Regularity of some invariant distributions on nice symmetric pairs

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Abstract

J. Sekiguchi determined the semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h})$, called nice symmetric pairs, on which there is no non-zero invariant eigendistribution with singular support. On such pairs, we study regularity of invariant distributions annihilated by a polynomial of the Casimir operator. We deduce that invariant eigendistributions on $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$ are locally integrable functions.

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Introduction

Let G be a reductive group such that $\mathrm{Ad}(G)$ is connected. Let σ be an involutive automorphism of G. We denote by the same letter σ the corresponding involution on the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition into $+1$ and -1 eigenspaces with respect to σ. Then $(\mathfrak{g}, \mathfrak{h})$ is called a reductive symmetric pair (or semisimple when $\mathfrak{g}$ is semisimple). Let H be the group of fixed points of σ in G.

In [7], J. Sekiguchi describes semisimple symmetric pairs on which there is no non-zero invariant eigendistribution with support in $\mathfrak{q} - \mathfrak{q}^{reg}$ where $\mathfrak{q}^{reg}$ is the set of semisimple regular elements of $\mathfrak{q}$. These pairs, called nice symmetric pairs, are characterized by a property on distinguished nilpotent elements and we can generalize this notion to reductive pairs (Definition 4.1). Our main result is the following . Let ω be the Casimir polynomial of $\mathfrak{q}$ and $\partial(\omega)$ the corresponding differential operator on $\mathfrak{q}$.

Theorem 0.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a nice reductive symmetric pair. Let $\mathcal{V}$ be an H- invariant open subset of $\mathfrak{q}$. Let Θ be an H-invariant distribution on $\mathcal{V}$ such that*

1. *There exists $P \in \mathbb{C}[X]$ such that $P\big(\partial(\omega)\big)\Theta = 0$,*
2. *There exists $F \in L^1_{loc}(\mathcal{V})^H$ such that $\Theta = F$ on $\mathcal{V} \cap \mathfrak{q}^{reg}$.*

Then $\Theta = F$ as distribution on $\mathcal{V}$.

In [2], E. Galina and Y. Laurent obtained stronger results on invariant distributions on nice symmetric pairs by different methods based on algebraic properties of $\mathcal{D}$-modules. They proved

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that any invariant distribution on a nice pair which is annihilated by a finite codimensional ideal of the algebra of H-invariant differential operators with constant coefficients on $\mathfrak{q}$ is a locally integrable function ([2] Corollary 1.7.6).

Our approach uses properties of distributions. Assuming that $S = \Theta - F$ is non-zero, we are led to a contradiction. By the work of G. van Dijk ([8]) and J. Sekiguchi ([7]), we can adapt the descent method of Harish-Chandra. Thus, we construct a non-zero distribution $\tilde{S}$ defined on a neighborhood W of 0 in $\mathbb{R}^r \times \mathbb{R}^m$ with support in $(\{0\} \times \mathbb{R}^m) \cap W$ such that there exist a locally integrable function $\tilde{F}$ on W and a differential operator D, which is obtained from radial parts of $\partial(\omega)$ near semisimple elements and nilpotent elements, satisfying $P(D)\tilde{S} = P(D)\tilde{F}$. Using the method developed by M. Atiyah in [1], one studies the degree of singularity along $\{0\} \times \mathbb{R}^m$ of different distributions in this equation. One deduces that $\tilde{S} = 0$ and thus a contradiction.

In the last section, we complete the results of [3] on the nice symmetric pair $(\mathfrak{g}l(4,\mathbb{R}), \mathfrak{g}l(2,\mathbb{R}) \times \mathfrak{g}l(2,\mathbb{R}))$ and deduce that any invariant eigensdistribution for a regular character on this pair is given by a locally integrable function.

1 Notation

Let M be a smooth variety. Let $C^\infty(M)$ be the space of smooth functions on M, $\mathcal{D}(M)$ the subspace of compactly supported smooth functions, $L^1_{loc}(M)$ the space of locally integrable functions on M, endowed with their standard topology and $\mathcal{D}'(M)$ the space of distributions on M.

For a group G acting on M, one denotes by $\mathcal{F}^G$ the points of $\mathcal{F}$ fixed by G for each space $\mathcal{F}$ defined as above.

If $N \subset M$ and if f is a function defined on M, one denotes by $f_{/N}$ its restriction to N.

If V is a finite dimensional real vector space then V^* is its algebraic dual and $V_\mathbb{C}$ is its complexified vector space.The symmetric algebra $S[V]$ of V can be identified to the space $\mathbb{R}[V^*]$ of polynomial functions on V^* with real coefficients and to the space of differential operators with real constant coefficients on V. Similary, one has $S[V_\mathbb{C}] = \mathbb{C}[V^*]$ and this algebra can be identified to the space of differential operators with complex constant coefficients on $V_\mathbb{C}$. If $u \in S[V]$ (resp. $S[V_\mathbb{C}]$), then $\partial(u)$ will denote the corresponding differential operator.

Let G be a reductive group such that $\mathrm{Ad}(G)$ is connected, and σ an involution on G. This defines an involution, denoted by the same letter σ on the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the direct decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces of σ. Then $(\mathfrak{g}, \mathfrak{h})$ is called a reductive symmetric pair. Let H be the subgroup of fixed points of σ in G.

Let $\mathfrak{c}_\mathfrak{g}$ be the center of $\mathfrak{g}$ and $\mathfrak{g}_s$ its derived algebra. We set

$$\mathfrak{c}_\mathfrak{q} = \mathfrak{c}_\mathfrak{g} \cap \mathfrak{q} \text{ and } \mathfrak{q}_s = \mathfrak{g}_s \cap \mathfrak{q}.$$

If x is an element of $\mathfrak{g}$ and $\mathfrak{r}$ is a subspace of $\mathfrak{g}$, we denote by $\mathfrak{r}_x$ the centralizer of x in $\mathfrak{r}$.

We fix a non-degenerate bilinear form B on $\mathfrak{g}$ which is equal to the Killing form on $\mathfrak{g}_s$. Then $\omega(X) = B(X, X)$ is the Casimir polynomial of $\mathfrak{q}$.

2 Transfer of distributions and differential operators

We recall results of ([8] sections 2 and 3) and ([7] section (3.2)) on restriction of distributions and radial parts of differential operators. Their proofs are similar to ([4] or [10] Part I, chapter 2).

Let $x_0 \in \mathfrak{q}_s$. Let U be a linear subspace of $\mathfrak{q}$ such that $\mathfrak{q} = U \oplus [x_0, \mathfrak{h}]$ and V be a linear subspace of $\mathfrak{h}$ such that $\mathfrak{h} = V \oplus \mathfrak{h}_{x_0}$. Consider the open subset $'U = \{Z \in U; U + [x_0+Z, \mathfrak{h}] = \mathfrak{q}\}$ containing 0. Then the map Ψ from $H \times 'U$ to $\mathfrak{q}$ defined by $\Psi(h,u) = h \cdot (x_0+u)$ is a submersion. In particular, $\Omega = \Psi(H \times' U)$ is an open H-invariant subset of $\mathfrak{q}$ containing x_0. We fix an Haar measure dh on H and we denote by du (respectively dx) the Lebesgue measure on U (respectively $\mathfrak{q}$). The submersion Ψ induces a continuous surjective map $\Psi_\star$ from $\mathcal{D}(H \times' U)$ onto $\mathcal{D}(\Omega)$ such that, for any $F \in L^1_{loc}(\mathfrak{q})$ and any $f \in \mathcal{D}(H \times' U)$, one has

$$\int_{H\times U} F \circ \Psi(h,u) f(h,u) dh\, du = \int_{\mathfrak{q}} F(x) \Psi_\star(f)(x) dx.$$

Theorem 2.1. *For $T \in \mathcal{D}'(\Omega)^H$ there exists a unique distribution $\mathcal{R}es_U T$ defined on $'U$, called the restriction of T to $'U$ with respect to Ψ, such that for any $f \in \mathcal{D}(H \times' U)$, one has*

$$< T, \Psi_\star(f) >=< \mathcal{R}es_U T, p_\star(f) >$$

where $p_\star(f) \in \mathcal{D}(U)$ is defined by $p_\star(f)(u) = \int_H f(h,u) dh$.

