by $V^\infty\subset V$ the space of C^∞ vectors of π endowed with its natural topology (For the topology of V^∞, see [4] section 2.4.3 and Lemma 2.15 where two equivalent definitions are given). We define the space of distribution vectors $V^{-\infty}$ as the topological dual of V^∞. Let π_∞ be the representation of G in V^∞, and $\pi_{-\infty}$ be the dual representation of π_∞ in $V^{-\infty}$.
If $f\in C_c^\infty(G)$ and $\xi\in V^{-\infty}$, we have $\pi_{-\infty}(f)\xi\in(V')^\infty$. Hence, if $\xi'\in(V')^{-\infty}$, we can define the distribution $m_{\xi,\xi'}$ by

$$m_{\xi,\xi'}(f)=\langle\pi_{-\infty}(f)\xi,\xi'\rangle,\quad f\in C_c^\infty(G).$$

If (π, V) is unitary for a scalar product $(\cdot,\cdot)$, then the map $j: v\mapsto(\cdot,v)$, intertwines the conjugate representation $(\overline{\pi},\overline{V})$ of (π,V) and its dual representation (π',V'). Let ξ_1 and ξ_2 be two elements of $V^{-\infty}$. We define $\overline{\xi_2}\in\overline{V}^{-\infty}$ by $\overline{\xi_2}(w)=\overline{\xi_2(w)}$. By the above identification, we can consider $\overline{\xi_2}$ as an element of $(V')^{-\infty}$. Thus we can define

$$c_{\xi_1,\xi_2}(f)=m_{\xi_1,\overline{\xi_2}}(f).$$

Let us explain what happens if ξ_1 and ξ_2 are elements of V'. Notice that V' inherits of a natural scalar product from the one of V. Then $\langle\xi_1,\overline{\xi_2}\rangle$ is just the scalar product (ξ_1,ξ_2). This leads to $c_{\xi_1,\xi_2}(f)=(\pi'(f)\xi_1,\xi_2)$. In other words, $c_{\xi_1,\xi_2}(f)$ is an ordinary matrix coefficient.

We will call in general c_{ξ_1,ξ_2} the generalized matrix coefficient of $\xi_1,\xi_2\in V^{-\infty}$ and write :

$$(\pi_{-\infty}(f)\xi_1,\xi_2):=c_{\xi_1,\xi_2}(f)=\langle\pi_{-\infty}(f)\xi_1,\overline{\xi_2}\rangle. \tag{2.2}$$

We consider the right regular representation R of G in $L^2(\Gamma\backslash G)$.
Then, for $f\in C_c(G)$, the corresponding operator $R(f)$ maps any function $\psi\in L^2(\Gamma\backslash G)$ to the function

$$[R(f)\psi](x)=\int_G f(g)\psi(xg)dg=\int_G f(x^{-1}y)\psi(y)dy$$

$$= \int_{\Gamma\backslash G} \psi(y) K_f(x,y)\, d\dot{y}$$

where

$$K_f(x,y) := \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y), \quad x, y \in G, \tag{2.3}$$

and this sum has only a finite number of nonzero terms for x, y contained in a compact subset of G since f is compactly supported and Γ is discrete.

Therefore, $R(f)$ is an operator with continuous kernel K_f.

We define the H-invariant linear form ξ_Γ on $C(\Gamma\backslash G)$, which contains $L^2(\Gamma\backslash G)^\infty = C^\infty(\Gamma\backslash G)$ (see [25] Theorem 5.1), by

$$\xi_\Gamma(\psi) = \int_{\Gamma_H\backslash H} \psi(h) d\dot{h}.$$

Then, the generalized matrix coefficient

$$c_{\xi_\Gamma,\xi_\Gamma}(f) := (R_{-\infty}(f)\xi_\Gamma, \xi_\Gamma), \quad f \in C_c^\infty(G)$$

associated to ξ_Γ according to (2.2) is an H-biinvariant distribution on G.

The relative trace formula in this context gives two expressions of the distribution $c_{\xi_\Gamma,\xi_\Gamma}$, the first one, called the spectral side, in terms of irreducible representations of G, and the second one, called the geometric side, in terms of orbital integrals.

We first deal with the spectral part. For this purpose, we consider the spectral decomposition of $L^2(\Gamma\backslash G)$:

$$L^2(\Gamma\backslash G) = \oplus_{\pi\in\hat{G}} \mathcal{H}_\pi \otimes \mathcal{M}_\pi,$$

where $\hat{G}$ is the set of equivalent classes of irreducible unitary representations $(\pi, \mathcal{H}_\pi)$ of G and $\mathcal{M}_\pi$ is a finite dimensional vector space whose dimension is the multiplicity of π in $L^2(\Gamma\backslash G)$ (For finite multiplicities, see [13] §2. 3 Theorem).

Then the space $V_\pi := \mathcal{H}_\pi \otimes M_\pi$ is the π- isotypic component of $L^2(\Gamma\backslash G)$. We denote by $\xi_{\Gamma,\pi}$ the restriction of ξ_Γ to V_π. Therefore, we obtain

$$c_{\xi_\Gamma,\xi_\Gamma}(f) = \sum_{\pi\in\hat{G}} c_{\xi_{\Gamma,\pi},\xi_{\Gamma,\pi}}(f), \quad f \in C_c^\infty(G). \tag{2.4}$$

For $\gamma \in \Gamma$, we define the groups

$$(H\times H)_\gamma = \{(h_1,h_2) \in H\times H;\, h_1^{-1}\gamma h_2 = \gamma\}, \quad \text{and} \quad (\Gamma_H\times\Gamma_H)_\gamma = (H\times H)_\gamma \cap (\Gamma\times\Gamma).$$

2.1 Proposition. *1. For $\gamma \in \Gamma$, the quotient $(\Gamma_H\times\Gamma_H)_\gamma\backslash(H\times H)_\gamma$ is of finite volume and for $f \in C_c^\infty(G)$, the orbital integral of f at γ*

$$I(f,\gamma) := \int_{(H\times H)_\gamma\backslash(H\times H)} f(h_1^{-1}\gamma h_2) dh_1\, dh_2$$

is absolutely convergent.

2. *We have the following relative trace formula*

$$\sum_{\gamma\in\Gamma_H\backslash\Gamma/\Gamma_H} vol((\Gamma_H\times\Gamma_H)_\gamma\backslash(H\times H)_\gamma)\, I(f,\gamma) = \sum_{\pi\in\hat{G}} c_{\xi_{\Gamma,\pi},\xi_{\Gamma,\pi}}(f), \tag{2.5}$$

where the left hand side is absolutely convergent.

Proof. The right hand side of (2.5) is just the expression of $c_{\xi_\Gamma,\xi_\Gamma}(f)$ given in (2.4). For the geometric side, we will express $c_{\xi_\Gamma,\xi_\Gamma}(f)$ in terms of the kernel K_f.

Let $f\in C_c(G)$. We first compute $R_{-\infty}(f)\xi_\Gamma$. For this, we use the bilinear duality bracket $\langle\cdot,\cdot\rangle$ between $V^{-\infty}$ and V^∞ where $V = L^2(\Gamma\backslash G)$. We define $\check{f}$ by $\check{f}(x) = f(x^{-1})$. Then, for $\psi\in C^\infty(\Gamma\backslash G)$, we have

$$\langle R_{-\infty}(f)\xi_\Gamma,\psi\rangle = \langle\xi_\Gamma, R(\check{f})\psi\rangle = \int_{\Gamma_H\backslash H}\Big(\int_{\Gamma\backslash G}\psi(y)K_{\check{f}}(h,y)d\dot{y}\Big)d\dot{h}.$$

The kernel $K_{\check{f}}$ is continuous and $|K_{\check{f}}|\leqslant K_{|\check{f}|}$. Applying the above equality to $|\psi|$ and $|f|$, one sees that the double integral on the right side is absolutely convergent and we can apply Fubini's Theorem. Thus we obtain

$$\langle R_{-\infty}(f)\xi_\Gamma,\psi\rangle = \int_{\Gamma\backslash G}\psi(y)\Big(\int_{\Gamma_H\backslash H}K_{\check{f}}(h,y)d\dot{h}\Big)d\dot{y}.$$

We deduce that $R_{-\infty}(f)\xi_\Gamma$ is the continuous function on $\Gamma\backslash G$ given by

$$\big(R_{-\infty}(f)\xi_\Gamma\big)(y) = \int_{\Gamma_H\backslash H}\overline{K_{\check{f}}(h,y)}d\dot{h}.$$

Therefore, we can extend the map $\varphi\in C_c^\infty(G)\mapsto c_{\xi_\Gamma,\xi_\Gamma}(\varphi) = (R_{-\infty}(\varphi)\xi_\Gamma,\xi_\Gamma)$ to $C_c(G)$. Since $K_{\check{f}}(x,y) = K_f(y,x)$, we obtain for $f\in C_c(G)$

$$c_{\xi_\Gamma,\xi_\Gamma}(f) = \int_{\Gamma_H\backslash H}\Big(\int_{\Gamma_H\backslash H}K_f(x,y)d\dot{x}\Big)d\dot{y} = \int_{\Gamma_H\backslash H}\Big(\int_{\Gamma_H\backslash H}\sum_{\gamma\in\Gamma}f(h_1^{-1}\gamma h_2)dh_1\Big)\,dh_2. \tag{2.6}$$

For $(h_1,h_2)\in H\times H$, we have

$$\sum_{\gamma\in\Gamma}f(h_1^{-1}\gamma h_2) = \sum_{[\gamma]\in\Gamma_H\backslash\Gamma/\Gamma_H}\ \sum_{(\gamma_1,\gamma_2)\in(\Gamma_H\times\Gamma_H)_\gamma\backslash(\Gamma_H\times\Gamma_H)} f(h_1^{-1}\gamma_1^{-1}\gamma\gamma_2h_2), \tag{2.7}$$

where the sum has only a finite number of nonzero terms.
Applying (2.6) and (2.7) to $|f|$ and using first Fubini's Theorem for positive functions and then for integrable functions, we obtain

$$c_{\xi_\Gamma,\xi_\Gamma}(f) = \sum_{[\gamma]\in\Gamma_H\backslash\Gamma/\Gamma_H}\int_{\Gamma_H\backslash H}\int_{\Gamma_H\backslash H}\sum_{(\gamma_1,\gamma_2)\in(\Gamma_H\times\Gamma_H)_\gamma\backslash(\Gamma_H\times\Gamma_H)} f(h_1^{-1}\gamma_1\gamma\gamma_2h_2)dh_1\,dh_2$$

$$= \sum_{[\gamma]\in\Gamma_H\backslash\Gamma/\Gamma_H} \int_{(\Gamma_H\times\Gamma_H)_\gamma\backslash H\times H} f(h_1^{-1}\gamma h_2)d(h_1,h_2), \tag{2.8}$$

the integral and the sum being absolutely convergent.

As by assumption the group $(H\times H)_\gamma$ is unimodular , using the transitivity property of invariant measures on homogeneous spaces (see [3] Chap. II, §3), we have

$$\begin{aligned} &\int_{(\Gamma_H\times\Gamma_H)_\gamma\backslash H\times H} f(h_1^{-1}\gamma h_2)d(h_1,h_2) \\ &= \int_{(H\times H)_\gamma\backslash H\times H}\Big(\int_{(\Gamma_H\times\Gamma_H)_\gamma\backslash(H\times H)_\gamma} f(h_1^{-1}u_1^{-1}\gamma u_2h_2)d(u_1,u_2)\Big)d(h_1,h_2) \\ &= vol((\Gamma_H\times\Gamma_H)_\gamma\backslash(H\times H)_\gamma)\int_{(H\times H)_\gamma\backslash H\times H} f(h_1^{-1}\gamma h_2)dh_1\,dh_2. \end{aligned} \tag{2.9}$$

We deduce from this that the volume $vol((\Gamma_H\times\Gamma_H)_\gamma\backslash(H\times H)_\gamma)$ is finite. Applying the above equality to $|f|$, we deduce that the orbital integral $I(\gamma,f)$ of $f\in C_c(G)$ at γ is absolutely converging. Thus we obtain the first assertion of the Proposition.

Therefore (2.8) and (2.9) give

$$c_{\xi_\Gamma,\xi_\Gamma}(f) = \sum_{[\gamma]\in\Gamma_H\backslash\Gamma/\Gamma_H} vol((\Gamma_H\times\Gamma_H)_\gamma\backslash(H\times H)_\gamma)\int_{(H\times H)_\gamma\backslash H\times H} f(h_1^{-1}\gamma h_2)d(h_1,h_2).$$

Then the relative trace formula follows from (2.4). □

3 Relative pseudo-coefficients with small support.

To define relative pseudo-coefficients for relative discrete series representation, we need to review the abstract Plancherel formula. Here, we assume that G is a real reductive group and H is a spherical subgroup, ie. G/H admits an open orbit for any minimal parabolic subgroups, which ensures finite multiplicities in the Plancherel formula (see [21]Theorem 3.2 and [19] Theorem A).

We denote by $\widehat{G}$ the unitary dual of G and pick for every equivalence class $[\pi]$ a representative $(\pi,\mathcal{H}_\pi)$. We keep the notations of section 2.

