LOCAL ZETA FUNCTIONS FOR A CLASS OF P-ADIC SYMMETRIC SPACES (I)

PASCALE HARINCK AND HUBERT RUBENTHALER

Part I: Structure and Orbits

Abstract: *This is an extended version of the first part of a forthcoming paper where we will study the local Zeta functions of the minimal spherical series for the symmetric spaces arising as open orbits of the parabolic prehomogeneous spaces of commutative type over a p-adic field. The case where the ground field is $\mathbb{R}$ has already been considered by Nicole Bopp and the second author ([7]). If F is a p-adic field of caracteristic 0, we consider a reductive Lie algebra $\widetilde{\mathfrak{g}}$ over F which is endowed with a short $\mathbb{Z}$-grading: $\widetilde{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We also suppose that the representation $(\mathfrak{g}_0, \mathfrak{g}_1)$ is absolutely irreducible. Under a so-called regularity condition we study the orbits of G_0 in $\mathfrak{g}_1$, where G_0 is an algebraic group defined over F, whose Lie algebra is $\mathfrak{g}_0$. We also investigate the P-orbits, where P is a minimal σ-split parabolic subgroup of G (σ being the involution which defines a structure of symmetric space on any open G_0-orbit in $\mathfrak{g}_1$).*

AMS classification: 22E46, 16S32

Table of Contents

Introduction

The ultimate purpose of this paper will be (in a final version) to define local Zeta functions for a class of reductive p-adic symmetric spaces (attached to the representations of the σ-minimal series) and to prove their explicit functional equation.

But in the present first part, we are only concerned with classification and structure theory for these symmetric spaces.

Let F be a p-adic field of characteristic 0 whose residue class field has characteristic $\neq 2$. The main object under consideration is a reductive Lie algebra $\widetilde{\mathfrak{g}}$ endowed with a short $\mathbb{Z}$-grading

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+ \quad (V^+ \neq \{0\}).$$

This means that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$, $[\mathfrak{g},V^\pm] \subset V^\pm$, $[V^+,V^+] = [V^-,V^-] = \{0\}$. We also suppose that the corresponding representation of $\mathfrak{g}$ on V^+ is absolutely irreducible. Then $(\mathfrak{g},V^+)$ is an infinitesimal prehomogeneous vector space. It is well known that such a grading is defined by a grading element H_0. We normalize it in such a way that $\mathrm{ad}(H_0)$ has eigenvalues -2,0, 2 on V^-, $\mathfrak{g}$, V^+, respectively.

We introduce a natural algebraic subgroup G of the group of automorphisms of $\widetilde{\mathfrak{g}}$ whose Lie algebra is $\mathfrak{g}$ (which was first used by Iris Muller ([15])). This group is defined at the beginning of section 1.7 as the centralizer of H_0 in the group of automorphisms of $\widetilde{\mathfrak{g}}$ which become elementary over a field extension. Then (G,V^+) is a prehomogeneous vector space.

Although a part of the structure theory can be carried out with no further assumption (section 1.1 up to section 1.7), we need to introduce the so-called *regularity condition*, and we will essentially only consider regular graded Lie algebras in this paper. A graded Lie algebra is said to be regular (Definition 1.7.11) if the grading element H_0 is the semi-simple element of an $\mathfrak{sl}_2$-triple. This condition is also equivalent to the existence of a non-trivial relative invariant polynomial Δ_0 of the prehomogeneous space (G,V^+), and also, as we will see in section 4, to the fact that the various open G-orbits in V^+ are symmetric spaces.

These open orbits of G in V^+ (and in V^-) are precisely the symmetric spaces we are interested in.

One can always suppose that $\widetilde{\mathfrak{g}}$ is semi-simple. Then the assumptions on the grading imply that $\overline{\mathfrak{g}}+\overline{V^+}$ (where the overline stands for scalar extension to an algebraic closure) is a maximal parabolic subalgebra of $\overline{\widetilde{\mathfrak{g}}}$, and hence it is defined by the single root which is removed from the root basis of $\overline{\widetilde{\mathfrak{g}}}$ to obtain the root system of $\mathfrak{g}$. It can be shown that this single root is a "white" root in the Satake-Tits diagram of $\widetilde{\mathfrak{g}}$. Therefore the gradings we are interested in are in one to one correspondence with "weighted" Satake-Tits diagrams where one "white" root is circled. The classification is done in section 2 and the list of the allowed diagrams is given in Table 1 (section 2.2).

A key tool in the orbital description of (G, V^+) is a kind of *principal diagonal*

$$\widetilde{\mathfrak{g}}^{\lambda_0} \oplus \ldots \oplus \widetilde{\mathfrak{g}}^{\lambda_k} \subset V^+$$

where $\lambda_0, \lambda_1, \ldots, \lambda_k$ is a maximal subset of strongly orthogonal roots living in V^+. This sequence starts with the root λ_0, the root defining the above mentioned parabolic, and is obtained by an induction process which we call "descent" (see sections 1.5 and 1.6).

A first step in the classification of the orbits is to prove that any G-orbits meets this principal diagonal. This is done in Theorem 3.2.2. Another step is the study of the so-called rank 1 case in section 1.12 (Theorem 1.12.4).

Finally, in order to classify the orbits, we need to distinguish three Types (see Definition 2.2.2) and the full classification of the orbits is obtained in Theorem 3.8.8 for Type I, Theorem 3.8.9 and Theorem 3.8.10 for Type II, and Theorem 3.9.8 for Type III.

Section 4 is devoted to the study of the associated symmetric spaces. Let $\Omega_1, \ldots, \Omega_m$ be the open orbits in V^+. As we said before these open orbits are symmetric spaces. More precisely this means that if we choose elements $I_j^+ \in \Omega_j$, and if $H_j = Z_G(I_j^+)$ then for all j there exists an involution σ_j of $\mathfrak{g}$ (in fact the restriction of an involution of $\widetilde{\mathfrak{g}}$ which stabilizes $\mathfrak{g}$) such that H_j is an open subgroup of the fixed point group G^{σ_j}. Therefore $\Omega_j \simeq G/H_j$ can be viewed as a symmetric space. A striking fact is that all these G-symmetric spaces have the same minimal σ_j-split parabolic subgroup P (i.e. $\sigma_j(P)$ is the opposite parabolic of P). Moreover (P, V^+) is again a prehomogeneous space. We define and study a family of polynomials $\Delta_0, \ldots, \Delta_k$ which are the fundamental relative invariants of (P, V^+). We also determine the open orbits of (P, V^+) in terms of the values of the Δ_j (Theorem 4.3.6). Finally we introduce an involution γ of $\widetilde{\mathfrak{g}}$ which exchanges V^+ and V^- and which allows to define the fundamental relative invariants of (P, V^-) and to determine its open orbits.

This paper follows the same lines as the corresponding paper which dealt with the real case by Nicole Bopp and the second author ([7]). But of course, due to the big difference between the structure of the base fields, the proofs (as well as the definition of the group which acts), are often rather different.

We must also mention that the study of graded algebras over a p-adic field was initiated by Iris Muller in a series of paper ([14], [15], [16], [17]), in a more general context (the so-called quasi-commutative case), but her results seem to us less precise and sometimes weaker than ours. Moreover she never considered the symmetric space aspects of these spaces.

Finally it should be noticed that, via the Kantor-Koecher-Tits construction (which is still valid over a p-adic field), there is a bijection between the regular graded Lie algebras we consider here and absolutely simple Jordan algebra structures on V^+ (see [8], and the references there). And, probably, the group G which is used in this paper is very closed to the "structure group" for the p-adic Jordan algebra V^+.

1. 3-graded Lie algebras

1.1. A class of graded algebras.

In this paper the ground field F is a p-adic field of characteristic 0, i.e. a finite extension of $\mathbb{Q}_p$. Moreover we will always suppose that the residue class field has characteristic $\neq 2$ (non dyadic case). We will denote by $\overline{F}$ an algebraic closure of F. In the sequel, if U is a F-vector space, we will set

$$\overline{U} = U \otimes_F \overline{F}$$

.

Definition 1.1.1. *Throughout this paper a reductive Lie algebra $\widetilde{\mathfrak{g}}$ over F satisfying the following two hypothesis will be called a graded Lie algebra:*

($\mathbf{H_1}$) *There exists an element $H_0 \in \widetilde{\mathfrak{g}}$ such that $\operatorname{ad} H_0$ defines a $\mathbb{Z}$-grading of the form*

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+ \quad (V^+ \neq \{0\}) \ ,$$

$$\textit{where} \quad [H_0, X] = \begin{cases} 0 & \textit{for } X \in \mathfrak{g} \ ; \\ 2X & \textit{for } X \in V^+ \ ; \\ -2X & \textit{for } X \in V^- \ . \end{cases}$$

(Therefore, in fact, $H_0 \in \mathfrak{g}$)

($\mathbf{H_2}$) *The (bracket) representation of $\mathfrak{g}$ on $\overline{V^+}$ is irreducible. (In other words, the representation $(\mathfrak{g}, V^+)$ is absolutely irreducible)*

The following relations are trivial consequences from ($\mathbf{H_1}$) :

$$[\mathfrak{g}, V^+] \subset V^+ \ ; \ [\mathfrak{g}, V^-] \subset V^- \ ; \ [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \ ; \ [V^+, V^-] \subset \mathfrak{g} \ ; \ [V^+, V^+] = [V^-, V^-] = \{0\}.$$

1.2. The restricted root system.

There exists a maximal split abelian Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ containing H_0. Then $\mathfrak{a}$ is also maximal split abelian in $\widetilde{\mathfrak{g}}$.

Denote by $\widetilde{\Sigma}$ the roots of $(\widetilde{\mathfrak{g}}, \mathfrak{a})$ and by Σ the roots of $(\mathfrak{g}, \mathfrak{a})$. (These are effectively root systems: see [28], p.10).

Let $\mathrm{Aut}_e(\mathfrak{g})$ denote the group of elementary automorphisms of $\mathfrak{g}$ ([4] VII, §3, $n°1$).

Two maximal split abelian subalgebras of $\mathfrak{g}$ are conjugated by $\mathrm{Aut}_e(\mathfrak{g})$ ([28], Theorem 2, page 27, or [27], Theorem 3.1.16 p. 27)

Theorem 1.2.1. (Cf. Th. 1.2 p.10 of [7])

(1) *There exists a system of simple roots $\widetilde{\Pi}$ in $\widetilde{\Sigma}$ such that*

$$\nu \in \widetilde{\Pi} \Longrightarrow \nu(H_0) = 0 \textit{ or } 2 \ .$$

(2) *There exists an unique root $\lambda_0 \in \widetilde{\Pi}$ such that $\lambda_0(H_0) = 2$.*

(3) *If the decomposition of a positive root $\lambda \in \widetilde{\Sigma}$ in the basis $\widetilde{\Pi}$ is given by*

$$\lambda = m_0\lambda_0 + \sum_{\nu \in \widetilde{\Pi} \setminus \{\lambda_0\}} m_\nu \nu \ , \ m_0 \in \mathbb{Z}^+, m_\nu \in \mathbb{Z}^+$$

then $m_0 = 0$ or $m_0 = 1$. Moreover λ belongs to Σ if and only if $m_0 = 0$.

Proof. Let S be the subset of $\widetilde{\Sigma}$ given by

$$S = \{\lambda \in \widetilde{\Sigma} \mid \lambda(H_0) = 0 \textit{ or } 2\} .$$

It is easily seen from $(\mathbf{H_1})$ that S is a parabolic subset of $\widetilde{\Sigma}$, i.e. $\mathfrak{g} \oplus V^+$ is a parabolic subalgebra of $\widetilde{\mathfrak{g}}$. It is well known (see [3] chap. 6 Prop. 20) that there exists an order on $\widetilde{\Sigma}$ such that, if a root $\lambda \in \widetilde{\Sigma}$ is positive for this order, then $\widetilde{\mathfrak{g}}^\lambda$ is a subspace of $\mathfrak{g} \oplus V^+$. If $\widetilde{\Pi}$ denotes the set of simple roots of $\widetilde{\Sigma}$ corresponding to this order, then $\widetilde{\Pi}$ satisfies (1).

There exists at least one root $\lambda_0 \in \widetilde{\Pi}$ such that $\lambda_0(H_0) = 2$ because $V^+ \neq \{0\}$. Moreover the commutativity of V^+ which is the nilradical of the parabolic algebra $\mathfrak{g} \oplus V^+$ implies (3).

Let us suppose that there exists in $\widetilde{\Pi}$ a root $\lambda_1 \neq \lambda_0$ such that $\lambda_1(H_0) = 2$. Let V_0 be the sum of the root spaces $\widetilde{\mathfrak{g}}^\lambda$ for the roots λ of the form

$$\lambda = \lambda_0 + \sum_{\nu \in \widetilde{\Pi}, \nu(H_0)=0} m_\nu \nu \quad (m_\nu \in \mathbb{Z}^+) .$$

Since V_0 does not contain $\widetilde{\mathfrak{g}}^{\lambda_1}$, it is a non-trivial subspace of V^+ which is invariant under the action of $\mathfrak{g}$. This gives a contradiction with $(\mathbf{H_2})$ and the uniqueness of λ_0 such that $\lambda_0(H_0) = 2$ follows, hence (2) is proved.

□

Let us fix once and for all such a set of simple roots $\widetilde{\Pi}$ of $\widetilde{\Sigma}$. Then

$$\widetilde{\Pi} = \Pi \cup \{\lambda_0\} ,$$

where $\Pi = \{\nu \in \widetilde{\Pi} \,|\, \nu(H_0) = 0\}$ is a set of simple roots of Σ. We denote by $\widetilde{\Sigma}^+$ (resp. Σ^+) the set of positive roots of $\widetilde{\Sigma}$ (resp. Σ) for the order defined by $\widetilde{\Pi}$ (resp. Π).

Then we have the following characterization of λ_0 :

Corollary 1.2.2. *The root λ_0 is the unique root in $\widetilde{\Sigma}$ such that*

- $\lambda_0(H_0) = 2$;
- $\lambda \in \Sigma^+ \Longrightarrow \lambda_0 - \lambda \notin \widetilde{\Sigma}$.

Proof. It is clear that λ_0 satisfies the two properties. Let $\mu \in \widetilde{\Sigma}$ a root satisfying the same properties. Then the first property implies that $\mu \in \widetilde{\Sigma}^+$. Suppose that $\mu \neq \lambda_0$. Then $\mu = \lambda_0 + \Sigma_{\nu \in A} m_\nu \nu$ with $A \subset \Pi$, $A \neq \emptyset$, and $m_\nu \neq 0$. This implies that $\mu = \nu_1 + \cdots + \nu_k + \lambda_0 + \nu_{k+1} + \cdots + \nu_p$, where each partial sum is a root. Then either $\mu - \nu_p \in \widetilde{\Sigma}$, or (if $\nu_{k+1} = \cdots = \nu_p = 0$) one has $\mu - (\nu_1 + \cdots + \nu_k) = \lambda_0 \in \widetilde{\Sigma}$. In both cases the second property would not be verified.

□

Remark 1.2.3. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ (which is also the centralizer of $\mathfrak{a}$ in $\widetilde{\mathfrak{g}}$). We have then the following decompositions:

$$\widetilde{\mathfrak{g}} = \mathfrak{m} \oplus \sum_{\lambda \in \widetilde{\Sigma}} \widetilde{\mathfrak{g}}^\lambda, \quad \mathfrak{g} = \mathfrak{m} \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}^\lambda, \quad V^+ = \sum_{\lambda \in \widetilde{\Sigma}^+ \setminus \Sigma^+} \widetilde{\mathfrak{g}}^\lambda.$$

The algebra $[\mathfrak{m}, \mathfrak{m}]$ is *anisotropic* (i.e., his unique split abelian subalgebra is $\{0\}$), and is called the *anisotropic kernel* of $\widetilde{\mathfrak{g}}$.

1.3. **Extension to the algebraic closure.**

Let us fix an algebraic closure $\overline{F}$ of F. Remember that for each vector space U over F we note $\overline{U}$ the vector space obtained by extension of the scalars:

$$\overline{U} = U \otimes_F \overline{F}.$$

We have then the decomposition

$$\overline{\widetilde{\mathfrak{g}}} = \overline{V^-} \oplus \overline{\mathfrak{g}} \oplus \overline{V^+}$$

and according to $(\mathbf{H_2})$, the representation $(\overline{\mathfrak{g}}, \overline{V^+})$ is irreducible. Let $\mathfrak{j}$ be a Cartan subalgebra of $\widetilde{\mathfrak{g}}$ which contains $\mathfrak{a}$. Then $\mathfrak{j} \subset \mathfrak{g}$ and $\mathfrak{j}$ is also a Cartan subalgebra of $\mathfrak{g}$. This implies that $\bar{\mathfrak{j}}$ is a Cartan subalgebra of $\overline{\widetilde{\mathfrak{g}}}$ (and also of $\overline{\mathfrak{g}}$) (see [4], chap.VII, §2, Prop.3)
Let $\widetilde{\mathcal{R}}$ (resp. $\mathcal{R}$) be the roots of the pair $(\overline{\widetilde{\mathfrak{g}}}, \bar{\mathfrak{j}})$ (resp. $(\overline{\mathfrak{g}}, \bar{\mathfrak{j}})$) and let $\overline{\widetilde{\mathfrak{g}}}^{\alpha}$ (resp. $\overline{\mathfrak{g}}^{\alpha}$) be the corresponding root spaces.
Let $X \in \overline{\widetilde{\mathfrak{g}}}^{\alpha}$. Let us write $X = \sum a_i X_i$ where $a_i \in \overline{F}^*$ and where the elements $X_i \in \widetilde{\mathfrak{g}}$ are $\overline{F}$-free eigenvectors of $\mathfrak{a}$. Then for $H \in \mathfrak{a}$, we have $[H, X] = \alpha(H)X = \sum a_i[H, X_i] = \sum a_i \alpha(H) X_i$. Hence $[H, X_i] = \alpha(H)X_i$. If $X_i = \sum_{\lambda \in \Sigma \cup \{0\}} X_\lambda$, we obtain $[H, X_i] = \sum \lambda(H) X_\lambda = \alpha(H) \sum X_\lambda$. Therefore $\alpha(H) \in F$, in other words the restrictions to $\mathfrak{a}$ of roots belonging to $\widetilde{\mathcal{R}}$, take values in F.
We denote by $\rho : \bar{\mathfrak{j}}^* \longrightarrow \mathfrak{a}^*$ the restriction morphism. One sees easily that $\rho(\widetilde{\mathcal{R}}) = \widetilde{\Sigma} \cup \{0\}$ and that $\rho(\mathcal{R}) = \Sigma \cup \{0\}$.
Let us recall the following well known result:

Lemma 1.3.1.
Let $\lambda \in \widetilde{\Sigma}$. Let $S_\lambda = \{\alpha \in \widetilde{\mathcal{R}},\ \rho(\alpha) = \lambda\}$. Then we have:

$$\overline{\widetilde{\mathfrak{g}}}^{\lambda} = \sum_{\alpha \in S_\lambda} \overline{\widetilde{\mathfrak{g}}}^{\alpha}$$

Proof. Let $\alpha \in S_\lambda$ and $X \in \overline{\widetilde{\mathfrak{g}}}^{\alpha}$. The element X can be written $X = \sum_{i=1}^n a_i X_i$ where $a_i \in \overline{F}^*$ and where the elements $X_i \in \widetilde{\mathfrak{g}}$ are free over $\overline{F}$ and are eigenvectors $\mathfrak{a}$ ($\mathfrak{a}$ is split). Then for $H \in \mathfrak{a}$ we have:
$[H, X] = \sum_{i=1}^n a_i [H, X_i] = \sum_{i=1}^n a_i \gamma_i(H) X_i = \alpha(H)X = \lambda(H)X = \sum_{i=1}^n a_i \lambda(H) X_i$. Therefore for each $H \in \mathfrak{a}$, $\gamma_i(H) = \lambda(H)$.This implies the inclusion $\sum_{\alpha \in S_\lambda} \overline{\widetilde{\mathfrak{g}}}^{\alpha} \subset \overline{\widetilde{\mathfrak{g}}^{\lambda}}$. Conversely let $X \in \overline{\widetilde{\mathfrak{g}}^{\lambda}}$ whose root space decomposition in $\overline{\widetilde{\mathfrak{g}}}$ is given by $X = \sum_{\beta \in \widetilde{\mathcal{R}} \cup \{0\}} X_\beta$. For $H \in \mathfrak{a}$ we have:

$$[H, X] = \lambda(H)X = \sum \lambda(H) X_\beta = \sum \beta(H) X_\beta,$$

and hence $\rho(\beta) = \lambda$. □

Set $\widetilde{P} = \{\alpha \in \widetilde{\mathcal{R}},\ \rho(\alpha) \in \widetilde{\Sigma}^+ \cup \{0\}\}$. One shows easily that $\widetilde{P}$ is a parabolic subset $\widetilde{\mathcal{R}}$. Therefore there is an order on $\widetilde{\mathcal{R}}$ such that if $\widetilde{\mathcal{R}}^+$ is the set of positive roots for this order one has $\widetilde{\mathcal{R}}^+ \subset \widetilde{P}$ ([3], chap. VI, §1, n°7, Prop. 20). Then:

$$\rho(\widetilde{\mathcal{R}}^+) = \widetilde{\Sigma}^+ \cup \{0\}$$

and hence

$$\rho(\mathcal{R}^+) = \Sigma^+ \cup \{0\}.$$

We will denote by $\widetilde{\Psi}$ the set of simple roots of $\widetilde{\mathcal{R}}$ corresponding to $\widetilde{\mathcal{R}}^+$.

Proposition 1.3.2.
There is a unique simple root $\alpha_0 \in \widetilde{\Psi}$ *such that* $\rho(\alpha_0) = \lambda_0$.

Proof. Suppose that there are two distinct simple roots α_0, β_0 such that $\rho(\alpha_0) = \rho(\beta_0) = \lambda_0$, and suppose that these two roots belong to the same irreducible component of $\widetilde{\mathcal{R}}$.
Then the highest root in this irreducible component will be of the form:

$$\gamma = m_0\alpha_0 + m_1\beta_0 + \sum_{\alpha \in \widetilde{\Psi} \setminus \{\alpha_0, \beta_0\}} m_\alpha \alpha, \;\; m_0, m_1 \geq 1, \, m_\alpha \geq 0,$$

and then

$$\rho(\gamma) = (m_0 + m_1)\lambda_0 + \sum_{\nu \in \Pi} m_\nu \nu.$$

And this is impossible according to Theorem 1.2.1.
Consequently each of the roots α_0 and β_0 belong to a different irreducible component of $\widetilde{\mathcal{R}}$. Let ω_0 (resp. ω_1) the highest root of the irreducible component of $\widetilde{\mathcal{R}}$ containing α_0 (resp. β_0). As $\omega_i(H_0) = 2$ (because $\alpha_0(H_0) = \beta_0(H_0) = 2$ and $\omega_i(H_0) = -2, 0, 2$) the linear forms ω_i are dominant weights of the irreducible representation $(\overline{\mathfrak{g}}, \overline{V^+})$. This implies the result.

□

Remark 1.3.3. If we had supposed that $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ was absolutely simple, then the second part of the proof would have been superfluous.

1.4. The highest root in $\widetilde{\Sigma}$.

Proposition 1.4.1.
There is a unique root $\lambda^0 \in \widetilde{\Sigma}$ *such that*

- $\lambda^0(H_0) = 2$;
- $\forall \lambda \in \Sigma^+, \lambda^0 + \lambda \notin \widetilde{\Sigma}$.

This root λ^0 *is the highest root of the irreducible component of* $\widetilde{\Sigma}$ *which contains* λ_0.

Proof. Let ω be the highest weight of the representation $(\mathfrak{g}, \overline{V^+})$. We will show that the restriction of ω to $\mathfrak{a}$ is the unique root in $\widetilde{\Sigma}$ satisfying the conditions of the proposition.
Define $\lambda^0 = \rho(\omega)$. Then $\lambda^0 \in \widetilde{\Sigma}^+$ and $\lambda^0(H_0) = \omega(H_0) = 2$. Let $\lambda \in \Sigma^+$. If $\lambda + \lambda^0 \in \widetilde{\Sigma}$ there exist two roots $\alpha, \beta \in \widetilde{\mathcal{R}}$ such that $\rho(\alpha) = \lambda$ and $\rho(\beta) = \lambda + \lambda^0$. As $\overline{\mathfrak{g}}^\beta \subset \overline{V^+}$, the root β will be a weight of $(\overline{\mathfrak{g}}, \overline{V^+})$ and therefore can be written : $\beta = \omega - \sum_{\gamma \in \widetilde{\Psi}} m_\gamma \gamma$ ($m_\gamma \in \mathbb{N}$). Restricting this equality to $\mathfrak{a}$ one gets:

$$\rho(\beta) = \lambda + \lambda^0 = \rho(\omega) - \sum_{\gamma \in \widetilde{\Psi}} m_\gamma \rho(\gamma) = \lambda^0 - \sum_{\gamma \in \widetilde{\Psi}} m_\gamma \rho(\gamma).$$

Hence

$$\lambda = -\sum_{\gamma \in \widetilde{\Psi}} m_\gamma \rho(\gamma). \qquad (*)$$

But $\lambda \in \Sigma^+ \subset \widetilde{\Sigma}^+$ and from the hypothesis $\rho(\gamma) \in \widetilde{\Sigma}^+ \cup \{0\}$; the preceding equation $(*)$ is therefore impossible and we have showed that $\lambda^0 = \rho(w)$ satisfies the required properties.
Let λ^1 be another root in $\widetilde{\Sigma}$ having these properties. In fact $\lambda^1 \in \widetilde{\Sigma}^+$. Set

$$S = \{\alpha \in \widetilde{\mathcal{R}}^+ \,|\, \rho(\alpha) = \lambda^1\}.$$

Each element in S is a weight of $(\mathfrak{g}, \overline{V^+})$ (as $\alpha(H_0) = \rho(\alpha)(H_0) = \lambda^1(H_0) = 2$, we have $\overline{\widetilde{\mathfrak{g}}}^{\alpha} \in \overline{V^+}$). Let ω^1 be a maximal element in S for the order induced by $\widetilde{\mathcal{R}}^+$.

- if $\beta \in \widetilde{\mathcal{R}}^+$ is such that $\rho(\beta) = 0$, then $\omega^1 + \beta$ is not a root, because we would have $\omega^1 + \beta \in S$ and this contradicts the maximality of ω^1.
- If $\beta \in \widetilde{\mathcal{R}}^+$ is such that $\rho(\beta) \neq 0$, then $\omega^1 + \beta$ is not a root, because in that case $\rho(\omega^1 + \beta) = \lambda^1 + \rho(\beta)$ would be a root of $\widetilde{\Sigma}^+$ and this contradicts the second property.

Therefore ω^1 is a highest weight of $(\overline{\mathfrak{g}}, \overline{V^+})$, hence $\omega^1 = \omega$. This implies $\lambda^1 = \lambda^0$.
The commutativity of V^+ implies that $\lambda^0 + \lambda \notin \widetilde{\Sigma}$ for $\lambda \in \widetilde{\Sigma}^+ \setminus \Sigma^+$. From the obtained characterisation of λ^0 we obtain the last assertion.

$\square$

1.5. **The first step in the descent.**

Let $\widetilde{\mathfrak{l}}_0$ be the algebra generated by the root spaces $\widetilde{\mathfrak{g}}^{\lambda_0}$ and $\widetilde{\mathfrak{g}}^{-\lambda_0}$. One has:

$$\widetilde{\mathfrak{l}}_0 = \widetilde{\mathfrak{g}}^{-\lambda_0} \oplus [\widetilde{\mathfrak{g}}^{-\lambda_0}, \widetilde{\mathfrak{g}}^{\lambda_0}] \oplus \widetilde{\mathfrak{g}}^{\lambda_0}.$$

(Just remark that $\widetilde{\mathfrak{g}}^{-\lambda_0} \oplus [\widetilde{\mathfrak{g}}^{-\lambda_0}, \widetilde{\mathfrak{g}}^{\lambda_0}] \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$ is a Lie algebra). This algebra is graded by the element $H_{\lambda_0} \in \mathfrak{a}$.
We will need the following result:

Lemma 1.5.1.
Let $\mathfrak{u} = \mathfrak{u}_{-1} \oplus \mathfrak{u}_0 \oplus \mathfrak{u}_1$ be a semi-simple graded Lie algebra over F. Suppose that $\mathfrak{u}_1$ is an absolutely simple $\mathfrak{u}_0$-module. Then the Lie algebra $\mathfrak{u}'$ generated by $\mathfrak{u}_1$ and $\mathfrak{u}_{-1}$ absolutely simple.

Proof. The algebra $\overline{\mathfrak{u}}$ is again graded and semi-simple:

$$\overline{\mathfrak{u}} = \overline{\mathfrak{u}_{-1}} \oplus \overline{\mathfrak{u}_0} \oplus \overline{\mathfrak{u}_1}$$

and from the hypothesis $\overline{\mathfrak{u}_1}$ is a simple $\overline{\mathfrak{u}_0}$-module. As before for $\widetilde{\mathfrak{l}}_0$, one has $\mathfrak{u}' = \mathfrak{u}_{-1} \oplus [\mathfrak{u}_{-1}, \mathfrak{u}_1] \oplus \mathfrak{u}_1$, and one verifies easily that $\mathfrak{u}' = \mathfrak{u}_{-1} \oplus [\mathfrak{u}_{-1}, \mathfrak{u}_1] \oplus \mathfrak{u}_1$ is an ideal of $\mathfrak{u}$, and therefore semi-simple. Then it is enough to prove that $\overline{\mathfrak{u}'} = \overline{\mathfrak{u}_{-1}} \oplus [\overline{\mathfrak{u}_{-1}}, \overline{\mathfrak{u}_1}] \oplus \overline{\mathfrak{u}_1}$ is a simple algebra over $\overline{F}$. Let J be an ideal of $\overline{\mathfrak{u}'}$. We will show that $J = \{0\}$ or $J = \overline{\mathfrak{u}'}$.
Note first that the ideal $\overline{\mathfrak{u}'}^{\perp} = \overline{\mathfrak{u}}''$, orthogonal of $\overline{\mathfrak{u}'}$ for the Killing form of $\overline{\mathfrak{u}}$, is a subset of $\overline{\mathfrak{u}_0}$. Hence

$$\overline{\mathfrak{u}} = \overline{\mathfrak{u}_{-1}} \oplus ([\overline{\mathfrak{u}_{-1}}, \overline{\mathfrak{u}_1}] \oplus \overline{\mathfrak{u}_0}'') \oplus \overline{\mathfrak{u}_1}$$

As J is an ideal of $\overline{\mathfrak{u}'}$, the space $J \cap \overline{\mathfrak{u}_1}$ is stable under $[\overline{\mathfrak{u}_{-1}}, \overline{\mathfrak{u}_1}]$. On the other hand $J \cap \overline{\mathfrak{u}_1}$ is also stable under $\overline{\mathfrak{u}_0}''$ because $[\overline{\mathfrak{u}_0}'', \overline{\mathfrak{u}_1}] = \{0\}$ (In a semi-simple Lie algebras orthogonal ideals commute). Therefore $J \cap \overline{\mathfrak{u}_1}$ is a sub-$\overline{\mathfrak{u}_0}$-module of $\overline{\mathfrak{u}_1}$. From the hypothesis, either $J \cap \overline{\mathfrak{u}_1} = \overline{\mathfrak{u}_1}$ or $J \cap \overline{\mathfrak{u}_1} = \{0\}$.

• If $J\cap\overline{\mathfrak{u}_1}=\overline{\mathfrak{u}_1}$, then $[\overline{\mathfrak{u}_{-1}},\overline{\mathfrak{u}_1}]\subset J$. Let U_0 be the grading element (which always exists). From the semi-simplicity of $\mathfrak{u}'$, one gets $U_0\in[\mathfrak{u}_{-1},\mathfrak{u}_1]$. Hence U_0 is in J. This implies that $\overline{\mathfrak{u}_{-1}}\subset J$, and finally $J=\overline{\mathfrak{u}'}$.
• If $J\cap\overline{\mathfrak{u}_1}=\{0\}$, then if $J^\perp$ is the ideal of $\overline{\mathfrak{u}'}$ orthogonal to J for the Killing form, we get $J^\perp=\overline{\mathfrak{u}'}$, hence $J=\{0\}$.
This proves that $\overline{\mathfrak{u}'}$ is simple.

□

Proposition 1.5.2.
The representation $([\widetilde{\mathfrak{g}}^{-\lambda_0},\widetilde{\mathfrak{g}}^{\lambda_0}],\widetilde{\mathfrak{g}}^{\lambda_0})$ *is absolutely simple and the algebra* $\widetilde{\mathfrak{l}}_0=\widetilde{\mathfrak{g}}^{-\lambda_0}\oplus[\widetilde{\mathfrak{g}}^{-\lambda_0},\widetilde{\mathfrak{g}}^{\lambda_0}]\oplus\widetilde{\mathfrak{g}}^{\lambda_0}$ *is absolutely simple of split rank* 1.

Proof. Let $\mathfrak{m}=\mathcal{Z}_{\mathfrak{g}}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Consider the graded algebra

$$\mathfrak{u}=\widetilde{\mathfrak{g}}^{-\lambda_0}\oplus\mathfrak{m}\oplus\widetilde{\mathfrak{g}}^{\lambda_0}$$

From Lemma 1.5.1, to prove that $\widetilde{\mathfrak{l}}_0$ is absolutely simple, it is enough to show that the representation $(\mathfrak{m},\widetilde{\mathfrak{g}}^{\lambda_0})$ is absolutely simple. One has:

$$\overline{\mathfrak{m}}=\bar{\mathfrak{j}}\oplus\sum_{\{\alpha\in\widetilde{\mathcal{R}}\,|\,\rho(\alpha)=0\}}\overline{\widetilde{\mathfrak{g}}}^{\alpha}.$$

The algebra $\overline{\mathfrak{m}}$ is reductive, and his root system for the Cartan subalgebra $\bar{\mathfrak{j}}$ is $\widetilde{\mathcal{R}}_0=\{\alpha\in\widetilde{\mathcal{R}}\,|\,\rho(\alpha)=0\}$. We put the order induced by $\widetilde{\mathcal{R}}^+$ on $\widetilde{\mathcal{R}}_0$.
We have to show that the module $(\overline{\mathfrak{m}},\overline{\widetilde{\mathfrak{g}}^{\lambda_0}})$ is simple. If it would not, this module would have a lowest weight α_1 distinct from α_0 (see Proposition 1.3.2). But from hypothesis $(\mathbf{H_2})$, the module $(\bar{\mathfrak{g}},\overline{V^+})$ is simple, and his lowest weight (with respect to $\mathcal{R}^+=\widetilde{\mathcal{R}}^+\cap\mathcal{R}$) is α_0. There exists then a sequence $\beta_1,\dots,\beta_k$ of simple roots in $\Psi=\widetilde{\Psi}\setminus\{\alpha_0\}$ such that $\alpha_1=\alpha_0+\beta_1+\cdots+\beta_k$, and such that each partial sum is a root. But as $\rho(\alpha_0)=\rho(\alpha_1)=\lambda_0$, and as the roots β_i are in $\widetilde{\mathcal{R}}^+$, we obtain $\rho(\beta_i)=0$, for $i=1,\dots,k$. Hence $\beta_i\in\widetilde{\mathcal{R}}_0^+$ and $\alpha_1-\beta_k=\alpha_0+\beta_1+\cdots+\beta_{k-1}$ is a root. As α_1 is a lowest weight, this is impossible.
The fact that this algebra is of split rank one is easy.

□

Let $\widetilde{\mathfrak{g}}_1=\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)$ be the centralizer of $\widetilde{\mathfrak{l}}_0$ in $\widetilde{\mathfrak{g}}$. This is a reductive subalgebra (see [4],chap.VII, §1, n°5, Prop.13). The same is then true for $\overline{\widetilde{\mathfrak{g}}_1}=\overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)}=\mathcal{Z}_{\overline{\widetilde{\mathfrak{g}}}}(\overline{\widetilde{\mathfrak{l}}_0})$ ([5], Chap. I, §6, n°10). (We will show the last equality in the proof of the next proposition).

Proposition 1.5.3.
If $\widetilde{\mathfrak{g}}_1\cap V^+\neq\{0\}$, *then* $\widetilde{\mathfrak{g}}_1$ *satisfies the hypothesis* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$, *where the grading is defined by the element* $H_1=H_0-H_{\lambda_0}$.

Proof. It is clear that $H_1\in\widetilde{\mathfrak{g}}_1$. As $H_{\lambda_0}\in\widetilde{\mathfrak{l}}_0$, the actions of $\operatorname{ad}H_0$ and of $\operatorname{ad}H_1$ on $\widetilde{\mathfrak{g}}_1$ are the same. As $\widetilde{\mathfrak{g}}_1\cap V^+\neq\{0\}$, the eigenvalues of $\operatorname{ad}H_1$ on $\widetilde{\mathfrak{g}}_1$ are $-2,0,2$. Therefore the hypothesis $\mathbf{H_1}$ is satisfied.
It remains to show that if $\mathfrak{g}_1=\mathfrak{g}\cap\widetilde{\mathfrak{g}}_1$ and if $\overline{V_1^+}=\overline{V^+}\cap\overline{\widetilde{\mathfrak{g}}_1}$, the the representation $(\mathfrak{g}_1,\overline{V_1^+})$ (or $(\overline{\mathfrak{g}_1},\overline{V_1^+})$) is irreducible over $\overline{F}$.