This restriction satisfies the following properties:

1. *If U is stable under the action of a subgroup H_0 of H then $\mathcal{R}es_U T$ is H_0-invariant.*
2. *$x_0 +$ supp $(\mathcal{R}es_U T) \subset$ supp $(T) \cap (x_0 +' U)$.*
3. *If $F \in L^1_{loc}(\Omega)^H$ then $\mathcal{R}es_U F$ is the locally integrable function on $'U$ defined by $\mathcal{R}es_U F(u) = F(x_0+u)$.*
4. *If $\mathcal{R}es_U T = 0$ then $T = 0$ on Ω.*

Theorem 2.2. *Let D be a H-invariant differential operator on $\mathfrak{q}$. Then there exists a differential operator $\mathcal{R}ad_U(D)$, called the radial part of D with respect to Ψ, defined on $'U$ such that for any $f \in \mathcal{D}(\Omega)^H$, one has $(D \cdot f)(x_0+u) = \mathcal{R}ad_U(D) \cdot \mathcal{R}es_U f(u)$ for $u \in' U$.*

Morever, for any $T \in \mathcal{D}'(\Omega)^H$, one has

$$\mathcal{R}es_U(D \cdot T) = \mathcal{R}ad_U(D) \cdot \mathcal{R}es_U(T).$$

3 Semisimple elements

We recall that a Cartan subspace of $\mathfrak{q}$ is a maximal abelian subspace of $\mathfrak{q}$ consisting of semisimple elements.

If $\mathfrak{r} = \mathfrak{q}$ or $\mathfrak{q}_s$, we denote by $\mathcal{S}(\mathfrak{r})$ the set of semisimple elements of $\mathfrak{r}$.

Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$. If $\lambda \in \mathfrak{g}^*_{\mathbb{C}}$, we set

$$\mathfrak{g}^\lambda_{\mathbb{C}} = \{X \in \mathfrak{g}_{\mathbb{C}}; [A, X] = \lambda(A)X \text{ for any } A \in \mathfrak{a}_{\mathbb{C}}\}$$

and
$$\Sigma(\mathfrak{a}) = \{\lambda \in \mathfrak{g}_{\mathbb{C}}^*; \mathfrak{g}_{\mathbb{C}}^{\lambda} \neq \{0\}\}.$$
Then $\Sigma(\mathfrak{a})$ is the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$.
An element X of $\mathcal{S}(\mathfrak{q})$ is $\mathfrak{q}$-regular (or regular) if its centralizer $\mathfrak{q}_X$ in $\mathfrak{q}$ is a Cartan subspace. If $X \in \mathfrak{a}$ then X is regular if and only if $\lambda(X) \neq 0$ for all $\lambda \in \Sigma(\mathfrak{a})$. We denote by $\mathfrak{q}^{reg}$ the open dense subset of semisimple regular elements of $\mathfrak{q}$.

Let $A_0 \in \mathcal{S}(\mathfrak{q})$. Its centralizer $\mathfrak{z} = \mathfrak{g}_{A_0}$ in $\mathfrak{g}$ is a reductive σ-stable Lie subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{c}$ its center and by $\mathfrak{z}_s$ its derived algebra. We set
$$\mathfrak{c}^- = \mathfrak{c} \cap \mathfrak{q}, \quad \mathfrak{c}^+ = \mathfrak{c} \cap \mathfrak{h}, \quad \mathfrak{z}_s^- = \mathfrak{z}_s \cap \mathfrak{q} \quad \text{and} \quad \mathfrak{z}_s^+ = \mathfrak{z}_s \cap \mathfrak{h}.$$
The pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ is a semisimple symmetric subpair of $(\mathfrak{g}_s, \mathfrak{h}_s)$ which is equal to $(\mathfrak{g}_s, \mathfrak{h}_s)$ if $A_0 \in \mathfrak{c}_{\mathfrak{q}}$. Let H_s^+ be the analytic subgroup of H with Lie algebra $\mathfrak{z}_s^+$.

We assume that $A_0 \notin \mathfrak{c}_{\mathfrak{q}}$. We take a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$ containing A_0 and consider the corresponding root system $\Sigma = \Sigma(\mathfrak{a})$. We fix a positive system Σ^+ of Σ. For any $\lambda \in \Sigma^+$, we choose a $\mathbb{C}$-basis $X_{\lambda,1}, \dots X_{\lambda,m_\lambda}$ of $\mathfrak{g}_{\mathbb{C}}^{\lambda}$ such that $B(X_{\lambda,i}, \sigma(X_{\lambda,j})) = -\delta_{i,j}$ for $i, j \in \{1, \dots, m_\lambda\}$. Let $\Sigma_1^+ = \{\lambda \in \Sigma^+; \lambda(A_0) \neq 0\}$. We set
$$V_{\mathbb{C}}^{\pm} = \sum_{\lambda \in \Sigma_1^+} \sum_{j=1}^{m_\lambda} \big(X_{\lambda,j} \pm \sigma(X_{\lambda,j})\big), \quad V^+ = V_{\mathbb{C}}^+ \cap \mathfrak{h}, \quad V^- = V_{\mathbb{C}}^- \cap \mathfrak{q}.$$
We have the decompositions $\mathfrak{h} = \mathfrak{z}^+ \oplus V^+$ and $\mathfrak{q} = \mathfrak{z}^- \oplus V^-$, with $dim\ V^+ = dim\ V^-$ and $[A_0, \mathfrak{h}] = V^-$.

If $Z_0 \in \mathfrak{z}^-$, we define the map η_{Z_0} from $V^+ \times \mathfrak{z}^-$ to $\mathfrak{q}$ by $\eta_{Z_0}(v, Z) = Z + [v, A_0 + Z_0]$. Then η_0 is a bijective map. We set $\xi(Z_0) = det(\eta_{Z_0} \circ \eta_0^{-1})$ and $'\mathfrak{z}^- = \{Z \in \mathfrak{z}^-; \xi(Z) \neq 0\}$. Then $'\mathfrak{z}^-$ is invariant under H_s^+.

Thus the map γ from $H \times' \mathfrak{z}^-$ to $\mathfrak{q}$ defined by $\gamma(h, Z) = h \cdot (A_0 + Z)$ is a submersion. By Theorem 2.1, for any H-invariant distribution Θ on $\mathfrak{q}$, there exists a unique H_s^+-invariant distribution $\mathcal{R}es_{\mathfrak{z}^-}\Theta$ defined on $'\mathfrak{z}^-$ such that, for any $f \in \mathcal{D}(H \times' \mathfrak{z}^-)$, one has $< \Theta, \gamma_\star(f) > = < \mathcal{R}es_{\mathfrak{z}^-}\Theta, p_\star(f) >$.

Let $\omega_{\mathfrak{z}^-}$ be the restriction of ω to $\mathfrak{z}^-$. Then, one has:

Lemma 3.1. *([7]) Lemma 4.4). Let $\mathcal{R}ad_{\mathfrak{z}^-}(\partial(\omega))$ be the radial part of $\partial(\omega)$ with respect to γ (Theorem 2.2). Then*
$$\mathcal{R}ad_{\mathfrak{z}^-}(\partial(\omega)) = \xi^{-1/2}\partial(\omega_{\mathfrak{z}^-}) \circ \xi^{1/2} - \mu$$
where $\mu(Z) = \xi(Z)^{-1/2}\big(\partial(\omega_{\mathfrak{z}^-})\xi^{1/2}\big)(Z)$ is an analytic function on $'\mathfrak{z}^-$.

4 Nilpotent and distinguished elements

Let $Z_0 \in \mathfrak{q}$. Let $Z_0 = A_0 + X_0$ be its Jordan decomposition ([7] Lemma 1.1). We construct the symmetric pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ related to A_0 as in 3.

We assume that X_0 is different from zero. From ([7] Lemma 1.7), there exists a normal sl_2-triple (B_0, X_0, Y_0) of $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ containing X_0, i.e. satisfying $B_0 \in \mathfrak{z}_s^+$ and $Y_0 \in \mathfrak{z}_s^-$ such that $[B_0, X_0] = 2X_0$, $[B_0, Y_0] = -2Y_0$ and $[X_0, Y_0] = B_0$.