The abstract Plancherel formula Theorem for the spherical variety $\mathcal{Z}:=G/H$ asserts the following. For every $[\pi]\in\hat{G}$, there exists a Hilbert space $\mathcal{M}_{\overline{\pi}}\subset(\overline{\mathcal{H}_\pi}^{-\infty})^H$ (note that as $\mathcal{M}_{\overline{\pi}}$ is finite dimensional, this induces a Hilbert space structure on $\mathrm{Hom}(\mathcal{M}_{\overline{\pi}},\mathcal{H}_\pi)\simeq\mathcal{M}_{\overline{\pi}}^*\otimes\mathcal{H}_\pi$ and $\mathcal{M}_\pi=\mathcal{M}_{\overline{\pi}}^*\subset(\mathcal{H}_\pi^{-\infty})^H$), such that the Fourier transform

$$\mathcal{F}:\begin{cases} C_c^\infty(G) & \to \int_{\hat{G}}^{\oplus}\mathrm{Hom}(\mathcal{M}_{\overline{\pi}},\mathcal{H}_\pi)d\mu(\pi), \\ F & \mapsto \mathcal{F}(F)=\big(\mathcal{F}(F)_\pi\big)_{\pi\in\hat{G}},\quad \mathcal{F}(F)_\pi(\overline{\xi})=\pi(F)\overline{\xi}\in\mathcal{H}_\pi^\infty \end{cases} \tag{3.1}$$

extends to a unitary isomorphism. Here μ is a certain Radon measure on $\hat{G}$ whose measure class is uniquely determined. The precise form of the measure depends on the chosen scalar products on the various stalks $\mathrm{Hom}(\mathcal{M}_{\overline{\pi}}, \mathcal{H}_\pi)$. We have

$$\|F\|_{L^2(\mathcal{Z})} = \int_{\hat{G}} H_\pi(F) d\mu(\pi)$$

where H_π are Hermitian forms which are defined as

$$H_\pi(F)^2 = \sum_{j=1}^{m_\pi} \|\overline{\pi}(F)\overline{\xi}_j\|^2_{\mathcal{H}_\pi}, \quad F \in C_c^\infty(\mathcal{Z}),$$

for $\overline{\xi}_1, \ldots, \overline{\xi}_{m_\pi}$ an orthonomal basis of $\mathcal{M}_{\overline{\pi}}$.

3.1 Definition. *Let $(\pi_0, \mathcal{H}_{\pi_0}) \in \hat{G}$ be a relative discrete series representation for $\mathcal{Z} = G/H$, ie. which admits an embedding in $L^2(G/H)$.*

1. *A function $f \in C_c^\infty(G/H)$ is a relative pseudo-coefficient for π_0 if*
   - *(a) there exists $\xi_0 \in \mathcal{M}_{\pi_0}$ such that $c_{\xi_0,\xi_0}(f) \neq 0$,*
   - *(b) for μ-almost all $\pi \in \hat{G}$ distinct from π_0 and for all $\xi \in \mathcal{M}_\pi$, then $c_{\xi,\xi}(f) = 0$.*
2. *Let $\xi_0 \in \mathcal{M}_{\pi_0}$. A function $f \in C_c^\infty(G/H)$ is a strong relative pseudo-coefficient for (π_0, ξ_0) if*
   - *(a) $c_{\xi_0,\xi_0}(f) \neq 0$ and $c_{a\xi_0+\xi, a\xi_0+\xi}(f) = a\bar{a}\, c_{\xi_0,\xi_0}(f)$, for $a \in \mathbb{C}$ and ξ in the orthogonal subspace of $\mathbb{C}\xi_0$ in $\mathcal{M}_{\pi_0}$,*
   - *(b) for any unitary irreducible representation $(\pi, \mathcal{H}_\pi)$ of G, non equivalent to π_0, and for all $\xi \in (\mathcal{H}_\pi^{-\infty})^H$, then $c_{\xi,\xi}(f) = 0$.*

The relative trace formula for $\Gamma\backslash G$ allows to determine, in some cases, if a relative discrete series representation π for G/H occurs in the spectral decomposition of $L^2(\Gamma\backslash G)$ and has a nonzero period (ie. $\xi_{\Gamma,\pi} \neq 0$).

3.2 Definition. *We say that $f \in C_c^\infty(G/H)$ has small support relative to Γ if $I(f, \gamma) \neq 0$ for $\gamma \in \Gamma$ implies that $\gamma \in \Gamma_H$.*

Let us assume that G/H has a relative discrete series representation $(\pi_0, \mathcal{H}_0)$. Then $\mathcal{H}_0$ can be realized as a subspace of $L^2(G/H)$ and the map $\xi_0 : \psi \in \mathcal{H}_0^\infty \to \psi(1)$ is an H-invariant distribution vector.

3.3 Proposition. *If there exists a strong relative pseudo-coefficient f for (π_0, ξ_0) with small support relative to Γ then π_0 occurs in $L^2(\Gamma\backslash G)$ with a nonzero period.*

Proof. By definition, if f is a strong relative pseudo-coefficient for (π_0, ξ_0) with small support relative to Γ, then the geometric side of the relative trace formula (2.5) is reduced to the term $f(1)$ and the spectral side to the term $c_{\xi_{\Gamma,\pi_0},\xi_{\Gamma,\pi_0}}(f) = (\pi_0(f)\xi_{\Gamma,\pi_0}, \xi_{\Gamma,\pi_0})$, hence we obtain the Proposition. □

We will precise the notion of small support relative to Γ in the case of symmetric spaces. We assume that H is the fixed point group of an involution σ of G.
Most of the results of harmonic analysis on real reductive symmetric spaces are available only when G is in the Harish-Chandra class (see [9] and [2]). From now, we will make this assumption on G.

3.4 Lemma. *If Γ is a σ-stable cocompact discrete subgroup of G then $\Gamma_H = \Gamma \cap H$ is a cocompact subgroup of H.*

Proof. Let (h_n) be a sequence of H. As $\Gamma\backslash G$ is compact, extracting possibly a subsequence of (h_n), we can find a sequence (γ_n) in Γ such that $(\gamma_n h_n)$ converges in G. Since Γ is σ-stable, the sequence of $\gamma_n\sigma(\gamma_n)^{-1} = \gamma_n h_n \sigma(\gamma_n h_n)^{-1}$ is a converging sequence in Γ. As Γ is discrete, there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$, we have $\gamma_n\sigma(\gamma_n)^{-1} = \gamma\sigma(\gamma)^{-1}$ where $\gamma := \gamma_{n_0} \in \Gamma$. This implies that $\gamma^{-1}\gamma_n \in \Gamma \cap H$ for $n \geqslant n_0$, and the sequence $(\gamma^{-1}\gamma_n h_n)$ converges. This proves that $\Gamma \cap H\backslash H$ is compact. □

Let θ be a Cartan involution of G which commutes with σ. Then $K := G^\theta$ is a maximal compact subgroup of G.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G in eigenspaces for θ and σ respectively.

We fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}\cap\mathfrak{q}$ and we denote by A the analytic subgroup of G with Lie algebra $\mathfrak{a}$. Then, we have the Cartan decompositions

$$G = K \exp \mathfrak{p} = KAH.$$

We fix a K-invariant norm $\|\cdot\|$ on $\mathfrak{p}$ and we define a K-invariant function τ on G by

$$\tau(k \exp X) = \|X\|, \quad k \in K, X \in \mathfrak{p}.$$

For $R > 0$, let $A_R := \{a \in A; \tau(a) < R\}$ be the ball of radius R in A.
We set

$$r_\Gamma := \inf_{g\in G, \gamma\in\Gamma-\{1\}} \tau(g^{-1}\gamma g).$$

3.5 Proposition. *Let G and σ be as above. Let Γ be a σ-stable cocompact discrete subgroup of G. Moreover, we assume that Γ is torsion-free. Then*

1. $r_\Gamma > 0$.
2. *Let $f \in C_c^\infty(G/H)$ be compactly supported in $KA_{r_\Gamma/2}H$. Then f has small support relative to Γ.*

Proof. 1. This property is asserted in [8]. We give a proof for sake of completeness.
If $r_\Gamma = 0$ then there would exist two sequences (g_n) of G and (γ_n) of Γ with $\gamma_n \neq 1$ for all $n \in \mathbb{N}$, such that $\tau(g_n^{-1}\gamma_n g_n)$ converges to 0. Then

$$g_n^{-1}\gamma_n g_n = k_n \exp X_n \text{ with } k_n \in K \text{ and } X_n \in \mathfrak{p} \text{ with } \lim_{n\to+\infty} \|X_n\| = 0. \tag{3.2}$$

As $\Gamma\backslash G$ is compact, possibly changing (g_n) and (γ_n) and extracting subsequences, we can assume that (g_n) converges to $g \in G$ and (k_n) converges to $k \in K$. Thus using (3.2), we see that the sequence (γ_n) converges, hence, as Γ is discrete, there exists n_0 such that γ_n is constant and equal to $\gamma := \gamma_{n_0}$ for $n \geqslant n_0$. Going to the limit in (3.2), we obtain $g^{-1}\gamma g = k$. Therefore γ belongs to the discrete compact, so finite, group $gKg^{-1} \cap \Gamma$. This implies that γ is a torsion element, thus $\gamma = 1$. This contradicts the hypothesis that all γ_n are distinct from 1. This proves the first assertion.

2. Let h_1, h_2 in H and $\gamma \in \Gamma$ such that $f(h_1^{-1}\gamma h_2) \neq 0$. Then, the point $g := h_1^{-1}\gamma h_2$ belongs to $KA_{r_\Gamma/2}H$. Therefore, we have $g\sigma(g)^{-1} \in KA_{r_\Gamma}K$, thus $\tau(g\sigma(g)^{-1}) = \tau(h_1^{-1}\gamma\sigma(\gamma)^{-1}h_1) < r_\Gamma$. By definition of r_Γ, this implies that $\gamma\sigma(\gamma)^{-1} = 1$, hence $\gamma \in \Gamma_H$.
□

4 Strong relative pseudo-coefficient for some hyperbolic spaces.

The aim of this part is to construct strong relative pseudo-coefficients associated to some relative discrete series representations of some hyperbolic spaces.

4.1 Preliminaries.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ be the classical field of real, complex numbers or quaternions respectively. Let $x \mapsto \bar{x}$ denotes the standard (anti-)involution of $\mathbb{K}$. Let $p > 2, q \geqslant 1$ be two integers. We consider the hermitian form $[\cdot,\cdot]$ on $\mathbb{K}^{p+q}$ given by

$$[x, y] = x_1\bar{y}_1 + \dots x_p\bar{y}_p - x_{p+1}\bar{y}_{p+1} - \dots - x_{p+1}\bar{y}_{p+q}, \quad (x, y) \in \mathbb{K}^{p+q}.$$

Let $G = U(p, q, \mathbb{K})$ denote the group of $(p+q) \times (p+q)$ matrices preserving $[\cdot,\cdot]$. Let $H = U(1, \mathbb{K}) \times U(p-1, q, \mathbb{K})$ be the stabilizer of $x_0 = (1, 0, \dots, 0) \in \mathbb{K}^{p+q}$ in G. Then H is the fixed-point group of the involution σ of G given by $\sigma(g) = JgJ$, where J is the diagonal matrix with entries $(-1, 1, \dots, , 1)$. The reductive symmetric space G/H (of rank 1) can be identified with the projective hyperbolic space $\mathbb{X} = \mathbb{X}(p, q, \mathbb{K})$ (see [1] §2.) :

$$\mathbb{X} = \{z \in \mathbb{K}^{p+q}; [z, z] = 1\}/\sim,$$

where $\sim$ is the equivalence relation $z \sim zu,\ u \in \mathbb{K}^*, |u| = 1$.

The group $K = K_1 \times K_2 = U(p, \mathbb{K}) \times U(q, \mathbb{K}) \subset G$ is the maximal compact subgroup of G consisting of elements fixed by the classical Cartan involution θ of G, $\theta(g) = (g^*)^{-1}$, which commutes with σ. Here g^* denotes the conjugate transpose of g.

Recall that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$ are the decompositions of the Lie algebra $\mathfrak{g}$ of G in eigenspaces for θ and σ respectively.

We define the one parameter abelian subgroup $A = \{a_t;\ t \in \mathbb{R}\}$ by

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{p+q-2} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$$

where I_j denotes the identity matrix of size j. Then, the Lie algebra $\mathfrak{a}$ of A is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Let W be the Weyl group of A in G. The nontrivial element of W acts on A by $a_t \mapsto a_{-t}$.

The Cartan decomposition $G = KAH$ holds and gives rise to the use of polar coordinates on $\mathbb{X}$ (see [1] §2):

$$(k,t) \in K \times \mathbb{R}^+ \mapsto ka_tH, \tag{4.1}$$

and the map $(k,t) \in K/K \cap M \times]0, +\infty[\mapsto ka_tH$ is a diffeomorphism on its image.

The centralizer M of A in H is the subgroup of matrices

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix}$$

where $u \in \mathbb{K}^*$, $|u| = 1$ and $v \in U(p-1, q-1, \mathbb{K})$.