From Lemma 1.3.1 one has $\overline{\widetilde{\mathfrak{l}}_0} = \overline{\widetilde{\mathfrak{g}}}^{-S_{\lambda_0}} \oplus [\overline{\widetilde{\mathfrak{g}}}^{-S_{\lambda_0}}, \overline{\widetilde{\mathfrak{g}}}^{S_{\lambda_0}}] \oplus \overline{\widetilde{\mathfrak{g}}}^{S_{\lambda_0}}$ where $\overline{\widetilde{\mathfrak{g}}}^{S_{\lambda_0}} = \overline{\widetilde{\mathfrak{g}}^{\lambda_0}} = \sum_{\alpha\in S_{\lambda_0}} \overline{\widetilde{\mathfrak{g}}}^{\alpha}$. Let us first describe $\overline{\widetilde{\mathfrak{g}}_1} = \overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)}$. It is easy to see that $\overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)} \subset \mathcal{Z}_{\overline{\widetilde{\mathfrak{g}}}}(\overline{\widetilde{\mathfrak{l}}_0})$. Let $X \in \mathcal{Z}_{\overline{\widetilde{\mathfrak{g}}}}(\overline{\widetilde{\mathfrak{l}}_0})$. Let us write $X = H + \sum_{\alpha\in\widetilde{\mathcal{R}}} X_\alpha$ whith $H \in \bar{\mathfrak{j}}$ and with $X_\alpha \in \overline{\widetilde{\mathfrak{g}}}^{\alpha}$. As $[X, X_\beta] = 0$ for $\beta \in \pm S_{\lambda_0}$, we obtain that $X \in \bar{\mathfrak{j}}_1 \oplus \sum_{\beta\in\widetilde{\mathcal{R}}_1} \overline{\widetilde{\mathfrak{g}}}^{\beta}$, where

$$\bar{\mathfrak{j}}_1 = \{H \in \bar{\mathfrak{j}} \,|\, \alpha(H) = 0\,, \forall\alpha \in S_{\lambda_0}\}$$

$$\widetilde{\mathcal{R}}_1 = \{\beta \in \widetilde{\mathcal{R}} \,|\; \beta \perp\!\!\!\perp \alpha,\, \forall\alpha \in S_{\lambda_0}\}.$$

We will now show that $\bar{\mathfrak{j}}_1 \oplus \sum_{\beta\in\widetilde{\mathcal{R}}_1} \overline{\widetilde{\mathfrak{g}}}^{\beta} \subset \overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)}$.

• Consider first $\overline{\widetilde{\mathfrak{g}}}^{\beta}$ for $\beta \in \widetilde{\mathcal{R}}_1$. Any element $X \in \overline{\widetilde{\mathfrak{g}}}^{\beta} \subset \widetilde{\mathfrak{g}}$ can be written $X = \sum_{i=1}^n e_i X_i$ where $X_i \in \widetilde{\mathfrak{g}}$ and where the e_i's are elements of $\overline{F}$ which are free over F. As $\beta \in \widetilde{\mathcal{R}}_1$ one has $[X, \overline{\widetilde{\mathfrak{g}}}^{\beta}] = \{0\}$ for each $\gamma \in S_{\lambda_0}$. As $\widetilde{\mathfrak{g}}^{\lambda_0} \subset \overline{\widetilde{\mathfrak{g}}}^{S_{\lambda_0}} \subset \sum_{\alpha\in S_{\lambda_0}} \overline{\widetilde{\mathfrak{g}}}^{\alpha}$, one has $[X, \widetilde{\mathfrak{g}}^{\lambda_0}] = \{0\} = \sum_{i=1}^n e_i [X_i, \widetilde{\mathfrak{g}}^{\lambda_0}]$. As $[X_i, \widetilde{\mathfrak{g}}^{\lambda_0}] \subset \widetilde{\mathfrak{g}}$ and as the e_i's are free over F, we obtain that $[X_i, \widetilde{\mathfrak{g}}^{\lambda_0}] = \{0\}$.
One would similarly prove that $[X_i, \widetilde{\mathfrak{g}}^{-\lambda_0}] = \{0\}$. Hence $X \in \overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)}$.

• Consider now $\bar{\mathfrak{j}}_1 = \{H \in \bar{\mathfrak{j}} \,|\, \alpha(H) = 0\,, \forall\alpha \in S_{\lambda_0}\}$. An element $u \in \bar{\mathfrak{j}}_1$ can be written $u = \sum_{i=1}^n a_i u_i$ where the elements $a_i \in \overline{F}$ are free over F and where $u_i \in \mathfrak{j}$. Then for $\alpha \in S_{\lambda_0}$ one has: $\alpha(u) = 0 = \sum_{i=1}^n a_i \alpha(u_i)$. This implies that $\alpha(u_i) = 0$, for any $\alpha \in S_{\lambda_0}$ and any i. Therefore the elements u_i belong to $\mathfrak{j}_1 = \{u \in \mathfrak{j}, \alpha(u) = 0, \forall\alpha \in S_{\lambda_0}\}$. But then $u = \sum_{i=1}^n a_i u_i \in \overline{(\mathfrak{j}_1)}$ and $[u_i, \widetilde{\mathfrak{g}}^{\pm\lambda_0}] \subset [u_i, \overline{\widetilde{\mathfrak{g}}}^{\pm S_{\lambda_0}}] = \{0\}$ (from the definition of $\mathfrak{j}_1$). Hence $u_i \in \mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)$, and therefore $u \in \overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)}$.
Finally we have proved that

$$\overline{\widetilde{\mathfrak{g}}_1} = \overline{\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)} = \mathcal{Z}_{\overline{\widetilde{\mathfrak{g}}}}(\overline{\widetilde{\mathfrak{l}}_0}) = \bar{\mathfrak{j}}_1 \oplus \sum_{\beta\in\widetilde{\mathcal{R}}_1} \overline{\widetilde{\mathfrak{g}}}^{\beta}.$$

If $\beta \in \widetilde{\mathcal{R}}_1$, then $\alpha(H_\beta) = 0$ for all root $\alpha \in S_{\lambda_0}$. Hence $H_\beta \in \bar{\mathfrak{j}}_1$, and the restriction of β to $\bar{\mathfrak{j}}_1$ is non zero. This implies that $\bar{\mathfrak{j}}_1$ is a Cartan subalgebra of the reductive algebra $\overline{\widetilde{\mathfrak{g}}_1}$ and $\widetilde{\mathcal{R}}_1$ can be seen as the root system of the pair $(\overline{\widetilde{\mathfrak{g}}_1}, \bar{\mathfrak{j}}_1)$. The order on $\widetilde{\mathcal{R}}$ defined by $\widetilde{\mathcal{R}}^+$ induces an order on $\widetilde{\mathcal{R}}_1$ and on the root system $\mathcal{R}_1 = \widetilde{\mathcal{R}}_1 \cap \mathcal{R}$ of the pair $(\overline{\mathfrak{g}_1}, \bar{\mathfrak{j}}_1)$ by setting:

$$\widetilde{\mathcal{R}}_1^+ = \widetilde{\mathcal{R}}_1 \cap \widetilde{\mathcal{R}}^+,\ \mathcal{R}_1^+ = \mathcal{R}_1 \cap \widetilde{\mathcal{R}}^+ = \widetilde{\mathcal{R}}_1 \cap \mathcal{R}^+.$$

Let ω_1 be the highest weight, for the preceding order, of one of the irreducible components of the representation $(\overline{\mathfrak{g}_1}, \overline{V_1^+})$. ω_1 is a root of $\widetilde{\mathcal{R}}_1^+$ and we will show that it is also the highest weight of the representation $(\overline{\mathfrak{g}}, \overline{V^+})$ which is irreducible from $(\mathbf{H_2})$. This will show that $(\overline{\mathfrak{g}_1}, \overline{V_1^+})$ is irreducible. To do this we will show that if $\beta \in \mathcal{R}^+$ then $\omega_1 + \beta \notin \widetilde{\mathcal{R}}$.

(1) If $\beta \in \mathcal{R}_1^+$, then from the definition of ω_1 we get $[\overline{\widetilde{\mathfrak{g}}}^{\omega_1}, \overline{\widetilde{\mathfrak{g}}}^{\beta}]$=$\{0\}$. Hence $\omega_1 + \beta \notin \widetilde{\mathcal{R}}$.

(2) If $\beta \notin \mathcal{R}_1^+$, there exists a root $\alpha \in S_{\lambda_0}$ such that α and β are not strongly orthogonal. We will show that in that case there is a root $\gamma \in S_{\lambda_0}$ such that

$$(*) \qquad \gamma + \beta \in \widetilde{\mathcal{R}} \text{ and } \gamma - \beta \notin \widetilde{\mathcal{R}}.$$

(2.1) If $\rho(\beta) \neq 0$, then $\alpha - \beta$ is not a root. If it would be the case, we would have

$$\rho(\alpha-\beta) = \rho(\alpha) - \rho(\beta) = \lambda_0 - \rho(\beta)$$

But $\rho(\beta) \neq \lambda_0$ as $\beta \in \Sigma^+$, so $\lambda_0 - \rho(\beta)$ would be a root, and this is impossible as λ_0 is a simple root in $\widetilde{\Sigma}^+$. In this case the root α satisfies $(*)$.

(2.2) If $\rho(\beta) = 0$, consider the root β-string through α. These roots are in S_{λ_0} and as α and β are not strongly orthogonal, this string contains at least two roots. This implies that there exists a root γ verifying $(*)$.

Therefore, from [3] (chap. 6, §1, Prop. 9) we obtain that $(\beta, \gamma) < 0$. As ω_1 and γ are strongly orthogonal, we have $(\omega_1, \gamma) = 0$ ($\omega_1 \in \widetilde{\mathcal{R}}_1^+$ and $\gamma \in S_{\lambda_0}$).

Now if $\omega_1 + \beta$ is a root, we would have:

$$(\omega_1 + \beta, \gamma) = (\omega_1, \gamma) + (\beta, \gamma) = (\beta, \gamma) < 0.$$

Then $\omega_1 + \beta + \gamma$ is a root and $(\omega_1 + \beta + \gamma)(H_0) = 4$ (as $\omega_1(H_0) = 2, \beta(H_0) = 0, \gamma(H_0) = 2$). Hence $\omega_1 + \beta$ is not a root.

This means that ω_1 is a highest weight of $(\overline{\mathfrak{g}}, \overline{V^+})$ which is irreducible from $(\mathbf{H_2})$. Therefore $(\mathfrak{g}_1, \overline{V_1^+})$ is irreducible (and $\omega_1 = \omega$).

□

Set $\mathfrak{a}_1 = \mathfrak{a} \cap \widetilde{\mathfrak{g}}_1$. Then $\mathfrak{a}_1$ is a maximal split abelian subalgebra of $\widetilde{\mathfrak{g}}_1$ included in $\mathfrak{g}_1$ (because the actions of the maximal split abelian subalgebras can be diagonalized in all finite dimensional representation).

Proposition 1.5.4.

1) *The root system $\widetilde{\Sigma}_1$ of the pair $(\widetilde{\mathfrak{g}}_1, \mathfrak{a}_1)$ is*

$$\widetilde{\Sigma}_1 = \{\lambda \in \widetilde{\Sigma}, \lambda \perp\!\!\!\perp \lambda_0\} = \{\lambda \in \widetilde{\Sigma}, \lambda \perp \lambda_0\}$$

(where $\perp$ means "orthogonal" and where $\perp\!\!\!\perp$ means "strongly orthogonal".)

2) *Consider the order on $\widetilde{\Sigma}_1$ defined by*

$$\widetilde{\Sigma}_1^+ = \widetilde{\Sigma}^+ \cap \widetilde{\Sigma}_1.$$

This order satisfies the properties of Theorem 1.2.1 for the graded algebra $\widetilde{\mathfrak{g}}_1$.

3) *The set of simple roots $\widetilde{\Pi}_1$ defined by $\widetilde{\Sigma}_1^+$ is given by:*

$$\widetilde{\Pi}_1 = (\Pi \cap \widetilde{\Sigma}_1) \cup \{\lambda_1\}$$

where λ_1 is the unique root of $\widetilde{\Pi}_1$ such that $\lambda_1(H_0 - H_{\lambda_0}) = \lambda_1(H_0) = 2$.

Proof.

1) Let us first show that a root $\lambda \in \widetilde{\Sigma}$ is strongly orthogonal to λ_0 if and only if it is orthogonal to λ_0.

Let $\lambda \in \widetilde{\Sigma}$ be a root orthogonal to λ_0. Let

$$\lambda - q\lambda_0, \ldots, \lambda - \lambda_0, \lambda, \lambda + \lambda_0, \ldots, \lambda + p\lambda_0$$

be the λ_0-string of roots through λ. From [3] (Chap. VI, §1, $n°3$, Prop. 9) one has $p-q=-2\frac{(\lambda,\lambda_0)}{(\lambda_0,\lambda_0)}=0$. Hence $p=q$, and the string is symmetric.

a) If $\lambda(H_0)=2$, then as $\lambda+\lambda_0\notin\widetilde{\Sigma}$ (from $(\mathbf{H_1})$), we get $\lambda-\lambda_0\notin\widetilde{\Sigma}$

b) If $\lambda(H_0)=-2$, then $\lambda-\lambda_0\notin\widetilde{\Sigma}$, and the same proof shows that $\lambda+\lambda_0\notin\widetilde{\Sigma}$.

c) If $\lambda(H_0)=0$ then $\lambda\in\Sigma$. Recall that $\widetilde{\Pi}=\Pi\cup\{\lambda_0\}$. If $\lambda\in\Sigma^+$ then $\lambda-\lambda_0\notin\widetilde{\Sigma}$, and if $\lambda\in\Sigma^-$ then $\lambda+\lambda_0\notin\widetilde{\Sigma}$. Again the same proof shows that, respectively, $\lambda+\lambda_0\notin\widetilde{\Sigma}$ and $\lambda-\lambda_0\notin\widetilde{\Sigma}$.

In all cases we have showed that $\lambda\perp\!\!\!\perp\lambda_0$.
For $\lambda\in\{\lambda\in\widetilde{\Sigma},\,\lambda\perp\!\!\!\perp\lambda_0\}$, it is clear that $\widetilde{\mathfrak{g}}^\lambda\in\widetilde{\mathfrak{g}}_1=\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0)$, that $H_\lambda\in\mathfrak{a}_1=\mathfrak{a}\cap\widetilde{\mathfrak{g}}_1$ and that $\lambda_{|\mathfrak{a}_1}$ is a root of the pair $(\widetilde{\mathfrak{g}}_1,\mathfrak{a}_1)$. Conversely any root of the pair $(\widetilde{\mathfrak{g}}_1,\mathfrak{a}_1)$ can be extended to a linear form on $\mathfrak{a}$ by setting $\lambda(H_0)=0$, and this extension is a root orthogonal to λ_0, and hence strongly orthogonal to λ_0 from above, and finally this extension is in $\widetilde{\Sigma}_1$.

2) The set $\widetilde{\Sigma}_1^+=\widetilde{\Sigma}^+\cap\widetilde{\Sigma}_1$ defines an order on $\widetilde{\Sigma}$. For $\lambda\in\widetilde{\Sigma}_1^+$ one has $\lambda(H_1)=\lambda(H_0)=0$ or 2, and this gives 2).

3) Let $\widetilde{\Pi}_1$ be the set of simple roots in $\widetilde{\Sigma}_1^+$. From Theorem 1.2.1, $\widetilde{\Pi}_1=\Pi_1\cup\{\lambda_1\}$ where $\Pi_1=\{\lambda\in\widetilde{\Pi}_1\,,\,\lambda(H_1)=0\}$. If $\lambda\in\Pi\cap\widetilde{\Sigma}_1$ then $\lambda\in\widetilde{\Pi}_1$ and $\lambda(H_1)=\lambda(H_0)-\lambda(H_{\lambda_0})=0$. Therefore $\Pi\cap\widetilde{\Sigma}_1\subset\Pi_1$. Conversely let $\mu\in\Pi_1$. Then $\mu\in\Sigma^+$. Hence μ can be written $\mu=\sum_{\nu\in\Pi}m_\nu\nu$, with $m_\nu\in\mathbb{N}$. One has $(\mu,\lambda_0)=0$ and $(\nu,\lambda_0)\leq 0$ (because $\Pi\cup\{\lambda_0\}=\widetilde{\Pi}$ is a set of simple roots). Therefore if $m_\nu\neq 0$, one has $(\nu,\lambda_0)=0$, and hence $\nu\in\widetilde{\Sigma}_1$. Finally we have proved that $\Pi_1=\Pi\cap\widetilde{\Sigma}_1$.

$\square$

Remark 1.5.5.
One may remark that $\Pi_1\subset\Pi$, but $\widetilde{\Pi}_1$ is not a subset of $\widetilde{\Pi}$. Indeed $\lambda_1(H_0)=2$, but there is only one root λ in $\widetilde{\Pi}$ such that $\lambda(H_0)=2$, namely λ_0. Hence $\lambda_1\notin\widetilde{\Pi}$.

1.6. The descent.

Theorem 1.6.1.
There exists a unique sequence of strongly orthogonal roots in $\widetilde{\Sigma}^+\setminus\Sigma^+$, denoted by $\lambda_0,\lambda_1,\dots,\lambda_k$ and a sequence of reductive Lie algebras $\widetilde{\mathfrak{g}}\supset\widetilde{\mathfrak{g}}_1\supset\cdots\supset\widetilde{\mathfrak{g}}_k$ such that

(1) $\widetilde{\mathfrak{g}}_j=\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0\oplus\widetilde{\mathfrak{l}}_1\oplus\cdots\oplus\widetilde{\mathfrak{l}}_{j-1})$ *where* $\widetilde{\mathfrak{l}}_i=\widetilde{\mathfrak{g}}^{-\lambda_i}\oplus[\widetilde{\mathfrak{g}}^{-\lambda_i},\widetilde{\mathfrak{g}}^{\lambda_i}]\oplus\widetilde{\mathfrak{g}}^{\lambda_i}$ *is the subalgebra generated by* $\widetilde{\mathfrak{g}}^{\lambda_i}$ *and* $\widetilde{\mathfrak{g}}^{-\lambda_i}$.

(2) *The algebra* $\widetilde{\mathfrak{g}}_j$ *is a graded Lie algebra verifying the hypothesis* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *with the grading element* $H_j=H_0-H_{\lambda_0}-\cdots-H_{\lambda_{j-1}}$.

(3) $V^+\cap\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0\oplus\widetilde{\mathfrak{l}}_1\oplus\cdots\oplus\widetilde{\mathfrak{l}}_k)=\{0\}$

Proof. The proof is done by induction on j, starting from Proposition 1.5.4, and using the fact that $\widetilde{\mathfrak{g}}_j$ is the centralizer of $\widetilde{\mathfrak{l}}_{j-1}$ in $\widetilde{\mathfrak{g}}_{j-1}$. It is worth noting that the construction stops for the index k such that $V^+\cap\mathcal{Z}_{\widetilde{\mathfrak{g}}_k}(\widetilde{\mathfrak{l}}_k)=\{0\}$, which amounts to saying that $\mathcal{Z}_{V^+}(\widetilde{\mathfrak{l}}_0\oplus\widetilde{\mathfrak{l}}_1\oplus\cdots\oplus\widetilde{\mathfrak{l}}_k)=\{0\}$.

□

Definition 1.6.2. *The number $k+1$ of strongly orthogonal roots appearing in the preceding Theorem will be called the* **rank** *of the graded algebra $\widetilde{\mathfrak{g}}$.*

Notation 1.6.3. Everything above also applies to the graded algebra $\widetilde{\mathfrak{g}}_j = V_j^- \oplus \mathfrak{g}_j \oplus V_j^+$ which is graded by $H_j = H_0 - H_{\lambda_0} - \cdots - H_{\lambda_{j-1}}$. The algebra $\mathfrak{a}_j = \mathfrak{a} \cap \widetilde{\mathfrak{g}}_j$ is a maximal split abelian subalgebra of $\widetilde{\mathfrak{g}}_j$ contained in $\mathfrak{g}_j$. The set $\widetilde{\Sigma}_j$ of roots of the pair $(\widetilde{\mathfrak{g}}_j, \mathfrak{a}_j)$ is the set of roots in $\widetilde{\Sigma}$ which are strongly orthogonal to $\lambda_0, \lambda_1, \ldots, \lambda_{j-1}$. We put an order on $\widetilde{\Sigma}_j$ by setting $\widetilde{\Sigma}_j^+ = \widetilde{\Sigma}^+ \cap \widetilde{\Sigma}_j$. The corresponding set of simple roots $\widetilde{\Pi}_j$ is given by $\widetilde{\Pi}_j = \Pi_j \cup \{\lambda_j\}$ where Π_j is the set of simple roots of Σ_j defined by $\Sigma_j^+ = \Sigma_j \cap \Sigma^+$ (where $\Sigma_j = \Sigma \cap \widetilde{\Sigma}_j$).

1.7. Generic elements in V^+ and maximal systems of long strongly orthogonal roots.

We define now the groups we will use. If $\mathfrak{k}$ is a Lie algebra over F, we will denote by $\mathrm{Aut}(\mathfrak{k})$ the group of automorphisms of $\mathfrak{k}$. The map $g \longmapsto g \otimes 1$ is an injective homomorphism from $\mathrm{Aut}(\mathfrak{k})$ in $\mathrm{Aut}(\mathfrak{k} \otimes_F \overline{F}) = \mathrm{Aut}(\overline{\mathfrak{k}})$. This allows to consider $\mathrm{Aut}(\mathfrak{k})$ as a subgroup of $\mathrm{Aut}(\overline{\mathfrak{k}})$.
From now on $\mathfrak{k}$ will be reductive.
We will denote by $\mathrm{Aut}_e(\mathfrak{k})$ the subgroup of elementary automorphisms of $\mathfrak{k}$, that is the automorphisms which are finite products of automorphisms of the form $e^{\mathrm{ad}\, x}$, where $\mathrm{ad}(x)$ is nilpotent in $\mathfrak{k}$. If $g \in \mathrm{Aut}_e(\mathfrak{k})$, then g fixes pointwise the elements of the center of $\mathfrak{k}$ and therefore $\mathrm{Aut}_e(\mathfrak{k})$ can be identified with $\mathrm{Aut}_e([\mathfrak{k},\mathfrak{k}])$.
Let us set $\mathrm{Aut}_0(\mathfrak{k}) = \mathrm{Aut}(\mathfrak{k}) \cap \mathrm{Aut}_e(\overline{\mathfrak{k}})$. The elements of $\mathrm{Aut}_0(\mathfrak{k})$ are the automorphisms of $\mathfrak{k}$ which become elementary after extension from F to $\overline{F}$. Again $\mathrm{Aut}_0(\mathfrak{k})$ can be identified with $\mathrm{Aut}_0([\mathfrak{k},\mathfrak{k}])$. Finally we have the following inclusions:

$$\mathrm{Aut}_e(\mathfrak{k}) \subset \mathrm{Aut}_0(\mathfrak{k}) \subset \mathrm{Aut}([\mathfrak{k},\mathfrak{k}]) \subset \mathrm{Aut}(\mathfrak{k}).$$

From [4] (Chap. VIII, §8, $n^\circ 4$, Corollaire de la Proposition 6, p.145), $\mathrm{Aut}_0(\mathfrak{k})$ is open and closed in $\mathrm{Aut}([\mathfrak{k},\mathfrak{k}])$, and $\mathrm{Aut}([\mathfrak{k},\mathfrak{k}])$ is closed in $\mathrm{End}([\mathfrak{k},\mathfrak{k}])$ for the Zariski topology ([4], Chap. VIII, §5, $n^\circ 4$, Prop. 8 p.111), therefore $\mathrm{Aut}_0(\mathfrak{k})$ is an algebraic group. As $\mathrm{Aut}_0(\mathfrak{k})$ is open in $\mathrm{Aut}([\mathfrak{k},\mathfrak{k}])$, its Lie algebra is the same as the Lie algebra of $\mathrm{Aut}(\mathfrak{k})$, namely $[\mathfrak{k},\mathfrak{k}]$. We know also that $\mathrm{Aut}_0(\overline{\mathfrak{k}}) = \mathrm{Aut}_e(\overline{\mathfrak{k}})$ is the connected component of the neutral element of $\mathrm{Aut}([\overline{\mathfrak{k}},\overline{\mathfrak{k}}])$ ([4], Chap. VIII,§5, $n^\circ 5$, Prop. 11, p. 113).

The group G we consider here is the following (this group was first introduced by Iris Muller in [15] and [14]):

- $G = \mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0) = \{g \in \mathrm{Aut}_0(\widetilde{\mathfrak{g}}),\, g.H_0 = H_0\}$ is the centralizer of H_0 in $\mathrm{Aut}_0(\widetilde{\mathfrak{g}})$.

The Lie algebra of G is then $\mathcal{Z}_{[\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}]}(H_0) = \mathfrak{g} \cap [\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}] = [\mathfrak{g},\mathfrak{g}] + [V^+,V^-] \supset FH_0 \oplus [\mathfrak{g},\mathfrak{g}]$.
As the elements of G fix H_0, we obtain that V^+ (which is the eigenspace of $\mathrm{ad}\, H_0$ for the eigenvalue 2) is stable under the action of G. Of course the same is true for V^-.

The representation (G,V^+) is a prehomogeneous vector space, or more precisely it is an F-form of a prehomogeneous vector space. This means just that this representation has a Zariski-open orbit. In fact G can be viewed as the F-points of an algebraic group (also noted G). Then if $\overline{G} = G(\overline{F})$ stands for the points over $\overline{F}$ of G, $(\overline{G}, \overline{V^+})$ is a prehomogeneous vector space from a well known result of Vinberg ([30]). As $\overline{F}$ is algebraically closed, this prehomogeneous vector space has of course only one open orbit. It is well known that then the representation (G, V^+) has only a finite number of open orbits. This is a consequence of a result of Serre (see [25], p.469, or [12], p. 1).

One has $G(\overline{F}) = \mathcal{Z}_{\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})}(H_0) = \mathcal{Z}_{\mathrm{Aut}_e(\overline{\widetilde{\mathfrak{g}}})}(H_0)$. (Because over an algebraically closed field one has $\mathrm{Aut}_e(\overline{\widetilde{\mathfrak{g}}}) = \mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})$). Moreover, as we mentioned before, $\mathrm{Aut}_0(\widetilde{\mathfrak{g}})$ and $\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})$ are closed and are the connected components of the neutral element in $\mathrm{Aut}(\widetilde{\mathfrak{g}})$ and $\mathrm{Aut}(\overline{\widetilde{\mathfrak{g}}})$ respectively (for the Zariski topology of $\mathrm{End}([\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}])$ and $\mathrm{End}([\overline{\widetilde{\mathfrak{g}}}, \overline{\widetilde{\mathfrak{g}}}])$.
The Lie subalgebra $\mathfrak{t} = FH_0$ is algebraic. Let us denote by T the corresponding one dimensional torus. Then as $G(\overline{F}) = \mathcal{Z}_{\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})}(H_0) = \mathcal{Z}_{\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})}(T)$, we obtain that $G(\overline{F})$ is connected ([11] Theorem 22.3 p.140).

Definition 1.7.1. *An element $X \in V^+$ is called generic if it satisfies one of the following equivalent conditions:*
(i) The G-orbit of X in V^+ is open.
(ii) $\mathrm{ad}(X) : \mathfrak{g} \longrightarrow V^+$ is surjective.

Lemma 1.7.2.
Let $X \in V^+$. If there exists $Y \in V^-$ such that (Y, H_0, X) is an $\mathfrak{sl}_2$-triple, then X is generic in V^+.

Proof. For $v \in V^+$ one has $2v = [H_0, v] = \mathrm{ad}([Y, X])v = -\mathrm{ad}(X)\,\mathrm{ad}(Y)v$. Hence $\mathrm{ad}(X)$ is surjective from $\mathfrak{g}$ onto V^+.

□

Note: Although, in the preceding proof we used the commutativity of V^+, the result is in fact true for any $\mathbb{Z}$-graded algebra.

For $i = 0, \ldots, k$, let us choose once and for all $X_i \in \widetilde{\mathfrak{g}}^{\lambda_i}$ and $Y_i \in \widetilde{\mathfrak{g}}^{-\lambda_i}$, such that $(Y_i, H_{\lambda_i}, X_i)$ is an $\mathfrak{sl}_2$-triple. This is always possible (see for example [28], Corollary of Lemma 6, p.6, or [27], Proposition 3.1.9 p.23)

Lemma 1.7.3.
The element X_k is generic in V_k^+.

Proof. The Lie algebra $\widetilde{\mathfrak{g}}_k$ is graded by $H_k = H_0 - H_{\lambda_0} - \cdots - H_{\lambda_{k-1}}$ and not by H_{λ_k}, therefore we cannot use Lemma 1.7.2.
We will show that $\mathrm{ad}(X_k) : \mathfrak{g}_k \longrightarrow V_k^+$ is surjective (Cf. definition 1.7.1). Let λ be a root such that $\widetilde{\mathfrak{g}}^\lambda \subset V_k^+$ and let $z \in \widetilde{\mathfrak{g}}^\lambda$. Then

$$\mathrm{ad}(X_k)\,\mathrm{ad}(Y_k)z = -\mathrm{ad}(H_k)z = -\lambda(H_k)z.$$

As $\mathrm{ad}(Y_k)z \in \mathfrak{g}_k$, it is enough to show that $\lambda(H_k) \neq 0$. If $\lambda = \lambda_k$, then of course $\lambda(H_k) = 2$. If $\lambda \neq \lambda_k$ and $\lambda(H_k) = 0$, then $\lambda \perp \lambda_k$, and hence $\lambda \perp\!\!\!\perp \lambda_k$, by Proposition 1.5.4 1). Then $\widetilde{\mathfrak{g}}^\lambda \subset V^+ \cap \mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_k) = \{0\}$. Contradiction.

□

Lemma 1.7.4.
If X is generic in V_j^+, then $X_0 + X_1 + \cdots + X_{j-1} + X$ is generic in V^+.

Proof. By induction we must just prove the Lemma for $j = 1$. Let X be be generic in V_1^+. We will prove that $[\mathfrak{g}, X_0 + X] = V^+$.

• One has $[\mathfrak{g}_1, X_0 + X] = [\mathfrak{g}_1, X] = V_1^+$, from the definition of $\mathfrak{g}_1$ and because X is generic in V_1^+.

• If $z \in \widetilde{\mathfrak{g}}^{\lambda_0}$, then $[Y_0, z] \in \mathfrak{g} \cap \widetilde{\mathfrak{l}}_0$ and hence $[Y_0, z]$ commutes with $X \in V_1^+$. But then $\operatorname{ad}(X_0 + X)[Y_0, z] = \operatorname{ad}(X_0)[Y_0, z] = [-H_0, z] = -2z$.

• If $z \in \widetilde{\mathfrak{g}}^{\lambda}$ with $\lambda \neq \lambda_0$ and $z \notin V_1^+$, then $\mu = \lambda - \lambda_0$ is a root (as λ is not strongly orthogonal to λ_0) which is positive, hence in Σ^+ (because $\lambda = \lambda_0 + \dots$). Therefore $\mu - \lambda_0$ is not a root. As $\mu + \lambda_0$ is a root, one has $(\mu, \lambda_0) < 0$ (see [3] (Chap. VI, §1, $n°3$, Prop. 9)).
Suppose first that $[\mathfrak{g}^{\mu}, V_1^+] \neq \{0\}$. Then there exists a root $\nu \in \widetilde{\Sigma}^+$ such $\widetilde{\mathfrak{g}}^{\nu} \subset V_1^+$ and $\mu + \nu \in \widetilde{\Sigma}^+$. One would have $(\mu + \nu, \lambda_0) = (\mu, \lambda_0) < 0$. Hence $\mu + \nu + \lambda_0$ would be a root, such that

$$(\mu + \nu + \lambda_0)(H_0) = \mu(H_0) + \nu(H_0) + \lambda_0(H_0) = 0 + 2 + 2 = 4$$

and this is impossible from the hypothesis $(\mathbf{H_1})$
Therefore $[\mathfrak{g}^{\mu}, V_1^+] = \{0\}$ and then, as $U = \operatorname{ad}(Y_0)z \in \mathfrak{g}^{\mu}$, we have $\operatorname{ad}(X_0 + X)U = \operatorname{ad}(X_0)U = \operatorname{ad}(X_0)\operatorname{ad}(Y_0)z = -\operatorname{ad}(H_0)z = -2z$.

□

The two preceding results imply:

Proposition 1.7.5. *An element of the form $X_0 + X_1 + \cdots + X_k$, where $X_i \in \widetilde{\mathfrak{g}}^{\lambda_i} \setminus \{0\}$, is generic in V^+.*

Proposition 1.7.6.
For $j = 0, \dots, k$ let us denote by W_j the weyl group of the pair $(\mathfrak{g}_j, \mathfrak{a}_i)$ (hence $W = W_0$). Let w_j be the unique element of W_j such that $w_j(\Sigma_j^+) = \Sigma_j^-$. Then $w_j(\lambda_j) = \lambda^0$ and the roots λ_j are long roots in $\widetilde{\Sigma}$.

Proof.
One has $w_0 \in W_0 \subset \mathcal{Z}_{\mathrm{Aut}_e(\widetilde{\mathfrak{g}})}(H_0) \subset G = \mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0)$. Hence

$$w_0(\lambda_0)(H_0) = \lambda_0(w_0(H_0)) = \lambda_0(H_0) = 2.$$

On the other hand, from Corollary 1.2.2, one has $\lambda_0 - \lambda \notin \widetilde{\Sigma}$ for $\lambda \in \Sigma^+$. Therefore $w_0(\lambda_0) - w_0(\lambda) \notin \widetilde{\Sigma}$. But $w_0(\lambda)$ takes all values in $-\Sigma^+$ when λ varies in Σ^+. Therefore $w_0(\lambda_0) + \lambda \notin \widetilde{\Sigma}$ if $\lambda \in \Sigma^+$. Proposition 1.4.1 implies then that $w_0(\lambda_0) = \lambda^0$. It is easy to see that λ^0 is also the highest root of the root systems $\widetilde{\Sigma}_j$. As λ_j is the analogue of λ_0 in $\widetilde{\mathfrak{g}}_j$, we obtain that $w_j(\lambda_j) = \lambda^0$. As the highest root λ^0 is a long root (see [3], Chap. VI,§1 $n°1$, Proposition 25 p.165), the roots λ_j are all long.

□

Proposition 1.7.7.
(1) *The set* $\lambda_0, \dots, \lambda_k$ *is a maximal system of strongly orthogonal long roots in* $\widetilde{\Sigma}^+ \setminus \Sigma^+$.
(2) *If* $\beta_0, \dots, \beta_m$ *is another maximal system of strongly orthogonal long roots in* $\widetilde{\Sigma}^+ \setminus \Sigma^+$ *then* $m = k$ *and there exists* $w \in W$ *such that* $w(\beta_j) = \lambda_j$ *for* $j = 0, \dots, k$.

Proof. The following proof is adapted from Théorème 2.12 in [18] which concerns the complex case.
(1) We have already seen that the roots λ_j are long and strongly orthogonal. Suppose that there exists a long root λ_{k+1}which is strongly orthogonal to each root λ_j $(j = 0, \dots, k)$. Then $\widetilde{\mathfrak{g}}^{\lambda_{k+1}}$ would be included in $\mathcal{Z}_{V^+}(\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_k) = \{0\}$ (Theorem 1.6.1). Contradiction.
(2) Set $\Phi = \{\gamma \in \Sigma \,|\, \gamma - \beta_0 \notin \widetilde{\Sigma}\}$. Let us show that Φ is a parabolic subset of Σ.
- We first show that Φ is a closed subset. Let $\gamma_1, \gamma_2 \in \Phi$ such that $\gamma_1 + \gamma_2 \in \Sigma$. Then $[\mathfrak{g}^{\gamma_1}, \mathfrak{g}^{\gamma_2}] \neq \{0\}$. Let $X_{\gamma_i} \in \mathfrak{g}^{\gamma_i} \setminus \{0\}$ $(i = 1, 2)$ such that $X_{\gamma_1+\gamma_2} = [X_{\gamma_1}, X_{\gamma_2}] \neq 0$. The Jacobi identity implies that $[X_{\gamma_1+\gamma_2}, X_{-\beta_0}] = 0$. Hence $\gamma_1 + \gamma_2 - \beta_0 \notin \Phi$. And hence Φ is closed.
- It remains to show that $\Phi \cup (-\Phi) = \Sigma$. If this is not the case, it exists $\gamma_0 \in \Sigma$ such that $\gamma_0 - \beta_0 \in \widetilde{\Sigma}$ and $\gamma_0 + \beta_0 \in \widetilde{\Sigma}$. Therefore the β_0-string of roots through γ_0 can be written:

$$\gamma_0 - \beta_0, \gamma_0, \gamma_0 + \beta_0$$

(remember that $V^{\pm}$ are commutative). From [3](chap. VI, §1, $n°3$, Corollaire de la proposition 9 p.149) one has

$$n(\gamma_0 - \beta_0, \beta_0) = -2 \qquad (*).$$

But from [3] (Chap. VI, p.148) this is only possible, as β_0 is long, if $\gamma_0 - \beta_0 = -\beta_0$, in other words if $\gamma_0 = 0$. Contradiction. Hence Φ is parabolic in Σ
Then ([3], Chap. VI, §1, $n°7$, Prop. 20, p.161) there exists a basis Π' of Σ such that $\Pi' \subset \Phi$. Hence it exists $w_0 \in W$ such that $w_0(\Pi') = \Pi$. Then $w_0(\beta_0)(H_0) = \beta_0(w_0(H_0)) = \beta_0(H_0) = 2$. One also has $\lambda - w_0(\beta_0) \notin \widetilde{\Sigma}$ for $\lambda \in \Sigma^+$. If one would have $\lambda - w_0(\beta_0) \in \widetilde{\Sigma}$, then as $\lambda = w_0\lambda'$ (with λ' positive for Π'), then $\omega(\lambda') - w_0(\beta_0) \in \widetilde{\Sigma}$, and hence $\lambda' - \beta_0 \in \widetilde{\Sigma}$, this is impossible from the definition of Φ. But we know from Corollary 1.2.2 that these properties characterize λ_0. Hence $w_0(\beta_0) = \lambda_0$.
Suppose first $k = 0$. In this case the set $\lambda_0 = w_0(\beta_0), w_0(\beta_1), \dots, w_0(\beta_m)$ is a set of strongly orthogonal roots. Hence $w_0(\beta_1), \dots, w_0(\beta_m) \in \mathcal{Z}_{V^+}(\widetilde{\mathfrak{l}}_0) = \{0\}$ This means that if $k = 0$ then $m = 0$ and the assertion (2) is proved in that case.
The general case goes by induction on k. Suppose that the result is true when the rank of the graded algebra is $< k$. In view of the above, there exists $w_0 \in W$ such that $w_0(\beta_0) = \lambda_0$. Then $w_0(\beta_1), \dots, w(\beta_m)$ is a maximal system of strongly orthogonal long roots in $\widetilde{\Sigma}_1^+ \setminus \Sigma_1^+$. As the graded algebra $\widetilde{\mathfrak{g}}_1$ is of rank $k-1$, we have $m = k$ by induction and there exists $w_1 \in W_1$ ($W_1 \subset W$ is the Weyl group of Σ_1) such that $w_1(w_0(\beta_i)) = \lambda_i$ for $i = 1, \dots, k$. The assertion (2) is then proved with $w = w_0 w_1$.