We set $\mathfrak{z}_0 = \mathbb{R}B_0 + \mathbb{R}X_0 + \mathbb{R}Y_0$. The Cartan involution θ_0 of $\mathfrak{z}_0$ defined by $\theta_0 : (B_0, X_0, Y_0) \to (-B_0, -Y_0, -X_0)$ extends to a Cartan involution of $\mathfrak{z}_s$, denoted by θ, which commutes with σ. ([8] Lemma 1). The bilinear form $(X, Y) \mapsto -B(\theta(X), Y)$ defines a scalar product on $\mathfrak{z}_s$.

We can decompose $\mathfrak{z}_s$ in an orthogonal sum $\mathfrak{z}_s = \sum_i \mathfrak{z}_i$ of irreducible representations $\mathfrak{z}_i$ under the adjoint action of $\mathfrak{z}_0$. One can choose a suitable ordering of the $\mathfrak{z}_i$ such that $(\mathfrak{z}_s^-)_{Y_0} = \sum_{i=1}^r \mathfrak{z}_i \cap (\mathfrak{z}_s^-)_{Y_0} = \theta((\mathfrak{z}_s^-)_{X_0})$ with $\mathfrak{z}_1 = \mathfrak{z}_0$ and dim $\mathfrak{z}_i \cap (\mathfrak{z}_s^-)_{Y_0} = 1$. We set $n_i + 1 =$ dim $\mathfrak{z}_i$. Hence, there exists an orthonormal basis $(w_1, \ldots, w_r)$ of $(\mathfrak{z}_s^-)_{Y_0}$ such that $w_1 = \dfrac{Y_0}{\|Y_0\|}$ and $[B_0, w_i] = -n_i w_i$ for $i \in \{1, \ldots, r\}$. In particular, one has $n_1 = 2$.

We set
$$\delta_{\mathfrak{q}}(Z_0) = \delta_{\mathfrak{z}_s^-}(X_0) = \sum_{i=1}^{r} (n_i + 2) - dim\ (\mathfrak{z}_s^-).$$

Let $\mathcal{N}(\mathfrak{z}_s^-)$ be the set of nilpotent elements of $\mathfrak{z}_s^-$.

Definition 4.1. *([7] Definitions 1.11 and 1.13)*

1. *An element X_0 of $\mathcal{N}(\mathfrak{z}_s^-)$ is a $\mathfrak{z}_s^-$-distinguished nilpotent element if $(\mathfrak{z}_s^-)_{X_0}$ contains no non-zero semisimple element.*
2. *An element Z_0 of $\mathfrak{q}$ with Jordan decomposition $Z_0 = A_0 + X_0$ is called $\mathfrak{q}$-distinguished if X_0 is a $\mathfrak{z}_s^-$-distinguished nilpotent element of $\mathfrak{z}_s^-$.*

Definition 4.2. *The symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is nice if for any $\mathfrak{q}$-distinguished element Z, one has $\delta_{\mathfrak{q}}(Z) > 0$.*

Let ω_s be the restriction of ω to $\mathfrak{z}_s^-$. Though ω_s is not the Casimir polynomial on $\mathfrak{z}_s^-$, one has the following result:

Lemma 4.3. *([8] Lemma 4) The following assertions are equivalent:*

1. *X_0 is a $\mathfrak{z}_s^-$-distinguished nilpotent element.*
2. *$\omega_s(X) = 0$ for all $X \in (\mathfrak{z}_s^-)_{X_0}$.*
3. *$\omega_s(X) = 0$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$.*
4. *$n_i > 0$.*
5. *$(\mathfrak{z}_s^-)_{X_0} \cap (\mathfrak{z}_s^-)_{Y_0} = \{0\}$.*

Thus, if X_0 is a $\mathfrak{z}_s^-$-distinguished nilpotent element then one has $\omega(X_0 + X) = 2B(X_0, X) = 2\|Y_0\| x_1$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$, where x_1 is the first coordinate of X in the basis $(w_1, \ldots, w_r)$ of $(\mathfrak{z}_s^-)_{Y_0}$.

For any $X_0 \in \mathcal{N}(\mathfrak{z}_s^-)$, one has $\mathfrak{z}_s^- = (\mathfrak{z}_s^-)_{Y_0} \oplus [\mathfrak{z}_s^+, X_0]$ and $\mathfrak{z}_s^+ = (\mathfrak{z}_s^+)_{X_0} \oplus [\mathfrak{z}_s^-, Y_0]$. From now on, we set
$$U = (\mathfrak{z}_s^-)_{Y_0}.$$
For $X \in U$, we consider the map ψ_X from $[\mathfrak{z}_s^-, Y_0] \times U$ to $\mathfrak{z}_s^-$ defined by $\psi_X(v, z) = z + [v, X_0 + X]$. The map ψ_0 is bijective.

We set $\kappa(X) = det(\psi_X \circ \psi_0^{-1})$ and ${}'U = \{X \in U; \kappa(X) \neq 0\}$. Hence, the map π from $H_s^+ \times {}' U$ to $\mathfrak{z}_s^-$ defined by $\pi(h, X) = h \cdot (X_0 + X)$ is a submersion.

We precise now some properties of π related to $\mathcal{N}(\mathfrak{z}_s^-)$.

By ([9] Theorem 23]), we can write $\mathcal{N}(\mathfrak{z}_s^-) = \mathcal{O}_1 \cup \ldots \mathcal{O}_\nu$ where the $\mathcal{O}_j$ are disjoints H_s^+-orbits with $\mathcal{O}_\nu = \{0\}$ and each $\mathcal{O}_j$ is open in the closed set $\mathcal{N}_j = \mathcal{O}_j \cup \ldots \mathcal{O}_\nu$. One assumes that $\mathcal{O}_j = H_s^+ \cdot X_0$.

Lemma 4.4. *([8] Lemma 17 and 18). There exists a neighborhood U_0 of 0 in U such that*

1. *π is a submersion on $H_s^+ \times U_0$,*
2. *$\Omega_0 = \pi(H_s^+ \times U_0)$ is an open neighborhood of X_0 in $\mathfrak{z}_s^-$ and $\Omega_0 \cap \mathcal{N}_j = \mathcal{O}_j$,*
3. *$\mathcal{O}_j \cap (X_0 + U_0) = \{X_0\}$*
4. *Let Θ be an H_s^+-invariant distribution on Ω_0. Let $\mathcal{R}es_U\Theta$ be its restriction to U with respect to π.*

   If supp $(\Theta) \subset \mathcal{N}_j$ then supp $(\mathcal{R}es_U\Theta) \subset \{0\}$.

We denote by $\omega_{\mathfrak{c}^-}$ and ω_s the restrictions of ω to $\mathfrak{c}^-$ and $\mathfrak{z}_s^-$ respectively. One has $\omega_{\mathfrak{z}^-} = \omega_{\mathfrak{c}^-} + \omega_s$. We precise now the radial part $\mathcal{R}ad_U(\partial(\omega_s))$ of $\partial(\omega_s)$ with respect to π. We denote by $\mathcal{R}ad_{U,X}(\partial(\omega_s))$ its local expression at $X \in U_0$.

Lemma 4.5. *([8] Lemma 13) The homogeneous part of degree 2 of $\mathcal{R}ad_{U,0}(\partial(\omega_s))$ is zero if and only if X_0 is $\mathfrak{z}_s^-$-distinguished.*

Theorem 4.6. *([8] Theorem 14) Let X_0 be a $\mathfrak{z}_s^-$-distinguished nilpotent element and $c_0 = \|X_0\|$. Then, there exist analytic functions $a_{i,j}$ $(2 \leq i,j \leq r)$ and a_i $(2 \leq i \leq r)$ on U_0 satisfying $a_{i,j}(0) = 0$ such that, for any H_s^+-invariant distribution T on Ω_0, one has*

$$\mathcal{R}es_U(\partial(\omega_s)T) = \mathcal{R}ad_U((\partial(\omega_s))\mathcal{R}es_U(T)$$
$$= \frac{1}{c_0}\Big(2x_1\frac{\partial^2}{\partial x_1^2} + (dim\, \mathfrak{z}_s^-)\frac{\partial}{\partial x_1} + \sum_{i=2}^{r}(n_i+2)x_i\frac{\partial^2}{\partial x_1 \partial x_i}$$
$$+ \sum_{2 \leq i \leq j \leq r} a_{i,j}(X)\frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=2}^{r} a_i(X)\frac{\partial}{\partial x_i}\Big)\mathcal{R}es_U(T)$$

where $x_1, \ldots, x_r$ are the coordinates of X in the basis $(w_1, \ldots, w_r)$.