Hence, the homogeneous space $K/K \cap M$ can be identified with the projective image $\mathbb{Y}$ of the product of unit spheres $\mathbb{S}^p(\mathbb{K}) \times \mathbb{S}^q(\mathbb{K})$:

$$K/K \cap M \simeq \mathbb{Y} = \{y \in \mathbb{K}^{p+q}; |y_1|^2 + \ldots + |y_p|^2 = |y_{p+1}|^2 + \ldots + |y_{p+q}|^2 = 1\}/\sim. \tag{4.2}$$

Let P be the subgroup of G which stabilizes the $\mathbb{K}$-line generated by $\gamma^0 = (1, 0, \ldots 0, 1)$. Then P is a maximal parabolic subgroup of G whose unipotent radical will be denoted by N, and we have $P = MAN$ ([11] V.1).

Let $d := \dim_{\mathbb{R}} \mathbb{K}$. We set $\rho := \frac{1}{2}(dq + dp) - 1$.

We recall some results about spherical distributions of positive type on $\mathbb{X}$ given in [11]. As $\mathbb{X}$ is a symmetric space of rank 1, the algebra of left G-invariant differential operators on $\mathbb{X}$ is generated by the Laplace-Beltrami operator Δ corresponding to the natural pseudo-Riemannian structure. The Laplace-Beltrami operator comes, up to a scalar, from the action of the Casimir of $\mathfrak{g}$ on $C_c^\infty(\mathbb{X})$.

We denote by $\mathcal{D}'_{s,H}(\mathbb{X})$ the space of spherical distributions Θ such that $\Delta\Theta = (s^2 - \rho^2)\Theta$.

We keep notation of §1 for representations.

For $s \in \mathbb{C}$, we define the character δ_s of P by $\delta_s(ma_tn) = e^{st}, m \in M, a_t \in A, n \in N$ and we denote by $(\pi_s := ind_P^G \delta_{s-\rho}, \mathcal{H}_s)$ the normalized induced representation.
For $s \in \mathbb{C}$, we denote by $\xi_s \in (\mathcal{H}_{-s}^{-\infty})^H$ the H-invariant distribution vector denoted by $u_s \in \mathscr{E}'_s(\Xi)$ in ([11] page 395). By (loc. cit. Proposition 5.3), the map $s \to \xi_s(\phi)(\phi \in \mathcal{H}_{-s}^\infty)$ is holomorphic on $\mathbb{C}$. Moreover, by (loc. cit. page 396), we have $(\pi_{-s})_{-\infty}(f)\xi_s \in \mathcal{H}_s^\infty$ for $f \in C_c^\infty(G/H)$. Thus, we can define the spherical distribution U_s by (see [11] Définition 5.4.)

$$U_s(f) := \langle (\pi_{-s})_{-\infty}(f)\xi_s, \xi_{-s} \rangle, \quad f \in C_c^\infty(\mathbb{X}). \tag{4.3}$$

Notice that for $s \in i\mathbb{R}$, we have $U_s = c_{\xi_s, \xi_s}$.

According to ([11] Proposition 5.4 and Théorème 7.3), the spherical distribution U_s satisfies the following properties:

1. for $f \in C_c^\infty(\mathbb{X})$, the map $s \to U_s(f)$ is holomorphic on $\mathbb{C}$,
2. $\Delta U_s = (s^2 - \rho^2)U_s$, (4.4)
3. $U_s = U_{-s}$ for $s \in \mathbb{C}$,

According to ([11] IX, (2 a), Proposition 9.1, Théorème 9.2 and Proposition 9.3), there exists up to a scalar at most one positive type in $\mathcal{D}'_{s,H}(\mathbb{X})$ for $s \in \mathbb{C}$, except when dq is even and $s = 0$.

If dq is odd, these distributions are the U_s for $s \in i\mathbb{R}$, and the $\varepsilon_s U_s$ with $\varepsilon_s = \pm 1$ for a set of real s.

If dq is even and $s \neq 0$, these distributions are the U_s for s in the union of $i\mathbb{R}$ and some set of real s, and the distributions denoted $(-1)^{r+1}\theta_r \in \mathcal{D}'_{\rho+2r,H}(\mathbb{X})$ for $r \in \mathbb{N}$.

When dq is even and $s = 0$, denoting by 1 the constant function equals to 1, the distributions of positive type in $\mathcal{D}'_{0,H}(\mathbb{X})$ are given by $A(-\theta_0) + B$ with $A, B \geqslant 0$.

Let (π, V) be an irreducible unitary representation of G and $\xi \in (V^{-\infty})^H$. Then the generalized matrix coefficient $c_{\xi,\xi}$ is a distribution of positive type. Moreover, as π has infinitesimal character and as the Laplace-Beltrami operator comes from the Casimir of $\mathfrak{g}$, the distribution $c_{\xi,\xi}$ is an eigendistribution for the Laplace-Beltrami operator, hence, up to a positive scalar, it is one of the distributions above.

Let us assume moreover that (π, V) is a relative discrete series representation of $\mathbb{X}$, that is a subrepresentation of $L^2(\mathbb{X})$. Let ξ_V be the evaluation at 1 of the elements of $V^\infty \subset C^\infty(\mathbb{X})$. We say that the distribution $T := c_{\xi_V,\xi_V}$ is associated to the relative discrete series representation (π, V) of $\mathbb{X}$.

By ([11] Théorème 10 and §X. (3) page 432), up to a positive scalar, the distributions associated to relative discrete series representations of $\mathbb{X}$ are

1. if $\mathbb{K} = \mathbb{R}$ and q is odd:
$$\varepsilon_r U_{\rho+2r+1} \quad \text{for} \quad r \in \mathbb{Z} \quad \text{such that} \quad \rho + 2r + 1 > 0,$$
where $\varepsilon_r = (-1)^{r+1}$ if $r \geqslant 0$ and $\varepsilon_r = 1$ if $0 < \rho + 2r + 1 < 2\rho$.
2. if dq is even:
$$U_{\rho+2r} \quad \text{for} \quad r \in -\mathbb{N}^* \quad \text{and} \quad 0 < \rho + 2r < \rho,$$
and the $(-1)^r\theta_r$ for $r \in \mathbb{N}$, which belong to $\mathcal{D}'_{s_r,H}(\mathbb{X})$ with $s_r = \rho + 2r$. (4.5)

4.1 Proposition. *Let (π, V) be a relative discrete series representation of $\mathbb{X}$ whose associated distribution of positive type is of the form εU_{s_0} for some $s_0 \in \mathbb{R}$ and $\varepsilon = \pm 1$. Then (π, ξ_V) admits no strong relative pseudo-coefficient.*

Proof. Let $f \in C_c^\infty(\mathbb{X})$ such that $\varepsilon U_{s_0}(f) \neq 0$. By the holomorphy of $s \mapsto U_s(f)$ (see (4.4)), the complex numbers $U_s(f)$ for $s \in i\mathbb{R}$ are not identically equal to 0. This implies the Proposition. □

4.2 K-types of relative discrete series representation

In this section, we assume that $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. Hence, in particular dq is even.

We want to use the results of [10] where the groups are connected and semisimple. For $\mathbb{K} = \mathbb{C}$, we have $G = ZG'$ where $Z := \{zI_{p+q}; z \in \mathbb{C}^*, |z| = 1\}$ is central and contained in $H \cap K$ and G' is equal to $SU(p,q)$ which is semisimple and connected. For $\mathbb{K} = \mathbb{H}$, we have $G \simeq Sp(p,q)$ which is semisimple and connected (see [20] Chap. I §1).

For $r \in \mathbb{N}$, we will denote by (ρ_r, V_r) the relative discrete series representation whose associated distribution is $(-1)^r\theta_r$, and by η_r the element of $(V_r^{-\infty})^H$ such that $c_{\eta_r,\eta_r} = (-1)^r\theta_r$. Note that η_r is equal to a positive multiple of the evaluation at 1 of the elements of $V_r^\infty \subset C^\infty(\mathbb{X})$. (4.6)

We will review the structure of K-module of V_r. For this, we introduce some notations.

By (4.2), the space $C^\infty(K/K \cap M)$ can be identified with the subspace of functions $f \in C^\infty(\mathbb{S}^p(\mathbb{K}) \times \mathbb{S}^q(\mathbb{K}))$ such that $f(u\zeta) = f(\zeta), \zeta \in \mathbb{S}^p(\mathbb{K}) \times \mathbb{S}^q(\mathbb{K}), u \in \mathbb{K}^*$ such that $|u| = 1$. According to ([11] page 399), for $l, m \in \mathbb{N}$, we set

$$\mathcal{Y}_{l,m} = \{f \in C^\infty(K/K \cap M); \Delta_1 f = -l(l + dp - 2)f,\ \Delta_2 f = -m(m + dq - 2)f\},$$

where Δ_1 and Δ_2 are the Laplace-Beltrami operators in the spheres $\mathbb{S}^p(\mathbb{K})$ and $\mathbb{S}^q(\mathbb{K})$ respectively.

Let E be the set of elements $(l,m) \in \mathbb{N} \times \mathbb{N}$ such that $\mathcal{Y}_{l,m} \neq \{0\}$. By ([11] page 399), $(l,0) \in E$ if and only if l is even.

Let $r \in \mathbb{N}$. We set

$$E_r := \{(l,m) \in E; l - m \geqslant dq + 2r\}. \tag{4.7}$$

Then by ([11] top of page 421), the decomposition of V_r as K-module is given by

$$V_r = \bigoplus_{(l,m)\in E_r} \mathcal{Y}_{l,m}. \tag{4.8}$$

Let $l \in 2\mathbb{N}$ be even. We consider the function $\omega_{l,0}$ defined in ([11] bottom of page 406) for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. Let $\zeta = (\zeta_1, \ldots, \zeta_{p+q})$ be the coordinates on $\mathbb{S}^p(\mathbb{K}) \times \mathbb{S}^q(\mathbb{K})$. Then $\omega_{l,0}$ is given by:

$$\omega_{l,0}(\zeta) = \int_{U(1,\mathbb{K})} F(\zeta_1 u)du, \tag{4.9}$$

for some function F.
Let us prove that

$$\omega_{l,0} \text{ is biinvariant by } K \cap H. \tag{4.10}$$

By definition, $\omega_{l,0}$ is right invariant by $K \cap M$. Recall that $K = K_1 \times K_2$ with $K_1 = U(p,\mathbb{K})$

and $K_2 = U(q, \mathbb{K})$. By loc. cit. top of page 407, we have $\omega_{l,0} \in \mathcal{Y}_{l,0}$, thus it is right invariant by K_2. But we have

$$K \cap M = \left\{ \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & u \end{pmatrix}, \quad u \in \mathbb{K}^*, |u| = 1, v_1 \in U(p-1, \mathbb{K}), v_2 \in U(q-1, \mathbb{K}) \right\}$$

and

$$K \cap H = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v_1 & 0 \\ 0 & 0 & w_2 \end{pmatrix}, \quad u \in \mathbb{K}^*, |u| = 1, v_1 \in U(p-1, \mathbb{K}), w_2 \in U(q, \mathbb{K}) \right\},$$

hence $K \cap H = (K \cap M)K_2$. Then $\omega_{l,0}$ is right invariant by $K \cap H$.

As $\mathbb{K}^* = \mathbb{R}^{+*}U(1, \mathbb{K})$ where $\mathbb{R}^{+*}$ is central in the multiplicative group $\mathbb{K}^*$, we deduce from (4.9) that $\omega_{l,0}$ is left invariant by $K \cap H$. Hence (4.10) follows.

We will determine the K-type of $\omega_{l,0}$.

Let $K' = K \cap SU(p, q)$ for $\mathbb{K} = \mathbb{C}$, or $K' = K$ for $\mathbb{K} = \mathbb{H}$. We denote by $\mathfrak{k}'$ the Lie algebra of K'. Then K-types with $K \cap H$-fixed vectors coincide with K'-types with $K' \cap H$-fixed vectors since, for $\mathbb{K} = \mathbb{C}$, we have $K = ZK'$ where $Z \subset K \cap H$ is central in K.

We fix a maximal abelian subspace $\mathfrak{t}$ of $i(\mathfrak{k}' \cap \mathfrak{q})$. As $K'/K' \cap H$ is of rank 1, the dimension of $\mathfrak{t}$ is equal to 1. We choose a short positive root γ of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{k}'_{\mathbb{C}}$. Then the roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{k}'_{\mathbb{C}}$ are of the form $\pm\gamma, \pm 2\gamma$. We identify $\mathbb{C}$ to $\mathfrak{t}^*_{\mathbb{C}}$ by the map $\lambda \mapsto \lambda\gamma$.
By the Cartan - Helgason Theorem,

if μ is a unitary irreducible representation of K then μ admits a nonzero $K \cap H$-invariant vector if and only if its highest weight is an even integer. In that case, μ admits a unique, up to a scalar, nonzero $K \cap H$-fixed vector. (4.11)

Let $\widehat{K}$ denotes the set of equivalence classes of unitary irreducible representations of K and $(\widehat{K})_{K \cap H}$ the subset of those representations having a non-trivial $(K \cap H)$-fixed vector. For $\mu \in \widehat{K}$, let χ_μ denotes its character and d_μ its dimension. We set

$$\chi_\mu^H(k) := \int_{K \cap H} \chi_\mu(kh)dh,$$

where Haar measures on compact groups are normalized so that their volume are equal to 1. Then,

the function χ_μ^H is, up to a scalar, the only function on K of type μ which is biinvariant by $K \cap H$. (4.12)

4.2 Lemma. *Let $\mu \in (\widehat{K})_{K \cap H}$ be the representation with highest weight $l \in 2\mathbb{N}$. Then*

1. *μ is the unique K-type of $\mathcal{Y}_{l,0}$ having a nonzero $K \cap H$-invariant vector.*

2. *The multiplicity of μ in $\mathcal{Y}_{l,0}$ is equal to 1 and $\omega_{l,0}$ is contained in this K-type.*

3. $\omega_{l,0} = C_l \chi_\mu^H$ *with* $C_l \neq 0$.