□

Corollary 1.7.8. (see [14] Lemme 2.1. p. 166, for the regular case defined below)

Let $\lambda_0, \lambda_1, \dots, \lambda_k$ be the maximal system of strongly orthogonal long roots obtained from the descent. Let $H_{\lambda_0}, H_{\lambda_1}, \dots, H_{\lambda_k}$ be the corresponding co-roots. For $i = 0, \dots, k$ we denote by $G_{H_{\lambda_i}}$ the stabilizer of H_{λ_i} in G. Then we have :

$$G = \mathrm{Aut}_e(\mathfrak{g}).(\bigcap_{i=0}^{k} G_{H_{\lambda_i}}).$$

Proof.
Let $g \in G$. The elements $g.H_{\lambda_i}$ belong to a maximal split torus $\mathfrak{a}'$. As the maximal split tori of $\mathfrak{g}$ are conjugated under $\mathrm{Aut}_e(\mathfrak{g}) \subset G$ ([28], Theorem 2, page 27, or [27], Theorem 3.1.16 p. 27) there exists $h \in \mathrm{Aut}_e(\mathfrak{g})$ such that $hg(\mathfrak{a}) = \mathfrak{a}$, hence $hg \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{a})$, the group of automorphisms of $\mathfrak{g}$ stabilizing $\mathfrak{a}$. But then hg acts on $\widetilde{\Sigma}$ and $hg(\lambda_0), \dots, hg(\lambda_k)$ is a sequence of strongly orthogonal long roots in $\widetilde{\Sigma}^+ \setminus \Sigma^+$. From the preceding proposition 1.7.7 , there exists $w \in W \subset \mathrm{Aut}_e(\mathfrak{g})$ such that $whg(\lambda_i) = \lambda_i$. Hence $whg \in \bigcap_{i=0}^{k} G_{H_{\lambda_i}}$. It follows that $g = h^{-1}w^{-1}whg \in \mathrm{Aut}_e(\mathfrak{g}).(\bigcap_{i=0}^{k} G_{H_{\lambda_i}})$.

□

Definition 1.7.9. *We set:*

$$\mathfrak{a}^0 = \oplus_{j=0}^{k} F H_{\lambda_j} \subset \mathfrak{a}$$

and

$$L = Z_G(\mathfrak{a}^0) = \bigcap_{i=0}^{k} G_{H_{\lambda_i}}$$

Hence, from the preceding Corollary, we have $G = \mathrm{Aut}_e(\mathfrak{g}).L$.

Remark 1.7.10. Let $j \in \{0, \dots, k\}$. Let us denote by G_j the analogue of the group G for the Lie algebra $\tilde{\mathfrak{g}}_j$ (hence $G_0 = G$). As $\overline{G_j} = \mathrm{Aut}_e(\tilde{\mathfrak{g}}_j \otimes \overline{F}) \subset \mathrm{Aut}_e(\tilde{\mathfrak{g}} \otimes \overline{F})$, any element g of G_j extends to an elementary automorphism of $\tilde{\mathfrak{g}} \otimes \overline{F}$, which we denote by $ext(g)$ and which acts trivially on $\oplus_{s=0}^{j-1} \mathfrak{l}_s$. Therefore $ext(g)$ centralizes H_0 and one has:

$$\mathrm{Aut}_e(\mathfrak{g}_j) \subset G_j \subset \overline{G_j} \subset \overline{G} = \mathcal{Z}_{\mathrm{Aut}_e(\tilde{\mathfrak{g}} \otimes \overline{F})}(H_0).$$

However, it may happen that, for $g \in G_j$, the automorphism $ext(g)$ does not stabilize $\tilde{\mathfrak{g}}$ and then $ext(g)$ does not define an automorphism of $\tilde{\mathfrak{g}}$. For example (see the proof of Theorem 3.8.9), when $\tilde{\mathfrak{g}}$ is the symplectic algebra $\mathfrak{s}p(2n, F)$, graded by $H_0 = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ where I_n is the identity matrix of size n, the group G is the group of elements $\mathrm{Ad}(g)$ for $g = \begin{pmatrix} \mathbf{g} & 0 \\ 0 & \mu\, {}^t\mathbf{g}^{-1} \end{pmatrix}$ where $\mathbf{g} \in GL(n, F)$ and $\mu \in F^*$ and where G_j, as a subgroup of $\overline{G}$, is the subgroups of elements of the form $\mathrm{Ad}(g_j)$ where

$$g_j = \left(\begin{array}{cc|cc} \mathbf{g}_j & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & I_j & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \mu\, {}^t\mathbf{g}_j^{-1} & 0 \\ & & 0 & I_j \end{array}\right)$$

with $\mathbf{g}_j \in GL(n-j, F)$ and $\mu \in F^*$.
This shows that G_j is not always included in G.

Definition 1.7.11. *A reductive graded Lie algebra $\widetilde{\mathfrak{g}}$ which verifies condition* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *is called* **regular** *if furthermore it satisfies:*

$(\mathbf{H_3})$ *There exist $I^+ \in V^+$ and $I^- \in V^-$ such that (I^-, H_0, I^+) is an $\mathfrak{sl}_2$-triple.*

Proposition 1.7.12.
In a regular graded Lie algebra $\widetilde{\mathfrak{g}}$, an element $X \in V^+$ is generic if and only if it exists $Y \in V^-$, such that (Y, H_0, X) is an $\mathfrak{sl}_2$-triple. Moreover, for a fixed generic element X , the element Y is unique.

Proof. If X can be put in such an $\mathfrak{sl}_2$-triple, then X is generic in V^+ from Lemma 1.7.2. Conversely, if X is generic in V^+, then $\mathrm{ad}(X) : \mathfrak{g} \longrightarrow V^+$ is surjective (cf. Definition 1.7.1), hence $\mathrm{ad}(X) : \overline{\mathfrak{g}} \longrightarrow \overline{V^+}$ is also surjective, therefore the $\overline{G}$-orbit of X in $\overline{V^+}$ is open and X is generic in $\overline{V^+}$. But there is only one open orbit in $\overline{V^+}$. Therefore X is in the $\overline{G}$-orbit of I^+ (Definition 1.7.11). Hence there exists $g \in \overline{G}$ such that $g.I^+ = X$. But then $(g.I^-, g.H_0 = H_0, g.I^+ = X)$ is an $\mathfrak{sl}_2$-triple. A standard tensor product argument (write $g.I^- = Y$ under the form $Y = \sum_{i=1}^n a_i Y_i$, with $a_1 = 1$, $a_2, \dots, a_n$ elements of $\overline{F}$ free over sur F, $Y_i \in V^-$), shows that $Y \in V^-$. Uniqueness is classical,([4], Chap VII, §11, $n°1$, Lemme 1). □

From now on we will always suppose that the graded Lie algebra $(\widetilde{\mathfrak{g}}, H_0)$ is regular.

1.8. Structure of the regular graded Lie algebra $(\widetilde{\mathfrak{g}}, H_0)$.

Let $\lambda_0, \lambda_1, \dots, \lambda_k$ be the sequence of strongly orthogonal roots defined in Theorem 1.6.1. Let $H_{\lambda_0}, H_{\lambda_1}, \dots, H_{\lambda_k}$ be the sequence of the corresponding co-roots.
Remember that we have defined:

$$\mathfrak{a}^0 = \oplus_{j=0}^k F H_{\lambda_j} \subset \mathfrak{a}$$

For $i, j \in \{0, 1, \dots, k\}$ and $p, q \in \mathbb{Z}$ we define the subspaces $E_{i,j}(p,q)$ of $\widetilde{\mathfrak{g}}$ by setting:

$$E_{i,j}(p,q) = \left\{ X \in \widetilde{\mathfrak{g}} \;\middle|\; [H_{\lambda_\ell}, X] = \begin{cases} pX & \text{if } \ell = i \;; \\ qX & \text{if } \ell = j \;; \\ 0 & \text{if } \ell \notin \{i,j\} \;. \end{cases} \right\}$$

Theorem 1.8.1.
If $(\widetilde{\mathfrak{g}}, H_0)$ is regular then

$$H_0 = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}.$$

Moreover one has the following decompositions:

$$\begin{aligned}
&(1) \qquad \mathfrak{g} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \left(\oplus_{i \neq j} E_{i,j}(1,-1)\right) \;; \\
&(2) \qquad V^+ = \left(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{\lambda_j}\right) \oplus \left(\oplus_{i<j} E_{i,j}(1,1)\right) \;; \\
&(3) \qquad V^- = \left(\oplus_{j=0}^k \widetilde{\mathfrak{g}}^{-\lambda_j}\right) \oplus \left(\oplus_{i<j} E_{i,j}(-1,-1)\right) \;.
\end{aligned}$$

Proof. For $j = \{0, 1, \dots, k\}$, we choose $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$. From Proposition 1.7.5 the element $X = X_0 + X_1 + \cdots + X_k$ is generic. Therefore X can be put in an $\mathfrak{sl}_2$-triple of the form (Y, H_0, X) with $Y \in V^-$ (Proposition 1.7.12). We choose also $X_{-j} \in \widetilde{\mathfrak{g}}^{-\lambda_j}$ such that $(X_{-j}, H_{\lambda_j}, X_j)$ is an $\mathfrak{sl}_2$-triple, and we set $Y' = X_{-0} + X_{-1} + \cdots + X_{-k}$.
Then $(Y', H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}, X)$ is again an $\mathfrak{sl}_2$-triple. On the other hand

$$\mathrm{ad}(X)^2 : V^- \longrightarrow V^+$$

is injective and as $\mathrm{ad}(X)^2 Y = 2X = \mathrm{ad}(X)^2 Y'$, we obtain that $Y = Y'$. But then $H_0 = \mathrm{ad}(X)Y = \mathrm{ad}(X)Y' = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}$. The first assertion is proved.

Let now X be an element of an eigenspace of $\mathrm{ad}(\mathfrak{a})$, i.e. either an element of a root space of $\widetilde{\mathfrak{g}}$ or an element of the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\widetilde{\mathfrak{g}}$ (cf. Remark 1.2.3). The representation theory of $\mathfrak{sl}_2$ implies the existence, for all $j = 0, 1, \dots, k$, of an integer $p_j \in \mathbb{Z}$ such that $[H_{\lambda_j}, X] = p_j X$. As $H_0 = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}$ one has

$$p_0 + p_1 + \cdots + p_k = \begin{cases} 2 & \text{if } X \in V^+ \ ; \\ 0 & \text{if } X \in \mathfrak{g} \ ; \\ -2 & \text{if } X \in V^- \ . \end{cases}$$

Define $w_j = e^{\mathrm{ad}\, X_j} e^{\mathrm{ad}\, X_{-j}} e^{\mathrm{ad}\, X_j}$. Hence w_j is the unique non trivial element of the Weyl group of the Lie algebra isomorphic to $\mathfrak{sl}_2$ generated by the triple $(X_{-j}, H_{\lambda_j}, X_j)$. As the λ_j are strongly orthogonal the elements w_j commute and

$$w_j.H_{\lambda_i} = \begin{cases} H_{\lambda_i} \text{ for } i \neq j \ ; \\ -H_{\lambda_j} \text{ for } i = j \ . \end{cases}$$

Let $J \subset \{p_0, p_1, \dots, p_k\}$ be the subset of i's such that $p_i < 0$. If we set $w = \prod_{i \in J} w_i$, one obtains that the sequence of eigenvalues of H_{λ_j} on wX is $|p_0|, |p_1|, \dots, |p_k|$. Hence $|p_0| + |p_1| + \cdots + |p_k| = 0$ or 2.
The case $|p_0| + |p_1| + \cdots + |p_k| = 0$ occurs if and only if $X \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0)$.
If $|p_0| + |p_1| + \cdots + |p_k| = 2$ then all p_j are zero, except for one which takes the value ± 2, or for two among them which take the value ± 1.
In the first case or if the two nonzero p_i's are equal then $X \in V^+$ or $X \in V^-$. Otherwise one p_i equals 1 and the other -1, and then $X \in \mathfrak{g}$.
To obtain the announced decompositions it remains to prove that

$$E_{i,j}(0,2) = \widetilde{\mathfrak{g}}^{\lambda_j} \text{ and } E_{i,j}(0,-2) = \widetilde{\mathfrak{g}}^{-\lambda_j} \text{ for } i \neq j \ .$$

A root space $\widetilde{\mathfrak{g}}^{\lambda}$occurs in $E_{i,j}(0,2)$ if $\lambda(H_j) = 2$ and $\lambda(H_\ell) = 0$ for $\ell \neq j$, and this means that $\lambda \perp \lambda_\ell$ if $\ell \neq j$. As $(\lambda + \lambda_\ell)(H_0) = 4$, $\lambda + \lambda_\ell$ is not a root. If $\lambda - \lambda_\ell$ is a root then $(\lambda, \lambda_\ell) \neq 0$ ([3] (Chap. VI, §1, $n°3$, Prop. 9)). Hence $\lambda \perp\!\!\!\perp \lambda_\ell$ for $\ell \neq j$ and $[\widetilde{\mathfrak{g}}^{\lambda}, \widetilde{\mathfrak{g}}^{\pm\lambda_\ell}] = 0$ for $\ell \neq j$. In particular $\widetilde{\mathfrak{g}}^{\lambda} \in \mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_{j-1})$, and hence $\lambda \in \widetilde{\Sigma}_j$. As $\lambda \geq 0$, one has $\lambda \in \widetilde{\Sigma}_j^+$. But λ_j is a simple root in $\widetilde{\Sigma}_j^+$, and therefore $\lambda - \lambda_j$ is not a root. If $\lambda \neq \lambda_j$, the equality $\lambda(H_{\lambda_j}) = 2$ would imply that $\lambda - \lambda_j$ is a root. Hence $\lambda = \lambda_j$ and $E_{i,j}(0,2) = \widetilde{\mathfrak{g}}^{\lambda_j}$. The same proof shows that $E_{i,j}(0,-2) = \widetilde{\mathfrak{g}}^{-\lambda_j}$.

□

Corollary 1.8.2.
Let $\lambda \in \widetilde{\Sigma}$. Then for $j = 0, 1, \ldots, k$, one has :

$$\lambda \perp \lambda_j \Longleftrightarrow \lambda \perp\!\!\!\perp \lambda_j$$

Proof.
Let $\lambda \perp \lambda_j$.
If $\lambda(H_0) = 2$, then $\lambda + \lambda_j$ is not a root, and if $\lambda - \lambda_j$ is a root one would have $(\lambda, \lambda_j) \neq 0$, see [3] (Chap. VI, §1, $n°3$, Prop. 9)). Therefore $\lambda \perp\!\!\!\perp \lambda_j$.
If $\lambda(H_0) = -2$, the same proof shows that $\lambda \perp\!\!\!\perp \lambda_j$.
If $\lambda(H_0) = 0$, then either $\mathfrak{g}^\lambda \subset \mathcal{Z}_\mathfrak{g}(\mathfrak{a}^0)$ or $\mathfrak{g}^\lambda \subset E_{r,s}(1,-1)$ for $r \neq j$, $s \neq j$.

a) If $\mathfrak{g}^\lambda \subset \mathcal{Z}_\mathfrak{g}(\mathfrak{a}^0)$, then $(\lambda + \lambda_j)(H_{\lambda_j}) = 2$ and if $\lambda + \lambda_j$ is a root the preceding Theorem 1.8.1 says that $\lambda + \lambda_j = \lambda_j$, this is not possible. Hence $\lambda + \lambda_j$ is not a root , and the same argument as before shows that $\lambda \perp\!\!\!\perp \lambda_0$.

b) If $\mathfrak{g}^\lambda \subset E_{r,s}(1,-1)$ and if $\lambda + \lambda_j$ is a root, then $(\lambda + \lambda_j)(H_{\lambda_j}) = 2$, $(\lambda + \lambda_j)(H_{\lambda_r}) = 1$, and $(\lambda + \lambda_j)(H_{\lambda_s}) = -1$, which is impossible. Again the same argument as before shows that $\lambda \perp\!\!\!\perp \lambda_0$. □

Remark 1.8.3. The decomposition of $\widetilde{\mathfrak{g}}$ using the subspaces $E_{i,j}(p,q)$ will be more useful that the root spaces decomposition. The bracket between two such spaces can be easily computed using the Jacobi identity. We will also show that this decomposition is also a "root space decomposition" with respect to another system of roots than $\widetilde{\Sigma}$ (see Remark 4.1.7 and Proposition 4.1.8).

The first part of Therorem 1.8.1 shows that the grading of $\widetilde{\mathfrak{g}}_j$ is defined by $H_{\lambda_j} + \cdots + H_{\lambda_k}$. By setting $I_j^+ = X_j + \cdots + X_k$, $I_j^- = X_{-j} + \cdots + X_{-k}$, where the elements $X_{\pm\ell} \in \widetilde{\mathfrak{g}}^{\pm\lambda_\ell}$ are chosen such that $(X_{-\ell}, H_{\lambda_\ell}, X_\ell)$ is an $\mathfrak{sl}_2$-triple, one obtains an $\mathfrak{sl}_2$-triple $(I_j^-, H_{\lambda_j} + \cdots + H_{\lambda_k}, I_j^+)$. Theorem 1.6.1 implies then the following decomposition of $\widetilde{\mathfrak{g}}_j$.

Corollary 1.8.4.
For $j = 0, \ldots, k$, the graded algebra $(\widetilde{\mathfrak{g}}_j, H_{\lambda_j} + \cdots + H_{\lambda_k})$ satisfies the hypothesis $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$. *One also has the following decompositions:*

$$\begin{aligned}
&(1) \qquad \mathfrak{g}_j = \Big(\mathfrak{z}_\mathfrak{g}(\mathfrak{a}^0) \cap \widetilde{\mathfrak{g}}_j\Big) \oplus \Big(\oplus_{r\neq s; j\leq r; j\leq s} E_{r,s}(1,-1)\Big) \ ;\\
&(2) \qquad V_j^+ = \Big(\oplus_{s=j}^k \widetilde{\mathfrak{g}}^{\lambda_s}\Big) \oplus \Big(\oplus_{j\leq r<s} E_{r,s}(1,1)\Big) \ ;\\
&(3) \qquad V_j^- = \Big(\oplus_{s=j}^k \widetilde{\mathfrak{g}}^{-\lambda_s}\Big) \oplus \Big(\oplus_{j\leq r<s} E_{r,s}(-1,-1)\Big) \ .
\end{aligned}$$

One can also extend the preceding decomposition to a subset A which is different from $\{j, j+1, \ldots, k\}$:

Corollary 1.8.5.

Let A be a non empty subset of $\{0,1,\ldots,k\}$. Define $H_A=\sum_{j\in A}H_{\lambda_j}$ and set:

$$\begin{aligned}\mathfrak{g}_A&=\{X\in\mathfrak{g}\mid[H_A,X]=0\}\ ;\\ V_A^+&=\{X\in V^+\mid[H_A,X]=2X\}\ ;\\ V_A^-&=\{X\in V^+\mid[H_A,X]=-2X\}\ ;\end{aligned}$$

Then the graded algebra $(\widetilde{\mathfrak{g}}_A=V_A^-\oplus\mathfrak{g}_A\oplus V_A^+,H_A)$ satisfies $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$.

Proof. Note that if A^c is the complementary set of A in $\{0,1,\ldots,k\}$ and if $H_{A^c}=\sum_{j\in A^c}H_{\lambda_j}$, then $\widetilde{\mathfrak{g}}_A=\mathcal{Z}_{\widetilde{\mathfrak{g}}}(H_{A^c})$. This implies that $\widetilde{\mathfrak{g}}_A$ is reductive ([4],chap.VII, §1, $n°5$, Prop.13). The hypothesis $(\mathbf{H_1})$ is then clearly verified. The regularity condition $(\mathbf{H_3})$ can be proved the same way as for $\widetilde{\mathfrak{g}}_j$.
It remains to show $(\mathbf{H_2})$, that is that the representation $(\mathfrak{g}_A,\overline{V_A^+})$ (or $(\overline{\mathfrak{g}_A},\overline{V_A^+})$) is irreducible. Let U be a subspace of $\overline{V_A^+}$ which is invariant by $\mathfrak{g}_A$. From Theorem 1.8.1 the algebra $\mathfrak{g}$ decomposes as follows:

$$\mathfrak{g}=\mathfrak{g}(-1)\oplus\mathfrak{g}_A\oplus\mathfrak{g}(1)\ ,$$

where $\mathfrak{g}(-1)=\oplus_{i\notin A,j\in A}E_{i,j}(1,-1)$ and $\mathfrak{g}(1)=\oplus_{i\in A,j\notin A}E_{i,j}(1,-1)$. Remark also that $\mathfrak{g}(-1)=\{X\in\mathfrak{g}\,,\ [H_A,X]=-X\}$ and $\mathfrak{g}(1)=\{X\in\mathfrak{g}\,,\ [H_A,X]=X\}$.

Note also that U is included in the eigenspace of $\mathrm{ad}(H_A)$ for the eigenvalue 2.

- $U_1=[\mathfrak{g}(-1),U]$ is then included in the eigenspace of $\mathrm{ad}(H_A)$ for the eigenvalue 1.
- $U_0=[\mathfrak{g}(-1),U_1]$ is then included in the eigenspace of $\mathrm{ad}(H_A)$ for the eigenvalue 0.
- One also has $[\mathfrak{g}(1),U]=\{0\}$.

We will now show that $U_0\oplus U_1\oplus U$ is $\mathfrak{g}$-invariant in $\overline{V^+}$.

a) One has : $[\mathfrak{g}_A,U]\subset U$, $[\mathfrak{g}(1),U]=\{0\}$, $[\mathfrak{g}(-1),U]=U_1$. Hence $[\mathfrak{g},U]\subset U_1\oplus U$.

One also shows easily that:

b) $[\mathfrak{g},U_1]\subset U_0\oplus U_1\oplus U$,

c) $[\mathfrak{g},U_0]\subset U_0\oplus U_1\oplus U$.

Therefore, using the hypothesis $(\mathbf{H_2})$ for $\widetilde{\mathfrak{g}}$, $U_0\oplus U_1\oplus U=\overline{V^+}$. But U is contained in the eigenspace of $\mathrm{ad}(H_A)$ for the eigenvalue 2, namely $\overline{V_A^+}$, and U_1 (resp. U_0) corresponds to the eigenvalue 1 (resp. 2). This implies $U=\overline{V_A^+}$.

□

Remark 1.8.6. Define $A_j=\{j,j+1,\ldots,k\}$. Then the reductive algebras $\widetilde{\mathfrak{g}}_j$ and $\widetilde{\mathfrak{g}}_{A_j}$ are graded by the same element $H_{\lambda_j}+\cdots+H_{\lambda_k}$. But these algebras are not equal. The obvious inclusion $\widetilde{\mathfrak{g}}_j\subset\widetilde{\mathfrak{g}}_{A_j}$ is strict as $H_{\lambda_0}\in\mathfrak{g}_{A_j}\setminus\mathfrak{g}_j$. More precisely one has

$$\begin{aligned}V_{A_j}^+&=\widetilde{\mathfrak{g}}_{A_j}\bigcap V^+=\left(\oplus_{s=j}^k\widetilde{\mathfrak{g}}^{\lambda_s}\right)\oplus\left(\oplus_{j\leq r<s}E_{r,s}(1,1)\right)=V_j^+\ ;\\ \mathfrak{g}_{A_j}&=\mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0)\oplus\left(\oplus_{r\neq s;j\leq r;j\leq s}E_{r,s}(1,-1)\right)\oplus\left(\oplus_{r\neq s;r<j;s<j}E_{r,s}(1,-1)\right)\\ &\supset_{\text{strict}}\mathfrak{g}_j=\mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0)\cap\mathfrak{g}_j\oplus\left(\oplus_{r\neq s;j\leq r;j\leq s}E_{r,s}(1,-1)\right)\ .\end{aligned}$$

Proposition 1.8.7.
The Lie algebra $\widetilde{\mathfrak{G}} = V^- \oplus [V^-, V^+] \oplus V^+$ generated by V^+ and V^- is a regular graded algebra which is an absolutely simple ideal of $\widetilde{\mathfrak{g}}$.

Proof. As $\widetilde{\mathfrak{g}}$ is regular, the element $H_0 \in [V^-, V^+]$, hence $(\mathbf{H_1})$ is satisfied. The hypothesis $(\mathbf{H_3})$ holds also clearly.
One easily verifies that $\widetilde{\mathfrak{G}}$ is an ideal of $\widetilde{\mathfrak{g}}$. Therefore $[V^-, V^+]$ is the only part of $\mathfrak{g}$ which acts effectively on V^+. Therefore the representation $([V^-, V^+], V^+)$ is absolutely simple. In other words $(\mathbf{H_2})$ is true.
The fact that $\widetilde{\mathfrak{G}}$ is simple is then a consequence of Lemma 1.5.1.

□

Remark 1.8.8. From the preceding Proposition 1.8.7, we obtain that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{G}} \oplus \mathfrak{G}'$ where the subalgebra $\tilde{\mathfrak{G}} = V^- \oplus [V^-, V^+] \oplus V^+$ is an abolutely simple graded Lie algebra and where $\mathfrak{G}'$ is the orthogonal of $\tilde{\mathfrak{G}}$ in $\tilde{\mathfrak{g}}$ with respect to the form $\tilde{B}$. Moreover, the subalgebra $\tilde{\mathfrak{G}}$ is an ideal of $\tilde{\mathfrak{g}}$ and hence $\tilde{\mathfrak{G}}'$ is an ideal of $\tilde{\mathfrak{g}}$ too. Therefore, if $X \in \tilde{\mathfrak{G}}'$ is nilpotent over an algebraic closure of F, then $e^{\operatorname{ad} X}$ acts trivially on $\tilde{\mathfrak{G}}$.
Hence, in order to classify the orbites of G in V^+, one can suppose that $\tilde{\mathfrak{g}}$ is simple.

1.9. **Properties of the spaces $E_{i,j}(p,q)$.**

Proposition 1.9.1.
Let $\lambda \in \Sigma$ such that $\mathfrak{g}^\lambda \subset E_{i,j}(1,-1)$. Then λ is positive if and only if $i > j$.

Proof. Let $\lambda \in \Sigma$ such that $\lambda(H_{\lambda_i}) = 1$ and $\lambda(H_{\lambda_j}) = -1$. From Theorem 1.8.1, $\lambda(H_{\lambda_s}) = 0$ if $s \neq i$ and $s \neq j$.
If $i > j$, then $\lambda \perp \lambda_s$ for $s < j$. But by Corollary 1.8.2, one has $\lambda \perp\!\!\!\perp \lambda_s$, for $s < j$. Hence $\lambda \in \widetilde{\Sigma}_j$. But as $(\lambda, \lambda_j) < 0$ $(\lambda(H_{\lambda_j}) = -1)$, $\lambda + \lambda_j$ is a root. We have also $\lambda \perp\!\!\!\perp \lambda_s$ for $s < j$ and $\lambda_j \perp\!\!\!\perp \lambda_s$ for $s < j$. Hence $\lambda + \lambda_j \in \widetilde{\Sigma}_j$. As $\lambda \in \Sigma_j$ and as $\lambda_j \in \widetilde{\Pi}_j \setminus \Pi_j$ (Notation 1.6.3),we obtain that $\lambda \in \Sigma_j^+$. But then $\lambda \in \Sigma^+$.
Conversely suppose that $\lambda \in \Sigma$ and $\mathfrak{g}^\lambda \subset E_{i,j}(1,-1)$ with $i < j$. Then $\mathfrak{g}^{-\lambda} \in E_{j,i}(1,-1)$, and from above $-\lambda \in \Sigma^+$, hence $\lambda \in \Sigma^-$. Hence if λ is positive and $\mathfrak{g}^\lambda \in E_{i,j}(1,-1)$, then $i > j$. □

We will denote by $\widetilde{W}$ and W the Weyl groups of $\widetilde{\Sigma}$ and Σ, respectively. W is the subgroup of $\widetilde{W}$ generated by the reflections with respect to the roots in Σ. In particular H_0 is fixed by each element of W.

Proposition 1.9.2.
Let s_0 be the unique element of W sending Σ^+ on Σ^-. Then

$$s_0.\lambda_j = \lambda_{k-j} \text{ for } j = 0, 1, \ldots, k\ .$$

Moreover the roots λ_i and λ_j are conjugated under W for all i and j in $\{0, 1, \ldots, k\}$.

Proof. Let us first prove that $s_0.\lambda_0 = \lambda_k$.

Recall (Corollary 1.2.2) that the root λ_0 is the unique root in $\widetilde{\Sigma}$ such that

$$(1) \qquad \begin{cases} \bullet & \lambda_0(H_0) = 2\ ; \\ \bullet & \lambda \in \Sigma^+ \Longrightarrow \lambda_0 - \lambda \notin \widetilde{\Sigma}\ . \end{cases}$$

Therefore the root $\mu = s_0.\lambda_0$ is characterized by the properties:

$$(2) \qquad \begin{cases} \bullet & \mu(H_0) = 2 \text{ (since } s_0.H_0 = 2); \\ \bullet & \lambda \in \Sigma^+ \Longrightarrow \mu + \lambda \notin \widetilde{\Sigma} \text{ (since } s_0.\Sigma^+ = \Sigma^-)\ . \end{cases}$$

From Proposition 1.4.1 (and its proof) we know that $s_0.\lambda_0$ is the root λ^0 which is the restriction to $\mathfrak{a}$ of the highest weight ω of $\mathfrak{g}$ on $\overline{V^+}$. From the proof of Proposition 1.5.3 we know also that ω is the highest weight of $\mathfrak{g}_1$ on $\overline{V_1^+}$, and by induction ω will be the highest weight of $\mathfrak{g}_k$ on $\overline{V_k^+}$. Hence $\omega_{|\mathfrak{a}}$ is a root of $\widetilde{\Sigma}$ which is strongly orthogonal to $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$. From Theorem 1.8.1, there is only one root having this property, namely λ_k. Hence $\lambda^0 = s_0.\lambda_0 = \lambda_k$.

We will now prove that $s_0.\lambda_1 = \lambda_{k-1}$. We first decide to take $s_0.\widetilde{\Sigma}^+$ as the set of positive roots in $\widetilde{\Sigma}$. The corresponding base will be $s_0.\widetilde{\Pi}$. We have $s_0.\Sigma^+ = \Sigma^- = -\Sigma^+$. As H_0 is fixed by W, the elements $\lambda \in s_0.\widetilde{\Pi}$ still verify $\lambda(H_0) = 0$ or 2 (condition (1) in Theorem 1.2.1). We apply now all we did before and we obtain by "descent" a sequence $\mu_0, \mu_1, \dots, \mu_k$ of strongly orthogonal roots. The root μ_0 is the unique root in $s_0.\widetilde{\Pi}$, such that $\mu_0(H_0) = 2$. Hence $\mu_0 = s_0.\lambda_0 = \lambda_k$.

Now we will prove that $s_0.\lambda_1 = \lambda_{k-1}$. The centralizer of $\widetilde{\mathfrak{l}}_k$ verifies again $(\mathbf{H_1})$ and $(\mathbf{H_2})$ and we will apply the preceding results to $\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_k)$. Corollary 1.2.2 applied to this graded algebra implies that the root μ_1 is characterized by

$$(3) \qquad \begin{cases} \bullet & \mu_1(H_0) = 2\ ; \\ \bullet & \mu_1 \perp\!\!\!\perp \lambda_k\ ; \\ \bullet & \lambda \in \Sigma^+ \text{ and } \lambda \perp\!\!\!\perp \lambda_k \Longrightarrow \mu_1 + \lambda \notin \widetilde{\Sigma}\ . \end{cases}$$

The same Corollary applied to the graded algebra $\widetilde{\mathfrak{g}}_1$ iimplies that the root λ_1 is characterized by

$$\begin{cases} \bullet & \lambda_1(H_0) = 2\ ; \\ \bullet & \lambda_1 \perp\!\!\!\perp \lambda_0\ ; \\ \bullet & \lambda \in \Sigma^+ \text{ and } \lambda \perp\!\!\!\perp \lambda_0 \Longrightarrow \lambda_1 - \lambda \notin \widetilde{\Sigma}\ . \end{cases}$$

As $s_0.\Sigma^+ = \Sigma^-$ and $s_0.\lambda_0 = \lambda_k$, we get $\mu_1 = s_0.\lambda_1$.

On the other hand the root λ_{k-1} appears in V^+ and is strongly orthogonal to λ_k. Let $\lambda \in \Sigma^+$ be a root strongly orthogonal to λ_k. If $\lambda(H_{\lambda_{k-1}}) = 0$, then λ is strongly orthogonal to λ_{k-1} from Corollary 1.8.2. Hence $\lambda + \lambda_{k-1}$ is not a root. If $\lambda(H_{\lambda_{k-1}}) \neq 0$, then by Proposition 1.9.1 there exists $j < k-1$ such that $\mathfrak{g}^\lambda \subset E_{k-1,j}(1,-1)$. As $(\lambda + \lambda_{k-1})(H_{\lambda_{k-1}}) = 3$, $\lambda + \lambda_{k-1}$ is not a root. This shows that λ_{k-1} verifies the properties (3). Hence $\lambda_{k-1} = \mu_1 = s_0.\lambda_1$.

The first assertion is then proved by induction on j.
For the second assertion one applies the preceding result to the graded algebras $\widetilde{\mathfrak{g}}_i$ and $\widetilde{\mathfrak{g}}_j$ where λ_i and λ_j play the role of λ_0. There exists an element $s_i \in W_i$ (W_i is the Weyl group of $(\widetilde{\mathfrak{g}}_i, \mathfrak{a}_i)$)

such that $s_i.\lambda_i = \lambda_k$ and an element $s_j \in W_j$ such that $s_j.\lambda_j = \lambda_k$. As W_i and W_j are subgroups of W, $s = s_j^{-1}s_i$ is an element of W which verifies $s.\lambda_i = \lambda_j$.

□

Proposition 1.9.3.
(1) *For $j = 0, \ldots, k$ the root spaces $\widetilde{\mathfrak{g}}^{\lambda_j}$ have the same dimension.*
(2) *More generally the Lie algebras $\widetilde{\mathfrak{l}}_i = \widetilde{\mathfrak{g}}^{-\lambda_i} \oplus [\widetilde{\mathfrak{g}}^{-\lambda_i}, \widetilde{\mathfrak{g}}^{\lambda_i}] \oplus \widetilde{\mathfrak{g}}^{\lambda_i}$ are two by two conjugated by G.*
(3) *For $i \neq j$, the spaces $E_{i,j}(1,1), E_{i,j}(-1,-1)$ and $E_{i,j}(1,-1)$ have the same dimension. This dimension is non zero and independant of the pair $\{i,j\} \in \{0, 1 \ldots, k\}^2$.*

Proof. (1) From Proposition 1.9.2 there exists an element $s \in W$ such that $s.\lambda_i = \lambda_j$. Let g be an element of G such that $g_{|\mathfrak{a}} = s$. It is the easy to see that $g.\widetilde{\mathfrak{g}}^{\lambda_i} = \widetilde{\mathfrak{g}}^{\lambda_j}$. Therefore the vector spaces $\widetilde{\mathfrak{g}}^{\lambda_i}$ and $\widetilde{\mathfrak{g}}^{\lambda_j}$ are isomorphic.
(2) Let $g \in G$ be the preceding element. Then one has also $g.\widetilde{\mathfrak{g}}^{-\lambda_i} = \widetilde{\mathfrak{g}}^{-\lambda_j}$. Hence $g.\widetilde{\mathfrak{l}}_i = \widetilde{\mathfrak{l}}_j$.
(3) Fix a pair (i,j) with $i \neq j$. We choose (X_i, Y_i) (resp. (X_j, Y_j)) in $\widetilde{\mathfrak{g}}^{-\lambda_i} \times \widetilde{\mathfrak{g}}^{\lambda_i}$ (resp. in $\widetilde{\mathfrak{g}}^{-\lambda_j} \times \widetilde{\mathfrak{g}}^{\lambda_j}$) such that$(Y_i, H_{\lambda_i}, X_i)$ (resp. $(Y_j, H_{\lambda_j}, X_j)$ is an $\mathfrak{sl}_2$-triple.
Then $\operatorname{ad} X_i : E_{i,j}(-1,-1) \longrightarrow E_{i,j}(1,-1)$ is an isomorphism whose inverse is -$\operatorname{ad} Y_i$ (if $u \in E_{i,j}(-1,-1)$ then $-\operatorname{ad} Y_i \operatorname{ad} X_i(u) = -[H_{\lambda_i}, u] + [X_i, [Y_i, u]] = u$).
Similarly $\operatorname{ad} X_j : E_{i,j}(1,-1) \longrightarrow E_{i,j}(1,1)$ is an isomorphism whose inverse is $-\operatorname{ad} Y_j$.
This implies that the spaces $E_{i,j}(\pm 1, \pm 1)$ are isomorphic when i and j are fixed.
In order to prove that the spaces $E_{i,j}(1,1)$ are isomorphic for distinct pairs (i,j) (with $i \neq j$), we will use the elements of the Weyl group which permute the λ_j.

We prove first that $E_{i,j}(1,1) \simeq E_{k,j}(1,1)$ for $j < i \leq k$ (recall that k is the final index in the descent). Proposition 1.9.2 applied to the graded algebra $\widetilde{\mathfrak{g}}_i$ implies the existence of $s_i \in W_i$ which permutes λ_i and λ_k. The group W_i is generated by the reflections defined by the roots λ strongly orthogonal to $\lambda_0, \lambda_1, \ldots, \lambda_{i-1}$. As $j < i$, λ_j is invariant under W_i. Hence

$$s_i : E_{i,j}(1,1) \longrightarrow E_{k,j}(1,1)$$

is an isomorphism. Indeed if $u \in E_{i,j}(1,1)$ then $[H_{\lambda_k}, s_i.u] = s_i.[s_i^{-1}.H_{\lambda_k}, u] = s_i.[H_{\lambda_i}, u] = s_i.u$ and $[H_{\lambda_j}, s_i.u] = s_i.[s_i^{-1}.H_{\lambda_j}, u] = s_i.[H_{\lambda_j}, u] = s_i.u$. Hence $s_i.E_{i,j}(1,1) \subset E_{k,j}(1,1)$ and the restriction of s_i^{-1} to $E_{k,j}(1,1)$ is the inverse.

Applying this to the triple $0 < k-j \leq k$ one obtains that

$$s_{k-j} : E_{k-j,0}(1,1) \longrightarrow E_{k,0}(1,1)$$

is an isomorphism.
One the other hand a similar proof (and Proposition 1.9.2) shows that

$$s_0 : E_{k,j}(1,1) \longrightarrow E_{k-j,0}(1,1)$$

is an isomorphism.
This will imply that all the spaces $E_{i,j}(1,1)$ are isomorphic. Indeed let us start from $E_{i,j}(1,1)$ and $E_{i',j'}(1,1)$ with $j < i$ and $j' < i'$.
From above we have the following isomorphisms:

$$E_{i,j}(1,1) \simeq E_{k,j}(1,1) \simeq E_{k-j,0}(1,1) \simeq E_{k,0}(1,1) \simeq E_{k-j',0}(1,1) \simeq E_{k,j'}(1,1) \simeq E_{i',j'}(1,1).$$

It remains to prove that these spaces are not reduced to $\{0\}$.
If they were trivial, the spaces $E_{i,j}(\pm 1, \pm 1)$ would all be trivial and one would have the following decompositions:

$$V^+ = \oplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\lambda_j} \quad \text{and} \quad \mathfrak{g} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0).$$

But then $\widetilde{\mathfrak{g}}^{\lambda_0}$ would be invariant under $\mathfrak{g}$. This is impossible by $(\mathbf{H_2})$.