5 The main Theorem

Our goal is to prove the following Theorem:

Theorem 5.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a nice reductive symmetric pair. Let $\mathcal{V}$ an H- invariant open subset of $\mathfrak{q}$. Let Θ be an H-invariant distribution on $\mathcal{V}$ such that*

1. *There exists $P \in \mathbb{C}[X]$ such that $P\big(\partial(\omega)\big)\Theta = 0$*

2. *There exists* $F \in L^1_{loc}(\mathcal{V})^H$ *such that* $\Theta = F$ *on* $\mathcal{V} \cap \mathfrak{q}^{reg}$.

Then $\Theta = F$ *as distribution on* $\mathcal{V}$.

We will use the method developed by M. Atiyah in [1]. First we recall some facts about distributions on $\mathbb{R}^r \times \mathbb{R}^m$. Let $\mathbb{N}$ be the set of non-negative integers. For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, we set $|\alpha| = \alpha_1 + \ldots + \alpha_r$ and

$$x^\alpha = x_1^{\alpha_1} \ldots x_r^{\alpha_r}, \quad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_r^{\alpha_r}}.$$

For $\varphi \in \mathcal{D}(\mathbb{R}^r \times \mathbb{R}^m)$ and $\varepsilon > 0$, we set $\varphi_\varepsilon(x, y) = \varphi(\frac{x}{\varepsilon}, y)$ for $(x, y) \in \mathbb{R}^r \times \mathbb{R}^m$. For $T \in \mathcal{D}'(\mathbb{R}^r \times \mathbb{R}^m)$ we denote by T_ε the distribution defined by $< T_\varepsilon, \varphi >=< T, \varphi_\varepsilon >$.

Definition 5.2. *Let* $V = \{0\} \times \mathbb{R}^m \subset \mathbb{R}^r \times \mathbb{R}^m$ *and* $T \in \mathcal{D}'(\mathbb{R}^r \times \mathbb{R}^m)$.

1. *The distribution* T *is regular along* V *if* $\lim\limits_{\epsilon \to 0} T_\epsilon = 0$.
2. *The distribution* T *has a degree of singularity along* V *smaller than* k *if for all* $\alpha \in \mathbb{N}^r$ *with* $|\alpha| = k$, *the distribution* $x^\alpha T$ *is regular.*

   We denote by $d_s^\circ T$ *the degree of singularity of* T *along* V *and we omit in what follows to precise "along* V*". Regularity corresponds to a degree of singularity equal to* 0.
3. *The degree of singularity of* T *is equal to* k *if* $d_s^\circ T \leq k$ *and* $d_s^\circ T \not\leq k-1$.

Lemma 5.3. 1. *If* $F \in L^1_{loc}(\mathbb{R}^{r+m})$ *then* $d_s^\circ F = 0$.

2. *If* $d_s^\circ T = k \geq 1$ *then* $d_s^\circ(x_i T) = k - 1$ *for* $i \in \{1, \dots r\}$.
3. *If* $d_s^\circ T \leq k$ *then* $\dfrac{\partial}{\partial x_i} T \leq k + 1$ *for* $i \in \{1, \dots r\}$.
4. *Let* δ_0 *be the Dirac measure at* $0 \in \mathbb{R}^r$ *and* $\delta_0^{(\alpha)} = \partial_x^\alpha \delta_0$. *If* $S \in \mathcal{D}'(\mathbb{R}^m)$ *then the degree of singularity of* $\delta_0^{(\alpha)} \otimes S$ *is equal to* $|\alpha| + 1$.

Proof. 1. Let $F \in L^1_{loc}(\mathbb{R}^{r+m})$ and $\phi \in \mathcal{D}(\mathbb{R}^{r+m})$ with $\mathrm{supp}(\phi) \subset K_1 \times K_2$ where K_1 (resp., K_2) is a compact subset of $\mathbb{R}^r$ (resp., $\mathbb{R}^m$). One has

$$|\int_{\mathbb{R}^r \times \mathbb{R}^m} F(x, y)\phi(\frac{x}{\varepsilon}, y)dxdy| \leq \sup_{(x,y)\in\mathbb{R}^{r+m}} |\phi(x, y)| \int_{(\varepsilon K_1)\times K_2} |F(x, y)|dxdy$$

and the first assertion follows.
2. is clear.
3. Let $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k + 1$. If $\alpha_j \geq 1$ for some $j \in \{1, \dots, r\}$, we set $\bar{\alpha}^j = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_r)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^{r+m})$.

If $\alpha_i \geq 1$, one has

$$< x^\alpha \frac{\partial}{\partial x_i} T, \varphi_\varepsilon >= - < T, \alpha_i x^{\bar{\alpha}^i} \varphi_\varepsilon + \frac{x^\alpha}{\varepsilon} (\frac{\partial}{\partial x_i}\varphi)_\varepsilon >$$

$$= -\alpha_i < x^{\bar{\alpha}^i} T, \varphi_\varepsilon > - < x^{\bar{\alpha}^i} T, (x_i \frac{\partial}{\partial x_i}\varphi)_\varepsilon >$$

thus $(x^\alpha T)_\varepsilon$ converges to 0 since $d_s^\circ T \leq k$.

If $\alpha_i = 0$, we choose j such that $\alpha_j \geq 1$. One has $< x^\alpha \dfrac{\partial}{\partial x_i} T, \varphi_\varepsilon > = - < x^{\bar{\alpha}^j} T, (x_j \dfrac{\partial}{\partial x_i}\varphi)_\varepsilon >$ which tends to 0 as before.

4. We recall that for $i \in \{1, \ldots, r\}$, one has

$$x_i^l \delta_0^{(\alpha)} = \begin{cases} (-1)^l \frac{(\alpha_i)!}{(\alpha_i - l)!} \delta_0^{(\alpha_1, \ldots, \alpha_i - l, \ldots \alpha_n)} & \text{if } \alpha_i \geq l \\ 0 & \text{if } \alpha_i < l. \end{cases}$$

Hence, one has $x^\alpha \delta_0^{(\alpha)} = (-1)^{|\alpha|} \alpha! \delta_0$ and for all $\beta \in \mathbb{N}^r$ with $|\beta| = |\alpha| + 1$, one has $x^\beta \delta_0^{(\alpha)} = 0$. The assertion follows. □

Definition 5.4. *Let $\Gamma = x^\beta \partial_x^\alpha D$ where D is a differential operator on $\mathbb{R}^m$. Then Γ increases the degree of singularity at most $|\alpha| - |\beta|$. The integer $|\alpha| - |\beta|$ is called the total degree of Γ in x.*

We can define the homogeneous part of highest total degree (in x) of an analytic differential operator developing its coefficients in Taylor series.

Proof of the Theorem. Let $\Theta \in \mathcal{D}'(\mathcal{V})^H$ and $F \in L^1_{loc}(\mathcal{V})^H$ such that $P(\partial(\omega))\Theta = 0$ for a unitary polynomial $P \in \mathbb{C}[X]$ and $\Theta = F$ on $\mathcal{V}^{reg} = \mathcal{V} \cap \mathfrak{q}^{reg}$. We write $\Theta = F + S$ where S is an H-invariant distribution with support contained in $\mathcal{V} - \mathcal{V}^{reg}$. We want to prove that $S = 0$, which is equivalent to supp $(S) = \emptyset$.