Proof. Let μ' be a representation of K contained in $\mathcal{Y}_{l,0}$ and having a nonzero fixed vector by $K \cap H$. Then by (4.11), it has a highest weight of the form $k\gamma$ where $k \in \mathbb{N}$ is even. The formula for the value of the Casimir operator acting on a highest weight representation implies that $k = l$, hence $\mu' = \mu$.

As $\omega_{l,0}$ is $K \cap H$-biinvariant by (4.10), we deduce easily *2.* and *3.* from (4.12). □

We come back to the structure of K-module of the relative discrete series representations $(\rho_r, V_r), r \in \mathbb{N}$.

$$\text{For } r \in \mathbb{N}, \text{ we denote by } \mu_r \in (\widehat{K})_{K\cap H} \text{ the representation with highest weight } l_r = dq + 2r \text{ and we set } \omega_{\mu_r} := \omega_{l_r,0}. \tag{4.13}$$

By (4.8), we have $\omega_{\mu_r} \in \mathcal{Y}_{l_r,0} \subset V_r$.

4.3 Lemma. *Let $r \in \mathbb{N}$. There exists a unique $K \cap H$- invariant function φ_r in V_r of type μ_r such that $\varphi_r(1) = 1$.*
Moreover, there exists a constant $C'_r \neq 0$ such that $\varphi_r(k) = C'_r \chi_{\mu_r}^H(k)$ for $k \in K$.

Proof. If φ_r satisfies the first assertion of the Lemma then the restriction of φ_r to K is a nonzero $K \cap H$- biinvariant function of type μ_r. Hence by (4.12), this restriction is proportional to $\chi_{\mu_r}^H$ and the second assertion follows.

Let us prove the first assertion. We first treat the case $\mathbb{K} = \mathbb{H}$. By ([12] Table 2), there is at most one relative discrete series representation for $L^2(\mathbb{X})$ with a given eigenvalue of the Laplace-Beltrami operator. As μ_r satisfies (2.6) of loc. cit. (where $\langle \lambda, \lambda \rangle$ has to be replaced by $\langle \alpha, \alpha \rangle$), Theorem 2.2 in loc. cit. implies that V_r contains a unique, up to a scalar, $K \cap H$ invariant function of type μ_r denoted there by ψ_λ with $\lambda = \rho + 2r$. By definition, it satisfies $\psi_\lambda(1) = \psi_\lambda^0(1)$ where ψ_λ^0 is given in loc. cit. (2.5). The formula defining ψ_λ^0 shows that $\psi_\lambda^0(1) \neq 0$. Thus the function $\varphi_r := \psi_\lambda / \psi_\lambda(1)$ satisfies the first assertion of the proposition.

For $\mathbb{K} = \mathbb{C}$, we proceed similarly by first going through the quotient by the center $Z \subset K \cap H$. □

Let $\mu \in (\widehat{K})_{K\cap H}$ with highest weight $l \in 2\mathbb{N}$. For $s \in \mathbb{C}$, the vector $(\pi_{-s})_{-\infty}(\chi_\mu)\xi_s = (\pi_{-s})_{-\infty}(\chi_\mu^H)\xi_s$ is an analytic vector for π_s. Thus, using Lemma 4.2 *3.*, we can define

$$\gamma_l(s) := \langle (\pi_{-s})_{-\infty}(\omega_{l,0})\xi_s, \xi_{-s} \rangle = U_s(\omega_{l,0}). \tag{4.14}$$

An explicit expression of $U_s(\Phi)$ for a K-finite function Φ in $C_c^\infty(\mathbb{X})$ is obtained in ([11] page 407). This expression allows us to calculate the function $\gamma_l(s)$ in the next Lemma, which is given for granted in [10].

4.4 Lemma. *Let $\mu \in (\widehat{K})_{K\cap H}$ be the representation with highest weight $l \in 2\mathbb{N}$. Let $s \in \mathbb{C}$. According to ([11] page 405), we define the function*

$$\beta_{l,0}(s) = b_l \frac{(s-\rho)(s-\rho-2)\dots(s-\rho-l_r+2)}{\Gamma((s-\rho+l+dp)/2)}, \qquad \text{where } b_l \text{ is a nonzero constant.}$$

Then, we have

$$\gamma_l(s) = U_s(\omega_{l,0}) = c_l' \beta_{l,0}(s)\beta_{l,0}(-s)$$

for some nonzero constant c_l'.

Proof. We consider the function $A(t) = (e^t + e^{-t})^{dp-1}(e^t - e^{-t})^{dq-1}$ according to ([11] page 403). We can find a sequence $(F_n)_{n\in\mathbb{N}}$ of $C_c^\infty(]0,+\infty[)$ such that $\mathrm{supp}(F_n) \subset]\frac{1}{2n}, \frac{1}{n}[$ and

$$\int_0^{+\infty} F_n(t)A(t)dt = 1.$$

Therefore, we have

$$\lim_{n\to+\infty} \int_0^{+\infty} F_n(t)A(t)g(t)dt = g(0), \qquad \text{for } g \in C^\infty(\mathbb{R}). \tag{4.15}$$

Using the Cartan decomposition (4.1), we define the function Φ_n on $\mathbb{X}$ by $\Phi_n(ka_tx^0) = \omega_{l,0}(k)F_n(t)$, $k \in K, t \in [0,+\infty[$. Hence, each Φ_n is of type μ and belongs to $C_c^\infty(\mathbb{X})$. By ([11] page 407), we have

$$U_s(\Phi_n) = c_l\beta_{l,0}(s)\beta_{l,0}(-s)\int_0^{+\infty} \Psi_{l,0}(t,s)F_n(t)A(t)dt \times \int_{K/K\cap M} \omega_{l,0}(k)^2dk,$$

where c_l is a nonzero constant and $\Psi_{l,0}$ is given in term of the hypergeometric function by

$$\Psi_{l,0}(t,s) = (\cosh t)^{s-\rho}\, {}_2F_1\big(\frac{\rho-s+l}{2}, \frac{\rho-s-dp+2-l}{2}, \frac{dq}{2}, \tanh^2 t\big).$$

Since $\Psi_{l,0}(0,s) = 1$, we deduce from (4.15) that

$$\lim_{n\to+\infty} U_s(\Phi_n) = c_l'\beta_{l,0}(s)\beta_{l,0}(-s),$$

with $c_l' = c_l\int_{K/K\cap M}\omega_{l,0}(k)^2dk$.
It remains to prove that $\lim_{n\to+\infty} U_s(\Phi_n) = U_s(\omega_{l,0})$. For $v \in \mathcal{H}_{-s}^\infty$, one has

$$\langle(\pi_{-s})_{-\infty}(\Phi_n)\xi_s, v\rangle = \int_{G/H}\Phi_n(g)\langle(\pi_{-s})_{-\infty}(g)\xi_s, v\rangle dg = \int_{K/K\cap M}\big(\int_0^{+\infty} F_n(t)A(t)\langle(\pi_{-s})_{-\infty}(ka_t)\xi_s, v\rangle dt\big)\omega_l$$

The map $g : (k,t) \mapsto \langle(\pi_{-s})_{-\infty}(ka_t)\xi_s, v\rangle$ belongs to $C^\infty(K/K\cap M \times \mathbb{R})$. Since $F_n \in C^\infty([0,1])$ for all $n \in \mathbb{N}$, the function $(k,t) \mapsto F_n(t)A(t)g(k,t)$ is integrable on $K/K\cap M \times \mathbb{R}$. Hence by Fubini's Theorem, we obtain

$$\langle(\pi_{-s})_{-\infty}(\Phi_n)\xi_s, v\rangle = \int_0^{+\infty} F_n(t)A(t)\big(\int_{K/K\cap M}\langle(\pi_{-s})_{-\infty}(ka_t)\xi_s, v\rangle dk\big)dt.$$

As the map $t \mapsto \int_{K/K\cap M} \langle (\pi_{-s})_{-\infty}(ka_t)\xi_s, v\rangle dk$ belongs to $C^\infty(\mathbb{R})$, we deduce by (4.15) that

$$\lim_{n\to+\infty} \langle (\pi_{-s})_{-\infty}(\Phi_n)\xi_s, v\rangle = \int_{K/K\cap M} \langle (\pi_{-s})_{-\infty}(k)\xi_s, v\rangle \omega_{l,0}(k)dk = \langle (\pi_{-s})_{-\infty}(\omega_{l,0})\xi_s, v\rangle. \tag{4.16}$$

Let d_μ denote the dimension of μ. Then $P_\mu := (\pi_{-s})_{-\infty}(d_\mu \chi_{\check{\mu}})$ is well defined as the projector of $\mathcal{H}_{-s}^{-\infty}$ onto the μ-isotypic component $(\mathcal{H}_{-s})_\mu$ in $\mathcal{H}_{-s}^\infty$. Hence we deduce that

$$\langle (\pi_{-s})_{-\infty}(\Phi_n)\xi_s, \xi_{-s}\rangle = \langle P_\mu(\pi_{-s})_{-\infty}(\Phi_n)\xi_s, \xi_{-s}\rangle = \langle (\pi_{-s})_{-\infty}(\Phi_n)\xi_s, P_\mu\xi_{-s}\rangle.$$

Applying (4.16) to $v = P_\mu \xi_{-s}$, we deduce that $\lim_{n\to+\infty} U_s(\Phi_n) = U_s(\omega_{l,0})$. Thus, we obtain the Lemma. □

4.3 Existence of strong relative pseudo-coefficients for certain relative discrete series representations.

In this section, we assume that dq is even and $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$.

Existence of strong relative pseudo-coefficients for relative discrete series representations $(\rho_r, V_r), r \in \mathbb{N}$ is an easy consequence of Proposition 4.7 below. This Proposition corresponds to ([10] Lemma 9), but the proof given in loc. cit. is slightly incomplete. We will give here a more precise and modified proof.

We first recall some results of [10] on the Paley-Wiener space of $\mathbb{X}$. Notice that for $\mathbb{K} = \mathbb{C}$, we have to go through the quotient by the center $Z \subset K \cap H$ to apply these results.

Let (π, V) a unitary irreducible representation of G in a Hilbert space V. If $\mu \in \widehat{K}$ then its contragredient is equivalent to μ and $P_\mu := \pi_{-\infty}(d_\mu \chi_\mu)$ is well defined as the projection of $V^{-\infty}$ onto the μ-isotypic component $V_\mu \subset V^\infty$. If ξ is an H-invariant distribution vector, then

$$P_\mu \xi = d_\mu \pi_{-\infty}(\chi_\mu^H)\xi \in V_\mu^{K\cap H}, \tag{4.17}$$

hence $P_\mu \xi = 0$ if $\mu \notin (\widehat{K})_{K\cap H}$.

Let $\mathfrak{a}_{\mathbb{C}}^*$ be the complexification of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$. Recall that W denotes the Weyl group of A in G, hence we can consider the action of W on $\mathfrak{a}$.

Let $R > 0$. Let $PW(\mathfrak{a})_R$ denotes the space of entire functions Ψ on $\mathfrak{a}_{\mathbb{C}}^*$ which satisfy

$$\forall N \in \mathbb{N}, \ \sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} (1 + \|\lambda\|)^N e^{-R\|\operatorname{Im}\lambda\|} |\Psi(\lambda)| < +\infty.$$

Then the classical Paley-Wiener space $PW(\mathfrak{a})$ is the union of $PW(\mathfrak{a})_R$ for $R \in]0, +\infty[$.

Restricting to the W-invariant functions, the classical Fourier transform is a bijection of $C_c^\infty(\mathfrak{a})^W$ to $PW(\mathfrak{a})^W$.

Recall that $A = \{a_t; t \in \mathbb{R}\}$ is the abelian subgroup of G corresponding to the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$. We fix $R > 0$ and we denote by $A_R := \{a_t; |t| < R\}$ the ball of radius R in A.

We have the following Paley-Wiener Theorem.