□

Notation 1.9.4. In the rest of the paper we will use the following notations:

$$\begin{aligned}
&\ell = \dim \widetilde{\mathfrak{g}}^{\lambda_j} \text{ for } j = 0, \dots, k\ ;\\
&d = \dim E_{i,j}(\pm 1, \pm 1) \text{ for } i \neq j \in \{0, \dots, k\}\\
&e = \dim \widetilde{\mathfrak{g}}^{(\lambda_i + \lambda_j)/2} \text{ for } i \neq j \in \{0, \dots, k\} \ (e \text{ may be equal to } 0).
\end{aligned}$$

From Theorem 1.8.1 giving the decomposition of $\dim V^+$, we obtain the following relation between, k, d and ℓ.

Proposition 1.9.5.

$$\dim V^+ = (k+1)\left(\ell + \frac{kd}{2}\right) \ .$$

1.10. Normalization of the Killing form.

Let $\widetilde{B}$ be a non degenerate extension to $\widetilde{\mathfrak{g}}$ of the Killing form of $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$. As $\widetilde{\mathfrak{g}} = \mathcal{Z}(\widetilde{\mathfrak{g}}) \oplus [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ where $\mathcal{Z}(\widetilde{\mathfrak{g}})$ is the center of $\widetilde{\mathfrak{g}}$, we have

$$\widetilde{B}(z_1 + u, z_2 + u') = \kappa(z_1, z_2) + B(u, u') \quad \text{for } z_1, z_2 \in \mathcal{Z}(\widetilde{\mathfrak{g}}) \text{ and } u, u' \in [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}],$$

where B is the Killing form of $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ and κ a non degenerate form on $\mathcal{Z}(\widetilde{\mathfrak{g}})$. We fix once and for all such a form $\widetilde{B}$.

Definition 1.10.1. *For X and Y in $\widetilde{\mathfrak{g}}$, we define the normalized Killing form by setting :*

$$b(X, Y) = -\frac{k+1}{4 \dim V^+} \widetilde{B}(X, Y) \ .$$

A first consequence of this definition is the following Lemma.

Lemma 1.10.2.
For $j \in \{0, \dots, k\}$ one has $b(H_{\lambda_j}, H_{\lambda_j}) = -2$. Moreover if $(Y_j, H_{\lambda_j}, X_j)$ is an $\mathfrak{sl}_2$-triple such that $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$ and $Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j}$, then $b(X_j, Y_j) = 1$.

Proof. As the elements H_{λ_j} are conjugated (Proposition 1.9.2), as the roots λ_j are strongly orthogonal, and as $H_0 = H_{\lambda_0} + H_{\lambda_1} + \cdots + H_{\lambda_k}$ (Theorem 1.8.1) one has:

$$\widetilde{B}(H_{\lambda_j}, H_{\lambda_j}) = \frac{1}{k+1} \widetilde{B}(H_0, H_0) = \frac{1}{k+1} \mathrm{tr}_{\widetilde{\mathfrak{g}}} (\mathrm{ad}\, H_0)^2 = 8\, \frac{\dim V^+}{k+1} \, .$$

And then from the definition of b, we obtain $b(H_{\lambda_j}, H_{\lambda_j}) = -2$.

On the other hand
$$\widetilde{B}(Y_j,X_j)=\frac{1}{2}\widetilde{B}(Y_j,[H_{\lambda_j},X_j])=-\frac{1}{2}\widetilde{B}(H_{\lambda_j},H_{\lambda_j})=-4\,\frac{\dim V^+}{k+1}\,,$$
and hence $b(Y_j,X_j)=1$.

□

Let A be a subset of $\{0,1,\dots,k\}$. Consider the graded algebra $\widetilde{\mathfrak{g}}_A$ defined in Corollary 1.8.5 which is graded by $H_A=\sum_{j\in A}H_{\lambda_j}$. We denote by b_A a normalized nondegenerate bilinear form $\widetilde{\mathfrak{g}}_A$ (defined as b on $\widetilde{\mathfrak{g}}$).

Lemma 1.10.3. *Let $\widetilde{\mathfrak{G}}_A$ be the subalgebra of $\widetilde{\mathfrak{g}}_A$ generated by V_A^+ and V_A^-. If X and Y belong to $\widetilde{\mathfrak{g}}_A$, then:*
$$b_A(X,Y)=b(X,Y)\ .$$

Proof. We know from Proposition 1.8.7 that $\widetilde{\mathfrak{G}}_A$ is absolutely simple. Then the dimension of the space of invariant bilinear forms on $\widetilde{\mathfrak{G}}_A$ is equal to 1 ([5], Exercice 18 a) of §6). Hence the restrictions of b and b_A to $\widetilde{\mathfrak{G}}_A$ are proportional. For $j\in A$, $X_j\in\widetilde{\mathfrak{g}}^{\lambda_j}$ and $Y_j\in\widetilde{\mathfrak{g}}^{-\lambda_j}$ such that (Y_j,H_{λ_j},X_j) is an $\mathfrak{sl}_2$-triple, we have $b_A(X_j,Y_j)=b(X_j,Y_j)=1$ (Lemma 1.10.2).

□

1.11. The relative invariant Δ_0.

Recall (§1.7) that the group we are interested in and which will act on V^+ is
$$G=\mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0)=\{g\in\mathrm{Aut}_0(\widetilde{\mathfrak{g}})\,,\ g.H_0=H_0\}$$
Recall also that (G,V^+) is a prehomogeneous vector space. Let S be the complementary set of the union of the open orbits in V^+.
Recall also the definition of a relative invariant:

Definition 1.11.1. *A rational function R on V^+ is a relative invariant under G if there exists a rational character χ of G such that*
$$R(g.X)=\chi(g)R(X)\ \textit{for all}\ g\in G\ \textit{and all}\ X\in V^+\setminus S.$$

Remark 1.11.2. From the density of G in $\overline{G}=G(\overline{F})=\mathcal{Z}_{\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})}(H_0)=\mathcal{Z}_{\mathrm{Aut}_e(\overline{\widetilde{\mathfrak{g}}})}(H_0)$ (§1.7), and from the density of V^+ in $\overline{V^+}$, the natural extension of R from V^+ to $\overline{V^+}$, is a relative invariant of $(\overline{G},\overline{V^+})$

Lemma 1.11.3. *For $t\in F^*$, one has $t\mathrm{Id}_{V^+}\in G_{|_{V^+}}$. More precisely, we have $t\mathrm{Id}_{V^+}\in L_{|_{V^+}}$.*

Proof. Consider the subalgebra $\mathfrak{u}\simeq\mathfrak{sl}_2(F)$ generated by an $\mathfrak{sl}_2$-triple (Y,H_0,X) where X is generic in V^+, $Y\in V^-$. Consider also the group $U=\mathrm{Aut}_0(\mathfrak{u})$. Extending the adjoint representation from $\mathfrak{u}$ to $\overline{\widetilde{\mathfrak{g}}}$, one sees that the group U can be injected into the group $\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})$. On the other hand, one verifies easily that, for $t\in F^*$, the map $\Psi_t:\widetilde{\mathfrak{g}}\longrightarrow\widetilde{\mathfrak{g}}$, defined by $\Psi_t(x)=tx$, $\Psi_t(y)=t^{-1}y$, $\Psi_t(h)=h$, for $x\in V^+,y\in V^-,h\in\mathfrak{g}$ is an automorphism of $\widetilde{\mathfrak{g}}$, which stabilizes $\mathfrak{u}$ and fixes H_0. From [4] (Chap. VIII,§5, $n°3$, Corollaire 2 de la Proposition

5, p.110), one has $\mathrm{Aut}_0(\mathfrak{u}) = \mathrm{Aut}(\mathfrak{u})$. Hence there exist nilpotent elements $u_1, u_2, \dots, u_p \in \overline{\mathfrak{u}}$ such that $\Psi_{t_{|\mathfrak{u}}} = (e^{\mathrm{ad}_{\overline{\mathfrak{u}}} u_1} e^{\mathrm{ad}_{\overline{\mathfrak{u}}} u_2} \dots e^{\mathrm{ad}_{\overline{\mathfrak{u}}} u_p})_{|\mathfrak{u}}$. Note that these elements are also nilpotent in $\overline{\widetilde{\mathfrak{g}}}$. Hypothesis $(\mathbf{H_1})$ implies that the irreducible components of the $\overline{\mathfrak{u}}$-module $\overline{\widetilde{\mathfrak{g}}}$ are either isomorphic to the trivial module (the isotypic component being $\mathcal{Z}_{\overline{\mathfrak{g}}}(X)$), or isomorphic to the adjoint representation, of dimension 3. In this last case the space of highest weight vectors is $\overline{V^+}$. Let $X' \in V^+$, $X' \neq 0$. The $\overline{\mathfrak{u}}$-module $\overline{\mathfrak{u}'}$ generated by X' is therefore isomorphic to the $\overline{\mathfrak{u}}$-module $\overline{\mathfrak{u}}$. Hence there exists an isomorphism $\alpha : \overline{\mathfrak{u}} \longrightarrow \overline{\mathfrak{u}'}$ such that $\alpha(X) = X'$ and such that for all $u \in \overline{\mathfrak{u}}$, one has $\alpha \circ \mathrm{ad}\, u = \mathrm{ad}\, u \circ \alpha$. Therefore α intertwines also $e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p}$:

$$\alpha \circ \Psi_{t_{|\mathfrak{u}}} = e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p} \circ \alpha.$$

Explicitly:
$\alpha \circ \Psi_{t_{|\mathfrak{u}}}(X) = \alpha(tX) = t\alpha(X) = tX' = e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p} \circ \alpha(X)$
$= e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p}(X')$.
Hence for all $X' \in V^+$, we have shown that $e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p}(X') = tX'$.
As $e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p} \in \mathcal{Z}_{\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})}(H_0) = G$, we get $t\mathrm{Id}_{V^+} \in G_{|_{V^+}}$. The same proof shows that if $(Y_j, H_{\lambda_j}, X_j)$ is an $\mathfrak{sl}_2$-triple such that $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$ and $Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j}$, then

$$e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p}(X_{\lambda_j}) = tX_{\lambda_j} \text{ and } e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p}(Y_{\lambda_j}) = t^{-1}Y_{\lambda_j}.$$

And hence $e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_1} e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_2} \dots e^{\mathrm{ad}_{\overline{\widetilde{\mathfrak{g}}}} u_p}(H_{\lambda_j}) = H_{\lambda_j}$. This implies that $t\mathrm{Id}_{V^+} \in L_{|_{V^+}}$.

□

Another proof:

Proof. Let us give another proof of the preceding Lemma, more explicit, but based on the same idea which is to use $\mathfrak{sl}_2$.
Let again (Y, H_0, X) be an $\mathfrak{sl}_2$-triple with X generic in V^+, $Y \in V^-$. For $t \in F^*$ consider the automorphism $\theta(t) \in \mathrm{Aut}_e(\widetilde{\mathfrak{g}})$ defined by

$$\theta(t) = e^{t\,\mathrm{ad}_{\widetilde{\mathfrak{g}}} X} e^{t^{-1}\,\mathrm{ad}_{\widetilde{\mathfrak{g}}} Y} e^{t\,\mathrm{ad}_{\widetilde{\mathfrak{g}}} X}. \tag{$*$}$$

Set then

$$h(t) = \theta(t)\theta(-1). \tag{$**$}$$

Recall that V^+ (resp. V^-, resp. $\mathfrak{g}$) is the space of weight vectors of weight 2 (resp. -2, resp. 0) of $\widetilde{\mathfrak{g}}$ for the adjoint action of the algebra $\mathfrak{u} \simeq \mathfrak{sl}_2(F)$. From [4] (Chap. VIII, §1, $n°5$, Prop. 6, p.75), $h(t)_{|_{V^+}}$ is scalar multiplication by t^2, $h(t)_{|_{V^-}}$ is scalar multiplication by t^{-2}, and $h(t)_{|_{\mathfrak{g}}}$ is the identity. Over $\overline{F}$, we can consider $\sqrt{t} \in \overline{F}^*$. Then the automorphism $h(\sqrt{t})$ belongs to $\mathrm{Aut}_e(\overline{\widetilde{\mathfrak{g}}})$ and stabilizes $\widetilde{\mathfrak{g}}$ as $h(\sqrt{t})_{|_{V^+}} = t\mathrm{Id}_{V^+}$, $h(\sqrt{t})_{|_{V^-}} = t^{-1}\mathrm{Id}_{V^-}$, $h(\sqrt{t})_{|_{\mathfrak{g}}} = \mathrm{Id}_{\mathfrak{g}}$. The preceding relations $(*)$ and $(**)$, (for $\theta(\sqrt{t})$and $h(\sqrt{t})$), imply then that the automorphism $h(\sqrt{t})$ belongs to $\mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0) = G$. The same argument as in the first proof shows that $t\mathrm{Id}_{V^+} \in L_{|_{V^+}}$.

□

Theorem 1.11.4.
(1) *There exists on V^+ a unique (up to scalar multiplication) relative invariant polynomial Δ_0 which is absolutely irreducible (i.e. irreducible as a polynomial on $\overline{V^+}$).*

(2) *Any relative invariant on V^+ is (up to scalar multiplication) a power of Δ_0.*
(3) *An element $X \in V^+$ is generic if and only if $\Delta_0(X) \neq 0$.*

Proof. We begin by constructing a non trivial relative invariant P of (G, V^+). We choose a base of V^+ and a base of V^-, for example the dual base if we identify V^- and $(V^+)^*$ by using the form b. One can then define a determinant for all linear map from V^- into V^+. Consider the following linear map:

$$(\operatorname{ad} X)^2 : V^- \longrightarrow V^+.$$

Set, for all $X \in V^+$:

$$P(X) = \det(\operatorname{ad} X)^2.$$

We have, for $g \in G$:

$$\begin{aligned} P(g.X) &= \det(\operatorname{ad}(g.X))^2) = \det(g.(\operatorname{ad} X)^2 g^{-1} \\ &= \det_{V^+}(g) \det_{V^-}(g^{-1}) \det(\operatorname{ad} X)^2 = \det_{V^+}(g)^2 P(X) \end{aligned}$$

If X is generic, it can be put in an $\mathfrak{sl}_2$-triple (Y, H_0, X) $(Y \in V^-)$, and then $(\operatorname{ad} X)^2$ is an isomorphism between V^- and V^+. Hence $P \neq 0$. By Lemma 1.11.3, $t\mathrm{Id}_{V^+} \in G$. Therefore the character of P is non trivial, hence P is non constant.
By Remark 1.11.2, the natural extension of P to $\overline{V^+}$ is a relative invariant of $(\overline{G}, \overline{V^+})$. Then (by [26], Proposition 12, p. 64), there exists a non trivial relative invariant Δ_0 of $(\overline{G}, \overline{V^+})$ which is an irreducible polynomial and any other relative invariant is of the form $c.\Delta_0^m$ $(m \in \mathbb{Z})$.

We will need the following Lemma.

Lemma 1.11.5. *There exists $\alpha \in \overline{F}^*$ such that $\alpha\Delta_0$ takes values in F on V^+.*

Proof of the Lemma: Let $\mathcal{G} = \mathrm{Gal}(\overline{F}, F)$ be the Galois group of $\overline{F}$ over F. Let $\sigma \in \mathcal{G}$. Then σ acts on $\overline{G}$ and fixes each point of G. It acts also on $\overline{V^+}$ and fixes V^+. Then σ acts on Δ_0 by $\Delta_0^\sigma(x) = \sigma(\Delta_0(\sigma^{-1}x))$. The action on characters of $\overline{G}$ is similarly defined. The polynomial Δ_0^σ is still irreducible. Let now $\overline{G}$ act on Δ_0^σ.
Let χ_0 be the character of Δ_0 (it is a character of $\overline{G}$ for the moment). Let $x \in \overline{V^+}$ and $g \in \overline{G}$. One has:
$\Delta_0^\sigma(g.x) = \sigma(\Delta_0(\sigma^{-1}(g.x))) = \sigma(\Delta_0(\sigma^{-1}(g).\sigma^{-1}(x))) = \sigma(\chi_0(\sigma^{-1}(g))\sigma(\Delta_0(\sigma^{-1}(x)))$
$= \chi_0^\sigma(g)\Delta_0^\sigma(x)$. Hence Δ_0^σ is an irreducible relative invariant of $(\overline{G}, \overline{V^+})$, with character χ_0^σ. As this representation is irreducible ($(\mathbf{H_3})$), there exists $c_\sigma \in \overline{F}$ such that

$$\Delta_0^\sigma = c_\sigma \Delta_0.$$

Let x_0 be a generic element of V^+, this implies that $\Delta_0(x_0) \neq 0$. Define $\alpha = \frac{1}{\Delta_0(x_0)}$. Then

$$c_\sigma = \frac{\Delta_0^\sigma(x_0)}{\Delta_0(x_0)} = \frac{\sigma(\Delta_0(x_0))}{\Delta_0(x_0)} = \frac{\alpha}{\sigma(\alpha)}.$$

For $x \in V^+$ one has:

$$\sigma(\alpha)\Delta_0^\sigma(x) = \sigma(\alpha)c_\sigma\Delta_0(x) = \sigma(\alpha)\frac{\alpha}{\sigma(\alpha)}\Delta_0(x) = \alpha\Delta_0(x).$$

As $x \in V^+$, this can also be written:

$$\forall \sigma \in \mathcal{G},\ \sigma(\alpha\Delta_0(x)) = \alpha\Delta_0(x)$$

One knows that the fixed points of $\mathcal{G}$ in $\overline{F}$ are exactly the points in F. Hence for all $x \in V^+$, one has $\alpha\Delta_0(x) \in V^+$.
The Lemma is proved.
From now on we will denote by Δ_0 the modified relative invariant of Lemma 1.11.5 which takes it values in F on V^+.
Let P be a relative invariant of (G, V^+). Its extension $\overline{P}$ to $\overline{V^+}$ is a relative invariant of $(\overline{G}, \overline{V^+})$ (use the density of G in $\overline{G}$ and of V^+ in $\overline{V^+}$). Hence it exists $a \in \overline{F}$ and $m \in \mathbb{Z}$ such that $\overline{P} = a\Delta_0^m$. As $P(x) = a\Delta_0(x)^m$ for a generic point x in V^+ and as $\Delta_0(V^+) \subset F$, one get that $a \in F$.
Hence assertions 1) and 2) are proved.
One knows that $X \in V^+$ is generic if and only if there exists an $\mathfrak{sl}_2$-triple (Y, H_0, X) $(Y \in V^-)$ (Proposition 1.7.12). In that case $P(X) \neq 0$ where P is the relative invariant defined at the beginning of the proof. Hence $\Delta_0(X) \neq 0$. Conversely if $\Delta_0(X) \neq 0$ for $X \in V^+$, then $P(X) \neq 0$, and therefore $(\operatorname{ad} X)^2 : V^- \longrightarrow V^+$ is an isomorphism. Hence $V^+ = \operatorname{Im}(\operatorname{ad} X_{|\mathfrak{g}})$, and X is generic.
Assertion 3) is now proved.

□

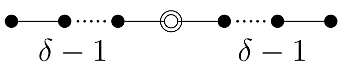
1.12. **The case $k = 0$.**

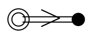
Recall that the case $k = 0$ corresponds to the graded algebra

$$\widetilde{\mathfrak{l}}_0 = \widetilde{\mathfrak{g}}^{-\lambda_0} \oplus [\widetilde{\mathfrak{g}}^{-\lambda_0}, \widetilde{\mathfrak{g}}^{\lambda_0}] \oplus \widetilde{\mathfrak{g}}^{\lambda_0}.$$

This algebra is an absolutely simple algebra of split rank 1 and it satisfies hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_2})$ (Proposition 1.5.2). As this algebra is graded by H_{λ_0} and as there exist $X \in \widetilde{\mathfrak{g}}^{\lambda_0}$ and $Y \in \widetilde{\mathfrak{g}}^{-\lambda_0}$ such that (Y, H_{λ_0}, X) is an $\mathfrak{sl}_2$-triple ([28], Corollaire du Lemme 6, p.6, or [27], Proposition 3.1.9 p.23), this algebra $\widetilde{\mathfrak{l}}_0$ satisfies also $(\mathbf{H_3})$.

Lemma 1.12.1.
The absolutely simple Lie algebras of split rank 1 graded by H_λ (λ being the unique restricted root) and which satisfy $(\mathbf{H_1}), (\mathbf{H_2})$ and $(\mathbf{H_3})$ (this last condition is automatically satisfied) have the following Satake-Tits diagrams ($d \in \mathbb{N}^*$):

$A_{2\delta-1}$ 

$B_2 = C_2$ 

Proof. The proof is a consequence of a careful reading of the tables of Tits ([29]). It can also be extracted from the recent work of T. Schoeneberg ([27]). For the convenience of the reader, we give some guidelines in connection with this last paper:
- The fact that there exist no diagram of type G_2, F_4, E_6 (inner forms), E_6 (outer forms), E_7, E_8, D_4 (with *trialitarian* action of the Galois group $\overline{F}/F$) is a consequence of, respectively:

Proposition 5.5.1 p.116, Proposition 5.5.3 p.118, Proposition 5.5.4 p.118, Proposition 5.5.13 p.134, Proposition 5.5.4 p.122, Proposition 5.5.7 p.126, and of Proposition 5.5.8 p.127.
- The case A_n is a consequence of Proposition 4.5.21 p. 93 (inner forms) and of pages 112-113 (outer forms).
- The case B_n is a consequence of Proposition 5.4.4 p. 113.
- The case C_n is a consequence of Proposition 5.4.5 p. 113.
- The case D_n (*non trialitarian*) is a consequence of Proposition 5.4.6 p. 114 and of the fact that the diagrams on p. 115-116 do not occur.
One can note that $(\mathbf{H_1})$ excludes the diagrams of split rank 1 where the the unique white root has a coefficient > 1 in the highest root of the underlying Dynkin diagram, and those which have two white roots connected by an arrow.

□

Corollary 1.12.2.
The graded algebras

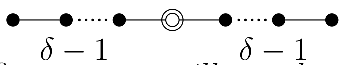
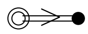
$$\widetilde{\mathfrak{l}}_0=\widetilde{\mathfrak{g}}^{-\lambda_0}\oplus[\widetilde{\mathfrak{g}}^{-\lambda_0},\widetilde{\mathfrak{g}}^{\lambda_0}]\oplus\widetilde{\mathfrak{g}}^{\lambda_0}$$

are either isomorphic to $\mathfrak{sl}_2(D)$ where D is a central division algebra over F, of degree δ, or isomorphic to $\mathfrak{o}(q,5)$ where q is a non degenerate quadratic form on F^5 which is the direct sum of an hyperbolic plane and an anisotropic form of dimension 3 *(in other word a form of index* 1*).*

Proof. These are the only Lie algebras over F whose Satake-Tits diagram is of the type given in the preceding Lemma ([29], [27]).

□

Definition 1.12.3. *By 1.9.3, all the algebras $\widetilde{\mathfrak{l}}_i$ are isomorphic either to*

$A_{2\delta-1}$  *or to* B_2 

In the first case we will say that $\widetilde{\mathfrak{g}}$ is of 1*-type A (or (A,δ) to be more precise), in the second case we wil say that $\widetilde{\mathfrak{g}}$ is of* 1*-type B.*

Theorem 1.12.4.
1) *If $\widetilde{\mathfrak{l}}_0=\mathfrak{sl}_2(D)$ where D is a central division algebra over F, of degree δ, the group G is the group of isomorphisms of $\mathfrak{sl}_2(D)$ of the form*

$$\mathfrak{sl}_2(D)\ni X=\begin{pmatrix}a & x\\ y & b\end{pmatrix}\longmapsto\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix}\begin{pmatrix}a & x\\ y & b\end{pmatrix}\begin{pmatrix}u^{-1} & 0\\ 0 & v^{-1}\end{pmatrix}$$

where $a,b,x,y\in D$, with $\mathrm{tr}_{D/F}(a+b)=0$ ($\mathrm{tr}_{D/F}$ is the reduced trace), and where $u,v\in D^*$. Therefore the action of G on $V^+\simeq D$ can be identified with the action of $D^*\times D^*$ on D given by

$$(u,v).x=uxv^{-1},$$

the group G being isomorphic to $(D^*\times D^*)/H$ where $H=\{(\lambda,\lambda),\ \lambda\in F^*\}$.
Hence there are two orbits: $\{0\}$ and D^*, and the fundamental relative invariant is the reduced norm $\nu_{D/F}$ of D over F. Its degree is δ.

2) *If* $\widetilde{\mathfrak{l}_0} = \mathfrak{o}(q, 5)$ *where* q *is a non degenerate quadratic form over* F^5 *which is the sum of an hyperbolic plane and an anisotropic form* Q *of dimension* 3*, then the group* G *can be identified with the group* $SO(Q) \times F^*$ *acting by the natural action on* F^3*. The fundamental relative invariant is then* Q*. There are four orbits, namely* $\{0\}$ *and three open orbits which are the sets* $\mathcal{O}_i = \{x \in F^3, Q(x) \in u_i\}$*,* $(i = 1, 2, 3)$*, where* u_i *runs over the three classes modulo* ${F^*}^2$ *distinct from* $-d(Q)$ *(*$d(Q)$ *being the discriminant of* Q*).*

Proof. 1) Let us make explicit the structure of $\mathfrak{sl}_2(D)$. For the material below, see [27], p. 93. It is well known that

$$\mathfrak{sl}_2(D) = \{\begin{pmatrix} a & x \\ y & b \end{pmatrix}, \text{ where } a, b, x, y \in D, \text{ with } \operatorname{tr}_{D/F}(a + b) = 0\}.$$

A maximal split abelian subalgebra is given by:

$$\mathfrak{a} = \{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, t \in F\}.$$

It is easy to see that

$$\mathfrak{m} = \mathcal{Z}_{\mathfrak{sl}_2(D)}(\mathfrak{a}) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in D, \operatorname{tr}_{D/F}(a + b) = 0\} = \mathfrak{a} \oplus \begin{pmatrix} [D, D] & 0 \\ 0 & [D, D] \end{pmatrix}$$

(recall that $[D, D] = \ker(\operatorname{tr}_{D/F})$, by [6], §17, $n°3$, Corollaire of proposition 5, p. A VIII.337). Therefore the anisotropic kernel of $\mathfrak{sl}_2(D)$ is given by

$$[\mathfrak{m}, \mathfrak{m}] = [\mathcal{Z}_{\mathfrak{sl}_2(D)}(\mathfrak{a}), \mathcal{Z}_{\mathfrak{sl}_2(D)}(\mathfrak{a})] = \begin{pmatrix} [D, D] & 0 \\ 0 & [D, D] \end{pmatrix} \simeq [D, D] \oplus [D, D].$$

Its Satake-Tits diagram is

$A_{d-1} \times A_{d-1}$ 

The grading of $\widetilde{\mathfrak{g}} = \mathfrak{sl}_2(D)$ is then defined by $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and this implies that $\widetilde{\mathfrak{g}} = \mathfrak{sl}_2(D) = V^- \oplus \mathfrak{g} \oplus V^+$ where

$$V^- = \{\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, y \in D\} \simeq D, \quad V^+ = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, x \in D\} \simeq D$$

and

$$\mathfrak{g} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in D, \operatorname{tr}_{D/F}(a + b) = 0\}.$$

We will now determine the group $G = \mathcal{Z}_{\operatorname{Aut}_0(\mathfrak{sl}_2(D))}(H_0) = \{g \in \operatorname{Aut}_0(\mathfrak{sl}_2(D)), g.H_0 = H_0\}$ and his action on $V^+ \simeq D$.

Let $g \in G$. As $g.H_0 = H_0$, one get $g.V^- \subset V^-$, $g.V^+ \subset V^+, g.\mathfrak{g} \subset \mathfrak{g}$. Let $\overline{g}$ be the natural extension of g as an automorphism of $\mathfrak{sl}_2(D) \otimes \overline{F} = \mathfrak{sl}_{2d}(\overline{F})$. From above, $\overline{g}$ stabilizes $\overline{V^-}, \overline{V^+}$

and $\overline{\mathfrak{g}}$. By [4] (Chap. VIII, §13, $n°1$, (VII), p.189), there exists $U \in GL(2d, \overline{F})$ such that $\overline{g}.x = UxU^{-1}, \forall x \in \mathfrak{sl}_{2d}(\overline{F})$. Let us write U in the form

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ \alpha, \beta, \gamma, \delta \in M_d(\overline{F}).$$

As $\overline{g}$ stabilizes $\overline{V^+}$, for all $x \in M_d(\overline{F})$, there exists $x' \in M_d(\overline{F})$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

It follows that $\gamma = 0$. Similarly the invariance of $\overline{V^-}$ implies that $\beta = 0$. Hence $U = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, with $\alpha, \delta \in GL_d(\overline{F})$. Let us now write down that the conjugation by U (that is, the action of $\overline{g}$) stabilizes $\mathfrak{sl}_2(D)$. For all $a, b \in D$, the element

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{pmatrix} = \begin{pmatrix} \alpha a \alpha^{-1} & 0 \\ 0 & \delta b \delta^{-1} \end{pmatrix}$$

belongs to $\mathfrak{sl}_2(D)$. Therefore the map $a \longmapsto \alpha a \alpha^{-1}$ (resp. $b \longmapsto \delta b \delta^{-1}$) will be an automorphism of the associative algebra D. By the Skolem-Noether Theorem ([1] (Théorème III-4 p.70), [19] (12.6 p.230) there exists u_0 (resp. v_0) in D^* such that $\alpha a \alpha^{-1} = u_0 a u_0^{-1}$ for all $a \in D$ (resp. $\delta b \delta^{-1} = v_0 b v_0^{-1}$ for all $b \in D$). Hence $u_0^{-1}\alpha$ (resp. $v_0^{-1}\delta$) is an element of $M_d(\overline{F}) = \overline{D}$ which commutes with any element of D, and hence it belongs to the center of $M_d(\overline{F}) = \overline{D}$. Therefore $u_0^{-1}\alpha = \lambda.1$ and $v_0^{-1}\delta = \mu.1$ ($\lambda, \mu \in \overline{F}$), i.e. $\alpha = \lambda u_0$ and $\delta = \mu v_0$. As $\overline{g}$ stabilizes V^+ and V^- and

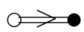
$$\begin{pmatrix} \lambda u_0 & 0 \\ 0 & \mu v_0 \end{pmatrix} \begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} \lambda^{-1} u_0^{-1} & 0 \\ 0 & \mu^{-1} v_0^{-1} \end{pmatrix} = \begin{pmatrix} u_0 a u_0^{-1} & \lambda \mu^{-1} u_0 x v_0^{-1} \\ \lambda^{-1} \mu v_0 y u_0^{-1} & v_0 b v_0^{-1} \end{pmatrix},$$

we deduce that $\lambda\mu^{-1} \in F^*$ and $\overline{g}$ is the conjugation by $V = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ with $u = \lambda\mu^{-1}u_0 \in D^*$ and $v = v_0 \in D^*$.

Therefore the assertions concerning the action and the orbits are clear. It is also clear that the conjugation by $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ induces the trivial automorphism if and only if $u = v = \lambda \in F^*$. This proves that $G = (D^* \times D^*)/H$.

As $\nu_{D/F}$ is a polynomial on D which takes values in F, it is also clear that $P(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \nu_{D/F}(x)$ is a relative invariant. As the reduced norm is an irreducible polynomial ([23]), this relative invariant is the fundamental one.

2) We will now give a realization [1] of the Lie algebra (there is only one up to isomorphism) whose Satake-Tits diagram is

B_2 

[1]We thank Marcus Slupinski for having indicated this realization to us.

For this, let D be a central division algebra of degree 2 over F. There is only one such algebra by [1] (Corollaire V-2, p. 130), and of course $D = (\frac{\pi,u}{F})$ is the unique quaternion division algebra over F ([13], Th. 2.2. p.152). Recall that π is a uniformizer of F and that u is a unit which is not a square. Let $a \longmapsto \overline{a}$ $(a \in D)$, be the usual conjugation in a quaternion algebra. The derived Lie algebra $[D,D]$ is then the space of pure quaternions, that is the set of $a \in D$ such that $\overline{a} = -a$.
Set

$$\widetilde{\mathfrak{g}} = \{X = \begin{pmatrix} a & b \\ c & -\overline{a} \end{pmatrix}, a \in D, b, c \in [D,D]\}.$$

It is easy to verify that $\widetilde{\mathfrak{g}}$ is a Lie algebra for the bracket $[X,X'] = XX' - X'X$, $X, X' \in \widetilde{\mathfrak{g}}$. Set also

$$\mathfrak{a} = \{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, t \in F\}.$$

It is clear that $\mathfrak{a}$ is a split torus in $\widetilde{\mathfrak{g}}$. If $\mathfrak{a}'$ is a split torus containing $\mathfrak{a}$, then $\mathfrak{a}'$ is contained in the eigenspace for the eigenvalue 0 of $\mathrm{ad}(\mathfrak{a})$, that is

$$\mathfrak{a}' \subset \{\begin{pmatrix} a & 0 \\ 0 & -\overline{a} \end{pmatrix}, a \in D\} \simeq D.$$

As $D = F.1 \oplus [D,D]$, and as $[D,D]$ is anisotropic (see for example [27] Corollaire 4.4.3. p.78), we obtain that $\mathfrak{a}' = \mathfrak{a}$, hence $\mathfrak{a}$ is a split maximal torus in $\widetilde{\mathfrak{g}}$. The Lie algebra $\widetilde{\mathfrak{g}}$ is graded by $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

$$\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+,$$

where

$$V^+ = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, b \in [D,D]\},$$

$$V^- = \{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, c \in [D,D]\},$$

$$\mathfrak{g} = \{\begin{pmatrix} a & 0 \\ 0 & -\overline{a} \end{pmatrix}, a \in D\} \simeq D.$$

Note that $[\mathfrak{g},\mathfrak{g}] \simeq [D,D]$ is anisotropic of dimension 3. Let us now show that $\widetilde{\mathfrak{g}}$ is simple. If I is an ideal of $\widetilde{\mathfrak{g}}$, as $[\mathfrak{a}, I] \subset I$, one has:

$$I = (V^- \cap I) \oplus (\mathfrak{g} \cap I) \oplus (V^+ \cap I).$$

If $(V^+ \cap I) \neq \{0\}$ and if $b \in (V^+ \cap I) \setminus \{0\}$ then the elements of the form

$$[\begin{pmatrix} a & 0 \\ 0 & -\overline{a} \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & ab + b\overline{a} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab - \overline{ab} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\mathrm{Im}(ab) \\ 0 & 0 \end{pmatrix}$$

run over V^+ if $a \in D$. An analogous statement is true if $(V^- \cap I) \neq \{0\}$. If $(\mathfrak{g} \cap I) \neq \{0\}$, then $(V^- \cap I) \neq \{0\}$ and $(V^+ \cap I) \neq \{0\}$. On the other hand, if $V^+ \subset I$ and $V^- \subset I$, then it is easy to see that $\mathfrak{g} \subset I$. Finally we have shown that $\widetilde{\mathfrak{g}}$ has no non trivial ideal. Hence $\widetilde{\mathfrak{g}}$ is a simple Lie algebra of dimension 10. Therefore $\overline{\widetilde{\mathfrak{g}}} = \widetilde{\mathfrak{g}} \otimes_F \overline{F}$ is a semi-simple Lie algebra of dimension 10 over $\overline{F}$. There is only one such algebra, it is the orthogonal algebra $\mathfrak{o}(5,\overline{F})$ whose Dynkin diagram is B_2. Therefore the algebra $\widetilde{\mathfrak{g}}$, with the grading described before, is indeed the algebra whose Satake-Tits diagram is

$$B_2 \qquad \circ\!\!\Rightarrow\!\!\bullet$$

(To avoid the preceding dimension argument, it is possible to compute explicitly $\overline{\widetilde{\mathfrak{g}}}$ by using the fact that $\overline{F}$ is a splitting field of D and show that

$$\overline{\widetilde{\mathfrak{g}}}=\{\begin{pmatrix} A & B\\ C & -\tau(A)\end{pmatrix}, A\in M_2(\overline{F}), B,C\in[M_2(\overline{F}),M_2(\overline{F})]=\mathfrak{sl}_2(\overline{F})\}$$

where the anti-involution τ of $M_2(\overline{F})$ is defined by $\tau(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix})=\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}$. This algebra is $\mathfrak{o}(5,\overline{F})$).

Remind that $[D,D]=\mathrm{Im}D=\{a\in D,\overline{a}=-a\}$. Set

$$W=\{\begin{pmatrix} h & \lambda\\ \mu & -h\end{pmatrix}, \text{ where } h\in \mathrm{Im}D=[D,D],\lambda,\mu\in F\}.$$

One sees easily that $[\widetilde{\mathfrak{g}},W]\subset W$, where [,] is the usual bracket of matrices. The corresponding representation is of dimension 5. An easy but a little tedious computation shows that the symmetric bilinear form

$$\Psi(\begin{pmatrix} h & \lambda\\ \mu & -h\end{pmatrix},\begin{pmatrix} h' & \lambda'\\ \mu' & -h'\end{pmatrix})=-\frac{1}{2}(hh'+h'h+\lambda'\mu+\lambda\mu')$$

is invariant under the action of $\widetilde{\mathfrak{g}}$. The form Ψ is the bilinear form associated to the quadratic form "determinant" $q(\begin{pmatrix} h & \lambda\\ \mu & -h\end{pmatrix})=-h^2-\lambda\mu$. This form is the direct sum of an hyperbolic plane and the anisotropic form $Q(h)=-h^2$ on $[D,D]$. The space F^3 in the statement of the Theorem is therefore $[D,D]$ and the algebra $\widetilde{\mathfrak{g}}$ is realized as the algebra $\mathfrak{o}(q,5)$ as in the statement.