Assuming S is non-zero, we are led to a contradiction. We will study S near an element $Z_0 \in$ supp (S) chosen as follows:

For $Z_0 \in$ supp (S) with Jordan decomposition $Z_0 = A_0 + X_0$, we construct the symmetric subpair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ related to A_0 and we set $\mathfrak{q}_{A_0} = \mathfrak{z}^- = \mathfrak{c}^- \oplus \mathfrak{z}_s^-$ as in section 3. Let $\mathcal{S}_k$ be the set of Z_0 in the support of S such that rank$(\mathfrak{z}_s^-) = k$. Since supp $(S) \subset \mathcal{V} - \mathcal{V}^{reg}$, if $Z_0 = A_0 + X_0$ belongs to supp (S) then A_0 is not $\mathfrak{q}$-regular. One deduces that $S_0 = \emptyset$. Let $k_0 > 0$ such that $S_0 = S_1 = \ldots = \mathcal{S}_{k_0 - 1} = \emptyset$ and $\mathcal{S}_{k_0} \neq \emptyset$.

For $Z_0 = A_0 + X_0$ in $\mathcal{S}_{k_0}$, we denote by $\mathcal{N}(\mathfrak{z}_s^-) = \mathcal{O}_1 \cup \ldots \mathcal{O}_\nu$ the set of nilpotent elements in $\mathfrak{z}_s^-$ as in section 4. Since supp $(S) \cap (A_0 + \mathcal{N}(\mathfrak{z}_s^-)) \neq \emptyset$, one can choose $j_0 \in \{1, \ldots, \nu\}$ such that supp $(S) \cap (A_0 + \mathcal{O}_i) = \emptyset$ for $i \in \{1, \ldots j_0 - 1\}$ and supp $(S) \cap (A_0 + \mathcal{O}_{j_0}) \neq \emptyset$.

From now on, we fix $Z_0 = A_0 + X_0$ in $\mathcal{S}_{k_0}$ such that $X_0 \in \mathcal{O}_{j_0}$.

For $\varepsilon > 0$, we denote by $\mathcal{W}_\varepsilon$ the set of x in $\mathfrak{z}_s^-$ such that, for any eigenvalue λ of ad$_\mathfrak{g} x$, one has $|\lambda| < \varepsilon$. The choice of k_0 implies that there exists $\varepsilon > 0$ such that supp$(S) \cap (Z_0 + \mathcal{W}_\varepsilon) \subset$ supp$(S) \cap (Z_0 + \mathfrak{c}^- + \mathcal{N}(\mathfrak{z}_s^-))$. Hence, we can choose an open neighborhood $\mathcal{W}_c$ of 0 in $\mathfrak{c}^-$ and an open neighborhood $\mathcal{W}_s$ of X_0 in $\mathfrak{z}_s^-$ such that

$$\text{supp}(S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset \text{ supp } (S) \cap (A_0 + \mathcal{W}_c + \mathcal{N}(\mathfrak{z}_s^-)). \tag{5.1}$$

First case. $A_0 \notin \mathfrak{c}_\mathfrak{q}$ and $X_0 \neq 0$.

We keep the notation of section 4. We fix a normal sl_2-triple (B_0, Y_0, X_0) in $(\mathfrak{z}_s, \mathfrak{z}_s^+)$. We choose an open neighborhood U_0 of 0 in U, the centralizer of Y_0 in $\mathfrak{z}_s^-$, as in Lemma 4.4. We keep the notation of this lemma. We recall that the map γ from $H \times^{\backprime} \mathfrak{z}^-$ to $\mathfrak{q}$ defined by $\gamma(h, Z) = h \cdot (A_0 + Z)$ is a submersion. Reducing U_0, $\mathcal{W}_c$ and $\mathcal{W}_s$ if necessary, we may assume that $\mathcal{W}_c + \Omega_0 \subset \mathcal{W}_c + \mathcal{W}_s \subset' \mathfrak{z}^-$ and that $V_0 = \gamma(H \times (\mathcal{W}_c + \Omega_0))$ is an open neighborhood of Z_0 contained in $\mathcal{V}$.

If T is an H-invariant distribution on $\mathcal{V}$, we denote by T_0 its restriction to V_0. By theorem 2.1, one can consider its restriction $T_1 = \mathcal{R}es_{\mathfrak{z}^-}T_0$ to $\mathcal{W}_c + \Omega_0$ with respect to γ. One has $A_0 +$ supp $(T_1) \subset$ supp $(T) \cap (A_0 + \mathcal{W}_c + \Omega_0)$.

We set $T_2 = \xi^{1/2}T_1$ where $\xi^{1/2}$ is the analytic function on $\mathcal{W}_c + \Omega_0$ defined in section 3.

Now, we consider the submersion π_0 from $H_s^+ \times U_0 \times \mathcal{W}_c$ to $\mathfrak{z}^-$ defined by $\pi_0(h, X, C) = h \cdot (X_0 + X) + C$. One denotes by T_3 the restriction on $U_0 \times \mathcal{W}_c$ of T_2 with respect to π_0 . We have $X_0 + \mathrm{supp}(T_3) \subset \mathrm{supp}(T_2) \cap (X_0 + U_0)$.

Since F is a locally integrable function, the distribution F_3 is the locally integrable function on $U_0 \times \mathcal{W}_c$ defined by $F_3(X, C) = \xi^{1/2}(C + X)F(C + X)$.

By assumption, the distribution S_3 is non-zero. By (5.1) and Lemma 4.4 (*2.*), one has supp $(S_2) =$ supp $(S_1) \subset \mathcal{W}_c + \Omega_0 \cap \mathcal{N}_{j_0} = \mathcal{W}_c + \mathcal{O}_{j_0}$. We deduce from Lemma 4.4 (*3.*) that supp $(S_3) \subset \{0\} \times \mathcal{W}_c$. By ([6], Lemma 3), there exists a family $(S_\alpha)_\alpha$ of $\mathcal{D}'(\mathcal{W}_c)$ such that $S_3 = \sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha$ where δ_0 is the Dirac measure at 0 of U_0 and for $\alpha \in \mathbb{N}^r$, the S_α with $|\alpha| = l$ are not all zero.

By assumption, the distribution Θ satisfies $P\big(\partial(\omega)\big)\Theta = 0$. By Lemma 3.1, one has

$$P\Big((\partial(\omega_s) + \partial(\omega_{\mathfrak{c}})) - \mu(Z)\Big)\Theta_2 = 0 \text{ on } \mathcal{W}_c + \Omega_0.$$

Using the restriction with respect to π_0, one obtains

$$P\Big(\mathcal{R}ad_U(\partial(\omega_s)) + \partial(\omega_{\mathfrak{c}}) - \tilde{\mu}\Big)\Theta_3 = 0 \text{ on } U_0 \times \mathcal{W}_c$$

where $\tilde{\mu}(X, C) = \mu(C + X)$ for $X \in U_0$ and $C \in \mathcal{W}_c$.

Let D_0 be the homogeneous part of highest total degree d of $\mathcal{R}ad_U(\partial(\omega_s))$. We set

$$P\Big(\mathcal{R}ad_U(\partial(\omega_s)) + \partial(\omega_{\mathfrak{c}}) - \tilde{\mu}\Big) = D_0^N + D_1$$

where N is the degree of P and D_1 is a differential operator with total degree in X strictly smaller than Nd. Since $\Theta_3 = F_3 + S_3$ with $S_3 = \sum_{a \in \mathbb{N}^r; \alpha_1 \leq l} \delta_0^{(\alpha)} \otimes S_\alpha$, we obtain the following relation on $U_0 \times \mathcal{W}_c$:

$$(D_0^N + D_1)S_3 = (D_0^N + D_1)(\sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha) = -(D_0^N + D_1)F_3 \tag{5.2}$$

We study now the degree of singularity along $\{0\} \times \mathcal{W}_c$ of the two members of (5.2).

If X_0 is not a $\mathfrak{z}_s^-$-distinguished nilpotent element then by Lemma 4.5, the homogeneous part of degree 2 of $\mathcal{R}ad_{U,0}(\partial(\omega_s)$ does not vanish and is a differential operator with constant coefficients of degree 2. Hence the total degree of D_0 is equal to $d = 2$. Since F_3 is a locally integrable function, it follows from Lemma 5.3 that one has $d_s^\circ F_3 = 0$ and $d_s^\circ((D_0^N + D_1)F_3) \leq 2N$. By the same Lemma, one has $d_s^\circ((D_0^N + D_1)S_3) = l + 1 + 2N$. Hence, we have a contradiction.