4.5 Theorem. *([10] Theorem 1 and its remark) Let $\Psi \in PW(\mathfrak{a})_R^W$. Let $\mu \in \widehat{K}_{K\cap H}$. Then, there exists a unique function $f \in C_c^\infty(G/H)$ of type μ, supported in KA_RH such that the following holds:*

For all unitary irreducible representations (π, V) of G and for all H-invariant distribution vectors $\xi \in V^{-\infty}$ such that $\pi_{-\infty}(\Delta)\xi = (\lambda^2 - \rho^2)\xi$, we have

$$\pi_{-\infty}(f)\xi = \Psi(i\lambda)P_\mu\xi.$$

4.6 Proposition. *Let $r \in \mathbb{N}$ and μ_r be the K-type of highest weight $l_r = dq + 2r$ (see (4.13)). We set $s_r := \rho + 2r$. Let $G \in PW(\mathfrak{a})_R^W$ such that $G(is_r) \neq 0$. Then, there exists $f_1 \in C_c^\infty(G/H)$ of type μ_r supported in KA_RH such that*

1. *$\theta_r(f_1) = 1$,*
2. *$\theta_{r'}(f_1) = 0$ for $r' \in \mathbb{N}$ such that $r' > r$,*
3. *$U_s(f_1) = C\gamma_{l_r}(s)G(is)$, $s \in \mathbb{C}$, for some nonzero constant C.*

Proof. As $c_{\eta_r,\eta_r} = (-1)^{r+1}\theta_r \in \mathcal{D}'_{s_r,H}(\mathbb{X})$ (see (4.5) and (4.6)), we have $\rho_r(\Delta)\eta_r = (s_r^2 - \rho^2)\eta_r$. We apply Theorem 4.5 to $\mu = \mu_r$. Then there exists $g_1 \in C_c^\infty(\mathbb{X})$ of type μ_r supported in KA_RH such that

$$(\rho_r)_{-\infty}(g_1)\eta_r = G(is_r)P_{\mu_r}\eta_r, \tag{4.18}$$

and

$$(\pi_{-s})_{-\infty}(g_1)\xi_s = G(is)P_{\mu_r}\xi_s, \text{ for } s \in \mathbb{C}. \tag{4.19}$$

By Lemma 4.2 and (4.17), we have $P_{\mu_r}\eta_r = d_{\mu_r}(\rho_r)_{-\infty}(\chi^H_{\mu_r})\eta_r$ and the analytic vector $(\rho_r)_{-\infty}(\chi^H_{\mu_r})\eta_r$ is a $K \cap H$-invariant function of type μ_r in V_r. By Lemma 4.3, we obtain that $(\rho_r)_{-\infty}(\chi^H_{\mu_r})\eta_r = C'\varphi_r$ for some constant C'.
Let us prove that $C' \neq 0$. By Lemma 4.3 again, the function φ_r coincides with $C'_r\chi^H_{\mu_r}$ on K. Since $\rho_r(\chi^H_{\mu_r})$ is, up to a scalar, the projection on the $K \cap H$-fixed vectors in V_r, we have $\rho_r(\chi^H_{\mu_r})\varphi_r = C_1\varphi_r$ for some nonzero constant C_1. Recall that there is a nonzero constant C_2 such that η_r is defined by $\eta_r(\phi) = C_2\phi(1)$ for $\phi \in V_r^\infty$(see (4.6)). Thus we deduce

$$C'\langle\varphi_r, \varphi_r\rangle = \langle(\rho_r)_{-\infty}(\chi^H_{\mu_r})\eta_r, \varphi_r\rangle = \langle\eta_r, \rho_r(\chi^H_{\mu_r})\varphi_r\rangle = C_1C_2\Big(\rho_r(\chi^H_{\mu_r})\varphi_r\Big)(1) = C_1C_2\varphi_r(1) = C_1C_2 \neq 0,$$

hence $C' \neq 0$. Then $\theta_r(g_1) = C'G(is_r) \neq 0$. We set

$$f_1 := \frac{g_1}{C'G(is_r)}.$$

Then f_1 satisfies

$$\theta_r(f_1) = 1.$$

and we obtain the first assertion of the Proposition.

Let $r' \in \mathbb{N}$ such that $r' > r$. As f_1 is of type μ_r and μ_r is not a K-type of $V_{r'}$ by (4.8), we have $\theta_{r'}(f_1) = 0$.

To prove the last assertion of the Proposition, we consider the property (4.19) for $s \in \mathbb{C}$. By (4.17), we have $P_{\mu_r}\xi_s = d_{\mu_r}(\pi_{-s})_{-\infty}(\chi_{\mu_r})\xi_s = d_{\mu_r}(\pi_{-s})_{-\infty}(\chi^H_{\mu_r})\xi_s$. Using Lemma 4.2 and Lemma 4.4, this leads to

$$U_s(f_1) = \frac{d_{\mu_r}}{C_r C' G(is_r)} G(is) U_s(\omega_{\mu_r}) = \frac{d_{\mu_r}}{C_r C' G(is_r)} G(is)\gamma_{l_r}(s).$$

This finishes the proof of the Proposition. □

4.7 Proposition. *Let $r \in \mathbb{N}$ and μ_r be the K-type of highest weight $l_r = dq + 2r$. Then, there exists a function $f \in C_c^\infty(\mathbb{X})$ with support in KA_RH, sum of a K-invariant function and of a function of type μ_r, such that*

1. *$\theta_r(f) = 1$,*
2. *$\theta_{r'}(f) = 0$ for $r' \in \mathbb{N}$ such that $r' > r$,*
3. *$U_s(f) = 0$ for $s \in \mathbb{C}$.*

Proof. We set $s_r = \rho + 2r$. Let $G \in PW(\mathfrak{a})^W_R$ such that $G(is_r) \neq 0$. Let $f_1 \in C_c^\infty(G)$ be a function satisfying the properties of Proposition 4.6. Then for $s \in \mathbb{C}^*$, we have $U_s(f_1) = C\gamma_{l_r}(s)G(is)$ where C is a nonzero constant.
By definition (see Lemma 4.4), we have

$$\gamma_{l_r}(s) = c'_{l_r}\beta_{l_r,0}(s)\beta_{l_r,0}(-s),$$

and

$$\beta_{l_r,0}(s) = b_{l_r}\frac{(s-\rho)(s-\rho-2)\dots(s-\rho-l_r+2)}{\Gamma((s-\rho+l_r+dp)/2)},$$

where $l_r = dq + 2r$.
Recall that $\rho = \frac{1}{2}(dp+dq) - 1$, then we have $-\rho + l_r + dp = \rho + 2r + 2 = s_r + 2$ with $s_r = \rho + 2r$. Thus, we can write

$$\Gamma((s-\rho+l_r+dp)/2) = 2^{-l_r}(s+s_r)(s+s_r-2)\dots(s-\rho+dp)\Gamma((s-\rho+dp)/2).$$

Hence we obtain

$$\gamma_{l_r}(s) = \frac{P(s)}{(s+s_r)(-s+s_r)Q(s)}\gamma_0(s)$$

where

$$P(s) = 2^{l_r}c'_{l_r}b^2_{l_r}(s-\rho)(s-\rho-2)\dots(s-\rho-(l_r-2))(-s-\rho)(-s-\rho-2)\dots(-s-\rho-(l_r-2))$$

and

$$Q(s) = (s+s_r-2)(s+s_r-4)\dots(s-\rho+dp)(-s+s_r-2)(-s+s_r-4)\dots(-s-\rho+dp).$$

By assumption we have $dq \geqslant 2$, hence $2r \leqslant dq + 2r - 2 = l_r - 2$. Then, $P_1(s) := \dfrac{P(s)}{(s+s_r)(-s+s_r)}$ and $Q(s)$ are even polynomials such that $Q(s_r) \neq 0$ and $P_1(s_r) \neq 0$.

By Proposition 4.6 *3.*, we obtain

$$U_s(f_1) = CG(is)\frac{P_1(s)}{Q(s)}\gamma_0(s), \quad \text{with } \gamma_0(s) = c_0' \frac{1}{\Gamma(\frac{s-\rho+dp}{2})\Gamma(\frac{-s-\rho+dp}{2})}.$$

Since Q is even and $Q(s_r) \neq 0$, we may choose an invariant differential operator $D \in \mathbb{D}(\mathbb{X})$ such that

$$U_s(Df_1) = \frac{Q(s)}{Q(s_r)} U_s(f_1), \quad s \in \mathbb{C},$$

and

$$\theta_{r'}(Df_1) = \frac{Q(s_{r'})}{Q(s_r)} \theta_{r'}(f_1), \quad r' \in \mathbb{N}.$$

Therefore the function $f_2 := Df_1$ satisfies the assertions *1.* and *2.* of the Proposition and

$$U_s(f_2) = C\frac{G(is)P_1(s)}{Q(s_r)}\gamma_0(s).$$

Since $PW(\mathfrak{a})_R^W$ is stable by multiplication by an even polynomial function, applying Theorem 4.5 to the trivial K-type, we can find a K-invariant function $f_3 \in C_c^\infty(\mathbb{X})$ supported in KA_RH such that

$$U_s(f_3) = C\frac{G(is)P_1(s)}{Q(s_r)}\gamma_0(s) = U_s(f_2).$$

By (4.8), the trivial representation is not a K-type of $V_{r'}$ for $r' \geqslant 0$, thus we have $\theta_{r'}(f_3) = 0$ for $r' \geqslant 0$. Therefore, the function $f = f_2 - f_3$ satisfies the properties of the Proposition. □

4.8 Theorem. *Let $R > 0$ and $A_R := \{a_t \in A; |t| < R\}$ be the ball of radius R in A. Then, for all $r \in \mathbb{N}$, there exists a strong relative pseudo-coefficient $f \in C_c^\infty(\mathbb{X})$ supported in KA_RH for (ρ_r, η_r). This means that the function f satisfies*

$$\theta_r(f) = 1, \quad \theta_{r'}(f) = 0 \text{ for } r' \in \mathbb{N},\ r' \neq r,$$

and

$$U_s(f) = 0, \text{ for } s \in \mathbb{C}$$

Proof. Proposition 4.7 gives the result for $r = 0$. Let $r > 0$ and assume we have a strong relative pseudo-coefficient $f_{r'}$ for $(\rho_{r'}, \eta_{r'})$ $(r' < r)$ supported in KA_RH. We denote by Ψ_r the function obtained in Proposition 4.7. Then, the function $f_r = \Psi_r - \sum_{r'=0}^{r-1} \theta_{r'}(\Psi_r) f_{r'}$ is a strong relative pseudo-coefficient for (ρ_r, η_r) with support in KA_RH. □

5 Existence of σ-stable torsion free cocompact discrete subgroups of $U(p, q, \mathbb{K})$.

This section is entirely due to the kind help of R. Beuzart-Plessis and J. P. Labesse.

Let F be a totally real number field of degree $[\mathrm{F} : \mathbb{Q}] = r > 1$ and let $V_\infty = \{v : \mathrm{F} \hookrightarrow \mathbb{R}\}$ denote the finite set of real places of F.

We consider the group $\mathbf{G}'$ defined in ([26] page 372), which depends on $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. By loc. cit., there exists a unique archimedean place $v_1 \in V_\infty$ such that

$$\mathbf{G}'(\mathrm{F}_{v_1}) = \begin{cases} SO(p,q) & \text{for } \mathbb{K} = \mathbb{R} \\ SU(p,q) & \text{for } \mathbb{K} = \mathbb{C} \\ Sp(p,q) & \text{for } \mathbb{K} = \mathbb{H} \end{cases} \tag{5.1}$$

and for $v \in V_\infty$, $v \neq v_1$,

$$\mathbf{G}'(\mathrm{F}_{v}) = \begin{cases} SO(p+q) & \text{for } \mathbb{K} = \mathbb{R} \\ SU(p+q) & \text{for } \mathbb{K} = \mathbb{C} \\ Sp(p+q) & \text{for } \mathbb{K} = \mathbb{H} \end{cases} \tag{5.2}$$

Let $\mathbf{G} = \mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbf{G}'$ be the group obtained by restriction of scalars. For $v \in V_\infty, v \neq v_1$, we have $\mathbf{G}(\mathbb{Q}) = \mathbf{G}'(\mathrm{F}) \subset \mathbf{G}'(\mathrm{F}_v)$ and $\mathbf{G}'(\mathrm{F}_v)$ is compact, hence each element of $\mathbf{G}(\mathbb{Q})$ is semisimple. Thus the group $\mathbf{G}(\mathbb{Q})$ is anisotropic.

Let $\mathbb{A}_\mathrm{F}$ and $\mathbb{A}_\mathbb{Q}$ be the rings of adeles of F and $\mathbb{Q}$ respectively. We denote by $\mathbb{A}_{\mathrm{F},f}$ and $\mathbb{A}_{\mathbb{Q},f}$ the subrings of finite adeles in $\mathbb{A}_\mathrm{F}$ and $\mathbb{A}_\mathbb{Q}$ respectively.

Then, $\mathbf{G}(\mathbb{Q})$ is diagonally embedded in $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ and by ([24] Theorem 5.5 (1)), the quotient $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_\mathbb{Q})$ is compact.

Let τ be the rational involution of $\mathbf{G}'$ denoted $\tau_{1,0}$ in ([26] §2.2). The involution of $\mathbf{G}'(\mathrm{F}_{v_1})$ induced by τ, again denoted by τ, is simply the restriction to $\mathbf{G}'(\mathrm{F}_{v_1})$ of the involution σ of $U(p, q, \mathbb{K})$ defined in section 3. The involution τ defines a continuous automorphism of $\mathbf{G}(\mathbb{A}_\mathbb{Q}) = \mathbf{G}'(\mathbb{A}_\mathrm{F})$ preserving $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f}) = \mathbf{G}'(\mathbb{A}_{\mathrm{F},f})$.
We have the following classical result.