We need also to consider $\overline{W}$. One has:

$$\overline{W}=\{\begin{pmatrix} U & x\mathrm{Id}_2\\ y\mathrm{Id}_2 & -U\end{pmatrix}, \text{ where } U\in[M_2(\overline{F}),M_2(\overline{F})]=\mathfrak{sl}_2(\overline{F}),x,y\in\overline{F}\}.$$

Let us denote by $\overline{q}$, $\overline{Q}$ and $\overline{\Psi}$ the lifts of q,Q and Ψ to $\overline{W}$, respectively. These are given by (remark that $UU'+U'U$ is a scalar matrix if U,U' are in $\mathfrak{s}l_2(\bar{F})$):

$$\overline{q}(\begin{pmatrix} U & x\mathrm{Id}_2\\ y\mathrm{Id}_2 & -U\end{pmatrix})=-U^2-xy$$

$$\overline{Q}(\begin{pmatrix} U & 0\\ 0 & -U\end{pmatrix})=-U^2$$

$$\overline{\Psi}(\begin{pmatrix} U & x\mathrm{Id}_2\\ y\mathrm{Id}_2 & -U\end{pmatrix},\begin{pmatrix} U' & x'\mathrm{Id}_2\\ y'\mathrm{Id}_2 & -U'\end{pmatrix})=-\frac{1}{2}(UU'+U'U+x'y+xy')$$

The Lie algebra $\overline{\widetilde{\mathfrak{g}}}$ acts on $\overline{W}$ by the adjoint action: $[\overline{\widetilde{\mathfrak{g}}},\overline{W}]\subset\overline{W}$. More explicitly a calculation shows that for $\begin{pmatrix}A & B\\ C & -\tau(A)\end{pmatrix}\in\overline{\widetilde{\mathfrak{g}}}$ and $\begin{pmatrix}U & x\mathrm{Id}_2\\ y\mathrm{Id}_2 & -U\end{pmatrix}\in\overline{W}$ one has

$$[\begin{pmatrix}A & B\\ C & -\tau(A)\end{pmatrix},\begin{pmatrix}U & x\mathrm{Id}_2\\ y\mathrm{Id}_2 & -U\end{pmatrix}]=\begin{pmatrix}[A,U]+yB-xC & -(UB+BU)+x(A+\tau(A))\\ CU+UC-y(A+\tau(A)) & [\tau(A),U]+xC-yB\end{pmatrix}$$

and this last matrix is effectively an element of $\overline{W}$.

This implies that the representation of $\overline{\widetilde{\mathfrak{g}}}$ in W is faithful, that the form $\overline{\Psi}$ is invariant under $\overline{\widetilde{\mathfrak{g}}}$ and therefore $\overline{\widetilde{\mathfrak{g}}}$ is realized as $\mathfrak{o}(\overline{W},\overline{q})$.
Let $\varphi\in\mathrm{Aut}(\overline{\widetilde{\mathfrak{g}}})$. As $\overline{\widetilde{\mathfrak{g}}}\simeq\mathfrak{o}(\overline{W},\overline{q})$ is a split simple Lie algebra, we know by [5], Chap. VIII, §13, $n°2$, (VII), p. 199, that

$$\mathrm{Aut}_0(\overline{\widetilde{\mathfrak{g}}})=\mathrm{Aut}_e(\overline{\widetilde{\mathfrak{g}}})=\mathrm{Aut}(\overline{\widetilde{\mathfrak{g}}})$$

and that there exists a unique $M\in SO(\overline{W},\overline{q})$ such that, for $X\in\overline{\widetilde{\mathfrak{g}}}$ one has

$$\varphi(X)=MXM^{-1},$$

the right hand side being a product of endomorphisms of $\overline{W}$. In the rest of the proof we will write $\varphi=M$, by abuse of notation, and we will consider that $\varphi\in SO(\overline{W},\overline{q})$.
Under the action of H_0 the space $\overline{W}$ decomposes into three eigenspaces corresponding to the eigenvalues $-2,0,2$:

$$\overline{W}_{-2}=\{\begin{pmatrix}0 & 0\\ y\mathrm{Id}_2 & 0\end{pmatrix},y\in\overline{F}\},\ \overline{W}_0=\{\begin{pmatrix}U & 0\\ 0 & -U\end{pmatrix},U\in\mathfrak{sl}_2(\overline{F})\},\ \overline{W}_2=\{\begin{pmatrix}0 & x\mathrm{Id}_2\\ 0 & 0\end{pmatrix},x\in\overline{F}\}.$$

We will need the following Lemma.

Lemma 1.12.5.
Let $\varphi\in\mathrm{Aut}(\overline{\widetilde{\mathfrak{g}}})=SO(\overline{W},\overline{q})$.

a) φ *fixes* $H_0\iff\begin{cases}\varphi \textit{ stabilizes } \overline{W}_{-2},\overline{W}_0,\overline{W}_2;\\ \varphi_{|\overline{W}_0}\in SO(\overline{W}_0,\overline{Q});\\ \textit{there exists } \alpha_\varphi\in\overline{F}^* \textit{ such that } \varphi_{|\overline{W}_{-2}}=\alpha_\varphi^{-1}\mathrm{Id}_{|\overline{W}_{-2}} \textit{ and } \varphi_{|\overline{W}_2}=\alpha_\varphi\mathrm{Id}_{|\overline{W}_2}.\end{cases}$

b) *Let* $g\in\mathrm{Aut}(\widetilde{\mathfrak{g}})$ *and let* $\overline{g}$ *be his natural extension to an element of* $\mathrm{Aut}(\overline{\widetilde{\mathfrak{g}}})$. *Then one has:*

$$g\in G=\mathcal{Z}_{Aut_0(\widetilde{\mathfrak{g}})}(H_0)\iff\begin{cases}\overline{g} \textit{ stabilizes } W_{-2},W_0,W_2;\\ \overline{g}_{|W_0}\in SO(W_0,Q);\\ \alpha_{\overline{g}}\in F^*.\end{cases}$$

Proof of the Lemma:
a) Suppose that φ commutes with H_0, then $H_0=\varphi H_0\varphi^{-1}$ (products of endomorphisms of $\overline{W}$). $(i=-2,0,2)$. Let $T\in\overline{W}_i$ $(i=-2,0,2)$. Then $H_0.\varphi(T)=\varphi H_0\varphi^{-1}\varphi(T)=\varphi H_0(T)=\varphi(iT)=i\varphi(T)$. Hence φ stabilizes the spaces $\overline{W}_i$.
As $\overline{W}_{-2}$ and $\overline{W}_2$ are 1-dimensional , there exist $\alpha_\varphi,\beta_\varphi\in\overline{F}^*$ such that

$$\varphi(\begin{pmatrix} 0 & 0 \\ y\mathrm{Id}_2 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ \beta_\varphi y\mathrm{Id}_2 & 0 \end{pmatrix},\ \varphi(\begin{pmatrix} 0 & x\mathrm{Id}_2 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & \alpha_\varphi x\mathrm{Id}_2 \\ 0 & 0 \end{pmatrix},\ x, y \in \overline{F}.$$

But as $\varphi \in SO(\overline{W}, \overline{q})$, one has

$$\overline{q}(\varphi(\begin{pmatrix} 0 & x\mathrm{Id}_2 \\ y\mathrm{Id}_2 & 0 \end{pmatrix})) = \overline{q}(\begin{pmatrix} 0 & \alpha_\varphi x\mathrm{Id}_2 \\ \beta_\varphi y\mathrm{Id}_2 & 0 \end{pmatrix}) = -\alpha_\varphi\beta_\varphi xy = \overline{q}(\begin{pmatrix} 0 & x\mathrm{Id}_2 \\ y\mathrm{Id}_2 & 0 \end{pmatrix}) = -xy.$$

Hence $\beta_\varphi = \alpha_\varphi^{-1}$. As $\overline{q}_{|_{\overline{W}_0}} = \overline{Q}$ one get that $\varphi_{|_{\overline{W}_0}} \in SO(\overline{W}_0, \overline{Q})$.
Conversely suppose that φ satisfies the conditions on the right hand side of a). Let us look how $\varphi H_0\varphi^{-1}$ acts on $\overline{W}$:

$$\begin{pmatrix} U & x\mathrm{Id}_2 \\ y\mathrm{Id}_2 & -U \end{pmatrix} \xmapsto{\varphi^{-1}} \begin{pmatrix} \varphi^{-1}(U) & \alpha_\varphi^{-1}x\mathrm{Id}_2 \\ \alpha_\varphi y\mathrm{Id}_2 & -\varphi^{-1}(U) \end{pmatrix} \xmapsto{H_0} \begin{pmatrix} 0 & 2\alpha_\varphi^{-1}x\mathrm{Id}_2 \\ -2\alpha_\varphi y\mathrm{Id}_2 & 0 \end{pmatrix} \xmapsto{\varphi} \begin{pmatrix} 0 & 2x\mathrm{Id}_2 \\ -2y\mathrm{Id}_2 & 0 \end{pmatrix}.$$

But one has:

$$\begin{pmatrix} U & x\mathrm{Id}_2 \\ y\mathrm{Id}_2 & -U \end{pmatrix} \xmapsto{H_0} \begin{pmatrix} 0 & 2x\mathrm{Id}_2 \\ -2y\mathrm{Id}_2 & 0 \end{pmatrix}.$$

Therefore $\varphi H_0\varphi^{-1} = H_0$, and this proves a).
b) is a consequence of a).

End of the proof of Lemma 1.12.5.

If $g \in G$, let α_g be the element $\alpha_{\overline{g}} \in F^*$ obtained in the preceding Lemma.

Lemma 1.12.6.
The action of $g \in G$ on $V^+ = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\ b \in [D, D]\}$ is as follows

$$g.\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_g g_{|_{[D,D]}}(b) \\ 0 & 0 \end{pmatrix}.$$

Proof of the Lemma:
In order to compute $g.\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = g\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}g^{-1}$, we will compute its action on the element $\begin{pmatrix} h & \lambda \\ \mu & -h \end{pmatrix} \in W$, using the preceding Lemma 1.12.5. One has:

$$\begin{pmatrix} h & \lambda \\ \mu & -h \end{pmatrix} \xmapsto{g^{-1}} \begin{pmatrix} g^{-1}(h) & \alpha_g^{-1}\lambda \\ \alpha_g\mu & -g^{-1}(h) \end{pmatrix} \xmapsto{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} \alpha_g\mu b & -bg^{-1}(h) - g^{-1}(h)b \\ 0 & -\alpha_g\mu b \end{pmatrix}$$

$$\xmapsto{g} \begin{pmatrix} \alpha_g\mu g(b) & -\alpha(bg^{-1}(h) + g^{-1}(h)b) \\ 0 & -\alpha_g\mu g(b) \end{pmatrix} (*).$$

Let us show now that the action of the element $\begin{pmatrix} 0 & \alpha_g g_{|_{[D,D]}}(b) \\ 0 & 0 \end{pmatrix}$ is the same. One has:

$$[\begin{pmatrix} 0 & \alpha_g g_{|_{[D,D]}}(b) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} h & \lambda \\ \mu & -h \end{pmatrix}] = \begin{pmatrix} \alpha_g \mu g(b) & -\alpha(g(b)h+hg(b)) \\ 0 & -\alpha_g \mu g(b) \end{pmatrix} (**)$$

As the bilinear form associated to Q is $B(b,h) = bh+hb$ ($b,h \in [D,D]$) and as $g_{|W_0} \in SO(W_0,Q)$ we obtain $bg^{-1}(h)+g^{-1}(h)b = g(b)h+hg(b)$, hence (*)=(**).

End of the proof of Lemma 1.12.6.

From Lemma 1.12.6 we know that the map

$$\begin{array}{ccc} G & \longrightarrow & SO(W_0,Q) \times F^* \\ g & \longmapsto & (g_{|W_0}, \alpha_g) \end{array}$$

is a bijection. We have therefore proved the first part of the statement 2) Theorem 1.12.4. From [13] (Théorème 2.2. 1) p. 152), there are four classes modulo F^{*2} in F^*, which are the classes of $1, u, \pi, u\pi$ (u is a non square unit and π is a uniformizer). And from [13] (Corollary 2.5 3), p.153-154) the anisotropic form Q represents all (non zero) classes except the class of $-d(Q)$. If for $b_1, b_2 \in V^+ \simeq [D,D]$, the elements $Q(b_1)$ and $Q(b_2)$ are in the same class then there exists $t \in F^*$ such that $Q(b_1) = t^2 Q(b_2) = Q(tb_2)$. Witt's Theorem implies then that there exists $g \in O(Q)$ such that $b_1 = tg(b_2)$. As the dimension is 3, $\det(-\mathrm{Id}_{V^+}) = -1$ and one can suppose that $g \in SO(Q)$. The Theorem is proved.

□

Remark 1.12.7. As the algebras $\widetilde{\mathfrak{l}}_0$ and $\widetilde{\mathfrak{l}}_j$ are isomorphic (Proposition 1.9.3), Corollary 1.12.2 and Theorem 1.12.4 will also be true for $\widetilde{\mathfrak{l}}_j$.

Lemma 1.12.8.

Let us fix an element $X^1 = X_1 + X_2 + \cdots + X_k$ $\quad (X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \backslash \{0\})$. Then for $X \in \widetilde{\mathfrak{g}}^{\lambda_0}$ one has

$$V^+ = [\mathfrak{g}, X^1 + X] + \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

The codimension of the G-orbit of $X + X^1$ in V^+ is given by

$$\mathrm{codim}[\mathfrak{g}, X + X^1] = \begin{cases} \ell \text{ if } X = 0 \ ; \\ 0 \text{ if } X \neq 0 \ . \end{cases}$$

Proof.
If $X \neq 0$, then by Proposition 1.7.5, $X + X^1$ is generic in V^+. Hence $V^+ = [\mathfrak{g}, X + X^1]$ and $\mathrm{codim}[\mathfrak{g}, X + X^1] = 0$.
If $X = 0$, we know by Corollary 1.8.4 that

$$V^+ = V_1^+ \oplus \left(\oplus_{j=1}^k E_{0,j}(1,1)\right) \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

X^1 is generic in V_1^+ by Proposition 1.7.5 applied to $\widetilde{\mathfrak{g}}_1$. Therefore:

$$V_1^+ = [\mathfrak{g}_1, X^1] \subset [\mathfrak{g}, X^1]. \qquad (*)$$

Let $Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j}$ be such that $\{Y_j, H_{\lambda_j}, X_j\}$ is an $\mathfrak{sl}_2$-triple. Let $A \in E_{0,j}(1,1)$, then $B = [Y_j, A]$ is an element of $E_{0,j}(1,-1) \subset \mathfrak{g}$ and one has

$$[B, X^1] = [B, X_1 + X_2 + \cdots + X_k] = [B, X_j] = [[Y_j, A], X_j] = [[Y_j, X_j], A] = [H_j, A] = A \ .$$

Hence $\oplus_{j=1}^{k} E_{0,j}(1,1) \subset [\mathfrak{g}, X^1]$ and from decomposition $(*)$ above we get

$$V^+ = [\mathfrak{g}, X^1] + \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

To obtain the result on the codimension, it is enough to prove that the preceding sum is direct. The relations

$$[\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0), X^1] \subset \oplus_{j=1}^{k} \widetilde{\mathfrak{g}}^{\lambda_j} \subset V_1^+ \ ,$$

$$[E_{i,j}(1,-1), X^1] \subset \begin{cases} \{0\} \text{ if } j=0, \\ E_{i,j}(1,1) \text{ if } j \neq 0 \ , \end{cases}$$

imply that

$$[\mathfrak{g}, X^1] \subset V_1^+ \oplus \left(\oplus_{j=1}^{k} E_{0,j}(1,1)\right) \ .$$

Hence

$$[\mathfrak{g}, X^1] = V_1^+ \oplus \left(\oplus_{j=1}^{k} E_{0,j}(1,1)\right) \ ,$$

and finally

$$V^+ = [\mathfrak{g}, X^1] \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \ .$$

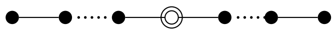
$\square$

1.13. **Properties of Δ_0.**

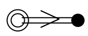
Let δ_i be the fundamental relative invariant of the prehomogeous vector space $\widetilde{\mathfrak{l}}_i$ $(0 \leq i \leq k)$. By Theorem 1.11.4, δ_i is absolutely irreducible. As all the algebras $\widetilde{\mathfrak{l}}_i$ are isomorphic (Remark 1.12.7), the δ_i's have all the same degree .

Notation 1.13.1. Let us denote by κ the common degree of the polynomial δ_i. By Theorem 1.12.4 there exists two types of graded algebras of rank 1. One has

$\kappa = \begin{cases} \delta \text{ in case (1) corresponding to the diagram} \\ 2 \text{ in case (2) corresponding to the diagram} \end{cases}$



$\delta - 1$ $\delta - 1$ $(\delta \in \mathbb{N}^*)$



Theorem 1.13.2.
(1) *Δ_0 is a homogeneous polynomial of degree $\kappa(k+1)$.*
(2) *For $j = 0, \dots, k$, let X_j be an element of $\widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and let $x_j \in F$. Then*

$$\Delta_0\Big(\sum_{j=0}^{k} x_j X_j\Big) = \prod_{j=0}^{k} x_j^{\kappa} \,.\Delta_0\Big(\sum_{j=0}^{k} X_j\Big) \ .$$

Proof.
(1) Consider the prehomogeneous vector space $\overline{\widetilde{\mathfrak{l}}_0} = \widetilde{\mathfrak{l}}_0 \otimes_F \overline{F} = \overline{\widetilde{\mathfrak{g}}^{-\lambda_0}} \oplus [\overline{\widetilde{\mathfrak{g}}^{-\lambda_0}}, \overline{\widetilde{\mathfrak{g}}^{\lambda_0}}] \oplus \overline{\widetilde{\mathfrak{g}}^{\lambda_0}}$. By [18] (Proposition 2.16) the degree of δ_0 $(= \kappa)$ is equal to the number of strongly orthogonal roots (over $\overline{F}$), $\beta_{0,1}, \dots, \beta_{0,\kappa}$, appearing in the descent applied to the graded algebra $\overline{\widetilde{\mathfrak{l}}_0}$ (see [18] for details). More generally let us denote by $\beta_{i,1}, \dots, \beta_{i,\kappa}$ the strongly orthogonal roots appearing in the descent applied to the graded algebra $\overline{\widetilde{\mathfrak{l}}_i}$. But then the set of $\kappa(k+1)$ roots

$\beta_{0,1},\ldots,\beta_{0,\kappa},\ldots,\beta_{k,1},\ldots,\beta_{k,\kappa}$ is a maximal set of strongly orthogonal roots in $\overline{V^+}$ (if not it would exist a root λ_{k+1} over F which is strongly orthogonal to $\lambda_0,\ldots,\lambda_k$). Then, again by Proposition 2.16 of [18], one obtain that the degree of Δ_0 is $\kappa(k+1)$.

(2) For $i=0,\ldots,k$, let us fix elements $X_i\in\widetilde{\mathfrak{g}}^{\lambda_i}\setminus\{0\}$. Consider the polynomial map on $\widetilde{\mathfrak{g}}^{\lambda_i}$ given by

$$\mu_i : X\mapsto \Delta_0(X+X^i) \text{ where } X^i=X_0+X_2+\cdots+X_{i-1}+X_{i+1}+\cdots+X_k\ .$$

Let L_i be the the group similar to G for the graded algebra $\widetilde{\mathfrak{l}}_i$, that is $L_i=\mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{l}}_i)}(H_{\lambda_i})$. By Corollary 1.8.4 one gets that $\widetilde{\mathfrak{l}}_k=\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0\oplus\cdots\oplus\widetilde{\mathfrak{l}}_{k-1})$. As all the algebras $\widetilde{\mathfrak{l}}_j$ are conjugated (Proposition 1.9.2) one has also $\widetilde{\mathfrak{l}}_i=\mathcal{Z}_{\widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{l}}_0\oplus\cdots\oplus\widetilde{\mathfrak{l}}_{i-1}\oplus\widetilde{\mathfrak{l}}_{i+1}\oplus\cdots\oplus\widetilde{\mathfrak{l}}_k)$. Therefore the elements of L_i, which are products of exponentials (of adjoints) of nilpotent elements of $\widetilde{\mathfrak{l}}_i\otimes\overline{F}$, fix all the element X^i.
Since $L_i\subset\overline{G}=G(\overline{F})$ and since Δ_0 is, by construction, the restriction to V^+ of a relative invariant polynomial of $(\overline{G},\overline{V^+})$, we deduce that for $g\in L_i$ and $X\in\widetilde{\mathfrak{g}}^{\lambda_i}$ one has :

$$\mu_i(g.X)=\Delta_0(g.X+X^i)=\Delta_0(g.(X+X^i))=\chi_0(g)\Delta_0(X+X^i)=\chi_0(g)\mu_i(X).$$

Hence μ_i is a relative invariant for the prehomogeneous space $(L_i,\widetilde{\mathfrak{g}}^{\lambda_i})$, with character ${\chi_0}_{|_{L_i}}$. This invariant is non zero, as $\mu_i(X_i)=\Delta_0(X_0+\cdots+X_k)\neq 0$. For $t\in F$, let $g_0(t)$ be an element of L_0 such that $g_0(t)_{|_{\widetilde{\mathfrak{g}}^{\lambda_0}}}=t\mathrm{Id}_{\widetilde{\mathfrak{g}}^{\lambda_0}}$ (Lemme 1.11.3). If $g\in G$ is such that $g.\widetilde{\mathfrak{l}}_0=\widetilde{\mathfrak{l}}_i$, then the element $g_i(t)=gg_0(t)g^{-1}\in L_i$ satisfies $g_i(t)_{|_{\widetilde{\mathfrak{g}}^{\lambda_i}}}=t\mathrm{Id}_{\widetilde{\mathfrak{g}}^{\lambda_i}}$ and $\chi_0(g_i(t))=\chi_0(g_0(t))$. Therefore the polynomials μ_i have the same homogeneous degree, say p. Then
$\Delta_0(g_0(t)\ldots g_k(t).(X_0+\cdots+X_k))=\Delta_0(t(X_0+\cdots+X_k))=t^{\kappa(k+1)}\Delta_0(X_0+\cdots+X_k)$
$=\chi_0(g_0(t))\Delta_0(X_0+g_1(t).X_1+g_2(t).X_2+\cdots+g_k(t).X_k)$
$=\ldots.$
$=\chi_0(g_0(t))\chi_0(g_1(t))\ldots\chi_0(g_k(t))\Delta_0(X_0+\cdots+X_k)$
$=t^{p(k+1)}\Delta_0(X_0+\cdots+X_k)$
Hence $\kappa=p$ is the common degree of the μ_i's, and $\mu_i=c_i\delta_i$, with $c_i\in F^*$ (remind that δ_i is the fundamental relative invariant of $(L_i,\widetilde{\mathfrak{g}}^{\lambda_i})$).
Also for $(x_0,x_1,\ldots,x_k)\in F^{k+1}$:
$\Delta_0(x_0X_0+\cdots+x_kX_k)=\prod_{j=0}^k x_j^\kappa\ .\Delta_0\ (X_0+\cdots+X_k)$

□

1.14. The polynomials Δ_j.

Let $j\in\{1,2,\ldots,k\}$. By 1.11.4 applied to the graded regular algebra $(\widetilde{g}_j,H_{\lambda_j}+\cdots+H_{\lambda_k})$, there exists an absolutely irreducible polynomial P_j on V_j^+ which is relatively invariant under the action of $G_j=\mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}}_j)}(H_{\lambda_j}+\cdots+H_{\lambda_k})\subset\overline{G}$. By Corollary 1.7.8 applied to $(\widetilde{g}_j,H_{\lambda_j}+\cdots+H_{\lambda_k})$, one has $G_j=\mathrm{Aut}_e(\mathfrak{g}_j).(\cap_{s=j}^k(G_j)_{H_{\lambda_s}})$.
Let χ_j be the corresponding character of G_j. We will define extensions of these polynomials to V^+, using the following decomposition:

$$V^+=V_j^+\oplus V_j^\perp \text{ where } V_j^\perp=\big(\underset{r<s,r<j}{\oplus}E_{r,s}(1,1)\big)\oplus\big(\underset{r<j}{\oplus}\widetilde{\mathfrak{g}}^{\lambda_r}\big)\ .$$

Definition 1.14.1. *We denote by Δ_j the unique polynomial (up to scalar multiplication) such that*

$$\Delta_j(X+Y) = P_j(X) \text{ for } X \in V_j^+, Y \in V_j^\perp \ ,$$

where P_j is an absolutely irreducible polynomial on V_j^+, under the action of G_j.

It may be noticed that, as P_j is the restriction to V_j^+ of an irreducible polynomial on $\overline{V_j^+}$, which is relatively invariant under the action of $\overline{G_j}$, the polynomial Δ_j is the restriction to V^+ of a polynomial defined on $\overline{V^+}$.

Theorem 1.14.2. *Let $j, s \in \{0, 1, \dots, k\}$ and let $X_s \in \widetilde{\mathfrak{g}}^{\lambda_s} \setminus \{0\}$.*
(1) *Δ_j is an absolutely irreducible polynomial of degree $\kappa(k+1-j)$.*
(2) *For $X \in V^+$ one has*

$$\Delta_j(g.X) = \chi_j(g)\Delta_j(X) \text{ for } g \in G_j \ ;$$
$$\Delta_j(g.X) = \Delta_j(X) \text{ for } g \in Aut_e(\mathfrak{g}_j) \ ;$$
$$\Delta_j(g.X) = \Delta_j(X) \text{ for } g = \exp(\operatorname{ad} Z) \text{ where } Z \in \underset{r<s}{\oplus} E_{r,s}(1,-1) \ .$$

(3) *For $s = 0, \dots, k$, let X_s be an element of $\widetilde{\mathfrak{g}}^{\lambda_s} \backslash \{0\}$ and let $x_s \in F$. Then, for $j = 0, \dots, k$,*

$$\Delta_j\Big(\sum_{s=0}^{k} x_s X_s\Big) = \prod_{s=j}^{k} x_s^\kappa \,.\Delta_j\Big(\sum_{s=j}^{k} X_s\Big) \ .$$

(4) *The polynomial $X \in V_j^+ \mapsto \Delta_0(X_0 + X_1 + \cdots + X_{j-1} + X)$ is non zero and equal (up to scalar multiplication) to the restriction of Δ_j to V_j^+.*

Proof.
Statements (1) and (3) are just Theorem 1.13.2 applied to the graded algebra $\widetilde{\mathfrak{g}}_j$.
As $\mathfrak{g}_j = \mathcal{Z}_{\mathfrak{g}}(\widetilde{\mathfrak{l}}_0 \oplus \widetilde{\mathfrak{l}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{l}}_{j-1})$, it is easy to see that $V_j^\perp$ is stable under $\operatorname{ad}(\mathfrak{g}_j)$. Hence $\overline{G}_j.\overline{V}_j^\perp \subset \overline{V}_j^\perp$.
The first assertion in (2) is a consequence of the definition of Δ_j.
We know from §1.7, that the groups $\mathrm{Aut}_e(\mathfrak{g}_j)$ and $\mathrm{Aut}_0(\mathfrak{g}_j)$ are respectively isomorphic to $\mathrm{Aut}_e([\mathfrak{g}_j, \mathfrak{g}_j])$ and $\mathrm{Aut}_0([\mathfrak{g}_j, \mathfrak{g}_j])$. Then, from ([4] Chap VIII §11 n°2, Proposition 3 page 163), the group $\mathrm{Aut}_e(\mathfrak{g}_j)$ is the derived group of $\mathrm{Aut}_0(\mathfrak{g}_j)$. It follows that the character χ_j is trivial on $\mathrm{Aut}_e(\mathfrak{g}_j)$, this is the second assertion of (2).

As any element of

$$\mathfrak{n}_j = \underset{j \leq r<s}{\oplus} E_{r,s}(1,-1) \subset \mathfrak{g}_j$$

is nilpotent, we obtain that Δ_j is invariant under the action of the group generated by the elements $\exp(\operatorname{ad} \mathfrak{n}_j)$.
One has the decomposition:

$$\mathfrak{n}_0 = \underset{r<s}{\oplus} E_{r,s}(1,-1) = \mathfrak{n}_j \oplus K(0) \oplus K(1)$$

where

$$K(0) = \underset{r<s\leq j-1}{\oplus} E_{r,s}(1,-1) \text{ and } K(1) = \underset{r\leq j-1,\ j\leq s}{\oplus} E_{r,s}(1,-1).$$

Note that

$$K(1) = \{X \in \mathfrak{n}, [H_{\lambda_0} + \ldots + H_{\lambda_{j-1}}, X] = X\}$$

and

$$\mathfrak{n}_j \oplus K(0) = \{X \in \mathfrak{n}, [H_{\lambda_0} + \ldots + H_{\lambda_{j-1}}, X] = 0\}.$$

Therefore, as $[\mathfrak{n}_j, K(0)] = 0$, we get $[\mathfrak{n}_j, K(0) \oplus K(1)] \subset K(1)$.
Hence any element of $\exp(\mathrm{ad}(\mathfrak{n}))$ can be written $\exp(\mathrm{ad}\, Z')\exp(\mathrm{ad}\, Z)$ with $Z' \in \mathfrak{n}_j$ and $Z \in K(0) \oplus K(1)$. Therefore it is enough to show that Δ_j is invariant by $\exp(\mathrm{ad}\, Z)$.
But $[K(0), V_j^+] = [\oplus_{r<s\leq j-1} E_{r,s}(1,-1), (\oplus_{j\leq r'<s'} E_{r',s'}(1,1)) \oplus (\oplus_{\ell=j}^{k} \widetilde{\mathfrak{g}}^{\lambda_\ell})] = \{0\}$. One has also $[K(1), V_j^+] \subset V_j^\perp$ as V_j^+ is the eigenspace of $\mathrm{ad}(H_{\lambda_j} + \ldots + H_{\lambda_k})$ for the eigenvalue 2 and as $K(1)$ is the eigenspace of $\mathrm{ad}(H_{\lambda_j} + \ldots + H_{\lambda_k})$ for the eigenvalue -1. Hence for $X \in V_j^+$ and for $Z \in K(0) \oplus K(1)$ one has $\exp(\mathrm{ad}\, Z).X = X + X'$, with $X' \in V_j^\perp$. This implies that $\Delta_j(\exp(\mathrm{ad}\, Z).X = \Delta_j(X)$.
It remains to prove (4). Let Q_j be the polynomial on V_j^+ defined by

$$Q_j(X) = \Delta_0(X_0 + \ldots + X_{j-1} + X).$$

The polynomial Q_j is non zero because $Q_j(X_j + \ldots + X_k) = \Delta_0(X_0 + \ldots + X_k) \neq 0$ (as $X_0 + \ldots + X_k$ is generic by the criterion of Proposition 1.7.12).
This polynomial Q_j is also relatively invariant under G_j (because $G_j \subset \bar{G}$ centralizes the elements $X_0, \ldots, X_{j-1}$). For $t \in F$, let g_t be the element of G_j whose action on V_j^+ is $t.\mathrm{Id}_{V_j^+}$ (Lemma 1.11.3). By Theorem 1.13.2 one has:

$$Q_j(g_t X) = \Delta_0(X_0 + \ldots + X_{j-1} + tX) = t^{\kappa(k-j+1)} Q_j(X).$$

Hence Q_j is a relative invariant of the same degree as Δ_j. Therefore $Q_j = \alpha\Delta_j$, with $\alpha \in F^*$. $\square$

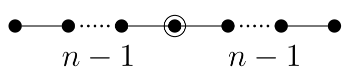
2. Classification of regular graded Lie algebras

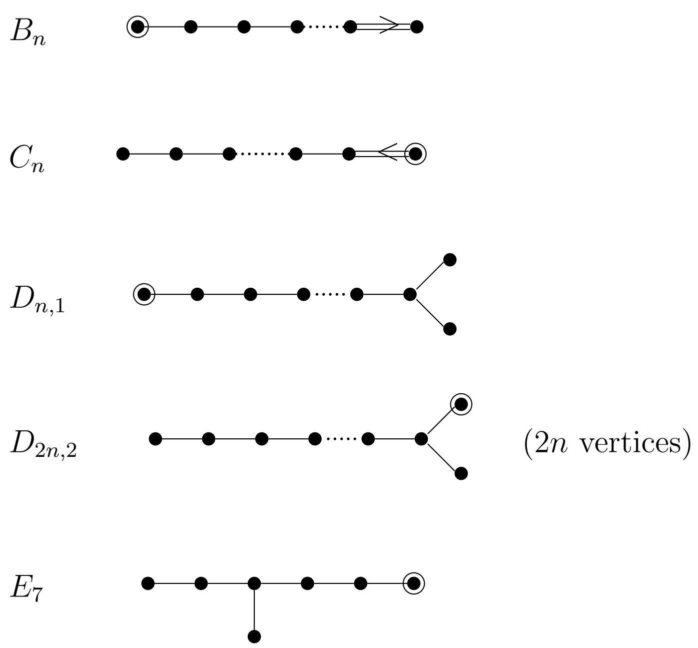
2.1. General principles for the classification.

Our aim is to classify the regular graded Lie algebras defined in Definition 1.7.11. The notations are those of section 1. We have seen in section 1.3 that the graded algebra

$$\overline{\widetilde{\mathfrak{g}}} = \overline{V^-} \oplus \overline{\mathfrak{g}} \oplus \overline{V^+}$$

is a regular prehomogeneous space of commutative type, over the algebraically closed field $\overline{F}$, defined by the data $(\widetilde{\Psi}, \alpha_0)$. We associate to such an object the Dynkin diagram $\widetilde{\Psi}$, on which the vertex corresponding to α_0 is circled. Such a diagram is called the weighted Dynkin diagram of the graded algebra $\overline{\widetilde{\mathfrak{g}}}$. The classification is the same as over $\mathbb{C}$. It was given in [18]. This is the list:

A_{2n-1} 



Remind that the circled root α_0 is the unique root in $\widetilde{\Psi}$ whose restriction to $\mathfrak{a}$ is the root λ_0. Therefore the Satake-Tits diagram of $\widetilde{\mathfrak{g}}$ is such that α_0 is a white root which is not connected by an arrow to another white root. We associate to the regular graded Lie algebra $(\widetilde{\mathfrak{g}}, H_0)$, the Satake-Tits diagram of $\widetilde{\mathfrak{g}}$ where the white root α_0 is circled. Such a diagram will be called the weighted Satake-Tits diagram of $(\widetilde{\mathfrak{g}}, H_0)$. Conversely if we are given a Satake-Tits diagram where the unique circled root, not connected by an arrow to another white root, such that the underlying weighted Dynkin diagram is in the list above, then this diagram defines uniquely a regular graded Lie algebra. The grading is just defined by the element $H_0 \in \mathfrak{a}$ satisfying the equations $\alpha_0(H_0) = 2$ (α_0 being the circled root) and $\beta(H_0) = 0$ if β is one of the other simple roots.

Remind that in the p-adic case, in which we are interested in here, and unlike the case of $\mathbb{R}$, the Satake-Tits diagram does not characterize $\widetilde{\mathfrak{g}}$ up to isomorphism. Two algebras having the same Satake-Tits diagram may have distinct anisotropic kernels ([29], [27]). However, as far as we are concerned, graded algebras having the same weighted Satake-Tits diagram will give rise to the same orbital decomposition of (G, V^+).

The Satake-Tits diagram of $\mathfrak{g}$ is obtained from the weighted diagram of $\widetilde{\mathfrak{g}}$ by removing the vertex α_0, and all the edges connected to α_0.

Although this is not needed here, let us note that the infinitesimal representations $(\mathfrak{g}, V^+)$ obtained this way exhaust all the F-forms of $(\overline{\mathfrak{g}}, \overline{V^+})$. By F-form we mean here a pair $(\mathfrak{u}, W)$, where the Lie algebra $\mathfrak{u}$ is an F-form of $\mathfrak{g}$, and where W is a F-form of $\overline{V^+}$ such that $[\mathfrak{u}, W] \subset W$. To prove this, one can remark that the results obtained over $\mathbb{R}$ in [21], are still true in the p-adic case (see also Proposition 4.1.2. p.66 of [27]).

We will now give a simple "diagrammatic" or "combinatorial" algorithm wich allows to determine the weighted Satake-Tits diagram of $\widetilde{\mathfrak{g}}_1$ (section 1.5) from the diagram of $\widetilde{\mathfrak{g}}$. By induction this algorithm will give the 1-type (Definition 1.12.3). This algorithm allows also to determine easily the rank of the graded algebra. (Remark 2.1.3 b) and c) below).

For the definition of the extended Dynkin diagram see [3], Chap. VI §4 $n°3$ p.198.

Proposition 2.1.1.
Let us make the following operations on the Satake-Tits diagram of the regular graded lie algebra $\widetilde{\mathfrak{g}}$:
1) *One extends the Satake-Tits diagram by considering the underlying extended Dynkin diagram where the additional root $-\omega$ is white (ω being the greatest root of $\widetilde{\mathcal{R}}^+$).*
2) *One removes the vertex α_0 (the circled root), as well as all the white vertices which are connected to α_0 through a chain of black vertices, and one removes also these black vertices.*
3) *One circles the vertex $-\omega$ which has been added.*
The diagram which is obtained after these three operations is the weighted Satake-Tits diagram of $\widetilde{\mathfrak{g}}_1$.

Proof. Remind that (Proposition 1.5.3)

$$\widetilde{\mathcal{R}}_1 = \{\beta \in \widetilde{\mathcal{R}} \mid \ \beta \perp\!\!\!\perp \alpha, \forall \alpha \in S_{\lambda_0}\}.$$

Hence

$$\mathcal{R}_1 = \widetilde{\mathcal{R}}_1 \cap \mathcal{R} = \{\beta \in \mathcal{R} \mid \ \beta \perp\!\!\!\perp \alpha, \forall \alpha \in S_{\lambda_0}\}.$$

Remind also that we have defined the following sets of positive roots:

$$\widetilde{\mathcal{R}}_1^+ = \widetilde{\mathcal{R}}_1 \cap \widetilde{\mathcal{R}}^+, \ \mathcal{R}_1^+ = \mathcal{R}_1 \cap \widetilde{\mathcal{R}}^+ = \widetilde{\mathcal{R}}_1 \cap \mathcal{R}^+,$$

which correspond to the basis $\widetilde{\Psi}_1$ and Ψ_1 of $\widetilde{\mathcal{R}}_1$ and $\mathcal{R}_1$ respectively.
Let us first prove a Lemma.

Lemma 2.1.2.
One has $\Psi_1 = \mathcal{R}_1 \cap \Psi$.