Assume that X_0 is a $\mathfrak{z}_s^-$-distinguished nilpotent element. Lemma 4.6 gives $c_0 D_0 = 2x_1 \dfrac{\partial^2}{\partial x_1^2} + (dim\ \mathfrak{z}_s^-)\dfrac{\partial}{\partial x_1} + \sum_{i=2}^{r}(n_i+2)x_i \dfrac{\partial^2}{\partial x_1 \partial x_i} + \sum_{2\leq i\leq j\leq r} a_{i,j}(X)\dfrac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=2}^{r} a_i(X)\dfrac{\partial}{\partial x_i})$ where $c_0 = \|X_0\|$. Since $a_{i,j}(0) = 0$, the total degree of D_0 is equal to 1.

For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, we set $\tilde{\alpha}^i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i{+}1, \alpha_{i+1} \dots \alpha_r)$ and $\bar{\alpha}^i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1} \dots \alpha_r)$. The relation $x_i \delta_0^{(\alpha)} = -\alpha_i \delta_0^{(\bar{\alpha}^i)}$ and the above expression of D_0 give

$$c_0 D_0 \cdot \delta_0^{(\alpha)} \otimes S_\alpha = \lambda_\alpha \delta^{(\tilde{\alpha}^1)} \otimes S_\alpha + \sum_{2\leq i\leq j\leq r} a_{i,j}(X)\delta^{(\tilde{\alpha}^{i,j})} \otimes S_\alpha + \sum_{i=2}^{r} a_i(X)\delta^{(\tilde{\alpha}^i)} \otimes S_\alpha$$

where

$$\lambda_\alpha = -2(\alpha_1 + 2) + dim\ \mathfrak{z}_s^- - \sum_{i=2}^{r}(n_i+2)(\alpha_i+1).$$

Since n_1 is equal to 2 and $(\mathfrak{g}, \mathfrak{h})$ is a nice pair, we obtain

$$\lambda_\alpha = -\delta_{\mathfrak{q}}(Z_0) - \big[2\alpha_1 + \sum_{i=2}^{r}(n_i+2)\alpha_i\big] < 0 \text{ for all } \alpha \in \mathbb{N}^r.$$

Consider $\alpha_0 = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ such that $|\alpha_0| = l$, $S_{\alpha_0} \neq 0$ and α_1 is maximal for these properties. One deduces that the coefficient of $\delta^{(\widetilde{\alpha_0}^1)} \otimes S_{\alpha_0}$ in $D_0 \cdot (\sum_{\alpha\in\mathbb{N}^r; |\alpha|=l} \delta_0^{(\alpha)} \otimes S_\alpha)$ is non-zero. Thus, the degree of singularity of $(D_0^N + D_1)S_3$ is equal to $1 + l + N$. Since F_3 is locally integrable and the total degree of D_0 is equal to 1, we have $d_s^\circ(D_0^N + D_1)F_3 \leq N$. This gives a contradiction in (5.2)

Second case. $A_0 \in \mathfrak{c}_{\mathfrak{q}}$ and $X_0 \neq 0$.

The symmetric pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ is equal to $(\mathfrak{g}_s, \mathfrak{h}_s)$. We just consider the submersion π_0 from $H \times U_0 \times \mathcal{W}_c$ to $\mathfrak{q}$ defined by $\pi_0(h, X, C) = h \cdot (X_0 + X) + A_0 + C$ where U_0 is defined as in Lemma 4.4 for the symmetric pair $(\mathfrak{g}_s, \mathfrak{h}_s)$.

For $T \in \mathcal{D}'(\mathfrak{q})^H$, we denote by T_1 the restriction of T to $U_0 \times \mathcal{W}_c$ with respect to π_0. As in the first case, we have $\Theta_1 = F_1 + S_1$ where F_1 is a locally integrable function on $U_0 \times \mathcal{W}_c$ and S_1 is a non-zero distribution such that supp $(S_1) \subset \{0\} \times \mathcal{W}_c$. Moreover the distribution Θ_1 satisfies the relation

$$P\Big(\mathcal{R}ad_U(\partial(\omega_s)) + \partial(\omega_{\mathfrak{c}})\Big)\Theta_1 = 0 \text{ on } U_0 \times \mathcal{W}_c.$$

The same arguments as in the first case lead to the contradiction $S_1 = 0$.

Third case. $X_0 = 0$.

The open sets $\mathcal{W}_c$ and $\mathcal{W}_s$ satisfy supp $(S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset$ supp $(S) \cap (A_0 + \mathcal{W}_c + \mathcal{N}(\mathfrak{z}_s^-))$. By the choice of j_0, we deduce that supp $(S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset$ supp $(S) \cap (A_0 + \mathcal{W}_c)$.

If $A_0 \in \mathfrak{c}_{\mathfrak{q}}$, then $V_0 = A_0 + \mathcal{W}_c + \mathcal{W}_s$ is an open neighborhood of A_0 in $\mathfrak{q}$. We identify $\mathfrak{q}$ with $\mathfrak{q}_s \times \mathfrak{c}_{\mathfrak{q}}$. Thus, the restriction S_0 of S to V_0 is different from zero and satisfies supp$(S_0) \subset \{0\} \times (A_0 + \mathcal{W}_c)$. On the other hand, one has $P(\partial(\omega))S_0 = -P(\partial(\omega))F_{|V_0}$. Since $\partial(\omega)$ is a second order operator with constant coefficients, we obtain a contradiction as above.

If $A_0 \notin \mathfrak{c}_\mathfrak{q}$, we may assume that $\mathcal{W}_c + \mathcal{W}_s \subset' \mathfrak{z}^-$. We denote by T_1 the restriction of an H-invariant distribution T to $\mathcal{W}_c + \mathcal{W}_s$ with respect to the submersion γ from $H \times' \mathfrak{z}^-$ to $\mathfrak{q}$ and we consider $T_2 = \xi^{1/2} T_1$ as distribution on $\mathcal{W}_s \times \mathcal{W}_c$. Thus, we have $S_2 \neq 0$ and supp $(S_2) = \{0\} \times \mathcal{W}_c$. Moreover, the distribution $\Theta_2 = F_2 + S_2$ satisfies $P\Big((\partial(\omega_s) + \partial(\omega_\mathfrak{c})) - \mu(Z)\Big)\Theta_2 = 0$ on $\mathcal{W}_s \times \mathcal{W}_s$ by Lemma 3.1. This is equivalent to

$$P\Big((\partial(\omega_s) + \partial(\omega_\mathfrak{c})) - \mu(Z)\Big)S_2 = -P\Big((\partial(\omega_s) + \partial(\omega_\mathfrak{c})) - \mu(Z)\Big)F_2.$$

Since $\partial(\omega_s)$ is a second order operator with constant coefficients, we obtain a contradiction as above.

This achieves the proof of the Theorem. $\square$

6 Application to $\big(\mathfrak{gl}(4,\mathbb{R}), \mathfrak{gl}(2,\mathbb{R}) \times \mathfrak{gl}(2,\mathbb{R})\big)$

On $G = GL(4,\mathbb{R})$ and its Lie algebra $\mathfrak{g} = \mathfrak{gl}(4,\mathbb{R})$, we consider the involution σ defined by $\sigma(X) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} X \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ where I_2 is the 2×2 identity matrix. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A, B \in \mathfrak{gl}(2,\mathbb{R}) \right\} \text{ and } \mathfrak{q} = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}; Y, Z \in \mathfrak{g}l((2,\mathbb{R}) \right\}.$$

By ([7] Theorem 6.3), the symmetric pair $\big(\mathfrak{g}l((4,\mathbb{R}), \mathfrak{g}l((2,\mathbb{R}) \times \mathfrak{g}l((2,\mathbb{R})\big)$ is a nice pair.