5.1 Lemma. *Let K_f be an open compact subgroup of $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$. Then the subgroup $\Gamma(K_f) := \mathbf{G}(\mathbb{R}) \cap \big(\mathbf{G}(\mathbb{Q})K_f\big)$ is a cocompact discrete subgroup of $\mathbf{G}(\mathbb{R})$.*

Proof. Let us give a proof for sake of completeness. Since $\mathbf{G}(\mathbb{Q})$ is discrete in $\mathbf{G}(\mathbb{A}_\mathbb{Q})$, the subgroup $\Gamma(K_f)$ is discrete. As $\mathbf{G}(\mathbb{Q})$ is diagonally embedded in $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$, we have also $\Gamma(K_f) = \mathbf{G}(\mathbb{Q}) \cap K_f$.

We consider the map ψ from $\mathbf{G}(\mathbb{R})K_f$ to $\Gamma(K_f)\backslash\mathbf{G}(\mathbb{R})$ given by $\psi(gk) = \Gamma(K_f)g$. If $gk = \gamma g'k'$ with $g, g' \in \mathbf{G}(\mathbb{R})$, $k, k' \in K_f$ and $\gamma \in \mathbf{G}(\mathbb{Q})$, we have $g = \gamma g'k'k^{-1}$. As $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ and $\mathbf{G}(\mathbb{R})$ commute, we obtain $g = \gamma k'k^{-1}g'$, hence $gg'^{-1} = \gamma k'k^{-1} \in \Gamma(K_f)$. Therefore ψ goes through the quotient $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{R})K_f/K_f$ and the induced map is a surjection from $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{R})K_f/K_f$ to $\Gamma(K_f)\backslash\mathbf{G}(\mathbb{R})$.

Thus it remains to prove that $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{R})K_f/K_f$ is compact.

The group $\mathbf{G}(\mathbb{R})K_f$ is an open subgroup of $\mathbf{G}(\mathbb{A}_\mathbb{Q})$. As $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_\mathbb{Q})$ is compact, the number of $(\mathbf{G}(\mathbb{Q}), K_f)$ double cosets, which are open, is finite. Then each of them is also closed, hence compact. We deduce that $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{R})K_f/K_f$ is compact $\square$

5.2 Lemma. *Let K_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$. Then there exists a compact open subgroup $K'_f \subset K_f$ of $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ such that $\Gamma(K'_f)$ is a torsion free subgroup of $\Gamma(K_f)$ of finite index.*

Proof. We fix an embedding of $\mathbf{G}$ in $GL(n)$ defined over $\mathbb{Q}$. We have

$$GL(n,\mathbb{Z}) = GL(n,\mathbb{Q}) \bigcap \prod_{p\,\mathrm{prime}} GL(n,\mathbb{Z}_p).$$

By the proof of ([5] Proposition 2.2), the group $GL(n,\mathbb{Z})$ contains a torsion free subgroup $\tilde{\Gamma}$ of finite index of the form $\tilde{\Gamma} = GL(n,\mathbb{Q}) \cap \tilde{K}_f$, where $\tilde{K}_f$ is a compact open subgroup of $\prod_{p\,\mathrm{prime}} GL(n,\mathbb{Z}_p)$.

Let $K'_f := \tilde{K}_f \cap K_f \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$. Then $\Gamma(K'_f) = \mathbf{G}(\mathbb{Q}) \cap K'_f \subset \tilde{\Gamma}$ is without torsion. As K'_f is a compact open subgroup of K_f, it is of finite index. It follows that $\Gamma(K'_f)$ is of finite index in $\Gamma(K_f)$. $\square$

5.3 Corollary. *For each open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$, there exists a τ-stable open compact subgroup $K^0_f \subset K_f$ such that $\Gamma(K^0_f)$ is a τ-stable torsion free cocompact discrete subgroup of $\mathbf{G}(\mathbb{R})$.*

Proof. Let K'_f be the subgroup obtained in the previous Lemma. As τ is a continuous involution of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ which preserves $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$, the subgroup $K^0_f := \tau(K'_f) \cap K'_f$ is a τ-stable compact open subgroup of K_f. It is clear that $\Gamma(K^0_f)$ is τ-stable since K^0_f is τ-stable. The properties of K'_f imply that $\Gamma(K^0_f)$ is a torsion free cocompact discrete subgroup of $\mathbf{G}(\mathbb{R})$. Hence we obtain the Corollary. $\square$

5.4 Lemma. *Let $G = G_1 \times G_2$ be the product of two locally compact groups with G_2 compact. Let Γ be a torsion free cocompact discrete subgroup of G. Then the projection Γ_1 of Γ to G_1 is a torsion free cocompact discrete subgroup of G_1.*

Proof. If Γ_1 was not discrete, there would exists a sequence $(\gamma_{1,n})$ of distinct elements of Γ_1 converging to a limit l. There exists a sequence $(\gamma_{2,n})$ in G_2 such that $\gamma_n = (\gamma_{1,n}, \gamma_{2,n})$ belongs to Γ. Since G_2 is compact, extracting a subsequence, we can assume that $(\gamma_{2,n})$ converges. Then the sequence (γ_n) convergences. As Γ is discrete, there is n_0 such that $\gamma_n = \gamma_{n_0}$ for $n \geqslant n_0$. This implies that $(\gamma_{1,n})$ is constant for $n \geqslant n_0$, which contradicts the fact that the $\gamma_{1,n}$'s are distinct. Thus Γ_1 is discrete.

Let us show that Γ_1 is a cocompact subgroup of G_1. Let (g_n) be a sequence in G_1. Since Γ is cocompact in G, there exist a subsequence (g'_n) of (g_n) and a sequence (γ_n) in Γ such that $(\gamma_n g'_n)$ converges. Writing $\gamma_n = (\gamma_{1,n}, \gamma_{2,n})$ with $\gamma_{i,n} \in G_i$, we deduce that $(\gamma_{1,n} g'_n)$ converges, hence Γ_1 is cocompact in G_1.

Let $\gamma_1 \in \Gamma_1$ and $r \in \mathbb{N}^*$ such that $\gamma_1^r = 1$. Let $\gamma_2 \in G_2$ such that $(\gamma_1, \gamma_2) \in \Gamma$. Then the sequence $\big((\gamma_1,\gamma_2)^n\big) = (\gamma_1^n, \gamma_2^n)$ remains in a compact set, thus it admits a converging subsequence $((\gamma_1, \gamma_2)^{k(n)})$. As Γ is discrete, this subsequence is constant for n large enough. Hence $(\gamma_2^{k(n)})$ is constant for n large enough. This implies that there exists $s \in \mathbb{N}^*$ such that $\gamma_2^s = 1$. For m a multiple of r and s, we have $(\gamma_1, \gamma_2)^m = 1$. As Γ is torsion free, this leads to $\gamma_1 = \gamma_2 = 1$, hence Γ_1 is torsion free. $\square$

5.5 Lemma. *If Γ is cocompact in $SO(p,q)$ (resp., in $SU(p,q)$) then Γ is cocompact in $O(p,q)$ (resp. in $U(p,q)$).*

Proof. This follows from the fact that $SO(p,q)$ (resp., $SU(p,q)$) is cocompact in $O(p,q)$ (resp., $U(p,q)$). $\square$

By (5.1) and (5.2), there is a compact group $\Omega_{\mathbb{K}}$, depending on $\mathbb{K}$, such that

$$\mathbf{G}(\mathbb{R}) = \begin{cases} SO(p,q) \times \Omega_{\mathbb{R}} & \text{for } \mathbb{K} = \mathbb{R} \\ SU(p,q) \times \Omega_{\mathbb{C}} & \text{for } \mathbb{K} = \mathbb{C} \\ Sp(p,q) \times \Omega_{\mathbb{H}} & \text{for } \mathbb{K} = \mathbb{H} \end{cases} . \tag{5.3}$$

We denote by $G_{1,\mathbb{K}}$ the first factor of this decomposition.

5.6 Proposition. *For sufficiently small τ-stable open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$, the projection $\Gamma_1(K_f)$ of $\Gamma(K_f) = \mathbf{G}(\mathbb{Q}) \cap K_f$ onto $G_{1,\mathbb{K}}$ according the decomposition (5.3) is a σ-stable torsion free cocompact discrete subgroup of $U(p,q,\mathbb{K})$.*

Proof. By Corollary 5.3, we can choose K_f sufficiently small so that $\Gamma(K_f)$ is a τ-stable torsion free cocompact discrete subgroup of $\mathbf{G}(\mathbb{R})$. By Lemma 5.4, the subgroup $\Gamma_1(K_f)$ is a τ-stable torsion free cocompact discrete subgroup of $G_{1,\mathbb{K}}$. Since the involution τ coincides with σ on $G_{1,\mathbb{K}}$, the Proposition follows from Lemma 5.5. $\square$

5.7 Theorem. *Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. Let K_f and $\Gamma_1(K_f)$ be as in Proposition 5.6. Then the relative discrete series representation (ρ_r, V_r) of $U(p,q,\mathbb{K})$ occurs with a nonzero period in $L^2(\Gamma_1(K_f)\backslash U(p,q,\mathbb{K}))$.*

Proof. To apply the relative trace formula (2.5), we have to verify that $\Gamma_1(K_f)$ and H satisfy assumptions (2.1). The group $H = U(1,\mathbb{K}) \times U(p-1,q,\mathbb{K})$ is unimodular. By (5.1) and (5.2), each element of $\mathbf{G}(\mathbb{Q})$ is semisimple, hence each element γ of $\Gamma_1(K_f)$ is semisimple. By ([27] Part II, chap. 2 Proposition 13) , the centralizer of $\gamma\sigma(\gamma)^{-1}$ in $G = U(p,q,\mathbb{K})$ is reductive since $\gamma\sigma(\gamma)^{-1}$ is semisimple. Moreover, this centralizer is σ-stable, hence the centralizer $Z_H(\gamma\sigma(\gamma)^{-1})$ of $\gamma\sigma(\gamma)^{-1}$ in H is reductive. As the identity map induces an isomorphism from $(H\times H)_\gamma$ to $Z_H(\gamma\sigma(\gamma)^{-1})\times H$, we deduce that $(H\times H)_\gamma$ is reductive, hence unimodular. The quotient $(\Gamma_1(K_F) \cap H)\backslash H$ is compact by Lemma 3.4. Therefore the assumptions (2.1) are all satisfied. By Theorem 4.8 there exists a strong relative pseudo-coefficient for (ρ_r, V_r), with arbitrary small support. As $\Gamma_1(K_f)$ is torsion free, Proposition 3.5 and Proposition 3.3 give the result. $\square$

6 Non existence of K-finite relative pseudo-coefficients for $G(\mathbb{C})/G(\mathbb{R})$.

Let G be a connected, simply connected complex semisimple Lie group. Let H be a real form of G and σ be the conjugation of G relative to H. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ relative to σ. Hence we have $\mathfrak{q} = i\mathfrak{h}$. Recall that a Cartan subspace of $\mathfrak{q}$ is a maximal abelian subspace made of semsimple elements. Then the map $\mathfrak{a} \to i\mathfrak{a}$ is an isomorphism from the set of Cartan subalgebras of $\mathfrak{h}$ to the set of Cartan subspaces of $\mathfrak{q}$ which preserves H-conjugacy classes.

We fix a Cartan involution θ of $\mathfrak{h}$ commuting with σ and we denote by K the maximal compact subgroup of G of fixed points under θ.

By ([23] Theorems 1 and 2), the symmetric space G/H has relative discrete series representations if and only if $\mathfrak{q}$ has a compact Cartan subspace, or equivalently, if $\mathfrak{h}$ is a split real form of $\mathfrak{g}$. The goal of this section is to establish that no discrete relative series representation of G/H admits K-finite relative pseudo-coefficient. (see Theorem 6.2 below). These results will follow from the inversion formula of orbital integrals (see [17] Théorème 6.15).

We assume that H is split and we fix a split Cartan subalgebra $\mathfrak{a}_0$ of $\mathfrak{h}$. Let $\Gamma_{\mathfrak{a}_0}$ be the lattice of elements $X \in \mathfrak{a}_0$ satisfying $\exp 2iX = 1$ and let $\Gamma^*_{\mathfrak{a}_0}$ be its dual lattice so that $\mu(X) \in 2\pi\mathbb{Z}$ for $\mu \in \Gamma^*_{\mathfrak{a}_0}$ and $X \in \Gamma_{\mathfrak{a}_0}$.

Let $P_0 = L_0 N_0$ be a σ-stable Borel subgroup of G with Levi subgroup $L_0 = \exp(\mathfrak{a}_0 + i\mathfrak{a}_0)$. For $\mu \in \Gamma^*_{\mathfrak{a}_0}$, we define the character δ_μ of P_0 by $\delta_\mu(\exp(X + iY)n) = e^{i\mu(Y)}$ for $X, Y \in \mathfrak{a}_0$ and $n \in N_0$. We denote by $(\pi_\mu, \mathcal{H}_\mu)$ the normalized induced representation $(ind_{P_0}^G \chi_\mu, \mathcal{H}_\mu)$.