Proof of the Lemma: Let $\langle \mathcal{R}_1 \cap \Psi\rangle^+$ be the set of positive linear combinations of elements in $\mathcal{R}_1 \cap \Psi$. It is enough to show that $\langle \mathcal{R}_1 \cap \Psi\rangle^+ = \mathcal{R}_1^+$.
The inclusion $\langle \mathcal{R}_1 \cap \Psi\rangle^+ \subset \mathcal{R}_1^+$ is obvious. Conversely let $\beta \in \mathcal{R}_1^+$. Let us write β as a positive linear combination of elements of Ψ:

$$\beta = \sum_{\beta_i \in \Psi} x_i \beta_i, \qquad x_i \in \mathbb{N}.$$

As β is orthogonal to any root which restricts to λ_0, one has

$$\langle \beta, \alpha_0\rangle = 0 = \sum_{\beta_i \in \Psi} x_i \langle \beta_i, \alpha_0\rangle.$$

As $\langle \beta_i, \alpha_0\rangle \leq 0$ (scalar product of two roots in the same base), one gets

$$\beta = \sum_{\beta_i \in \Psi, \beta_i \perp \alpha_0} x_i \beta_i.$$

Let now $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be the set of black roots which are connected to α_0 through a chain of black roots (in the Satake-Tits diagram). Moreover we suppose that this set of roots is ordered in such a way that $\alpha_0 + \gamma_1 + \ldots + \gamma_p \in \widetilde{\mathcal{R}}^+$ for all $p = 1, \ldots, m$. This is always possible.
We will now show by induction on j that if β_i belongs to the support of β then $\beta_i \perp \gamma_j$ for $j = 0, \ldots, m$ (where we have set $\gamma_0 = \alpha_0$) . What we have done before is the first step of the

induction. Suppose that $\beta_i \perp \gamma_j$, for all $j \leq p$ ($p \in \{0, \dots, m-1\}$). As $\beta \in \mathcal{R}_1^+$, using the induction hypothesis, one gets:

$$\langle \beta, \alpha_0 + \gamma_1 + \ldots + \gamma_{p+1} \rangle = 0 = \sum_{\beta_i \in \Psi, \beta_i \perp \gamma_j \, (j=0,\dots,m-1)} x_i \langle \beta_i, \gamma_{p+1} \rangle.$$

Again, as $\langle \beta, \gamma_{p+1} \rangle \leq 0$, one obtains

$$\beta = \sum_{\beta_i \in \Psi, \beta_i \perp \gamma_j \, (j=0,\dots,m)} x_i \beta_i.$$

End of the proof of the Lemma.

Hence $\widetilde{\Psi}_1 = \Psi_1 \cup \{\alpha_1\}$ where α_1 is the unique root of $\widetilde{\Psi}_1$ such that $\alpha_1(H_1) = 2$ (Proposition 1.3.2). Let W_1 be the Weyl group of $\mathcal{R}_1$, and let w_1 be the unique element in W_1 such that $w_1(\Psi_1) = -\Psi_1$. Then Proposition 1.9.2 (in his "absolute" version over $\overline{F}$, whose proof is exactly the same) implies that $w_1(\alpha_1) = \omega$ where ω is the greatest root of $\widetilde{\mathcal{R}}_1$, which is also the greatest root of $\widetilde{\mathcal{R}}$.

Let us denote by Dyn(.) the Dynkin diagram of the basis "." of a root system. One has:

$$\mathrm{Dyn}(\widetilde{\Psi}_1) = \mathrm{Dyn}(\Psi_1 \cup \{\alpha_1\}) = \mathrm{Dyn}(w_1(\Psi_1 \cup \{\alpha_1\}) = \mathrm{Dyn}(-\Psi_1 \cup \{\omega\}) = \mathrm{Dyn}(\Psi_1 \cup \{-\omega\}).$$

The preceding Lemma implies that the Dynkin diagram of $\widetilde{\mathfrak{g}}_1$ is the underlying Dynkin diagram of the Satake-Tits diagram described in the statement. It remains to show that the "*colors* " (black or white) are the right one.

For this, let X be the root lattice of $\widetilde{\mathcal{R}}$ (that is the $\mathbb{Z}$-module generated by $\widetilde{\mathcal{R}}$). The restriction morphism ρ extends to a surjective morphism:

$$\rho : X \longrightarrow \overline{X},$$

where $\overline{X}$ is the root lattice of Σ (in $\mathfrak{a}^*$). Consider also the sublattice

$$X_0 = \{\chi \in X \,|\, \rho(\chi) = 0\}.$$

The map ρ induces a bijective morphism: $\overline{\rho} : X/X_0 \longrightarrow \overline{X}$. Set

$$\mathcal{R}_0 = \{\alpha \in \mathcal{R} \,|\, \rho(\alpha) = 0\} = \{\alpha \in \widetilde{\mathcal{R}} \,|\, \rho(\alpha) = 0\}.$$

The order on $\mathcal{R}$ induces an order on $\mathcal{R}_0$ by setting $\mathcal{R}_0^+ = \mathcal{R}^+ \cap \mathcal{R}_0$. We choose an additive order $\leq$ on X_0 in such a way that $\mathcal{R}_0^+ \subset X_0^+$ (for the definition of an additive order on a lattice and for the notation we refer to the paper by Schoeneberg ([27], p.37), who call it "group linear order"). For this it is enough to consider an hyperplane in the vector space generated by X_0, whose intersection with X_0 is reduced to $\{0\}$ and such that $\mathcal{R}_0^+$ is contained in one of the half-spaces defined by this hyperplane. X_0^+ is then defined as the intersection of X_0 with this half-space.

Similarly we choose an additive order on the lattice $\overline{X}$ such that $\Sigma^+ \subset \overline{X}^+$ and we set :

$$(X/X_0)^+ = \{\overline{\chi} \in X/X_0 \,|\, \overline{\rho}(\overline{\chi}) \in \overline{X}^+\}$$

.

Let Γ be the Galois group of the finite Galois extension on which $\widetilde{\mathfrak{g}}$ splits ([27], p. 29). The data $(X/X_0)^+$ and X_0^+ define what is called by Schoeneberg a Γ-order on X ([27], Definitions 3.1.37 and 3.1.38 p. 37). This additive order is defined by

$$X^+ = (X/X_0)^+ \cup X_0^+$$

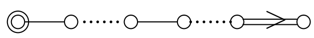
where $(X/X_0)^+$ stands here for the set of elements which are in a strictly positive class. If $\alpha \in \widetilde{\mathcal{R}}^+$, then either $\rho(\alpha) = 0$, and in this case $\alpha \in X_0^+ \subset X^+$, or $\rho(\alpha) \in \Sigma^+$ and then $\alpha \in X^+$. This shows that the order chosen at the beginning of this paper and which was defined by $\widetilde{\mathcal{R}}^+$ comes from a Γ-order in the sense of Schoeneberg on the corresponding root lattices. Similarly, at step 1 of the descent, the order defined by $\widetilde{\mathcal{R}}_1^+$ comes from a Γ-order. Then from the proof of Lemma 4.3.1 p. 72 of [27], we obtain that $w_1 \in (W_1)_\Gamma = \{w \in W(\mathcal{R}_1), w(X_0^1) = X_0^1\}$ (where X^1 is the root lattice of $\mathcal{R}_1$, and X_0^1 is the subset of X^1 which vanish on $\mathfrak{a}_1$), and that w_1 sends a black root on a black root and a white root on a white root. This ends the proof.

$\square$

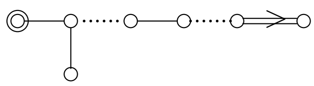
Remark 2.1.3.
a) As expected, if one applies the procedure of Proposition 2.1.1 to the Satake-Tits diagrams of Lemma 1.12.1, one obtains the empty diagram.
b) It is worth noting that the rank of $\widetilde{\mathfrak{g}}$ (cf. Definition 1.6.2) is the number of times one must apply the procedure of Proposition 2.1.1 until one obtains the empty diagram.
c) The last diagram, obtained before the empty diagram, when one iterates this procedure, is necessarily one of the two diagrams of Lemma 1.12.1. It defines therefore the 1-type de $\widetilde{\mathfrak{g}}$.
d) It may happen that the iteration of the procedure of Proposition 2.1.1 gives a non-connected Satake-Tits diagram (see example below). In that case the next iteration is made only on the connected component containing the circled root.

Exemple 2.1.4.

The following split diagram corresponds to a graded algebra $\widetilde{\mathfrak{g}}$ verifying the hypothesis $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$:

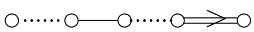
 B_n

The extended diagram is:



The diagram obtained by applying the procedure of Proposition 2.1.1 is :

 A_1

 B_{n-2}

When one applies again the procedure to the diagram ◎ , one obtains the empty diagram. Hence the rank is 2.

2.2. Table.

Notation 2.2.1. (Notations for Table 1)

We first define the *type* of the graded Lie algebra $\tilde{\mathfrak{g}}$ according to possible values which can be taken by e and ℓ. This notion of type allows to split the classification of graded Lie algebras according to the number of G-orbits in V^+.

Definition 2.2.2. [2] .

- *$\tilde{\mathfrak{g}}$ is said to be of type I if $\ell=\delta^2$, $\delta\in\mathbb{N}^*$ and $e=0$ or 4.*
- *$\tilde{\mathfrak{g}}$ is said to be of type II if $\ell=1$ and $e=1{,}2$ or 3,*
- *$\tilde{\mathfrak{g}}$ is said to be of type III if $\ell=3$.*

- We denote always by D a central division algebra over F. Its degree is denoted by δ (remind that this means that the dimension is δ^2). If its degree is 2, then D is necessarily the unique quaternion division algebra over F.

- This quaternion division algebra over F is denoted by $\mathbb{H}$ and its canonical anti-involution is denoted by $\gamma : x\mapsto \bar{x}$.

- $M(m,D)=M_m(D)$ is the algebra of $m\times m$ matrices with coefficients in D.

- $\mathfrak{sl}(m,D)$ is the derived Lie algebra of $M(m,D)$. It is also the space of matrices in $M(m,D)$ whose reduced trace is zero. (Recall that if $x=(x_{i,j})\in M(m,D)$, $Tr_{red}(x)=\sum\tau(x_{i,i})$ where τ is the reduced trace of D. ([31], IX,§2, Corollaire 2 p.169). Recall also that the reduced trace of the quaternion division algebra is $\tau(x)=x+\overline{x}$.

- $E=F(y)$ is a quadratic extension of F. Then σ is the canonical conjugation in E: $\sigma(a+by)=a-by$.

- H_{2n} is the hermitian form on E^{2n} defined by $H_{2n}(u,v)={}^tuS_n\sigma(v)$ where u,v are columns vectors of E^{2n} and where $S_n=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}$.

- $\mathfrak{u}(2n,E,H_{2n})=\{X\in\mathfrak{sl}(2n,E), XS_n+S_n{}^t(\sigma(X))=0\}$ (this is the so-called unitary algebra of the form H_{2n})

- $\mathrm{Herm}_\sigma(n,E)$ is defined as follows:

$$\mathrm{Herm}_\sigma(n,E)=\{U\in M(n,E),\ {}^t\sigma(U)=U\}$$

- $q_{(p,q)}$, with $p\geq q$ is a quadratic form of Witt index q on F^{p+q}. $\mathfrak{o}(q_{(p,q)})$ is the corresponding orthogonal algebra.

- $\mathfrak{sp}(2n,F)$ is the usual symplectic Lie algebra (the matrices in it being of type $2n\times 2n$, with coefficients in F).

- $\mathrm{Sym}(n,F)$ is the space of symmetric matrices of type $n\times n$ with coefficients in F.

[2] We caution the reader that our notion of type in the non archimedean case is not related to the notion of type in the archimedean($F=\mathbb{R}$) case done in [7]. Our definition of type is related to the structure of the open G-orbits (see Theorem 3.6.3 below).

- On $\mathbb{H}^{2n}$ we denote also by H_{2n} the hermitian form defined by $H_{2n}(u,v) = {}^t\gamma(u)S_n v$ where u, v are columns vectors of $\mathbb{H}^{2n}$, and where, as above, $S_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.

- $\mathfrak{u}(2n, \mathbb{H}, H_{2n}) = \{A \in M(2n, \mathbb{H}),\ {}^t\overline{A}S_n + S_n A = 0\}$

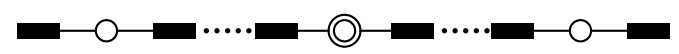
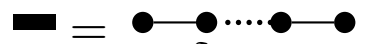
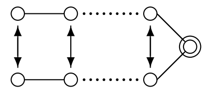
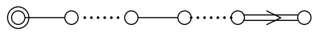
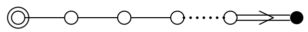
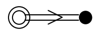
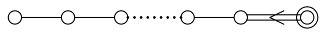
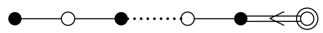
- $\mathrm{SkewHerm}(n, \mathbb{H}) = \{A \in M(n, \mathbb{H}),\ {}^t\overline{A} + A = 0\}$

- $\mathrm{Herm}(n, \mathbb{H}) = \{A \in M(n, \mathbb{H}),\ {}^t\overline{A} = A\}$

- $\mathrm{Skew}(2n, F)$ is the space of skew-symmetric matrices of type $2n \times 2n$ with coefficients in F.

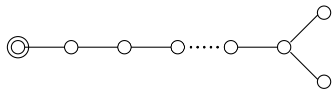
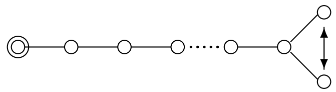
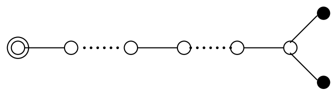
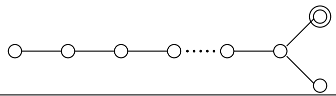
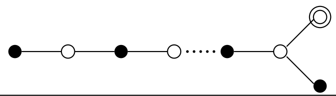
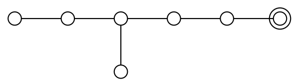
- K_{2n} is the γ-skewhermitian form on $\mathbb{H}^{2n}$ defined by $K_{2n}(u,v) = {}^t\gamma(u)K_{2n}v$ where u, v are columns vectors of $\mathbb{H}^{2n}$ and where, by abuse of notation,we also set $K_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. - $\mathfrak{u}(2n, \mathbb{H}, K_{2n}) = \{A \in M(2n, \mathbb{H}),\ {}^t\overline{A}K_{2n} + K_{2n}A = 0\}$

Table 1 Simple Regular Graded Lie Algebras over a p-adic field

	$\widetilde{\mathfrak{g}}$	$\mathfrak{g}'$	V^+	$\widetilde{\mathcal{R}}$	$\widetilde{\Sigma}$	Satake-Tits diagram	$rank(=k+1)$	ℓ	d	e	Type	1-type
(1)	$\mathfrak{sl}(2(k+1),D)$	$\mathfrak{sl}(k+1,D)$ $\oplus$ $\mathfrak{sl}(k+1,D)$	$\mathrm{M}(k+1,D)$	A_{2n-1} $n=(k+1)\delta$	A_{2k+1}	 where  $(\delta\in\mathbb{N}^*)$ $\delta-1$	$k+1$	δ^2	$2\delta^2$	0	I	(A,δ)
(2)	$\mathfrak{u}(2n,E,H_n)$	$\mathfrak{sl}(n,E)$	$\mathrm{Herm}_\sigma(n,E)$	A_{2n-1} $n\geqslant 1$	C_n		n	1	2	2	II	$(A,1)$
(3)	$\mathfrak{o}(q_{(n+1,n)})$	$\mathfrak{o}(q_{(n,n-1)})$	F^{2n-1}	B_n $n\geq 3$	B_n		2	1	$2n-3$	1	II	$(A,1)$
(4)	$\mathfrak{o}(q_{(n+2,n-1)})$	$\mathfrak{o}(q_{(n+1,n-2)})$	F^{2n-1}	B_n $n\geqslant 3$	B_{n-1}		2	1	$2n-3$	3	II	$(A,1)$
(5)	$\mathfrak{o}(q_{(4,1)})$	$\mathfrak{o}(3)$	F^3	B_2	$B_1=A_1$		1	3	$--$	$--$	III	B
(6)	$\mathfrak{sp}(2n,F)$	$\mathfrak{sl}(n,F)$	$\mathrm{Sym}(n,F)$	C_n $n\geqslant 2$	C_n		n	1	1	1	II	$(A,1)$
(7)	$\mathfrak{u}(2n,\mathbb{H},H_{2n})$	$\mathfrak{sl}(n,\mathbb{H})$	SkewHerm(n, $\mathbb{H}$)	C_{2n}	C_n		n	3	4	4	III	B

(continued next page)

Table 1 (continued) Simple Regular Graded Lie Algebras over a p-adic field

	$\widetilde{\mathfrak{g}}$	$\mathfrak{g}'$	V^+	$\widetilde{\mathcal{R}}$	$\widetilde{\Sigma}$	Satake diagram	$rank(=k+1)$	ℓ	d	e	Type	1-type
(8)	$\mathfrak{o}(q_{(m,m)})$	$\mathfrak{o}(q_{(m-1,m-1)})$	F^{2m-2}	D_m $m\geqslant 4$	D_m		2	1	$2m-4$	0	I	$(A,1)$
(9)	$\mathfrak{o}(q_{(m+1,m-1)})$	$\mathfrak{o}(q_{(m,m-2)})$	F^{2m-2}	D_m $m\geqslant 4$	B_{m-1}		2	1	$2m-4$	2	II	$(A,1)$
(10)	$\mathfrak{o}(q_{(m+2,m-2)})$	$\mathfrak{o}(q_{(m+1,m-3)})$	F^{2m-2}	D_m $m\geqslant 4$	B_{m-2}		2	1	$2m-4$	4	I	$(A,1)$
(11)	$\mathfrak{o}(q_{(2n,2n)})$	$\mathfrak{sl}(2n,F)$	$\mathrm{Skew}(2n,F)$	D_{2n} $n\geqslant 3$	D_{2n}		n	1	4	0	I	$(A,1)$
(12)	$\mathfrak{u}(2n,\mathbb{H},K_{2n})$	$\mathfrak{sl}(n,\mathbb{H})$	$\mathrm{Herm}(n,\mathbb{H})$	D_{2n} $n\geqslant 3$	C_n		n	1	4	**4**	I	$(A,1)$
(13)	split E_7	split E_6	$\mathrm{Herm}(3,\mathbb{O}_s)$	E_7	E_7		3	1	8	0	I	(A,1)

3. The G-orbits in V^+

3.1. Representations of $\mathfrak{sl}(2,F)$.

(For the convenience of the reader we recall here some classical facts about $\mathfrak{sl}_2$ modules, see [4], chap.VIII, §1, Proposition and Corollaire).

Let $\{Y,H,X\}$ be an $\mathfrak{sl}_2$-triple (ie. $[H,X]=2X,[H,Y]=-2Y,[Y,X]=H$) in $\widetilde{\mathfrak{g}}$ and let E be a subspace of $\widetilde{\mathfrak{g}}$ which is invariant under this $\mathfrak{sl}_2$-triple.
Let M be an irreducible submodule of dimension $m+1$ of E, then M is generated by a primitive element e_0 (ie. $X.e_0=0$) of weight m under the action of H and the set of elements $e_p:=\dfrac{(-1)^p}{p!}(Y)^p.e_0$, of weight $m-2p$ under the action of H, is a basis of$M$.
We have also the weight decomposition $E=\oplus_{p\in\mathbb{Z}}E_p$ into weight spaces of weight p under the action of H. The following properties are classical:

(1) The non trivial element of the Weyl group of $SL(2,F)$ acts on E by

$$w=e^Xe^Ye^X,$$

(2) If, as before, M is an irreducible submodule of E of dimension $m+1$, then $w:M_{m-2p}\to M_{2p-m}$ is an isomorphism. More precisely, on the base (e_p) defined above, one has

$$w.e_p=(-1)^{m-p}e_{m-p}=\frac{1}{(m-p)!}(Y)^{m-p}.e_0.$$

(3) For $Z\in E_p$, one has $w^2Z=(-1)^pZ$.

(4) $X^j:E_p\to E_{p+2j}$ is $\begin{cases}\text{injective for} & j\leq -p\\ \text{bijective for} & j=-p\\ \text{surjective for} & j\geq -p\end{cases}$

$Y^j:E_p\to E_{p-2j}$ is $\begin{cases}\text{injective for} & j\leq p\\ \text{bijective for} & j=p\\ \text{surjective for} & j\geq p\end{cases}$

3.2. First reduction.

Remind the definition of V_1^+ (cf. Corollaire 1.8.4):

$$V_1^+=\widetilde{\mathfrak{g}}_1\cap V^+=\{X\in V^+|[H_{\lambda_0},X]=0\}.$$

The decomposition of V^+ into eigenspaces of H_{λ_0} is then given by

$$V^+=V_1^+\oplus\widetilde{\mathfrak{g}}^{\lambda_0}\oplus W_1^+,\text{ where }W_1^+=\{X\in V^+|[H_{\lambda_0},X]=X\}=\{X\in V^+|[H_{\lambda_1}+\ldots+H_{\lambda_k},X]=X\}.$$

Proposition 3.2.1.
Let $X\in V^+$.

(1) *If $\Delta_0(X)=0$ then X is conjugated under $\mathrm{Aut}_e(\mathfrak{g})\subset G$ to an element of V_1^+.*
(2) *if $\Delta_1(X)\neq 0$ then X is conjugated under $\mathrm{Aut}_e(\mathfrak{g})\subset G$ to an element of $\widetilde{\mathfrak{g}}^{\lambda_0}\oplus V_1^+$.*

Proof. (we give the proof for the convenience of the reader although it is the same as in the real case, see Prop 2.1 page 38 of [BR]),

(1) Let X be a non generic element of V^+. Then X belongs to the semi-simple part of $\widetilde{\mathfrak{g}}$ and satisfies $(\operatorname{ad} X)^3 = 0$. By the Jacobson-Morozov Theorem ([4] chap. VIII, §11 Corollaire of Lemme 6), there exists an $\mathfrak{sl}_2$-triple $\{Y, H, X\}$. Decomposing these elements according to the decomposition $\widetilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$, one sees easily that one can suppose that $Y \in V^-$ and $H \in \mathfrak{g}$.

The eigenvalue of $\operatorname{ad} H$ are the weights of the representation of this $\mathfrak{s}l_2$-triple in $\widetilde{\mathfrak{g}}$, they are therefore integers. Hence H belongs to a maximal abelian split subalgebra of $\mathfrak{g}$. By ([27] Theorem 3.1.16 or [28], I.3), there exists $g \in \operatorname{Aut}_e(\mathfrak{g}) \subset G$ such that $g.H \in \mathfrak{a}$. One can choose $w \in N_{\operatorname{Aut}_e(\mathfrak{g})}(\mathfrak{a})$ such that $wg.H$ belongs to the Weyl chamber $\overline{\mathfrak{a}}^+ := \{Z \in \mathfrak{a}; \lambda(Z) \geq 0 \, \text{for} \, \lambda \in \Sigma^+\}$. Let $\{Y', H', X'\} = wg\{Y, H, X\}$. We show first that

$$\lambda_0(H') = 0.$$

Any element in $\widetilde{\mathfrak{g}}^{\lambda_0} \backslash \{0\}$ is primitive with weight $\lambda_0(H')$ under the action of this $\mathfrak{s}l_2$-triple, hence $\lambda_0(H') \geq 0$.

Suppose that $\lambda_0(H') > 0$. Let $\lambda \in \widetilde{\Sigma}^+ \backslash \Sigma$. By Theorem 1.2.1 we obtain $\lambda = \lambda_0 + \sum_i m_i \lambda_i$ with $m_i \in \mathbb{N}$ and $\lambda_i \in \Sigma^+$ and hence $\lambda(H') > 0$. Therefore, for all $Z \in \widetilde{\mathfrak{g}}^\lambda \subset V^+$, we get

$$Z = -\frac{1}{\lambda(H')} \operatorname{ad} X'(\operatorname{ad} Y'(Z)).$$

It follows that $\operatorname{ad} X' : \mathfrak{g} \to V^+$ is surjective, and this is not possible because X' is not generic. Therefore $\lambda_0(H') = 0$.

Let us show that $X' \in V_1^+$. Let $Y_0 \in \widetilde{\mathfrak{g}}^{-\lambda_0} \backslash \{0\}$. The $\mathfrak{s}l_2$-module generated by Y_0 under the action of $\{Y', H', X'\}$ has lowest weight 0 ($\operatorname{ad} Y'.Y_0 = 0$ and $\operatorname{ad} H'.Y_0 = 0$) and hence it is the trivial module. It follows that $\operatorname{ad} X'.Y_0 = 0$, and then X' commutes with $\widetilde{\mathfrak{g}}^{\lambda_0}$ and with $\widetilde{\mathfrak{g}}^{-\lambda_0}$. Hence X' commutes with H_{λ_0} and therefore $X' \in V_1^+$. Since $wg \in \operatorname{Aut}_e(\mathfrak{g})$, statement (1) is proved.

(2) Let $X \in V^+$ such that $\Delta_1(X) \neq 0$. This element decomposes as follows

$$X = X_0 + X_1 + X_2, \quad \text{with} \, X_0 \in \widetilde{\mathfrak{g}}^{\lambda_0}, X_1 \in W_1^+, X_2 \in V_1^+.$$

From the definition of Δ_1 (cf. Definition 1.14.1), one has

$$\Delta_1(X) = \Delta_1(X_2).$$

Therefore X_2 is generic in V_1^+. By Proposition 1.7.12 applied to $\widetilde{\mathfrak{g}}_1$, there exists $Y_2 \in V_1^-$ such that $\{Y_2, H_{\lambda_1} + \ldots + H_{\lambda_k}, X_2\}$ is a $\mathfrak{s}l_2$-triple. The weights of this triple on V^+ are 2 on V_1^+, 1 on W_1^+ and 0 on $\widetilde{\mathfrak{g}}^{\lambda_0}$. Let us note:

$$\mathfrak{g}_{-1} := \{X \in \mathfrak{g} | [H_{\lambda_1} + \ldots + H_{\lambda_k}, X] = -X\}.$$

The map $\operatorname{ad} X_2$ is a bijection from $\mathfrak{g}_{-1}$ onto W_1^+, hence there exists $Z \in \mathfrak{g}_{-1}$ such that $[X_2, Z] = X_1$. If we write the decomposition of $e^{\operatorname{ad} Z} X$ according to the weight spaces of V^+, we obtain

$$e^{\operatorname{ad} Z} X = X_0 + [Z, X_1] + \frac{1}{2}[Z, [Z, X_2]] + X_1 + [Z, X_2] + X_2$$

$$= X_0 + [Z, X_1] + \frac{1}{2}(\operatorname{ad} Z)^2.X_2 + X_2.$$

Hence $e^{\operatorname{ad} Z}X$ belongs to $X_2 \oplus \widetilde{\mathfrak{g}}^{\lambda_0} \subset V_1^+ \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$ and this gives (2). □

Theorem 3.2.2. *Any element of V^+ is $\mathrm{Aut}_e(\mathfrak{g})$-conjugated to an element of $\widetilde{\mathfrak{g}}^{\lambda_0} + \ldots \oplus \widetilde{\mathfrak{g}}^{\lambda_k}$.*

Proof. Let us show that any element of V^+ is conjugated under $\mathrm{Aut}_e(\mathfrak{g})$ to an element of $V_1^+ \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$.

Let $X \in V^+$. If X is not generic, then X is $\mathrm{Aut}_e(\mathfrak{g})$-conjugated to an element of V_1^+ (Proposition 3.2.1, (1)).

Suppose now that X is generic. As the Lie algebra of $\mathrm{Aut}_e(\mathfrak{g})$ is equal to $[\mathfrak{g}, \mathfrak{g}]$, the Lie algebra of $F^* \times \mathrm{Aut}_e(\mathfrak{g})$ is $F \times [\mathfrak{g}, \mathfrak{g}]$, which is the Lie algebra of $\mathfrak{g}$, (see Remark 1.8.8, remember also that, as $\mathfrak{g} \oplus V^+$ is a maximal parabolic subalgebra, the center of $\mathfrak{g}$ has dimension one). Therefore the orbit of X under the group $F^*.\mathrm{Aut}_e(\mathfrak{g})$ is open. Suppose that $\mathrm{Aut}_e(\mathfrak{g}).X \cap \{Y \in V^+, \Delta_1(Y) = 0\} = \emptyset$. Then, as $\{Y \in V^+, \Delta_1(Y) = 0\}$ is a cone, we would have $F^*.\mathrm{Aut}_e(\mathfrak{g}).X \subset \{Y \in V^+, \Delta_1(Y) = 0\}$. This is impossible, as a Zariski open set is never a subset of a closed one. Hence $\mathrm{Aut}_e(\mathfrak{g}).X \cap \{Y \in V^+, \Delta_1(Y) = 0\} \neq \emptyset$. From Proposition 3.2.1 (2), we obtain that X is conjugated under $\mathrm{Aut}_e(\mathfrak{g})$ to an element of $\widetilde{\mathfrak{g}}^{\lambda_0} \oplus V_1^+$.

The same argument applied to $\widetilde{\mathfrak{g}}_j$ ($j = 1, \ldots, k$) shows that if $X \in V_j^+$ then the $\mathrm{Aut}_e(\mathfrak{g}_j)$-orbit of X_j meets $\widetilde{\mathfrak{g}}^{\lambda_j} \oplus V_{j+1}^+$. Any element of $\mathrm{Aut}_e(\mathfrak{g}_j)$ stabilizes $\widetilde{\mathfrak{g}}^{\lambda_0} + \ldots \oplus \widetilde{\mathfrak{g}}^{\lambda_{j-1}}$ and, by Corollary 1.8.4, one has $V_k^+ = \widetilde{\mathfrak{g}}^{\lambda_k}$. The result is then obtained by induction.

□

3.3. An involution which permutes the roots in $E_{i,j}(\pm 1, \pm 1)$.

Define $\widetilde{G} := \mathrm{Aut}_0(\widetilde{\mathfrak{g}})$.

For $i = 1, \ldots, k$, we fix $X_i \in \widetilde{\mathfrak{g}}^{\lambda_i}$. There exist then $Y_i \in \widetilde{\mathfrak{g}}^{-\lambda_i}$ such that $\{Y_i, H_{\lambda_i}, X_i\}$ is an $\mathfrak{s}l_2$-triple. The action of the non trivial Weyl group element of this triple is given by

$$w_i := e^{\operatorname{ad} X_i} e^{\operatorname{ad} Y_i} e^{\operatorname{ad} X_i} = e^{\operatorname{ad} Y_i} e^{\operatorname{ad} X_i} e^{\operatorname{ad} Y_i} \in N_{\widetilde{G}}(\mathfrak{a}).$$

Lemma 3.3.1. *Let $j \neq i$ and $p = \pm 1$. One has*

1. *If $H \in \mathfrak{a}$ then $w_i.H = H - \lambda_i(H)H_{\lambda_i}$;*
2. *If $X \in E_{i,j}(1, p)$ then $w_i.X = \operatorname{ad} Y_i.X \in E_{i,j}(-1, p)$;*
3. *If $X \in E_{i,j}(-1, p)$ then $w_i.X = \operatorname{ad} X_i.X \in E_{i,j}(1, p)$;*
4. *If $X \in E_{i,j}(\pm 1, p)$ then $w_i^2.X = -X$.*

Proof. The first statement is obvious as w_i acts as the reflection on $\mathfrak{a}$ associated to λ_i.

As each $E_{i,j}(\pm 1, p)$ is included in a weight space for the action of the $\mathfrak{s}l_2$-triple $\{Y_i, H_{\lambda_i}, X_i\}$, the other statements are immediate consequences of the properties of $\mathfrak{s}l_2$-modules given in section 3.1. □

For $i \neq j$, we set

$$w_{i,j} := w_i w_j = w_j w_i.$$

The preceding Lemma implies that $w_{i,j}$ satisfies the following properties.

Corollary 3.3.2. *For $i \neq j$ and $p, q \in \{\pm 1\}$, one has*

(1) *$w_{i,j}$ is an isomorphism from $E_{i,j}(p,q)$ onto $E_{i,j}(-p,-q)$*
(2) *The restriction of $w_{i,j}^2$ to $E_{i,j}(p,q)$ is the identity.*

Remark 3.3.3. The involution $w_{i,j}$ permutes the roots λ such that $\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(\pm 1, \pm 1)$. More precisely one has

$$\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(1,-1) \Longrightarrow w_{i,j}(\lambda) = \lambda - \lambda_i + \lambda_j,$$
$$\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(1,1) \Longrightarrow w_{i,j}(\lambda) = \lambda - \lambda_i - \lambda_j,$$
$$\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(-1,-1) \Longrightarrow w_{i,j}(\lambda) = \lambda + \lambda_i + \lambda_j,$$

3.4. **Construction of elements interchanging λ_i and λ_j.**

Let i and j be two distinct elements of $\{0,\dots,k\}$. By Proposition 1.9.2, the roots λ_i and λ_j are conjugated by the Weyl group W. The aim of this section is to construct explicitly an element of G which exchanges λ_i and λ_j.

Lemma 3.4.1. *Let λ be a root such that $\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(1,-1)$. Then $\lambda + w_{i,j}(\lambda)$ is not a root.*

Proof. If $\lambda = \dfrac{\lambda_i - \lambda_j}{2}$ then $w_{i,j}(\lambda) = -\lambda$, this implies the statement.
Let now $\lambda \neq \dfrac{\lambda_i - \lambda_j}{2}$. Suppose that $\mu = \lambda + w_{i,j}(\lambda)$ is a root.
As $\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(1,-1)$, one has $w_{i,j}(\lambda) = \lambda - \lambda_i + \lambda_j$ (Remark 3.3.3) and $\mu = 2\lambda - \lambda_i + \lambda_j$. It follows that for $s \in \{0,\dots,k\}$, the root μ is orthogonal to λ_s, and hence strongly orthogonal to λ_s (Corollaire 1.8.2). Let us write

$$\lambda = \frac{\mu}{2} + \frac{\lambda_i - \lambda_j}{2}, \quad \text{and} \quad w_{i,j}(\lambda) = \lambda - \lambda_i + \lambda_j = \frac{\mu}{2} - \frac{\lambda_i - \lambda_j}{2}.$$

If α and β are two roots, we set, as usually $n(\alpha,\beta) = \alpha(H_\beta) = 2\frac{\langle \alpha,\beta\rangle}{\langle \beta,\beta\rangle}$. As $\lambda - w_{i,j}(\lambda) = \lambda_i - \lambda_j$ is not a root, and as $\lambda \neq w_{i,j}(\lambda)$, we obtain that $n(\lambda, w_{i,j}(\lambda)) \leq 0$. Remind that we have supposed that $\mu = \lambda + w_{i,j}(\lambda)$ is a root. As $w_{i,j}(\lambda) - \lambda$ is not a root, the λ-chain through $w_{i,j}(\lambda)$ cannot be symmetric with respect to $w_{i,j}(\lambda)$. This implies that $n(\lambda, w_{i,j}(\lambda)) < 0$. As λ and $w_{i,j}(\lambda)$ have the same length (and hence $n(\lambda, w_{i,j}(\lambda)) = n(w_{i,j}(\lambda), \lambda) = -1$), we get

$$-1 = n(w_{i,j}(\lambda), \lambda) = 2 - n(\lambda_i, \lambda) + n(\lambda_j, \lambda).$$

Consider the root $\delta = \lambda + \lambda_j = w_j(\lambda) = \dfrac{\mu}{2} + \dfrac{\lambda_i + \lambda_j}{2}$. Then $n(\delta, \lambda_i) = n(\delta, \lambda_j) = 1$ and the preceding relation gives

$$3 = n(\lambda_i, \delta) + n(\lambda_j, \delta).$$

It follows that either $n(\lambda_i, \delta)$ or $n(\lambda_j, \delta)$ is ≥ 2. Suppose for example that $n(\lambda_i, \delta) \geq 2$. Consider the δ-chain through $-\lambda_i$. As $(-\lambda_i - \delta)(H_0) = -4$, $-\lambda_i - \delta$ is not a root. It follows that $-\lambda_i + 2\delta = \mu + \lambda_j$ is a root. This is impossible as μ is strongly orthogonal to λ_j.

Hence $\mu = \lambda + w_{i,j}(\lambda)$ is not a root.