We first recall some results of [3]. Let $\kappa(X, X') = \frac{1}{2} tr(XX')$. The restriction of κ to the derived algebra of $\mathfrak{g}$ is a multiple of the Killing form. Let $S(\mathfrak{q}_\mathbb{C})^{H_\mathbb{C}}$ be subalgebra of $S(\mathfrak{q}_\mathbb{C})$ of all elements invariant under $H_\mathbb{C}$. We identify $S(\mathfrak{q}_\mathbb{C})^{H_\mathbb{C}}$ with the algebra of $H_\mathbb{C}$-invariant differential operators on $\mathfrak{q}_\mathbb{C}$ with constant coefficients. Using κ, we identify $S(\mathfrak{q}_\mathbb{C})^{H_\mathbb{C}}$ with the algebra $\mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H_\mathbb{C}}$ of $H_\mathbb{C}$-invariant polynomials on $\mathfrak{q}_\mathbb{C}$. A basis of $\mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H_\mathbb{C}}$ is given by $Q(X) = \frac{1}{2} tr(X^2)$ and $S(X) = det(X)$. The Casimir polynomial is just a multiple of Q.

By ([3] Lemma 1.3.1), the H-orbit of a semisimple element $X = \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}$ of $\mathfrak{q}$ is characterized by $(Q(X), S(X))$ or by the set $\{\nu_1(X), \nu_2(X)\}$ of eigenvalues of YZ, where the functions ν_1 and ν_2 are defined as follows: let Y be the Heaviside function. Let $S_0 = Q^2 - 4S$ and $\delta = \iota^{Y(-S_0)}\sqrt{|S_0|}$. We set

$$\nu_1 = (Q + \delta)/2 \quad \text{and} \quad \nu_2 = (Q - \delta)/2.$$

Regular elements of $\mathfrak{q}$ are semisimple elements with 2 by 2 distinct eigenvalues or equivalently, semisimple elements X of $\mathfrak{q}$ such that $\nu_1(X)\nu_2(X)(\nu_1(X) - \nu_2(X)) \neq 0$ ([3] Remarque 1.3.1).

Let χ be the character of $\mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H_\mathbb{C}}$ defined by $\chi(Q) = \lambda_1 + \lambda_2$ and $\chi(S) = \lambda_1\lambda_2$ where λ_1 and λ_2 are two complex numbers satisfying $\lambda_1\lambda_2(\lambda_1 - \lambda_2) \neq 0$.

For an open H-invariant subset $\mathcal{V}$ in $\mathfrak{q}$, we denote by $\mathcal{D}'(\mathcal{V})^H_\chi$ the set of H-invariant distributions T with support in $\mathcal{V}$ such that $\partial(P)T = \chi(P)T$ for all $P \in \mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H_\mathbb{C}}$. Let $\mathcal{N}$ be the set of nilpotent elements of $\mathfrak{q}$ and $\mathcal{U} = \mathfrak{q} - \mathcal{N}$ its complement. In [3], we describe a basis of the subspace of $\mathcal{D}'(\mathcal{U})^H_\chi$ consisting of locally integrable functions. More precisely, we obtain the following result.

We consider the Bessel operator $L_c = 4\left(z\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\right)$ on $\mathbb{C}$ and its analogous $L = 4\left(t\frac{d^2}{dt^2} + \frac{d}{dt}\right)$ on $\mathbb{R}$. Let $\mathcal{S}ol(L_c, \lambda)$ (resp., $\mathcal{S}ol(L, \lambda)$) be the set of holomorphic (resp., real analytic) functions f on $\mathbb{C} - \mathbb{R}_-$ (resp., $\mathbb{R}^*$) such that $L_c f = \lambda f$ (resp., $Lf = \lambda f$). For $\lambda \in \mathbb{C}^*$, we set

$$\Phi_\lambda(z) = \sum_{n\geq 0} \frac{(\lambda z)^n}{4^n (n!)^2} \quad \text{and} \quad w_\lambda(z) = \sum_{n\geq 0} \frac{a(n)(\lambda z)^n}{4^n (n!)^2},$$

where $a(x) = -2\frac{\Gamma'(x+1)}{\Gamma(x+1)}$. Then $(\Phi_\lambda, W_\lambda = w_\lambda + \log(\cdot)\Phi_\lambda)$ form a basis of $\mathcal{S}ol(L_c, \lambda)$, where log is the principal determination of the logarithm function on $\mathbb{C} - \mathbb{R}_-$ and $(\Phi_\lambda, W^r_\lambda = w_\lambda + \log|\cdot|\Phi_\lambda)$ form a basis of $\mathcal{S}ol(L, \lambda)$.

For two functions f and g defined over $\mathbb{C}$, we set

$$S^+(f,g)(X) = f(\nu_1(X))g(\nu_2(X)) + f(\nu_2(X))g(\nu_1(X))$$

and

$$[f,g](X) = f(\nu_1(X))g(\nu_2(X)) - f(\nu_2(X))g(\nu_1(X)).$$

We define the following functions on $\mathfrak{q}^{reg}$:

1. $$F_{ana} = \frac{[\Phi_{\lambda_1}, \Phi_{\lambda_2}]}{\nu_1 - \nu_2}$$

2. $$F_{sing} = \frac{[\Phi_{\lambda_1}, w_{\lambda_2}] + [w_{\lambda_1}, \Phi_{\lambda_2}] + \log|\nu_1\nu_2|[\Phi_{\lambda_1}, \Phi_{\lambda_2}]}{\nu_1 - \nu_2}$$

3. For $(A,B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}),\ (\Phi_{\lambda_1}, W^r_{\lambda_2}),\ (W^r_{\lambda_1}, \Phi_{\lambda_2}),\ (W^r_{\lambda_1}, W^r_{\lambda_2})\}$, we set

$$F^+_{A,B} = Y(S_0)\frac{S^+(A,B)}{|\nu_1 - \nu_2|}$$

where $S_0 = Q^2 - 4S \in \mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H_\mathbb{C}}$ and Y is the Heveaside function.

Theorem 6.1. *([3] Theorem 5.2.2 and Corollary 5.3.1).*

1. *The functions F_{ana} and F_{sing} are locally integrable on $\mathfrak{q}$.*
2. *For $(A,B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}),\ (\Phi_{\lambda_1}, W^r_{\lambda_2}),\ (W^r_{\lambda_1}, \Phi_{\lambda_2}),\ (W^r_{\lambda_1}, W^r_{\lambda_2})\}$, the functions $F^+_{A,B}$, are locally integrable on $\mathcal{U}$.*
3. *The family F_{ana}, F_{sing} and $F^+_{A,B}$, with (A,B) as above form a basis $\mathcal{B}$ of the subspace of $\mathcal{D}'(\mathcal{U})^H_\chi$ consisting of distributions given by a locally integrable function.*

Corollary 6.2. *Any invariant distribution of $\mathcal{D}'(\mathcal{U})^H_\chi$ is given by a locally integrable function on $\mathcal{U}$. In particular, the family $\mathcal{B}$ defined in the previous Theorem is a basis of $\mathcal{D}'(\mathcal{U})^H_\chi$.*

Proof. Let $T \in \mathcal{D}'(\mathcal{U})^H_\chi$. We denote by F its restriction to $\mathcal{U}^{reg}$. By ([7] Theorem 5.3 (i)), F is an analytic function on $\mathcal{U}^{reg}$ satisfying $(*)\quad \partial(P)F = \chi(P)F$ on $\mathcal{U}^{reg}$ for all $P \in \mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H_\mathbb{C}}$.