By ([15] Corollaire 2.6), the linear form ξ_μ defined by the integration over $H/H \cap P_0$

$$\xi_\mu(\psi) = \int_{H/H\cap P_0} \psi(h) d\dot{h}, \quad \psi \in \mathcal{H}_\mu^\infty, \tag{6.1}$$

where $d\dot{h}$ is a semi-invariant measure on $H/H \cap P_0$, is an H-invariant distribution vector of π_μ.

Then by (eg. [9] Proposition 5 and [15] §3. Application 1.), the relative discrete series representations of G/H are given by the representations $(\pi_\mu, \mathcal{H}_\mu)$ where $\mu \in \Gamma^*_{\mathfrak{a}_0}$ is regular. Moreover, these representations occur with multiplicity one in the Plancherel formula and the space $\mathcal{M}_{\pi_\mu}$ (defined in (3.1)) is equal to $\mathbb{C}\xi_\mu$. We denote by $\Gamma^*_{\mathfrak{a}_0,reg}$ the set of regular elements in $\Gamma^*_{\mathfrak{a}_0}$ and by $\Gamma^*_{\mathfrak{a}_0,sing}$ its complementary in $\Gamma^*_{\mathfrak{a}_0}$.

We recall some facts about regular elements in G/H and orbital integrals (see [22] and [17] §1 and §2).

Let φ be the map from G/H to G defined by $\varphi(gH) = g\sigma(g)^{-1}$. A semisimple element $x \in G/H$ is regular if $\varphi(x)$ is semisimple and regular in G in the usual sense. Let $(G/H)_{reg}$ be the open dense subset of semisimple regular elements of G/H. To a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{h}$, we associate the Cartan subset A of G/H consisting of elements x such that $\varphi(x)$ centralizes $\mathfrak{a}$. If $x \in (G/H)_{reg}$ then the centralizer $\mathfrak{a} := Z_{\mathfrak{h}}(\varphi(x))$ of $\varphi(x)$ in $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$ and x belongs to the Cartan subset A associated to $\mathfrak{a}$.

If $Y \in \mathfrak{h}$, we denote by $\mathrm{Exp} Y$ the class of $\exp iY$ in G/H.
If V is a subset of G/H (respectively, a subset of $\mathfrak{h}$ or of its dual space $\mathfrak{h}^*$), then V_{reg} will denote the set of regular elements in V.

For $x \in G/H$, we consider the polynomial function

$$\det_{\mathbb{C}}(1 - t - \mathrm{Ad}\,\varphi(x)) = t^n + q_{n-1}(x)t^{n-1} + \ldots + q_l(x)t^l, \tag{6.2}$$

where l is the rank of G and n is its dimension as a complex group. We set $D_{G/H}(x) := q_l(x)$.

The orbital integral $\mathcal{M}(f)$ of $f \in C_c^\infty(G)$ is the function $\mathcal{M}(f) \in C^\infty((G/H)_{reg})$ defined by

$$\mathcal{M}(f)(x) = |D_{G/H}(x)|^{1/2} \int_{H/Z_H(\mathfrak{a})} f(h \cdot x) dh,$$

where $\mathfrak{a} = Z_{\mathfrak{h}}(\varphi(x))$ and $Z_H(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in H. As in the group case (see[27] Part II, 10.2. Proposition 2), orbital integrals satisfy the following property.

6.1 Lemma. *([6] §8) Let $f \in C_c^\infty(G/H)$. Let A be a Cartan subset of G/H. Then there exists a compact subset $U \subset A$, depending on the support of f, such that for all $x \in (A - U)_{reg}$, we have $\mathcal{M}(f)(x) = 0$.*

Proof. For sake of completeness we give a complete proof of this Lemma. Let ω be the support of f. We consider the set ω_A of elements a in A which are in the closure of $H\omega$. For $x \in G/H$, as in (6.2) we consider the polynomial function $\det_{\mathbb{C}}(1 - t - \mathrm{Ad}\,\varphi(x)) = t^n + q_{n-1}(x)t^{n-1} + \ldots + q_l(x)t^l$. Each q_j is an H-invariant regular function on G/H and thus is bounded on ω_A. Therefore, the roots of $\det(1 - t - \mathrm{Ad}\,\varphi(x))$ are bounded on ω_A.

Let Φ be the root system of $(\mathfrak{g}, \mathfrak{a}_{\mathbb{C}})$. Since $\varphi(A) = \exp(\mathfrak{a}_{\mathbb{C}}) \cap \varphi(G/H)$, we can define the functions ξ_α, $\alpha \in \Phi$, on A by $\xi_\alpha(a) = e^{\alpha(X)}$ for $\varphi(a) = \exp X$ with $X \in \mathfrak{a}_{\mathbb{C}}$.

Then the roots of $\det(1 - t - \mathrm{Ad}\,\varphi(a))$ are the numbers $1 - \xi_\alpha(a)$ for $\alpha \in \Phi$. Since these roots are bounded on ω_A, we obtain that the maps $a \to \xi_\alpha(a)$, $\alpha \in \Phi$, are bounded on ω_A. This implies that ω_A is bounded, and hence the closure U of ω_A satisfies the Lemma. $\square$

We say that a closed H-invariant subset U of G/H is a compact modulo H if $U \cap A$ is compact for all Cartan subset A of G/H. Let $C_{comp}^\infty(G/H)$ be the subspace of functions $f \in C^\infty(G/H)$ such that the intersection of the support of f with a compact subset modulo H is compact. Then the orbital integral $\mathcal{M}(f)$ for $f \in C_{comp}^\infty(G/H)$ is well-defined.

According to ([17] Proposition 2.3), we define the space $\mathcal{I}^\infty(G/H)$ of orbital functions on G/H as the image of $C_{comp}^\infty(G/H)$ by $\mathcal{M}$. By ([6] §8 and [17] §2), the space of orbital integrals, that is the image of $C_c^\infty(G/H)$ by $\mathcal{M}$, is the subspace of $\mathcal{I}^\infty(G/H)$ made of functions which satisfy the support property of Lemma 6.1.

6.2 Theorem. *No relative discrete series representation of G/H admits K-finite relative pseudo-coefficient.*

Proof. We first recall the Plancherel formula for G/H.

We fix a system $[Car(\mathfrak{h})]$ of θ-stable representatives of H-conjugacy classes of Cartan subalgebras of $\mathfrak{h}$. We may and will assume that $\mathfrak{a}_0 \in [Car(\mathfrak{h})]$.

Let $\mathfrak{a} \in [Car(\mathfrak{h})]$. Let $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}_R$ be its decomposition with respect to θ. We denote by $\Gamma_\mathfrak{a}$ the lattice made of elements $X \in \mathfrak{a}_R$ such that $\exp 2iX = 1$ and by $\Gamma^*_\mathfrak{a}$ its dual lattice. Let $\mathfrak{a}^*_{I,reg}$ be the set of $\lambda \in \mathfrak{a}^*_I$ such that $(\lambda, \alpha) \neq 0$ for all imaginary roots α of $\mathfrak{a}_\mathbb{C}$ in $\mathfrak{g}$. We choose a positive system Δ of the root system of $(\mathfrak{g}, \mathfrak{a}_\mathbb{C})$ such that for all non imaginary roots $\alpha \in \Delta$, one has $\sigma(\alpha) \in \Delta$. Let $P = \exp(\mathfrak{a}_\mathbb{C})N$ be the Borel subgroup corresponding to Δ. For $\lambda \in \Gamma^*_\mathfrak{a} + \mathfrak{a}^*_{I,reg}$, we define the character δ_λ of $\exp(\mathfrak{a}_\mathbb{C})$ by $\delta_\lambda(\exp(X+iY)) = e^{i\lambda(Y)}$ for $X, Y \in \mathfrak{a}$. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ denotes the normalized induced representation $ind_P^G \delta_\lambda$.

For $M = G$ or H, we denote by $W_M(\mathfrak{a})$ the quotient of the normalizer of $\mathfrak{a}$ in M by the centralizer of $\mathfrak{a}$ in M and we set $\mathcal{W}_\mathfrak{a} := W_G(\mathfrak{a})/W_H(\mathfrak{a})$. Then, by ([17] Théorème 7.4 and [9] §4.2) for allmost $\lambda \in \Gamma^*_\mathfrak{a} + \mathfrak{a}^*_{I,reg}$, there exist linearly independent H-invariant distribution vectors $(\xi^w_\lambda)_{w \in \mathcal{W}_\mathfrak{a}}$ in $(\mathcal{H}^{-\infty}_{\pi_\lambda})^H$, such that the Dirac measure has the following spectral decomposition:

$$f(eH) = \sum_{\mathfrak{a} \in [Car(\mathfrak{h})]} d_\mathfrak{a} \sum_{\mu \in \Gamma^*_\mathfrak{a}} \int_{\mathfrak{a}^*_I} c_{\xi^w_{\lambda+\mu}, \xi^w_{\lambda+\mu}}(f) |\det \mathrm{ad}(\lambda + \mu)_{/\mathfrak{h}^*/\mathfrak{a}^*}|^{1/2} d_I\lambda, \; f \in C^\infty_c(G/H), \tag{6.3}$$

where $d_I\lambda$ is a Haar measure on a^*_I and the $d_\mathfrak{a}$'s are constants depending only on the choice of measures.

For $\mathfrak{a} = \mathfrak{a}_0$, the space $\mathfrak{a}^*_{0I}$ is reduced to zero, and $\mathcal{W}_{\mathfrak{a}_0}$ is reduced to the trivial element which we denote by 1. Then ξ^1_μ coincides with the vector ξ_μ defined in (6.1).

Let $\mu_0 \in \Gamma^*_{\mathfrak{a}_0, reg}$. We assume that there exists a K-finite relative pseudo-coefficient $f_0 \in C^\infty_c(G/H)$ for $(\pi_{\mu_0}, \mathcal{H}_{\mu_0})$, hence $c_{\xi_{\mu_0}, \xi_{\mu_0}}(f_0) \neq 0$, and we want to obtain a contradiction.

Let us first prove that, in that case, there exists a relative pseudo-coefficient $f \in C^\infty_c(G/H)$ for $(\pi_{\mu_0}, \mathcal{H}_{\mu_0})$ such that

$$c_{\xi_\mu, \xi_\mu}(f) = 0 \quad \text{for all} \;\; \mu \in \Gamma^*_{\mathfrak{a}_0, sing}. \tag{6.4}$$

Let $\mathcal{S}$ be the finite set of K-types occurring in $L(K)f$, where L is the left regular representation. Let $\mu \in \Gamma^*_{\mathfrak{a}_0}$ such that $c_{\xi_\mu, \xi_\mu}(f_0) \neq 0$. We consider the projection $P_\mathcal{S}$ of $\mathcal{H}^{-\infty}_\mu$ onto $\oplus_{\tau \in \mathcal{S}} (\mathcal{H}_\mu)_\tau$, where $(\mathcal{H}_\mu)_\tau$ is the τ-isotypic component of $\mathcal{H}_\mu$. Therefore we have

$$c_{\xi_\mu, \xi_\mu}(f_0) = ((\pi_\mu)_{-\infty}(f_0)\xi_\mu, \xi_\mu) = (P_\mathcal{S}(\pi_\mu)_{-\infty}(f_0)\xi_\mu, \xi_\mu) = ((\pi_\mu)_{-\infty}(f_0)\xi_\mu, P_\mathcal{S}\xi_\mu) \neq 0.$$

Thus, the space $\mathrm{Hom}_K(\oplus_{\tau \in \mathcal{S}} \tau, \pi_\mu)$ is non trivial. As $\pi_\mu = ind_P^G \delta_\mu = ind_{L \cap K}^K \delta_{\mu/L \cap K}$, we deduce from Frobenius reciprocity that $\mathrm{Hom}_{K \cap L}(\oplus_{\tau \in \mathcal{S}} (\tau_{/L \cap K}), \delta_{\mu/L \cap K})$ is non trivial. Thus, we deduce that there exists a finite set F in $\Gamma^*_{\mathfrak{a}_0, sing}$ such that for $\mu \in \Gamma^*_{\mathfrak{a}_0, sing} \backslash F$, we have $c_{\xi_\mu, \xi_\mu}(f_0) = 0$.

By ([14] (2.1)), the algebra of G-invariant differential operators on G/H is isomorphic to the center $Z(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$. We identify these two algebras. Let $S(\mathfrak{a}_{0,\mathbb{C}})$ be the symmetric algebra of $\mathfrak{a}_{0,\mathbb{C}}$ and $W(\mathfrak{g}, \mathfrak{a}_{0,\mathbb{C}})$ be the Weyl group of $(\mathfrak{g}, \mathfrak{a}_{0,\mathbb{C}})$. We denote by $\gamma_{\mathfrak{a}_0}$ the isomorphism of Harish-Chandra from $Z(\mathfrak{g})$ to the subalgebra $S(\mathfrak{a}_{0,\mathbb{C}})^{W(\mathfrak{g}, \mathfrak{a}_{0,\mathbb{C}})}$ of $W(\mathfrak{g}, \mathfrak{a}_{0,\mathbb{C}})$-invariant elements in $S(\mathfrak{a}_{0,\mathbb{C}})$.