□

Lemma 3.4.2. *There exist $X \in E_{i,j}(1,-1)$ and $Y \in E_{i,j}(-1,1)$ such that $\{Y, H_{\lambda_i} - H_{\lambda_j}, X\}$ is an $\mathfrak{sl}_2$-triple.*

Proof. By definition, for a root λ, the element H_λ is the unique element such that $\beta(H_\lambda) = n(\beta,\lambda)$, for all $\beta \in \widetilde{\Sigma}$. If $\lambda = (\lambda_i - \lambda_j)/2$ is a root it is then easy to see that $H_\lambda = H_{\lambda_i} - H_{\lambda_j}$. Any non zero element $X \in \widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(1,-1)$ can be completed in an $\mathfrak{s}l_2$-triple $\{Y, H_{\lambda_i} - H_{\lambda_j}, X\}$ where $Y \in \widetilde{\mathfrak{g}}^{-\lambda} \subset E_{i,j}(-1,1)$.
If $(\lambda_i - \lambda_j)/2$ is not a root, let us fix a root λ such that $\widetilde{\mathfrak{g}}^\lambda \subset E_{i,j}(1,-1)$ and set $\lambda' = -w_{i,j}(\lambda)$. One has $\widetilde{\mathfrak{g}}^{\lambda'} \subset E_{i,j}(1,-1)$. Also $\lambda' \neq \lambda$ ($\lambda' = -w_{i,j}(\lambda) = -\lambda + \lambda_i - \lambda_j = \lambda$ would imply that $\lambda = \frac{\lambda_i - \lambda_j}{2}$) . Let $X_0 \in \widetilde{\mathfrak{g}}^\lambda \setminus \{0\}$. Choose $Y_0 \in \widetilde{\mathfrak{g}}^{-\lambda} \subset E_{i,j}(-1,1)$ such that $\{Y_0, H_\lambda, X_0\}$ is an $\mathfrak{s}l_2$-triple. Then $\{w_{i,j}(X_0), H_{\lambda'}, w_{i,j}(Y_0)\}$ is an $\mathfrak{s}l_2$-triple as $w_{i,j}(H_\lambda) = -H_{\lambda'}$ and as $w_{i,j}(X_0) \in \widetilde{\mathfrak{g}}^{-\lambda'}$ and $w_{i,j}(Y_0) \in \widetilde{\mathfrak{g}}^{\lambda'}$. Define

$$X = X_0 + w_{i,j}(Y_0) \quad \text{and} \quad Y = Y_0 + w_{i,j}(X_0) = w_{i,j}(X).$$

Then one has

$$[H_{\lambda_i} - H_{\lambda_j}, X] = 2X \quad \text{and} \quad [H_{\lambda_i} - H_{\lambda_j}, Y] = 2Y.$$

It remains to prove that $[Y, X] = H_{\lambda_i} - H_{\lambda_j}$. Let $Z = [Y, X]$. By the preceding lemma, $\lambda - \lambda'$ is not a root, hence $[Y_0, w_{i,j}(Y_0)] = 0$ and $[w_{i,j}(X_0), X_0] = 0$ and this implies

$$Z = [Y_0, X_0] - w_{i,j}([Y_0, X_0]) = H_\lambda - w_{i,j}(H_\lambda) \in \mathfrak{a}.$$

This shows that $Z \neq 0$ ($Z = 0 \Leftrightarrow H_\lambda = w_{i,j}(H_\lambda) \Leftrightarrow \lambda = w_{i,j}(\lambda) = \lambda - \lambda_i + \lambda_j \Leftrightarrow \lambda_i = \lambda_j$). Lemma 3.3.1 implies that $w_{i,j}(Z) = Z - \lambda_i(Z)H_{\lambda_i} - \lambda_j(Z)H_{\lambda_j}$. As $w_{i,j}Z = -Z$, we obtain

$$Z = \frac{\lambda_i(Z)H_{\lambda_i} + \lambda_j(Z)H_{\lambda_j}}{2}.$$

Therefore $Z \in \mathfrak{a}^0 = \oplus_{i=0}^k F\, H_{\lambda_i}$.
Let $H \in \mathfrak{a}^0$. Then $H = \sum_{i=0}^k \frac{\lambda_i(H)}{2} H_{\lambda_i}$ and an easy calculation shows that $[H, Y] = \frac{\lambda_j(H) - \lambda_i(H)}{2} Y$. Therefore

$$\widetilde{B}(H, Z) = \frac{\lambda_j(H) - \lambda_i(H)}{2} \widetilde{B}(Y, X).$$

On the other hand, the roots λ_i and λ_j are W-conjugate (Proposition 1.9.2), hence $\widetilde{B}(H_{\lambda_i}, H_{\lambda_i}) = \widetilde{B}(H_{\lambda_j}, H_{\lambda_j})$ for all i, j. Define $C_1 := \widetilde{B}(H_{\lambda_i}, H_{\lambda_i}) \in F^*$. Then

$$\widetilde{B}(H, H_{\lambda_i} - H_{\lambda_j}) = C_1 \frac{\lambda_i(H) - \lambda_j(H)}{2}, \quad \text{for } H \in \mathfrak{a}^0.$$

As $\widetilde{B}$ is nondegenerate on $\mathfrak{a}^0$, if we set $C_2 := -\widetilde{B}(X, Y) \in F^*$, we obtain

$$Z = \frac{C_2}{C_1}(H_{\lambda_i} - H_{\lambda_j}).$$

If we replace Y by $\dfrac{C_1}{C_2}Y$, then $\{Y, H_{\lambda_i} - H_{\lambda_j}, X\}$ is an $\mathfrak{s}l_2$-triple.

□

Let $\{Y, H_{\lambda_i} - H_{\lambda_j}, X\}$ be the $\mathfrak{s}l_2$-triple obtained in the preceding Lemma. The action of the non trivial element of the Weyl group of this $\mathfrak{s}l_2$-triple is given by

$$\gamma_{i,j} = e^{\operatorname{ad} X} e^{\operatorname{ad} Y} e^{\operatorname{ad} X} = e^{\operatorname{ad} Y} e^{\operatorname{ad} X} e^{\operatorname{ad} Y} \in \operatorname{Aut}_e(\mathfrak{g}).$$

Proposition 3.4.3. *For $i \neq j \in \{0, \dots, k\}$, the elements $\gamma_{i,j}$ belong to $N_{\operatorname{Aut}_e(\mathfrak{g})}(\mathfrak{a}^0)$ and*

(1)

$$\gamma_{i,j}(H_{\lambda_s}) = \begin{cases} H_{\lambda_i} & \text{for } s = j \\ H_{\lambda_j} & \text{for } s = i \\ H_{\lambda_s} & \text{for } s \notin \{i,j\} \end{cases}$$

(2) *The action of $\gamma_{i,j}$ is trivial on each root space $\widetilde{\mathfrak{g}}^{\lambda_s}$ for $s \notin \{i,j\}$ and it is a bijective involution from $\widetilde{\mathfrak{g}}^{\lambda_i}$ onto $\widetilde{\mathfrak{g}}^{\lambda_j}$.*

Proof. As $X \in E_{i,j}(1,-1)$, for $H \in \mathfrak{a}^0$ one has

$$\gamma_{i,j}(H) = H - \frac{\lambda_i(H) - \lambda_j(H)}{2}(H_{\lambda_i} - H_{\lambda_j}),$$

and this gives the relations (1).
For $\{s,l\} \cap \{i,j\} = \emptyset$ and $p, q \in \{\pm 1\}$, one has $[E_{s,l}(p,q), E_{i,j}(p,q)] = \{0\}$, therefore the action $\gamma_{i,j}$ is trivial on the spaces $E_{s,l}(p,q)$, and in particuliar of $\widetilde{\mathfrak{g}}^{\lambda_s}$ for $s \notin \{i,j\}$. The relations (1) imply that $\gamma_{i,j}$ is an isomorphism from $\widetilde{\mathfrak{g}}^{\lambda_i}$ onto $\widetilde{\mathfrak{g}}^{\lambda_j}$. It is an involution because the action of $\gamma_{i,j}^2$ on the even weight spaces of $H_{\lambda_i} - H_{\lambda_j}$ is trivial (section 3.1).

□

It is worth noting that the action of $\gamma_{i,j}^2$ on $\widetilde{\mathfrak{g}}$ is not trivial . Indeed, if X is in an odd weight space for $H_{\lambda_i} - H_{\lambda_j}$ then $\gamma_{i,j}^2(X) = -X$. Therefore, on order to obtain an involution, we will modify $\gamma_{i,j}$. This is the purpose of the next proposition.

Proposition 3.4.4. *For $i \neq j \in \{0, \dots, k\}$, the element $\widetilde{\gamma_{i,j}} = \gamma_{i,j} \circ w_i^2$ belongs to $N_{\widetilde{G}}(\mathfrak{a}^0)$ and it verifies the following relations:*

$$\widetilde{\gamma_{i,j}}(H_{\lambda_s}) = \begin{cases} H_{\lambda_i} & \text{for } s = j \\ H_{\lambda_j} & \text{for } s = i \\ H_{\lambda_s} & \text{for } s \notin \{i,j\} \end{cases}$$

and

$$\widetilde{\gamma_{i,j}}^2 = \mathrm{Id}_{\widetilde{\mathfrak{g}}}.$$

Proof. By section 3.1, the action of w_i^2 on the spaces $E_{i,s}(\pm 1, \pm 1)$ (for $s \neq i$) is the scalar multiplication by -1, and is trivial on the on the sum of the other spaces. Moreover, the action of $\gamma_{i,j}$ on the spaces $E_{s,l}(\pm 1, \pm 1)$ with $\{s,l\} \cap \{i,j\} = \emptyset$ is trivial, and is an isomorphism from $E_{i,s}(\pm 1, \pm 1)$ onto $E_{j,s}(\pm 1, \pm 1)$. Therefore one obtains

$$\widetilde{\gamma_{i,j}}^2(X) = \begin{cases} -{\gamma_{i,j}}^2(X) & \text{for } X \in \oplus_{s \notin \{i,j\}} E_{i,s}(\pm 1, \pm 1) \oplus E_{j,s}(\pm 1, \pm 1) \\ {\gamma_{i,j}}^2(X) & \text{for } X \in \oplus_{\{s,l\} \cap \{i,j\} = \emptyset} E_{s,l}(\pm 1, \pm 1). \end{cases}$$

As the subspace $\oplus_{s \notin \{i,j\}} E_{i,s}(\pm 1, \pm 1) \oplus E_{j,s}(\pm 1, \pm 1)$ of the odd weightspaces for $H_{\lambda_i} - H_{\lambda_j}$, the element $\gamma_{i,j}^2$ acts by -1 on it, and this ends the proof. □

3.5. **Quadratic forms.**

Remind that the quadratic form b is a normalization of the Killing form (Definition 1.10.1). For $X \in V^+$, let Q_X be the quadratic form on V^- defined by

$$Q_X(Y) = b(e^{\operatorname{ad} X} Y, Y), \quad Y \in V^-.$$

If $g \in G$, the quadratic forms Q_X and $Q_{g.X}$ are equivalent.
Therefore we will study the quadratic form Q_X for $X \in \oplus_{j=0}^{k}\widetilde{\mathfrak{g}}^{\lambda_j}$.
The grading of $\widetilde{\mathfrak{g}}$ is orthogonal for $\widetilde{B}$ and hence also for b. One obtains

$$Q_X(Y) = \frac{1}{2}b((\operatorname{ad} X)^2Y, Y) = -\frac{1}{2}b([X,Y],[X,Y])$$

Let

$$X = \sum_{j=0}^{k} X_j, \quad X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}.$$

Let $E_{s,l}(-1,-1) \subset V^-$. The action of $\operatorname{ad} X_i \operatorname{ad} X_j$ on $E_{s,l}(-1,-1)$ is non zero if and only if $(s,l) = (i,j)$ where $i \neq j$ or $s = l = i = j$. The quadratic forms q_{X_i,X_j} on $E_{i,j}(-1,-1)$ (*resp.* $\widetilde{\mathfrak{g}}^{\lambda_j}$) for $i \neq j$ (*resp.* $i = j$) are defined by

$$q_{X_i,X_j}(Y) = -\frac{1}{2}b([X_i,Y],[X_j,Y]), \quad \text{for } Y \in E_{i,j}(-1,-1)(\textit{resp. } \widetilde{\mathfrak{g}}^{\lambda_j}).$$

The decomposition of V^- implies that the quadratic form Q_X is equal to

$$(\oplus_{j=0}^{k} q_{X_j,X_j}) \oplus (\oplus_{i<j}^{k} 2\, q_{X_i,X_j}).$$

Theorem 3.5.1. *Let $X = \sum_{j=0}^{k} X_j$ where $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$. Let $i,j \in \{0,\dots,k\}$.*

(1) *If $X_i \neq 0$ and $X_j \neq 0$, then q_{X_i,X_j} is non degenerate.*
(2) *Let m be the number of indices i such that $X_i \neq 0$. Then*

$$\operatorname{rank} Q_X = m\ell + \frac{m(m-1)}{2}d,$$

where $\ell = \dim \widetilde{\mathfrak{g}}^{\lambda_i}$ and $d = \dim E_{i,j}(-1,1)$ for $i \neq j$.

Proof. The bilinear form associated to q_{X_i,X_j} is given by

$$L_{X_i,X_j}(u,v) = -\frac{1}{2}b([X_i,u],[X_j,v]), \quad u,v \in E_{i,j}(-1,-1)(\textit{resp. } \widetilde{\mathfrak{g}}^{\lambda_j}) \text{ for } i \neq j (\textit{resp. } i = j).$$

If $i = j$, $(\operatorname{ad} X_i)^2$ is an isomorphism from $\widetilde{\mathfrak{g}}^{-\lambda_i}$ onto $\widetilde{\mathfrak{g}}^{\lambda_i}$. As the form b (proportional to $\widetilde{B}$) is non degenerate on $\widetilde{\mathfrak{g}}^{-\lambda_i} \times \widetilde{\mathfrak{g}}^{\lambda_i}$, the form L_{X_i,X_i} is non degenerate on $\widetilde{\mathfrak{g}}^{\lambda_i}$.

If $i \neq j$, let us consider two roots λ and μ in the decomposition of $E_{i,j}(-1,-1)$. As $\operatorname{ad} X_j(\widetilde{\mathfrak{g}}^{\mu}) \subset \widetilde{\mathfrak{g}}^{\mu+\lambda_j}$ and $\operatorname{ad} X_i(\widetilde{\mathfrak{g}}^{\lambda}) \subset \widetilde{\mathfrak{g}}^{\lambda+\lambda_i}$, the restriction of L_{X_i,X_j} to $\widetilde{\mathfrak{g}}^{\lambda} \times \widetilde{\mathfrak{g}}^{\mu}$ is non zero if and only if $\lambda + \lambda_i = -(\mu + \lambda_j)$, that is if and only if $\mu = -w_{i,j}(\lambda)$.
Let $u \in \widetilde{\mathfrak{g}}^{\lambda}$ such that, for all $v \in \widetilde{\mathfrak{g}}^{-w_{i,j}(\lambda)}$, one has $L_{X_i,X_j}(u,v) = 0$. By section 3.1, $\operatorname{ad} X_i$ is an isomorphism from $\widetilde{\mathfrak{g}}^{\lambda} \subset E_{i,j}(-1,-1)$ onto $\widetilde{\mathfrak{g}}^{\lambda+\lambda_i} \subset E_{i,j}(1,-1)$ and $\operatorname{ad} X_j$ is an isomorphism from $\widetilde{\mathfrak{g}}^{-w_{i,j}(\lambda)} \subset E_{i,j}(-1,-1)$ onto $\widetilde{\mathfrak{g}}^{-(\lambda+\lambda_i)} \subset E_{i,j}(-1,1)$. As the restriction of b to $\widetilde{\mathfrak{g}}^{\lambda+\lambda_i} \times \widetilde{\mathfrak{g}}^{-(\lambda+\lambda_i)}$ is non degenerate, we get $\operatorname{ad} X_i(u) = 0$ and hence $u = 0$. This proves the first statement
The second statement is an immediate consequence of the formula $Q_X = (\oplus_{j=0}^{k} q_{X_j,X_j}) \oplus (\oplus_{i<j}^{k} 2\, q_{X_i,X_j})$ seen before.

□

Proposition 3.5.2. *There exist $\mathfrak{sl}_2$-triples $\{Y_s, H_{\lambda_s}, X_s\}$, $s \in \{0,\dots,k\}$, such that, for $i \neq j$, the quadratic forms q_{X_i,X_j} are all G-equivalent (this means that there exists $g \in G$ such that $q_{X_0,X_1} = q_{X_i,X_j} \circ g$), and such that each of the forms q_{X_i,X_j} represents 1 (i.e. there exists $u \in E_{i,j}(-1,-1)$ such that $q_{X_i,X_j}(u) = 1$).*

Moreover, if $\ell = 1$, these forms satisfy the following conditions (remember that $e = \dim \tilde{\mathfrak{g}}^{(\lambda_i+\lambda_j)/2}$ and $d = \dim E_{i,j}(-1,1)$ for $i \neq j$):

(1) *If $e \neq 0$, then the restriction of q_{X_i,X_j} to $\tilde{\mathfrak{g}}^{-(\lambda_i+\lambda_j)/2}$ is anisotropic of rank e.*

(2) *If $d-e \neq 0$, and if $W_{i,j}(-1,-1)$ denotes the direct sum of the spaces $\tilde{\mathfrak{g}}^{-\mu} \subset E_{i,j}(-1,-1)$ where $\mu \in \tilde{\Sigma}$ and $\mu \neq (\lambda_i+\lambda_j)/2$, then the restriction of q_{X_i,X_j} to $W_{i,j}$ is hyperbolic of rank $d-e$ (and therefore $d-e$ is even).*

Proof. Let $X_0 \in \widetilde{\mathfrak{g}}^{\lambda_0}$. We fix a $\mathfrak{sl}_2$-triple $\{Y_0, H_{\lambda_0}, X_0\}$. Let $j \in \{1,\dots,k\}$. We choose the maps $\gamma_{0,j}$ such that Proposition 3.4.3 is satisfied and we set $Y_j = \gamma_{0,j}Y_0$ and $X_j = \gamma_{0,j}X_0$. Then $\{Y_j, H_{\lambda_j}, X_j\}$ is an $\mathfrak{sl}_2$- triple and for $i \neq j$, we have $\gamma_{0,i}(X_j) = X_j$. Then, for $Y \in E_{i,j}(-1,-1)$, we obtain

$$q_{X_i,X_j}(Y) = -\frac{1}{2}b([\gamma_{0,i}(X_0), Y], [X_j, Y])$$
$$= -\frac{1}{2}b([X_0, \gamma_{0,i}^{-1}(Y)], [X_j, \gamma_{0,i}^{-1}(Y)]) = q_{X_0,X_j}(\gamma_{0,i}^{-1}(Y)).$$

Therefore q_{X_i,X_j} is equivalent to $q_{X_0,X_j}$for all $j \neq 0$ and $j \neq i$.

Let $j \geq 2$. One has $X_j = \gamma_{0,1}(X_j) = \gamma_{0,1}\gamma_{0,j}(X_0)$. As the restriction of $\gamma_{0,1}^2$ to $\widetilde{\mathfrak{g}}^{\lambda_0}$ is the identity, one has $X_0 = \gamma_{0,1}^2(X_0) = \gamma_{0,1}(X_1)$ and hence $X_j = \gamma_{0,1}\gamma_{0,j}\gamma_{0,1}(X_1)$. As $g = \gamma_{0,1}\gamma_{0,j}\gamma_{0,1}$ fixes X_0, one obtains, for $Y \in E_{0,j}(-1,-1)$,

$$q_{X_0,X_j}(Y) = -\frac{1}{2}b([X_0, g^{-1}Y], [X_1, g^{-1}Y]) = q_{X_0,X_1}(g^{-1}Y).$$

Therefore q_{X_0,X_j} is equivalent to q_{X_0,X_1}.

Let us now prove that q_{X_0,X_1} represents 1. We fix an $\mathfrak{sl}_2$-triple $\{Y, H_{\lambda_0} - H_{\lambda_1}, X\}$ with $X \in E_{0,1}(1,-1)$, $Y \in E_{0,1}(-1,1)$ such that $\gamma_{0,1} = e^{\operatorname{ad} X}e^{\operatorname{ad} Y}e^{\operatorname{ad} X}$. As X_0 is of weight 2 for the action of this $\mathfrak{sl}_2$-triple , one has $\gamma_{0,1}(X_0) = \dfrac{1}{2}(\operatorname{ad} Y)^2(X_0)$. From the normalization of b (Lemme 1.10.2), we get

$$1 = b(X_1, Y_1) = b(\gamma_{0,1}(X_0), Y_1) = -\frac{1}{2}b([Y, X_0], [Y, Y_1]).$$

Set $Z = [Y_1, Y] \in E_{0,1}(-1,-1)$. Using the Jacobi identity, one has $[X_0, Z] = \operatorname{ad}(Y_1)([X_0, Y])$ and $[X_1, Z] = -[Y_1, [Y, X_1]] - [Y, [X_1, Y_1]] = -Y$. Therefore

$$q_{X_0,X_1}(Z) = -\frac{1}{2}b([X_0, Z], [X_1, Z]) = -\frac{1}{2}b(\operatorname{ad}(Y_1)([X_0, Y]), -Y)$$
$$= -\frac{1}{2}b([X_0, Y], [Y_1, Y]) = 1.$$

Suppose now that $\ell = 1$.

If $e \neq 0$ then $\mu = (\lambda_i + \lambda_j)/2$ is a root, and its coroot is $H_{\lambda_i} + H_{\lambda_j}$. Let Y be a non zero element in $\tilde{\mathfrak{g}}^{-(\lambda_i+\lambda_j)/2}$. Let $\{Y, H_{\lambda_i} + H_{\lambda_j}, X\}$ be an $\mathfrak{sl}_2$-triple with $X \in \tilde{\mathfrak{g}}^{(\lambda_i+\lambda_j)/2}$ and denote by $w = e^{\operatorname{ad} X}e^{\operatorname{ad} Y}e^{\operatorname{ad} X}$ the non trivial Weyl group element associated to this $\mathfrak{sl}_2$-triple . As X_i is if weight 2 for the action of this $\mathfrak{sl}_2$-triple, one has $wX_i = \dfrac{(\operatorname{ad} Y)^2}{2}X_i$ and wX_i is a non zero element of $\tilde{\mathfrak{g}}^{\lambda_j}$. As $\ell = 1$, there exists $a \in F^*$ such that $wX_i = aY_j$. Therefore, we get

$$q_{X_i,X_j}(Y) = \frac{1}{2}b(\operatorname{ad}(Y))^2X_i, X_j) = b(w.X_i, X_j) = a\, b(Y_j, X_j) = a \neq 0.$$

Hence the restriction of q_{X_i,X_j} to $\tilde{\mathfrak{g}}^{-(\lambda_i+\lambda_j)/2}$ is anisotropic.

We prove now the last assertion. We have $d > e$. For $s = i$ or j, we note $w_s = e^{\operatorname{ad} X_s}e^{\operatorname{ad} Y_s}e^{\operatorname{ad} X_s}$

and $w_{i,j} = w_i w_j = w_j w_i$. Let $\mu \in \tilde{\Sigma}$ such that $\mu \neq (\lambda_i + \lambda_j)/2$ and such that $\tilde{\mathfrak{g}}^{\mu} \subset E_{i,j}(1,1)$. From Remark 3.3.3, we have $w_{i,j}\mu = \mu - \lambda_i - \lambda_j$. Hence $\mu' = -w_{i,j}\mu$ is a root, distinct from μ (because $\mu' = \mu$ would imply $\mu = (\lambda_i + \lambda_j)/2$ and this is not the case) such that $\mathfrak{g}^{-\mu'} \subset E_{i,j}(-1,-1)$. Let us fix an $\mathfrak{s}l_2$-triple $\{X_{-\mu}, H_\mu, X_\mu\}$ where $X_{\pm\mu} \in \tilde{\mathfrak{g}}^{\pm\mu}$. Applying $w_{i,j}$ we obtain the $\mathfrak{s}l_2$-triple $\{w_{i,j}X_\mu, H_{\mu'}, w_{i,j}X_{-\mu}\}$ where $w_{i,j}X_\mu \in \tilde{\mathfrak{g}}^{-\mu'}$. Using Lemma 3.3.1, we obtain

$$q_{X_i,X_j}(X_{-\mu}) = -\frac{1}{2}b([X_i, X_{-\mu}], [X_j, X_{-\mu}]) = -\frac{1}{2}b(w_iX_{-\mu}, w_jX_{-\mu}) = -\frac{1}{2}b(X_{-\mu}, w_{i,j}X_{-\mu}) = 0$$

and

$$q_{X_i,X_j}(w_{i,j}X_\mu) = -\frac{1}{2}b([X_i, X_\mu], [X_j, X_\mu]) = -\frac{1}{2}b(w_iX_\mu, w_jX_\mu) = -\frac{1}{2}b(X_\mu, w_{i,j}X_\mu) = 0.$$

This implies that the restriction of q_{X_i,X_j} to the vector space generated by $X_{-\mu}$ and $w_{i,j}X_\mu$ is a hyperbolic plane. This ends the proof.

□

Remark 3.5.3. Rather than the forms q_{X_i,X_j} which were already used in the real case by N. Bopp and H. Rubenthaler ([7]), I . Muller ([14],[17]) introduced the quadratic form f_{λ_i,λ_j} on $E_{i,j}(-1,1)$ defined as follows: if $(Y_i, H_{\lambda_i}, X_i)$ is an $\mathfrak{s}l_2$-triple then

$$f_{\lambda_i,\lambda_j}(u) = -\frac{1}{2}b([u, X_i], [u, Y_j]).$$

From the preceding proof we see that

$$f_{\lambda_i,\lambda_j}(u) = q_{X_i,X_j}([Y_j, u]), \quad u \in E_{i,j}(-1,1).$$

3.6. **Reduction to the diagonal. Rank of an element.**

Proposition 3.6.1. *For any element Z in V^+, there exists a unique $m \in \mathbb{N}$, and for $j = 0, \ldots, m-1$ there exist non zero elements $Z_j \in \tilde{\mathfrak{g}}^{\lambda_j}$ (non unique) such that Z is G-conjugated to $Z_0 + Z_1 + \ldots + Z_{m-1}$, or, equivalently, Z is G-conjugated to a generic element of V^+_{k-m+1}.*

Proof. By Theorem 3.2.2, any non zero element of V^+ is G-conjugated to an element of the form

$$Y = Y_0 + Y_1 + \ldots + Y_k, \quad \text{with}\, Y_j \in \tilde{\mathfrak{g}}^{\lambda_j}.$$

Let m be the number of indices j such that $Y_j \neq 0$. For $i \neq j$, the element $\gamma_{i,j} \in N_G(\mathfrak{a}^0)$ obtained in Proposition 3.4.3 exchanges $\tilde{\mathfrak{g}}^{\lambda_i}$ and $\tilde{\mathfrak{g}}^{\lambda_j}$ and fixes $\tilde{\mathfrak{g}}^{\lambda_s}$ for $s \notin \{i,j\}$. Therefore Y is conjugated either to an element of the form

$$Z_0 + Z_1 + \ldots + Z_{m-1}, \quad \text{where}\, Z_j \in \tilde{\mathfrak{g}}^{\lambda_j} \backslash \{0\},$$

or, equivalently, to an element of the form $Y_{k-m+1} + \ldots + Y_k$ where $Y_j \in \tilde{\mathfrak{g}}^{\lambda_j} \backslash \{0\}$, that is to a generic element of V^+_{k-m+1}.

Let us now show that for $m \neq m'$, the elements $Z = Z_0 + Z_1 + \ldots + Z_m$ and $Z' = Z'_0 + Z'_1 + \ldots + Z'_{m'}$ where the Z_j's and Z'_j's are non zero in $\tilde{\mathfrak{g}}^{\lambda_j}$, are not G-conjugate. If they were, the quadratic forms Q_Z and $Q_{Z'}$ would have the same rank. But, according to Theorem 3.5.1, one has

$$\operatorname{rang} Q_Z - \operatorname{rang} Q_{Z'} = (m - m')\Big(\ell + \frac{m - m'}{2}d\Big).$$

Hence $\operatorname{rang} Q_Z \neq \operatorname{rang} Q_{Z'}$ if $m \neq m'$.

□

Definition 3.6.2. *For $Z \in V^+$, the rank of Z is defined to be the integer m appearing in the preceding Lemma.*

Remember from Notation 1.9.4 that ℓ is the common dimension of the spaces $\tilde{\mathfrak{g}}^{\lambda_i}$ and that e is the common dimension of the spaces $\tilde{\mathfrak{g}}^{(\lambda_i+\lambda_j)/2}$. The structure of the G-orbits in V^+ depends on the integers (ℓ, e).
In the next (sub)sections, we will completely describe the G-orbits in V^+, not only the open one, see Theorem 3.7.1, Theorem 3.8.8, Theorem 3.8.9, Theorem 3.8.10 and Theorem 3.9.8 below.
But as the open orbits will be of particular interest for our purpose, let us summarize here our results concerning the number of these orbits (this is a Corollary of the results obtained in the following sections):

Theorem 3.6.3.
(1) *If $\ell = \delta^2$, $\delta \in \mathbb{N}^*$ and $e = 0$ or 4 (i.e. if $\tilde{\mathfrak{g}}$ is of type I), the group G has a unique open orbit in V^+,*
(2) *if $\ell = 1$ and $e \in \{1, 2, 3\}$ (i.e. if $\tilde{\mathfrak{g}}$ is of type II), the number of open G-orbits in V^+ depends on e and on the parity of k:*
(a) *if $e = 2$ then G has a unique open orbit in V^+ if k is even and 2 open orbits if k is odd,*
(b) *if $e = 1$ then G has a unique open orbit in V^+ if $k = 0$, it has 4 open orbits if $k = 1$, it has 2 open orbits if $k \geq 2$ is even , and 5 open orbits if $k \geq 2$ is odd .*
(c) *if $e = 3$, then G has 4 open orbits.*
(3) *If $\ell = 3$ (i.e. if $\tilde{\mathfrak{g}}$ is of type III, in that case $e = d = 4$), the group G has 3 open orbits in V^+ if $k = 0$ and 4 open orbits if $k \geq 1$.*

We know from Remark 1.8.8 that we can always assume that $\tilde{\mathfrak{g}}$ is simple. This will be the case in the sequel of the paper.

3.7. G-orbits in the case where $(\ell, d, e) = (\delta^2, 2\delta^2, 0)$ (Case (1) in Table 1).

Theorem 3.7.1. *If $(\ell, d, e) = (\delta^2, 2\delta^2, 0)$, then $\chi_0(G) = F^*$ and the group G has exactly $k+1 =$ rank$(\tilde{\mathfrak{g}})$ non zero orbits in V^+. These orbits are characterized by the rank of their elements, and a set of representatives is given by the elements $X_0 + \ldots + X_j$ ($j = 0, \ldots, k$) where the X_j's are non zero elements of $\tilde{\mathfrak{g}}^{\lambda_j}$. Any two generic elements of $\oplus_{j=0}^{k} \tilde{\mathfrak{g}}^{\lambda_j}$ are conjugated by the subgroup $L = Z_G(\mathfrak{a}^0)$.*

Proof. From the classification (cf. Table 1 - (1)), we can suppose that $\tilde{\mathfrak{g}} = \mathfrak{sl}(2(k+1), D)$ where D is a central division algebra of degree δ over$F$, graded by the element $H_0 = \begin{pmatrix} I_{k+1} & 0 \\ 0 & -I_{k+1} \end{pmatrix}$.
Then V^+ is isomorphic to the matrix space $M(k+1, D)$ through the map

$$B \mapsto X(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

The maximal split abelian subalgebra $\mathfrak{a}$ is the the set $H(\phi_0,\dots,\phi_{2k+1}) = diag(\phi_0,\dots,\phi_{2k+1})$ where the ϕ_j's belong to F and the maximal set of strongly orthogonal roots associated to this grading is given by $\lambda_j(H(\phi_0,\dots,\phi_{2k+2})) = \phi_{k+1-j} - \phi_{k+2+j}$ for $j \in \{0,\dots k+1\}$.

Let us denote by ν the reduced norm of the simple central algebra $M(k+1,D)$. Remember that if E is a splitting field for $M(k+1,D)$, then $M(k+1,D)\otimes E \simeq M((k+1)\delta,E)$, and if $\varphi : M(k+1,D) \longrightarrow M((k+1)\delta,E)$ is the canonical embedding, then $\nu(x) = \det(\varphi(x))$, for $x \in M(k+1,D)$ (see for example, Proposition IX.6 p.168 in [31]). Also if $x = (x_{i,j})$ is a triangular matrix in $M(k+1,D)$ and if ν_0 denotes the reduced norm of D, then $\nu(x) = \prod_{i=1}^{i=k+1}\nu_0(x_{i,i})$ ([31], Corollary IX.2 p.169).

Let us now describe the group $G = \mathcal{Z}_{\mathrm{Aut}_0(\mathfrak{sl}(2(k+1),D))}(H_0)$.
Consider first an element $g \in \mathrm{Aut}_0(\mathfrak{sl}(2(k+1),D))$ and denote by g_E the natural extension of g to $\mathfrak{sl}(2(k+1),D)\otimes E = \mathfrak{sl}(2(k+1)\delta,E)$. We know from [4] (Chap. VIII, §13, $n°1$, (VII), p.189), that there exists $U \in GL(2(k+1)\delta,E)$ such that $g_E.x = UxU^{-1}$ for all $x \in \mathfrak{sl}(2(k+1)\delta,E)$. Let us write U in the form

$$U = \begin{pmatrix} u_1 & u_3 \\ u_4 & u_2 \end{pmatrix}, \quad u_j \in M((k+1)\delta,E).$$

Let now $g \in G$. As $g.H_0 = H_0$, we have $g.V^- \subset V^-$, $g.V^+ \subset V^+$and $g.\mathfrak{g} \subset \mathfrak{g}$. Then g_E stabilizes $V^+\otimes E$, $V^-\otimes E$ and $\mathfrak{g}\otimes E$, and a simple computation shows that $u_3 = u_4 = 0$. On the other hand, the algebra $\mathfrak{g}$ is the set of matrices $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathfrak{sl}(2(k+1),D)$ with $X,Y \in M(k+1,D)$. As g_E also stabilizes $\mathfrak{g}$, the maps $X \mapsto u_1Xu_1^{-1}$ and $Y \mapsto u_2Yu_2^{-1}$ are automorphisms of $M(k+1,D)$, for the ordinary associative product.
By the Skolem-Noether Theorem, any automorphism of $M(k+1,D)$ is inner, hence it exists v_1 and v_2 in $GL(k+1,D)$ such that $u_1Xu_1^{-1} = v_1Xv_1^{-1}$ and $u_2Xu_2^{-1} = v_2Xv_2^{-1}$ for all $X \in M(k+1,D)$. Therefore $v_1^{-1}u_1$ and $v_2^{-1}u_2$ belong to the center of $M((k+1)\delta,E)$. It follows that there exist λ_1 and λ_2 in E^* such that $u_1 = \lambda_1v_1$ and $u_2 = \lambda_2v_2$. Hence the automorphism g is given by the conjugation by $U = \begin{pmatrix} \lambda_1v_1 & 0 \\ 0 & \lambda_2v_2 \end{pmatrix}$ and its action on V^+ is given by $g.X(Z) = X(\lambda_1\lambda_2^{-1}v_1Zv_2^{-1})$ for $Z \in M(k+1,D)$. As g stabilizes V^+ this implies that $\lambda_1\lambda_2^{-1} \in F^*$ and g is given by the conjugation by $diag(g_1,g_2) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ where $g_1 = \lambda_1\lambda_2^{-1}v_1 \in GL(k+1,D)$ and $g_2 = v_2 \in GL(k+1,D)$.

The polynomial Δ_0 defined by $\Delta_0(X(Z)) = \nu(Z)$ is then relatively invariant under G and it character is $\chi_0(g) = \Delta_0(g_1g_2^{-1})$ for $g = diag(g_1,g_2)$. Therefore $\chi_0(G) = F^*$.

The group $L = \cap_{j=0}^k G_{H_{\lambda_j}}$ corresponds to the action of the elements $diag(g_1,g_2)$ where g_1 and g_2 are diagonal matrices with coefficients in D^*.

From Proposition 3.6.1, any element in V^+ is conjugated to an non zero element $Z \in \oplus_{j=0}^k\tilde{\mathfrak{g}}^{\lambda_j}$. Such an element corresponds to a matrix of $M(k+1,D)$ whose coefficients are zero except those on the 2nd diagonal. This shows that the group L acts transitively on $\oplus_{j=0}^k\tilde{\mathfrak{g}}^{\lambda_j}$. □

3.8. **G-orbits in the case $\ell = 1$.**

In this section, we will always assume that $\ell = 1$.
If $k = 0$ the G-orbits in V^+ were already described in Theorem 1.12.4. Therefore we suppose that $k \geq 1$.

If $\ell = 1$, the G-orbits in V^+ were studied by I. Muller in a more general context (the so-called quasi-commutative prehomogeneous vector spaces), see [14]. Our results, are more precise than hers in the sense that we obtain explicit representatives for the orbits, and also we give detailed proofs.

For $j \in \{0, \ldots, k\}$, we fix $\mathfrak{sl}_2$-triples $\{Y_j, H_{\lambda_j}, X_j\}$ which satisfy the conditions of proposition 3.5.2. The quadratic forms q_{X_i,X_j} are then G-equivalent. We take $q = q_{X_0,X_1}$ as a representative of this equivalence class. Then $q = q_{an}^e + q_{hyp}$ where q_{an}^e is an anisotropic quadratic form of rank e ($q_{an}^0 = 0$), and where q_{hyp} is a hyperbolic quadratic form of rank $d - e$.
We set $Im(q)^* = Im(q) \cap F^*$. Let $a \in F^*$. As q represents 1, if q is equivalent to aq, (which will be denoted $aq \sim q$), then $a \in Im(q)^*$.

Remember (Corollary 1.7.8) that

$$G = \mathcal{Z}_{\mathrm{Aut}_0(\tilde{\mathfrak{g}})}(H_0) = \mathrm{Aut}_e(\mathfrak{g}).L \quad \text{where} \quad L = Z_G(\mathfrak{a}^0).$$

Definition 3.8.1. *For $t \in F^*$ and $j \in \{0, \ldots, k\}$, we define the following elements of G:*

$$\theta_{X_j}(t) = e^{t\ \mathrm{ad}_{\tilde{\mathfrak{g}}} X_j} e^{t^{-1}\ \mathrm{ad}_{\tilde{\mathfrak{g}}} Y_j} e^{t\ \mathrm{ad}_{\tilde{\mathfrak{g}}} X_j}, \quad \theta_{X_j} = \theta_{X_j}(1) \quad \text{and} \quad h_{X_j}(t) = \theta_{X_j}(t)\theta_{X_j}(-1)$$

If $\{Y, H_0, X\}$ is an $\mathfrak{sl}_2$-triple where X is generic in V^+ and $Y \in V^-$, we set also

$$\theta_X(t) = e^{t\ \mathrm{ad}_{\tilde{\mathfrak{g}}} X} e^{t^{-1}\ \mathrm{ad}_{\tilde{\mathfrak{g}}} X} e^{t\ \mathrm{ad}_{\tilde{\mathfrak{g}}} X}, \quad \text{and} \quad h_X(t) = \theta_X(t)\theta_X(-1).$$

Then $h_X(\sqrt{t}) \in L$ and acts by $t.Id_{V^+}$ on V^+ (see Lemma 1.11.3).

Lemma 3.8.2. *([14], Proposition 3.2 page 175).*

1. *$F^{*2} \subset \chi_0(G) = \chi_0(L)$.*
2. *If $k + 1$ is odd then $\chi_0(G) = F^*$.*
3. *If $k + 1$ is even then $\chi_0(G) \subset \{a \in F^*; aq \sim q\} \subset Im(q)^*$.*

Proof.
(1) We know from Theorem 1.14.2 (2), that the character χ_0 is trivial on $\mathrm{Aut}_e(\mathfrak{g})$ and hence $\chi_0(G) = \chi_0(L)$.

Let $g = h_{X_0}(t)$. From [4] (Chap VIII §1 $n°1$, Proposition 6 p. 75), we have $g(X_0 + \ldots + X_k) = t^2 X_0 + X_1 + \ldots + X_k$. As $\ell = 1$, we obtain by Theorem 1.13.2 that $\Delta_0(t^2 X_0 + X_1 + \ldots + X_k) = t^2 \Delta_0(X_0 + X_1 + \ldots + X_k)$. Therefore $\chi_0(g) = t^2 \in F^{*2}$, and this proves the first assertion.

(2) Consider the generic element $X = X_0 + \ldots + X_k \in V^+$ and $g = h_X(\sqrt{t})$. Theorem 1.13.2 implies $\chi_0(g) = t^{k+1}$ and this proves the second assertion.

(3) Let $g \in L$. Then g stabilizes each of the spaces $\tilde{\mathfrak{g}}^{\lambda_j}$ and $E_{i,j}(\pm 1, \pm 1)$. As $\ell = 1$, there exist scalars $a_j(g) \in F^*$ such that $g.X_j = a_j(g)X_j$. Theorem 1.13.2 implies then $\chi_0(g) = \prod_{j=0}^{k} a_j(g)$.