In ([3] section 4.), we describe the analytic solutions of $(*)$ in terms of Φ_λ, W_λ and W^r_λ for $\lambda = \lambda_1$ and λ_2. By the asymptotic behaviour of orbital integrals near non-zero semisimple elements ([3] Theorems 3.3.1 and 3.4.1), and the Weyl integration formula ([3] Lemma 3.1.2), one deduces that $F \in L^1_{loc}(\mathcal{U})^H$. Theorem 5.1 gives the result. $\square$

Corollary 6.3. *Any invariant distribution of $\mathcal{D}'(\mathfrak{q})^H_\chi$ is given by a locally integrable function on $\mathfrak{q}$.*

Proof. Let $T \in \mathcal{D}'(\mathfrak{q})^H_\chi$. By Corollary 6.2, the restriction of T to $\mathcal{U}$ is a linear combination of elements of $\mathcal{B}$. By Theorem 5.1 and Theorem 6.1, it is enough to prove that the functions $F^+_{A,B}$, with $(A,B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}),\ (\Phi_{\lambda_1}, W^r_{\lambda_2}),\ (W^r_{\lambda_1}, \Phi_{\lambda_2}),\ (W^r_{\lambda_1}, W^r_{\lambda_2})\}$ are locally integrable on $\mathfrak{q}$ or equivalently, that the integral $\displaystyle\int_\mathfrak{q} |F^+_{A,B}(X)f(X)|dX$ is finite for all positive function $f \in \mathcal{D}(\mathfrak{q})$. For this, we will use the Weyl integration formula ([5] Proposition 1.8 and Theorem 1.27).

For $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_j = \pm$, we define

$$\mathfrak{a}_\varepsilon = \left\{ X_\varepsilon(u_1,u_2) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & u_1 & 0 \\ & & 0 & u_2 \\ \hline \varepsilon_1 u_1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \varepsilon_2 u_2 & & \end{array}\right) ; (u_1,u_2) \in \mathbb{R}^2 \right\}.$$

and

$$\mathfrak{a}_2 = \left\{ \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \tau & -\theta \\ & & \theta & \tau \\ \hline \tau & -\theta & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \theta & \tau & & \end{array}\right) ; (\theta,\tau) \in \mathbb{R}^2 \right\}$$

By ([3], Lemma 1.2.1), the subspaces $\mathfrak{a}_{++}, \mathfrak{a}_{+-}, \mathfrak{a}_{--}$ and $\mathfrak{a}_2$ form a system of representatives of H-conjugaison classes of Cartan subspaces in $\mathfrak{q}$. By ([3] Remark 1.3.1), an element $X \in \mathfrak{q}$ satisfies $S_0(X) \geq 0$ if and only if X is H-conjugate to an element of $\mathfrak{a}_\varepsilon$ for some ε. Furthermore, one has $\{\nu_1(X_\varepsilon(u_1,u_2)), \nu_2(X_\varepsilon(u_1,u_2))\} = \{\varepsilon_1 u_1^2, \varepsilon_2 u_2^2\}$.

Let f be a positive function in $\mathcal{D}(\mathfrak{q})$. We define the orbital integral of f on $\mathfrak{q}^{reg}$ by

$$\mathcal{M}(f)(X) = |\nu_1(X) - \nu_2(X)| \int_{H/Z_H(X)} f(h.X)dX$$

where $Z_H(X)$ is the centralizer of X in H and dh is an invariant measure on $H/Z_H(X)$.

By ([5] Theorem 1.23), the orbital integral $\mathcal{M}(f)$ is a smooth function on $\mathfrak{q}^{reg}$ and there exists a compact subset Ω of $\mathfrak{q}$ such that $\mathcal{M}(f)(X) = 0$ for all regular element X in the complement of Ω.

Since $F^+_{A,B}$ is zero on $\mathfrak{a}_2^{reg}$, one deduces from the Weyl integration formula that there exist positive constants C_ε (only depending of the choice of measures), such that one has

$$\int_{\mathfrak{q}} F^+_{A,B}(X)f(X)dX = \sum_{\varepsilon\in\{(++),(+-),(--)\}} C_\varepsilon \int_{\mathbb{R}^2} F^+_{A,B}(X_\varepsilon(u_1,u_2))$$
$$\times\mathcal{M}(f)(X_\varepsilon(u_1,u_2))|u_1u_2(\varepsilon_1u_1^2-\varepsilon_2u_2^2)|du_1du_2.$$

By definition of $F^+_{A,B}$, there exist positive constants C, C_1 and C_2 such that, for all $X_\varepsilon(u_1,u_2)\in\Omega^{reg}$, one has

$$|(\varepsilon_1u_1^2-\varepsilon_2u_2^2)F^+_{A,B}(X_\varepsilon(u_1,u_2))|\leq C(C_1+|\log|u_1||)(C_2+|\log|u_2||).$$

One deduces easily the corollary from the following Lemma. □

Lemma 6.4. *Let $f\in\mathcal{D}(\mathfrak{q})$. Then there exist positive contants C', C'_1, C'_2 such that, for all $X_\varepsilon(u_1,u_2)\in\mathfrak{q}^{reg}$ one has*

$$|\mathcal{M}(f)(X_\varepsilon(u_1,u_2))|\leq C'(C'_1+\big|\log|u_1|\big|)(C'_2+\big|\log|u_2|\big|).$$

Proof. Let $H=KNA$ be the Iwasawa decomposition of H with $K=O(2)\times O(2)$, $N=N_0\times N_0$ where N_0 consists of 2 by 2 unipotent upper triangular matrices and A is the set of diagonal matrices in H. It is easy to see that the centralizer of X in H is the set of diagonal matrices $diag((\alpha,\beta,\alpha,\beta)$ with $(\alpha,\beta)\in(\mathbb{R}^*)^2$. Hence $H/Z_H(X)$ is isomorphic to $K\times N\times\{diag(e^x,e^y,1,1);x,y\in\mathbb{R}\}$.

For $\xi\in\mathbb{R}$, we set $n_\xi=\begin{pmatrix}1&\xi\\0&1\end{pmatrix}$. We define the function $\tilde{f}$ by $\tilde{f}(X)=\int_K f(k\cdot X)dk$. Then, one has

$$\mathcal{M}(f)(X_\varepsilon(u_1,u_2))=|\varepsilon_1u_1^2-\varepsilon_2u_2^2|\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\tilde{f}(Y(u,\varepsilon,x,y,\xi,\eta))d\xi d\eta\Big)dxdy$$

with

$$Y(u,\varepsilon,x,y,\xi,\eta)=\Big(\begin{pmatrix}n_\xi&0\\0&n_\eta\end{pmatrix}diag(e^x,e^y,1,1)\Big)\cdot X_{\varepsilon,u}.$$

Writing $Y(u,\varepsilon,x,y,\xi,\eta)=\begin{pmatrix}0&Y\\Z&0\end{pmatrix}$, one has

$$Y=\begin{pmatrix}u_1e^x&-\eta u_1e^x+e^y\xi u_2\\0&u_2e^y\end{pmatrix}\text{ and }Z=\begin{pmatrix}\varepsilon_1u_1e^{-x}&-\xi\varepsilon_1u_1e^{-x}+\eta\varepsilon_2u_2e^{-y}\\0&\varepsilon_2u_2e^{-y}\end{pmatrix}.$$

Since $f\in\mathcal{D}(\mathfrak{q})$, the function $\tilde{f}$ has compact support in $\mathfrak{q}$. Identify $\mathfrak{q}$ with $\mathbb{R}^8$, there exists $T>0$ such that $\text{supp}(\tilde{f})\subset[-T,T]^8$. If $\tilde{f}(Y(u,\varepsilon,x,y,\xi,\eta))\neq0$ then we have the following inequalities:

1. $|u_1e^{\pm x}|\leq T$ and $|u_2e^{\pm y}|\leq T$,
2. $|-\eta u_1e^x+e^y\xi u_2|\leq T$,
3. $|-\xi\varepsilon_1u_1e^{-x}+\eta\varepsilon_2u_2e^{-y}|\leq T$.

Changing the variables (ξ,η) in $(r,s)=(\xi u_2 e^y-\eta u_1 e^x, -\xi\varepsilon_1 u_1 e^{-x}+\eta\varepsilon_2 u_2 e^{-y})$, we obtain the result. $\square$

Remark. By ([3] Corollary 5.3.1), the function F_{ana} defines an invariant eigendistribution on $\mathfrak{q}$. At this stage, we don't know if it is the case for the functions F_{sing} and $F^+_{A,B}$. Indeed, the proof of Theorem 6.1 of [3] is based on integration by parts using estimates of orbital integrals and some of their derivates near non-zero semisimple elements of $\mathfrak{q}$. To determine if F_{sing} and $F^+_{A,B}$ are eingendistributions using the same method, we have to know the behavior of derivates of orbital integrals near 0.

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