By ([15] Application 1.), each element $z \in Z(\mathfrak{g})$ acts by $\gamma_{\mathfrak{a}_0}(i\mu)$ on $c_{\xi_\mu, \xi_\mu}(\cdot)$. Since μ_0 is regular, we have $\gamma_{\mathfrak{a}_0}(i\mu_0) \neq \gamma_{\mathfrak{a}_0}(i\mu)$ for all $\mu \in \Gamma^*_{\mathfrak{a}_{0, sing}}$. As F is finite, we can find $z \in Z(\mathfrak{g})$ such that $\gamma_{\mathfrak{a}_0}(i\mu_0) \neq 0$ and $\gamma_{\mathfrak{a}_0}(z)(i\mu) = 0$ for all $\mu \in F$.

Since $c_{\xi_\mu,\xi_\mu}(z.f_0) = \gamma_{\mathfrak{a}_0}(i\mu)c_{\xi_\mu,\xi_\mu}(f_0)$ for $\mu \in \Gamma^*_{\mathfrak{a}_0}$, and $Z(\mathfrak{g})$ acts by a scalar on the generalized matrix coefficients associated to the principal series representations $(\pi_\lambda, \mathcal{H}_\lambda)$ for $\mathfrak{a} \in Car[(\mathfrak{h})]$ and $\lambda \in \Gamma^*_{\mathfrak{a}} + \mathfrak{a}^*_{I,reg}$, we deduce that the function $f := z.f_0$ is a relative pseudo-coefficient for $(\pi_{\mu_0}, \xi_{\mu_0})$ which satisfies (6.4).

From now, we assume that $f \in C^\infty_c(G/H)$ is a relative pseudo-coefficient for $(\pi_{\mu_0}, \mathcal{H}_{\mu_0})$ which satisfies (6.4).

To prove the Theorem, we recall the inversion formula of orbital integrals (see [17] Théorème 6.15).

For $\mathfrak{a} \in [Car(\mathfrak{h})]$, we fix a positive system ψ of imaginary roots of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{g}$.

By ([16] Théorème 6.1) and ([17] Théorème 5.3), to each $\mathfrak{a} \in [Car(\mathfrak{h})]$ and $(\lambda, y) \in (\Gamma^*_{\mathfrak{a}} + \mathfrak{a}^*_{I,reg}) \times \mathcal{W}_{\mathfrak{a}}$, we can associate an H-invariant eigendistribution $\Theta(\lambda, y, \psi)$, which is a generalized matrix coefficient $c_{\xi,\xi'}$ for two H-invariant distribution vectors ξ, ξ' of the principal series representation $(\pi_\lambda, \mathcal{H}_\lambda)$, and an orbital function $F(\lambda, y, \psi)$ such that, for $f \in C^\infty_c(G/H)$ and $x \in (G/H)_{reg}$, we have

$$\mathcal{M}(f)(x) = \sum_{\mathfrak{a} \in [Car\mathfrak{h}]} c_{\mathfrak{a}} \sum_{\mu \in \Gamma^*_{\mathfrak{a}}} \int_{\mathfrak{a}^*_I} \sum_{w \in \mathcal{W}^{-1}_{\mathfrak{a}}} \sum_{y \in \mathcal{W}_{\mathfrak{a}}} F(w(\mu + \lambda), y, \psi)(x) \, \langle \Theta(-w(\mu + \lambda), y, \psi), f \rangle d\lambda, \tag{6.5}$$

where the $c_{\mathfrak{a}}$'s are constants depending only on the choices of measures.

Let $\mathfrak{a} = \mathfrak{a}_0$ and $\mu \in \Gamma^*_{\mathfrak{a}_0}$. We have $\mathcal{W}_{\mathfrak{a}_0} = \{1\}$ and $\psi = \emptyset$. We set $\Theta_\mu := \Theta(\mu, 1, \emptyset)$ and $F_\mu := F(\mu, 1, \emptyset)$. By ([15] Corollaire 2.6 and Corollaire 3.1), the distribution Θ_μ is equal, up to a scalar, to the generalized matrix coefficient c_{ξ_μ,ξ_μ} associated to (π_μ, ξ_μ) and $\Theta_{s\mu} = \Theta_\mu$ for $s \in W_H(\mathfrak{a}_0)$. By ([17] Théorème 4.1), we have also $F_{s\mu} = F_\mu$ for $s \in W_H(\mathfrak{a}_0)$.

Recall that $f \in C^\infty_c(G/H)$ is assumed to be a relative pseudo-coefficient for $(\pi_{\mu_0}, \mathcal{H}_{\mu_0})$ which satisfies (6.4).

We will consider $\mathcal{M}(f)(\mathrm{Exp}X)$ for $\mathrm{Exp}X \in (G/H)_{reg}$.

Let $\mathfrak{a} \in [Car(\mathfrak{h})]$ with $\mathfrak{a} \neq \mathfrak{a}_0$. Let $(y, w) \in \mathcal{W}_{\mathfrak{a}} \times \mathcal{W}^{-1}_{\mathfrak{a}}$ and $\lambda \in \Gamma^*_{\mathfrak{a}} + \mathfrak{a}^*_{I,reg}$.
By ([17] Théorème 5.8), if $y \neq 1$ then $F(w\lambda, y, \psi)(\mathrm{Exp}X) = 0$ for all $\mathrm{Exp}X \in (\mathrm{Exp}\ \mathfrak{h})_{reg}$.
By ([16] Proposition 6.4), for allmost $\lambda \in \Gamma^*_{\mathfrak{a}} + \mathfrak{a}^*_{I,reg}$ and for all $w \in \mathcal{W}_{\mathfrak{a}}$, the distribution $\Theta(-w^{-1}\lambda, 1, \psi)$ is equal, up to a scalar, to the generalized matrix coefficient $c_{\xi^w_\lambda,\xi^w_\lambda}$ where $(\xi^w_\lambda)_{w \in \mathcal{W}_{\mathfrak{a}}}$ is the family of $(\mathcal{H}^{-\infty}_\lambda)^H$ which occurs in Plancherel formula (6.3). Hence, since f is a relative pseudo-coefficient for $(\pi_{\mu_0}, \mathcal{H}_{\mu_0})$, we have $\langle \Theta(-w\lambda, 1, \psi), f \rangle = 0$ for all $w \in \mathcal{W}^{-1}_{\mathfrak{a}}$.
Moreover, since f satisfies (6.4), we have $\Theta_\mu(f) = 0$ for $\mu \in \Gamma^*_{\mathfrak{a}_0}$ with $\mu \neq s\mu_0$ for all $s \in W_H(\mathfrak{a}_0)$.

Therefore, by the above properties of $F(w^{-1}\lambda, y, \psi)$ for $y, w \in \mathcal{W}_{\mathfrak{a}}$, and the inversion formula (6.5), we obtain

$$\mathcal{M}(f)(\mathrm{Exp}X) = C_0 F_{\mu_0}(\mathrm{Exp}X) \langle \Theta_{\mu_0}, f \rangle, \quad \mathrm{Exp}X \in (\mathrm{Exp}\ \mathfrak{h})_{reg}. \tag{6.6}$$

where $C_0 := c_{\mathfrak{a}_0} |W_H(\mathfrak{a}_0)|$.

To obtain our contradiction about the existence of the pseudo-coefficient f, we need to come back to the definition of F_{μ_0} on a open subset of $(\mathrm{Exp}\ \mathfrak{h})_{reg}$.

For $\varepsilon > 0$, we denote by $\mathcal{V}_\varepsilon$ the open subset of elements $X \in \mathfrak{h}$ such that the real part $\mathrm{Re}(\lambda)$ of each eigenvalue λ of $\mathrm{ad}X$ satisfies $|\mathrm{Re}(\lambda)| < \varepsilon$. For ε small enough, the map Exp is a diffeomorphism from $\mathcal{V}_\varepsilon$ to $\mathrm{Exp}\mathcal{V}_\varepsilon$ (see ([17] Lemma 4.4). Thus, $X \in \mathcal{V}_{\varepsilon,reg}$ if and only if $\mathrm{Exp}X \in (\mathrm{Exp}\mathcal{V}_\varepsilon)_{reg}$.

Then by construction of F_{μ_0} (see [17] §4, page 76), we have

$$F_{\mu_0}(\mathrm{Exp}X) = \hat{\beta}_{H\cdot X}(\mu_0) \mid \det(\mathrm{ad}\,\mu_0)_{\mathfrak{h}^*/\mathfrak{a}_0^*} \mid^{1/2}, \qquad X \in \mathcal{V}_{\varepsilon,reg}, \tag{6.7}$$

where $\hat{\beta}_{H\cdot X}$ is the Fourier transform of the Liouville measure on the orbit $H.X$.

Classical results of Harish-Chandra on $\hat{\beta}_{H\cdot X}$ implies that F_{μ_0} does not satisfy the support property of Lemma 6.1. We give below a proof of this result for sake of completeness.

Let α be a (real) root of $\mathfrak{a}_0$ in $\mathfrak{h}$. Let $X_{\pm\alpha} \in \mathfrak{h}$ be root vectors in $\mathfrak{h}$ such that $H_\alpha := [X_{-\alpha}, X_\alpha]$ is the coroot of α. Then $\mathfrak{a}_\alpha := \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \mathrm{Ker}\,\alpha$ is a Cartan subalgebra of $\mathfrak{h}$ and $c_\alpha.\mathfrak{a}_{0\mathbb{C}} = \mathfrak{a}_{\alpha\mathbb{C}}$ where $c_\alpha := \mathrm{Ad}(\exp -i\frac{\pi}{4}(X_\alpha + X_{-\alpha}))$ is the usual Cayley transform. Then the imaginary roots of $\mathfrak{a}_\alpha$ in $\mathfrak{h}$ are $\beta = c_\alpha(\alpha)$ and $-\beta$ and we have $\mathfrak{a}_\alpha = \mathbb{R}iH_\beta + \mathrm{Ker}\,\beta$ with $\mathrm{Ker}\,\beta = \mathrm{Ker}\,\alpha$. We choose a basis Δ of the root system of $(\mathfrak{g}, \mathfrak{a}_{\alpha,\mathbb{C}})$. Each $\gamma \in \Delta$ can be written $\gamma = \gamma_I + \gamma_R$ according to the decomposition $\mathfrak{a}^* = \mathfrak{a}_I^* + \mathfrak{a}_R^*$. We denote by $\mathcal{C}$ the connected component of $\mathfrak{a}_{\alpha,reg}$ made of elements X such that $(i\gamma_I + \gamma_R)(X) > 0$ for $\gamma = \gamma_I + \gamma_R \in \Delta$.

Then, by the properties of the Fourier transform of orbits (see [27] Theorem I.7.7), there exist constants $c(w)$, $w \in W_H(\mathfrak{a}_0)$, such that, for all $X \in \mathcal{C}$, we have

$$\hat{\beta}_{H\cdot X}(\mu_0) \mid \det(\mathrm{ad}\,\mu_0)_{\mathfrak{h}^*/\mathfrak{a}_0^*} \mid^{1/2} = \sum_{w \in W_H(\mathfrak{a}_0)} c(w) e^{i\langle c_\alpha w\mu_0, X\rangle},$$

with $c(w) \neq 0$ if and only if $\mathrm{Im}\langle c_\alpha w\mu_0, X\rangle \geqslant 0$.

We set $\mathcal{C}_\varepsilon = \mathcal{C} \cap \mathcal{V}_\varepsilon$. By (6.7) we deduce that for all $X \in \mathcal{C}_\varepsilon$, we have

$$F_{\mu_0}(\mathrm{Exp}X) = \sum_{w \in W_H(\mathfrak{a}_0)} c(w) e^{i\langle c_\alpha w\mu_0, X\rangle}. \tag{6.8}$$

Each element of $\mathcal{C}_\varepsilon$ can be written $X = -itH_\beta + Y$ with $t > 0$ and $Y \in \mathrm{Ker}\beta \cap \mathcal{C}_\varepsilon$. Since $e^{2i\beta(X)} = e^{2t}$, the subset of elements $\mathrm{Exp}X$, for $X \in \mathcal{C}_\varepsilon$ is not included in a compact subset of the Cartan subset A_α associated to $\mathfrak{a}_\alpha$.

Then (6.6) and the support property of Lemma 6.1 would imply that there is an open subset $U \subset \mathcal{C}_\varepsilon$ such that $F_{\mu_0}(\mathrm{Exp}X) = 0$ for all $X \in U$. This would imply that the right hand side of (6.8), which is an analytic function of X, is equal to 0 for all $X \in \mathcal{C}$. Since μ_0 is regular, the linear forms $X \mapsto i\langle c_\alpha w\mu_0, X\rangle$ for $w \in W_H(\mathfrak{a}_0)$ are distinct elements. Thus we would deduce that $c(w) = 0$ for all $w \in W_H(\mathfrak{a}_0)$, which is impossible. This contradiction achieves the proof of the Theorem. □

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