On the other hand one gets easily that $q_{g.X_j,g.X_{k-j}} = a_j(g)a_{k-j}(g)q_{X_j,X_{k-j}}$. As $q_{g.X_j,g.X_{k-j}}$ and $q_{X_j,X_{k-j}}$ are G - equivalent to q, we get $a_j(g)a_{k-j}(g)q \sim q$. As $k+1$ is even, the scalar $\chi_0(g) = \prod_{j=0}^{(k-1)/2} a_j(g)a_{k-j}(g)$ is such that $\chi_0(g)q \sim q$. This gives the third assertion. $\square$

Lemma 3.8.3. *(compare with [14], lemme 2.2.2 page 168). Let $A \in E_{i,j}(-1,1)$. Then there exists $B \in E_{i,j}(1,-1)$ such that $\{B, H_{\lambda_j} - H_{\lambda_i}, A\}$ is an $\mathfrak{sl}_2$-triple if and only if the map $\mathrm{ad}(A)^2 : \tilde{\mathfrak{g}}^{\lambda_i} \to \tilde{\mathfrak{g}}^{\lambda_j}$ is injective (remember from Lemma 3.4.2 that such an $\mathfrak{sl}_2$-triple always exists).*

In that case, the element $\theta_A = e^{\mathrm{ad}\,A}e^{\mathrm{ad}\,B}e^{\mathrm{ad}\,A}$ of $\mathrm{Aut}_e(\mathfrak{g}) \subset G$ satisfies $\theta_A(X_s) = X_s$ for $s \neq i,j$, $\theta_A(X_i) = aX_j$ and $\theta_A(X_j) = a^{-1}X_i$ where $a = q_{X_i,X_j}([Y_j, A])$.

Proof. It is a well known property of $\mathfrak{sl}_2$-triples, that if such a triple exists the map $\mathrm{ad}(A)^2 : \tilde{\mathfrak{g}}^{\lambda_i} \to \tilde{\mathfrak{g}}^{\lambda_j}$ is injective.

Conversely, suppose that the map $\mathrm{ad}(A)^2 : \tilde{\mathfrak{g}}^{\lambda_i} \to \tilde{\mathfrak{g}}^{\lambda_j}$ is injective, and hence bijective. As A is nilpotent in $\mathfrak{g}$ (more precisely $(\mathrm{ad}\,A)^3 = 0$), the Jacobson-Morosov Theorem gives the existence of an $\mathfrak{sl}_2$-triple $\{B, u, A\}$ in $\mathfrak{g}$. If one decomposes the element B and u according to $\mathfrak{g} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus (\oplus_{r\neq s} E_{r,s}(1,-1))$, we easily see that one can suppose that $B \in E_{i,j}(1,-1)$ and $u \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}^0)$.
We will show that $u = H_{\lambda_j} - H_{\lambda_i}$.
As u commutes with the elements H_{λ_s}, the endomorphism $\mathrm{ad}\,u$ stabilizes all eigenspaces of these elements. Therefore, as $\ell = 1$, there exists $\alpha \in F$ such that $[u, X_i] = \alpha X_i$.

As $[B, X_i] = 0 = (\mathrm{ad}\;\; A)^3 X_i$ and as by hypothesis $\mathrm{ad}(A)^2 X_i \neq 0$, the $\mathfrak{sl}_2$ - module generated by X_i under the action of $\{B, u, A\}$ is irreducible of dimension 3 and a base of this module is $(X_i, [A, X_i], \mathrm{ad}(A)^2 X_i)$. This implies that $\alpha = -2$.
Let $\theta_A = e^{\mathrm{ad}\,A}e^{\mathrm{ad}\,B}e^{\mathrm{ad}\,A}$ be the non trivial element of the Weyl group of the $\mathfrak{sl}_2$-triple $\{B, u, A\}$. Then (cf. §3.1):

$$\theta_A(X_i) = \frac{1}{2}\mathrm{ad}(A)^2 X_i \in \tilde{\mathfrak{g}}^{\lambda_j}.$$

Hence, there exists $a \in F^*$ such that $\theta_A(X_i) = aX_j$. As $0 \neq B(X_i, Y_i) = B(\theta_A(X_i), \theta_A(Y_i))$, we have $\theta_A(Y_i) = a^{-1}Y_j$. And as $\{\theta_A(Y_i), \theta_A(H_{\lambda_i}), \theta_A(X_i)\}$ is again an $\mathfrak{sl}_2$-triple, we obtain that $\theta_A(H_{\lambda_i}) = H_{\lambda_j}$.

On the other hand, a simple computation shows that $\theta_A(H_{\lambda_s}) = H_{\lambda_s}$ if $s \neq i,j$ and $\theta_A(H_{\lambda_i}) = H_{\lambda_i} + u$. Hence $u = H_{\lambda_j} - H_{\lambda_i}$, which gives the first assertion of the Lemma.

By Remark 3.5.3 and the normalization of b (Lemma 1.10.2), we get

$$q_{X_i,X_j}([Y_j, A]) = -\frac{1}{2}b([A, X_i], [A, Y_j]) = b(\frac{1}{2}(\mathrm{ad}\,A)^2 X_i, Y_j) = b(\theta_A(X_i), Y_j) = a\, b(X_j, Y_j) = a.$$

This ends the proof. $\square$

Corollary 3.8.4. *(Compare with [14] Corollaire 4.2.2 and Remarques 4.1.6)*
Let $a \in Im(q)^*$. Let $i \neq j \in \{0, \dots k\}$.

(1) *There exists $g_{i,j}^a \in L \cap \mathrm{Aut}_e(\mathfrak{g})$ such that $g_{i,j}^a(X_i) = aX_i$, $g_{i,j}^a(X_j) = a^{-1}X_j$ and $g_{i,j}^a(X_s) = X_s$ for $s \neq i,j$.*
(2) *If either*

(a) $k+1 = rank(\tilde{\mathfrak{g}})$ *is odd,*

or

(b) $k+1 = rank(\tilde{\mathfrak{g}})$ *is even and if there exists a regular graded Lie algebra* $(\tilde{\mathfrak{r}}, \tilde{H}_0)$ *satisfying the hypothesis* $(\mathbf{H}_1), (\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *such that the algebra* $\tilde{\mathfrak{r}}_1$ *obtained at the first step of the descent (cf. Theorem 1.6.1) is equal to* $\tilde{\mathfrak{g}}$ *(in other words, the algebra* $\tilde{\mathfrak{g}}$ *is the first step in the descent from a bigger graded Lie algebra),*

then there exists $g_i^a \in L$ *such that* $g_i^a(X_i) = aX_i$ *and* $g_i^a(X_s) = X_s$ *for* $s \neq i$*. In particular we have* $\chi_0(G) = Im(q)^*$*.*

Remark 3.8.5. From the classification (cf. Table 1) and from Proposition 2.1.1, the condition on the descent in case *2 (b)* occurs if $e = d \in \{1, 2\}$ (Table 1, (2) and (6)) or $(d, e) = (2, 0)$ (Table 1, (1)) or $e = 0, 4$, and $k + 1 \geq 4$ (Table 1, (11) and (12)).

Proof.
(1) By hypothesis, there exists $Z \in E_{i,j}(-1,-1)$ such that $q_{X_i,X_j}(Z) = a$. As $\mathrm{ad}(Y_j) : E_{i,j}(-1,1) \to E_{i,j}(-1,-1)$ is an isomorphism, there exists $A \in E_{i,j}(-1,1)$ such that $[Y_j, A] = Z$. Then, as we have already seen at the end of the preceding proof, one has

$$a = q_{X_i,X_j}([Y_j, A]) = \frac{1}{2} b(\mathrm{ad}(A)^2 X_i, Y_j).$$

As $a \neq 0$, the map $\mathrm{ad}(A)^2 : \tilde{\mathfrak{g}}^{\lambda_i} \to \tilde{\mathfrak{g}}^{\lambda_j}$ is non zero, and hence injective because $\ell = 1$ and the preceding Lemma says that there exists an $\mathfrak{sl}_2$-triple $\{B, H_{\lambda_j} - H_{\lambda_i}, A\}$ with $B \in E_{i,j}(1,-1)$. Moreover the element $\theta_A \in \mathrm{Aut}_e(\mathfrak{g}) \subset G$ fixes each X_s and each H_{λ_s} for $s \neq i, j$ and we have $\theta_A(H_{\lambda_i}) = H_{\lambda_j}$, $\theta_A(X_i) = aX_j$ and $\theta_A(X_j) = a^{-1}X_j$. From the proof of Proposition 3.5.2, let us set $f_{i,j} = \gamma_{0,i}\gamma_{0,j}\gamma_{0,i}$. Then the automorphisms $f_{i,j} \in \mathrm{Aut}_e(\mathfrak{g}) \subset G$ satisfy the properties of Proposition 3.4.3 and we have $f_{i,j}(X_i) = X_j$ and $f_{i,j}(X_s) = X_s$ for $s \neq i, j$. Then, from the preceding Lemma, the automorphism $g_{i,j}^a = f_{i,j} \circ \theta_A \in L \cap \mathrm{Aut}_e(\mathfrak{g})$ and has the other required properties.

(2) As the involutions $f_{0,i}$ exchange X_0 and X_i and fix X_s if $s \neq i$, we can suppose that $i = 0$.

Let $k+1$ be odd. The element $g = g_{0,1}^a \dots g_{0,k}^a \in L$ satisfies $g(X_0) = a^k X_0$ and $g(X_s) = a^{-1}X_s$ if $s > 0$. From Definition 3.8.1, the element $h_{X_0+\ldots+X_k}(\sqrt{a}) \in L$ acts by multiplication by a on V^+ and $h_{X_0}(a^{-k/2})$ fixes X_s if $s \neq 0$, and also $h_{X_0}(a^{-k/2})(X_0) = a^{-k}X_0$. It follows that the element $g_0^a = h_{X_0+\ldots+X_k}(\sqrt{a}) \circ h_{X_0}(a^{-k/2}) \circ g$ has the required properties.

Let now $k+1$ be even and suppose that there exists a regular graded Lie algebra $(\tilde{\mathfrak{r}}, \tilde{H}_0)$, such that the algebra $\tilde{\mathfrak{r}}_1$ obtained by performing one step in the descent (cf. Theorem 1.6.1.) is equal to $\tilde{\mathfrak{g}}$. As we are only concerned by the action of an element of G on V^+, we can always suppose that $\tilde{\mathfrak{r}}$ is simple (cf. Remark 1.8.8).
Let $(\tilde{\lambda}_0, \dots \tilde{\lambda}_{k+1})$ the maximal set of stronly orthogonal roots associated to $\tilde{\mathfrak{r}}$. Then $(\tilde{\lambda}_1, \dots \tilde{\lambda}_{k+1}) = (\lambda_0, \dots \lambda_k)$. Let us fix the elements $\tilde{X}_i \in \tilde{\mathfrak{r}}^{\tilde{\lambda}_i}$ satisfying the conditions of Proposition 3.5.2 in such a way that $(\tilde{X}_1, \dots \tilde{X}_{k+1}) = (X_0, \dots, X_k)$.

Let R be the analogue of the group G for the algebra $\tilde{\mathfrak{r}}$. Set $\mathfrak{r} = \mathcal{Z}_{\tilde{\mathfrak{r}}}(\tilde{H}_0)$ and $\tilde{\mathfrak{a}}^0 = \oplus_{i=0}^{k+1} F H_{\tilde{\lambda}_i}$. Therefore $R = \mathrm{Aut}_e(\mathfrak{r})\mathcal{Z}_R(\tilde{\mathfrak{a}}^0)$. The first assertion of the Corollary gives the existence of an element $r \in \mathcal{Z}_R(\tilde{\mathfrak{a}}^0) \cap \mathrm{Aut}_e(\mathfrak{r})$ such that $r.\tilde{X}_0 = a\tilde{X}_0$, $r.\tilde{X}_1 = a^{-1}\tilde{X}_1$ and $r.\tilde{X}_s = \tilde{X}_s$ for $s \neq 0$. As the automorphism r also fixes H_{λ_0}, it stabilizes $\tilde{\mathfrak{r}}_1 = \tilde{\mathfrak{g}}$. Let $r_1 \in \mathrm{Aut}(\tilde{\mathfrak{g}})$ be the restriction of r to $\tilde{\mathfrak{g}}$. As r centralizes $\tilde{\mathfrak{a}}^0$, it is clear that r_1 centralizes $\mathfrak{a}^0$. Moreover, one has $r_1.X_0 = a^{-1}X_0$ and $r_1.X_s = X_s$ if $s \neq 0$, from our choice of the elements $\tilde{X}_j$. Then, if we set $g = r_1 \circ h_{X_0}(a)$, we have $g.X_0 = aX_0, g.X_s = X_s$ for $s \neq 0$ and also $\chi_0(g) = a$.

It remains to prove that $r_1 \in \mathrm{Aut}_0(\tilde{\mathfrak{r}}_1)$. But as $r_1 \in \mathrm{Aut}(\tilde{\mathfrak{r}}_1)$, it suffices to prove that r_1 belongs to $\mathrm{Aut}_e(\tilde{\mathfrak{r}}_1 \otimes \bar{F})$. As we have supposed that $k+1$ is even, the rank $k+2$ of $\tilde{\mathfrak{r}}$ is ≥ 3. Using the classification (Proposition 2.1.1 and Table 1), one sees easily that the weighted Dynkin diagram of $\tilde{\mathfrak{r}}$ is of type A_{2n-1} (corresponding to the cases (1) (with $\delta = 1$) and (2) in Table 1), C_n (corresponding to the case (6) in Table 1) or $D_{2n,2}$ (corresponding to the cases (11) and (12) in Table 1). Here in all cases $n = k+2$.

- If $\tilde{\mathfrak{r}}$ is of type A_{2n-1}, then $\tilde{\mathfrak{r}} \simeq \mathfrak{sl}(2(k+2)+1, F)$. From the description of the roots λ_j given in the proof of Theorem 3.7.1 and also from [4] (Chap. VIII, §13, $n°1$, (VII), p.189), it is easy to see that the group $\mathcal{Z}_{\mathrm{Aut}_e(\tilde{\mathfrak{r}}\otimes\bar{F})}(\tilde{\mathfrak{a}}^0)$ is the group of conjugations by invertible diagonal matrices Therefore the restriction r_1 of r to $\tilde{\mathfrak{g}}$ belongs effectively to $\mathrm{Aut}_e(\tilde{\mathfrak{r}}_1 \otimes \bar{F})$.

- If $\tilde{\mathfrak{r}}$ is of type C_n, then, by [4](Chap. VIII §13 $n°3$ (VII) page 205.), one has $\mathrm{Aut}_e(\tilde{\mathfrak{g}} \otimes \bar{F}) = \mathrm{Aut}(\tilde{\mathfrak{g}} \otimes \bar{F})$, and this implies the result.

- If $\Psi = D_{2n,2}$ then the algebra $\tilde{\mathfrak{r}} \otimes \bar{F}$ is isomorphic to the orthogonal algebra $\mathfrak{o}(q_{(2n,2n)}, \bar{F})$. We realize it as in ([4] Chap VIII §13 $n°4$ page 207). One fixes a Witt bases in which the matrix of $q_{(2n,2n)}$ is the square matrix s_{4n} of size $4n$ whose coefficients are all zero except those of the second diagonal which are equal to 1. The algebra $\tilde{\mathfrak{r}} \otimes \bar{F}$ is then the set of matrices $Z = \begin{pmatrix} A & B \\ C & -s_{2n}\,{}^t A s_{2n} \end{pmatrix}$ with $B = -s_{2n}\,{}^t B s_{2n}$ and $C = -s_{2n}\,{}^t C s_{2n}$. This algebra is then graded by the element $H_0 = \begin{pmatrix} I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}$ where I_{2n} is the identity matrix of size $2n$ and one can choose λ_0 in such a way that $H_{\lambda_0} = \left(\begin{array}{cc|c} 0_{2n-2} & 0 & \\ 0 & I_2 & 0 \\ \hline & & \begin{array}{cc} -I_2 & 0 \\ 0 & 0_{2n-2} \end{array} \\ \multicolumn{2}{c|}{0} & \end{array}\right)$.

Recall that a *similarity* of a quadratic form Q is a linear isomorphism g of the underlying space E such that $Q(gX) = \lambda(g)Q(X)$, where the scalar $\lambda(g)$ is called the ratio of g. If $\dim E = 2l$, then a similarity g is said to be *direct* if $\det(g) = \lambda(g)^l$.

But we know from [4] (Chap VIII §13 $n°4$ page 211), that the group $\mathrm{Aut}_e(\tilde{\mathfrak{r}} \otimes \bar{F})$ is the group of automorphisms of the form $\varphi_s : Z \mapsto sZs^{-1}$ where s is a direct similarity of $q_{(2n,2n)}$. It is easy to see that a similarity s commutes with H_0 if and only if $s = \begin{pmatrix} g & 0 \\ 0 & \mu s_{2n}\, {}^t g^{-1} s_{2n} \end{pmatrix}$ with $\mu \in \bar{F}^*$ and $g \in GL(2n, \bar{F})$. Moreover, if s commutes with H_{λ_0}, then g is of the form $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ with $g_1 \in GL(2n-2, \bar{F})$ and $g_2 \in GL(2, \bar{F})$. It is then clear that φ_s restricts to a direct similarity of $q_{(2n-2,2n-2)}$ and hence the restriction belongs to $\mathrm{Aut}_e(\tilde{\mathfrak{r}}_1 \otimes \bar{F})$.

□

The following result on anisotropic quadratic forms of rank 2 will be used later.

Lemma 3.8.6. *Let Q be an anisotropic quadratic form of rank 2. Let $Im(Q)^*$ be the set of non zero scalars which are represented by Q. Then*
(1) *$Im(Q)^*$ is the union of exactly 2 classes a and b in F^*/F^{*2}.*
(2) *Let Q' be another anisotropic quadratic form of rank 2. Then $Q' \sim Q$ if and only if $Im(Q')^* = Im(Q)^*$. In particular, one has $\mu Q \sim Q$ if and only if $\mu = 1$ or $\mu = ab$ modulo F^{*2}.*

Proof. Let π be a uniformizer of F and let u be a unit of F^* which is not a square. Then $F^*/F^{*2} = \{1, u, \pi, u\pi\}$. If $-1 \notin F^{*2}$ we suppose moreover that $u = -1$.

(1) As Q is anisotropic of rank 2, there exist $v, w \in F^*/F^{*2}$ with $v \neq -1$ such that $Q \sim w(x^2 + vy^2)$. Therefore it is enough to prove the assertion for $Q_v = x^2 + vy^2$.
From [13] (Chapter I, Corollary 3.5 page 11), we know that $\mu \in Im(Q_v)^*$ if and only if the quadratic form $x^2 + vy^2 - \mu z^2$ is isotropic. On the other hand the quadratic form $Q_{an} = x^2 - uy^2 - \pi z^2 + u\pi t^2$ is the unique anisotropic form of rank 4, up to equivalence (see [13] Chapter VI, Theorem 2.2 (3) page 152).
If $-1 \in F^{*2}$ then $v \in \{u, \pi, u\pi\}$. Suppose that for example that $v = u$. Then Q_v cannot represent π or $u\pi$, because in that case the forms $x^2 + vy^2 - \pi z^2$ or $x^2 + vy^2 - u\pi z^2$ would be isotropic, and hence and then $Q_{an} = x^2 - uy^2 - \pi z^2 + u\pi t^2$ would be isotropic too. Finally Q_v represents exactly the classes of 1 and v. The same argument works for the other possible values of v.
If $u = -1 \notin F^{*2}$ then -1 is sum of two squares ([13] Chapter VI, Corollary 2.6 page 154) and $v \in \{1, \pi, -\pi\}$ (as the form Q_v is anisotropic, v cannot be equal to $u = -1$). The forms $x^2 + y^2$ and $-(x^2 + y^2)$ have the same discriminant and represent both the element -1. They are therefore equivalent ([13] Chapter I, Proposition 5.1 page 15). Then, as $Q_{an} = x^2 + y^2 - \pi z^2 - \pi t^2 \sim -(x^2 + y^2) - \pi z^2 - \pi t^2 \sim x^2 + y^2 + \pi z^2 + \pi t^2$, the same argument as above shows that the form Q_1 represents exactly the classes of 1 and -1. By the same way, if $v = \pm\pi$ one shows that Q_v represents exactly the classes of 1 and v.

(2) From above we know that an anisotropic quadratic form $Q = ax^2 + by^2$ with $a, b \in F^*/F^{*2}$ represents a and b if $a \neq b$ and it represents $\pm a$ if $a = b$ and if $-1 \notin F^{*2}$ (because then -1 is a sum of two squares, the case where $-1 \in F^{*2}$ has not to be considered because the form Q would be isotropic). From [13] (Chapter I, Proposition 5.1 page 15), we know that $Q' = cx^2 + dy^2 \sim Q$ if and only if $ab = cd$ modulo F^{*2} and $\{a, b\} \cap \{c, d\} \neq \emptyset$. This implies the second assertion.

□

Proposition 3.8.7. *Suppose $k = 1$ (i.e. $\operatorname{rank}(\tilde{\mathfrak{g}}) = 2$). Let us denote by $[F^* : \chi_0(G)]$ the index of $\chi_0(G)$ in F^* (equal to $1, 2$ or 4 according to Lemma 3.8.2 (1)). Then*

(1) *The group G has $1 + [F^* : \chi_0(G)]$ non zero orbits in V^+: the non open orbit of X_0 and the open orbits of $X_0 + vX_1$ where $v \in F^*/\chi_0(G)$.*

(2) *Two generic elements in $\tilde{\mathfrak{g}}^{\lambda_0} \oplus \tilde{\mathfrak{g}}^{\lambda_1}$ are G-conjugated if and only if they are L-conjugated.*

(3) (a) *If $e = 1$ or 3, then $\chi_0(G) = F^{*2}$,*
(b) *if $e = 0$ or 4, then $\chi_0(G) = F^*$,*
(c) *if $e = 2$ then $\chi_0(G)$ is a subgroup of index 2 of F^*.*

Proof. By Proposition 3.6.1, any non zero element in V^+ is G-conjugated to an element $Z = x_0X_0 + x_1X_1$ in V^+ with $(x_0, x_1) \neq (0, 0)$.

If $\operatorname{rank}(Z)= 1$ (i.e. $x_0x_1 = 0$), as we can use the element $\gamma_{0,1}$ of Proposition 3.4.3 which exchanges $\tilde{\mathfrak{g}}^{\lambda_0}$ and $\tilde{\mathfrak{g}}^{\lambda_1}$, we can suppose $x_1 = 0$ and $x_0 \neq 0$. The element $h_{X_0+X_1}(\sqrt{x_0}^{-1})$ associated to $X_0 + X_1$ (see Definition 3.8.1) belongs to L and acts by $x_0^{-1}Id_{V^+}$ on V^+. It follows that Z is L-conjugated to X_0.

Suppose now that $\operatorname{rank}(Z)= 2$ (i.e. $x_0x_1 \neq 0$). Then, as above, we obtain that Z is L-conjugated to $Z_v = X_0 + vX_1$ with $v = x_0^{-1}x_1 \neq 0$.

By Theorem 1.13.2, one has $\Delta_0(Z_v) = v\Delta_0(Z_1)$. It follows easily that if $v \notin w\chi_0(G)$, then the elements Z_v and Z_w are not G-conjugated.

Suppose that $v = w\mu$ with $\mu \in \chi_0(G) = \chi_0(L)$ (cf. Lemma 3.8.2). Let $g \in L$ such that $\chi_0(g) = \mu$. As $g \in L$, it stabilizes the spaces $\tilde{\mathfrak{g}}^{\lambda_j}$ for $j = 0$ or 1. As $\dim\tilde{\mathfrak{g}}^{\lambda_j} = \ell = 1$, there exist α and β in F^* such that $g.X_0 = \alpha X_0$ and $g.X_1 = \beta X_1$. Therefore $\Delta_0(g.(X_0 + X_1)) = \Delta_0(\alpha X_0 + \beta X_1) = \alpha\beta\Delta_0(X_0 + X_1)$, and hence $\alpha\beta = \chi_0(g) = \mu$.

Then $g.(X_0 + wX_1) = \alpha X_0 + \beta wX_1 = \alpha^{-1}(\alpha^2X_0 + vX_1)$. The element $h_{X_0}(\alpha)$ associated to X_0 belongs to L and satisfies $h_{X_0}(\alpha)X_0 = \alpha^2X_0$ and $h_{X_0}(\alpha)X_1 = X_1$. The element $h_{X_0+X_1}(\sqrt{\alpha}^{-1})$ also belongs to L and acts by multiplication by α^{-1} on V^+. As $g.(X_0+wX_1) = h_{X_0+X_1}(\sqrt{\alpha}^{-1})\circ h_{X_0}(\alpha)(X_0+vX_1)$, the elements X_0+wX_1 and X_0+vX_1 are L-conjugated. This ends the proof of (1) and (2).

It remains to prove the last assertion.
By Lemma 3.8.2, one has $F^{*2} \subset \chi_0(G) \subset \{a \in F^*; aq \sim q\} \subset Im(q)^*$.
If $e = 1$, then q is the sum of the form q^e_{an} of rank 1 and of a hyperbolic quadratic form q_{hyp} of rank $d - 1$ (which may be zero). See Proposition 3.5.2. As $\mu q_{hyp} \sim q_{hyp}$ for all $\mu \in F^*$ ([13] Chapter I, Theorem 3.2 page 9), Witt's decomposition Theorem ([13] Chapter I, Theorem 4.2 page 12), implies that $aq \sim q$ if and only if $aq^e_{an} \sim q^e_{an}$. As q^e_{an} is of rank 1, we obtain $\chi_0(G) = F^{*2}$, and this is the assertion *3 (a)*, in the case where $e = 1$.

If $e = 3$, then q is the sum of an anisotropic form q^e_{an} of rank 3 and of a hyperbolic quadratic form q_{hyp} of rank $d - 3$. As above $aq \sim q$ if and only if $aq^e_{an} \sim q^e_{an}$. But from [13], Chap. VI, Corollary 2.5, p.152-153, we have $aq^e_{an} \sim q^e_{an} \iff -disc(aq^e_{an}) = -adisc(q^e_{an}) = -disc(q^e_{an})$.

This means that $aq^e_{an} \sim q^e_{an} \iff a \in F^{*2}$. As above this again implies that $\chi_0(G) = F^{*2}$. Hence (3) (a) is proved.
Let us suppose now that e even and hence $e \in \{0, 2, 4\}$. The proofs will depend on the values of e and d.
If $(d, e) = (2, 0)$ or $(2, 2)$ (cases (1) and (2) in Table 1) then the algebra $\tilde{\mathfrak{g}}$ satisfies the condition *2 (b)* of corollary 3.8.4 and hence $\chi_0(G) = Im(q)^*$.

- If $(d, e) = (2, 2)$, the quadratic form q is anisotropic of rank 2, and therefore q two classes in F^*/F^{*2} (cf. Lemme 3.8.6). From above we obtain $[F^* : \chi_0(G)] = 2$.
- $(d, e) = (2, 0)$ then q is hyperbolic, hence universal and therefore $\chi_0(G) = Im(q)^* = F^*$ (cf. [13] Chapter I, Theorem 3.4 page 10).

It remains to study the cases where $e \in \{0, 2, 4\}$ and $d \geq 4$, corresponding to the cases (8), (9) and (10) of Table 1. From Remark 1.8.8, one can suppose that $\tilde{\mathfrak{g}} = \mathfrak{o}(q_{(m+r,m-r)})$ with $2r = e$ and $m \geq 4$. We will describe precisely the group G is this case.
The quadratic form $q_{(m+r,m-r)}$ is the sum of $m - r$ hyperbolic planes and of an anisotropic quadratic form $q_{an,2r}$ of rank $2r$, where $q_{an,0} = 0$. It exists a basis $(e_1, \dots, e_m, e_{-m}, \dots e_{-1})$ in which the matrix of $q_{(m+r,m-r)}$ is given by

$$S_{2m,2r} = \begin{pmatrix} 0 & 0 & s_{m-r} \\ 0 & J_{an,2r} & 0 \\ s_{m-r} & 0 & 0 \end{pmatrix},$$

where s_n stands for the square matrix of size n whose coefficients are all zero except those on the second diagonal which are equal to 1 and where $J_{an,2r}$ is the square matrix of size $2r$ of the form $q_{an,r}$ which is supposed to be diagonal for $r \neq 0$ and equal to the empty block for $r = 0$.
Hence

$$\tilde{\mathfrak{g}} = \{X \in M_{2m}(F),\ {}^tXS_{2m,2r} + S_{2m,2r}X = 0\}$$

Then the maximal split abelian subalgebra $\mathfrak{a}$ of $\tilde{\mathfrak{g}}$ is the set of diagonal elements $\sum_{i=1}^{m-r} \varphi_i(E_{i,i} - E_{2m-i+1,2m-i+1})$ with $\varphi_i \in F$ (as usual $E_{i,j}$ is the matrix whose coefficients are all zero except the coefficient of index (i, j) which is equal to 1) and the algebra $\tilde{\mathfrak{g}}$ is graded by the element $H_0 = 2E_{1,1} - 2E_{2m,2m}$. It is then easy to see that V^+ is isomorphic F^{2m-2} by the map

$$y \in F^{2m-2} \mapsto X(y) = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & -S^{-1}_{2(m-1),2r}\,{}^ty \\ 0 & 0 & 0 \end{pmatrix} \in V^+.$$

Set $X_0 = X(1, 0, \dots, 0) \in \tilde{\mathfrak{g}}^{\lambda_0}$ and $X_1 = X(0, 0, \dots, 1) \in \tilde{\mathfrak{g}}^{\lambda_1}$.

We denote by $\mathcal{S}im_0(q_{(m+r,m-r)})$ the group of direct similarities of $q_{(m+r,m-r)}$, that is the group of elements $A \in GL(2m, F)$ such there exists $\mu \in F^*$ satisfying ${}^tAS_{m,r}A = \mu S_{m,r}$ and such that $\det(A) = \mu^m$. We will denote by $\mu(A) = \mu$ the ratio of A.

If $e = r = 0$, the algebra $\tilde{\mathfrak{g}}$ is split and from ([Bou] Chap VIII §13 $n°4$ page 211), we know that the group $\mathrm{Aut}_0(\tilde{\mathfrak{g}})$ is the group of automorphisms of the form $Z \mapsto AZA^{-1}$ where $A \in \mathcal{S}im_0(q_{(m,m)})$. It is easy to see that an element $A \in \mathcal{S}im_0(q_{(m,m)})$ commutes with

H_0 if and only if there exists $b \in F^*$ and $g_1 \in \mathcal{S}im_0(q_{(m-1,m-1)})$ of ratio $\mu(g_1)$ such that $A = \begin{pmatrix} \mu(g_1)b^{-1} & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & b \end{pmatrix} = b \begin{pmatrix} \mu(g_1)b^{-2} & 0 & 0 \\ 0 & b^{-1}g_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. As $b^{-1}g_1$ is a direct similarity of ratio $\mu(g_1)b^{-2}$, we obtain that the group G is isomorphic to the group of direct similarities $\mathcal{S}im_0(q_{(m-1,m-1)})$ of the quadratic form $q_{(m-1,m-1)}$. Its action on V^+ is given by $g.X(y) = X(\mu(g)\, yg^{-1})$ for $g \in \mathcal{S}im_0(q_{(m-1,m-1)})$ with ratio $\mu(g)$.

Suppose now that $r \neq 0$. Let $g \in G$ and denote by $\bar{g}$ its natural extension to $\tilde{\mathfrak{g}} \otimes \bar{F}$. As g commutes with H_0, the same is true for $\bar{g}$ and therefore , $\bar{g}$ stabilizes $\bar{V}^+$ and $\bar{V}^-$ (ie. $\bar{g}\bar{V}^+ \subset \bar{V}^+$ and $\bar{g}\bar{V}^- \subset \bar{V}^-$). From the split case above, the action of $\bar{g}$ on $\tilde{\mathfrak{g}} \otimes \bar{F}$ is given by the conjugation of an element $\bar{A} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \bar{g}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\mu \in \bar{F}^*$, ${}^t\bar{g}_1 S_{2(m-1),2r}\bar{g}_1 = \mu S_{2(m-1),2r}$ and $\det(\bar{g}_1) = \mu^{m-1}$.

Such an element acts on V^+ by

$$\bar{A}X(y)\bar{A}^{-1} = X(\mu\, y\bar{g}_1^{-1}) = \begin{pmatrix} 0 & \mu\, y\bar{g}_1^{-1} & 0 \\ 0 & 0 & -\bar{g}_1 S_{m-1,r}^{-1}\, {}^t y \\ 0 & 0 & 0 \end{pmatrix}, \quad y \in F^{2m-2}.$$

As $g \in G$ stabilizes V^+ and V^-, we see that the coefficients of $\bar{g}_1$ are in F and that $\mu \in F$. Moreover the restriction of $\bar{g}_1$ to F^{2m-2}, which we denote by g_1, is a direct similarity of $q_{(m+r-1,m-r-1)}$ with ratio μ, i.e. $g_1 \in \mathcal{S}im_0(q_{(m-1+r,m-1-r)})$.

Hence, in each case, the group G is isomorphic to the group $\mathcal{S}im_0(q_{(m+r-1,m-1-r)})$ of direct similarities of $q_{(m+r-1,m-1-r)}$. Its action on V^+ is given by $g.X(y) = \mu(g)X(yg^{-1})$ where $\mu(g)$ is the ratio of g. Then the polynomial $\Delta_0(X(y)) = q_{(m+r-1,m-1-r)}(y)$ is relatively invariant under the action of G and its character is given by $\chi_0(g) = \mu(g)$.

All the anisotropic quadratic forms of rank 4 are equivalent ([13] Chapter VI, Theorem 2.2 page 152) and all the hyperbolic quadratic forms are equivalent ([13] Chapter I, Theorem 3.2 page 9). Therefore, if $e = r = 0$ or $e = 4 = 2r$, for any $\mu \in F^*$, there exists $g \in \mathcal{S}im_0(q_{(m+r-1,m-1-r)})$ whose ratio is μ. Hence $\chi_0(G) = F^*$.

If $e = 2 = 2r$, then by Lemma 3.8.6, the anisotropic form $q_{an,2}$ represents exactly 2 classes a and b in F^*/F^{*2} and $\mu q_{an,2} \sim q_{an,2}$ if and only if $\mu = 1$ or $\mu = ab$ modulo F^{*2}. It follows that if $g \in \mathcal{S}im_0(q_{(m+r-1,m-1-r)})$ then $\mu(g) \in \{1, ab\}$ modulo F^{*2}. Hence the subgroup $\chi_0(G)$ is of index 2 in F^*.

This ends the proof of Proposition 3.8.7.

□

Theorem 3.8.8. *In the case where $e = 0$ or 4 we have $\chi_0(G) = F^*$ and the group G has exactly $k + 1 =$ rank$(\tilde{\mathfrak{g}})$ non zero orbits in V^+. These orbits are characterized by their rank and a representative of the unique open orbit is $X_0 + \ldots + X_k$.*
Moreover two generic elements in $\oplus_{j=0}^{k}\tilde{\mathfrak{g}}^{\lambda_j}$ are conjugated under the subgroup $L = Z_G(\mathfrak{a}^0)$.

Proof. From Proposition 3.6.1, any non zero element in V^+ is G-conjugated to an element $Z = x_0X_0 + \ldots + x_{m-1}X_{m-1}$ of V^+ such that $\prod_{s=0}^{m-1} x_s \neq 0$. It suffices to prove that Z is L-conjugated to $X_0 + \ldots X_{m-1}$.
If $k+1 = 2$, the result is a consequence of proposition 3.8.7.
Suppose now that $k+1 \geq 3$. As $e = 0$ or 4, the form q is either isotropic or anisotropic of dimension 4, and hence $Im(q)^* = F^*$ (an anisotropic form of dimension 4 represents any nonzero element because any form of dimension 5 is isotropic ([13], Chap VI, Theorem 2.2 p. 152)).
If $d \leq 4$ (as the conditions are then $\ell = 1$, $e = 0$ or 4, and $k+1 \geq 3$, this corresponds to the cases 1 (with $\delta = 1$), (11) and (12) in the Table 1) then the algebra $\tilde{\mathfrak{g}}$ satisfies one of the two hypothesis in Corollary 3.8.4 (2). Hence, for $i \in \{0,,\ldots m-1\}$, there exists $g_i^{x_i} \in L$ such that $g_i^{x_i}X_i = x_iX_i$ and $g_i^{x_i}X_s = X_s$ for $s \neq i$. It follows that if $g = \prod_{i=0}^{m-1} g_i^{x_i}$, then $g.(X_0 + \ldots X_{m-1}) = Z$, and this proves the required result.

In the case where $e \in \{0,4\}$, $k+1 \geq 3$ and $d > 4$ (and $\ell = 1$ of course), the classification shows that this corresponds to the only case (13) in Table 1, and that $k+1 = 3$. Again Corollary 3.8.4 gives the result.

$\square$

The next Theorem gives the G-orbits in V^+ in the case where $e = 1$ or $e = 3$. If $e = d = 1$, this classification coincides with the classification of similarity classes of quadratic forms over F: a quadratic form Q is *similar* to a quadratic form Q' if and only if there exists $a \in F^*$ such that Q is equivalent to aQ'.

Theorem 3.8.9. *Suppose that $e = 1$ or $e = 3$.*

(1) *If $k+1 =$ rank$(\tilde{\mathfrak{g}}) = 2$ (this corresponds to the cases (3), (4) and (6) with $n = 2$ in Table 1), then $\chi_0(G) = F^{*2}$. The group G has 5 non zero orbits, for which 4 are open. A set of representatives of the open orbits is given by the elements $X_0 + vX_1$ for $v \in F^*/F^{*2}$.*
Two generic elements of $\tilde{\mathfrak{g}}^{\lambda_0} \oplus \tilde{\mathfrak{g}}^{\lambda_1}$ are L-conjugated if and only if they are G-conjugated.
(2) *If $k+1 =$ rank$(\tilde{\mathfrak{g}}) \geq 3$ then $e = d = 1$. This corresponds to the case (6) in Table 1, namely the symplectic algebra. Remember also that $F^*/F^{*2} = \{1, u, \pi, u\pi\}$ where π is a uniformizer of F and where u is a unit which is not a square.*
    (a) *If $k+1 =$ rank$(\tilde{\mathfrak{g}}) \geq 3$ is odd then $\chi_0(G) = F^*$ and the group G has 2 open orbits with representatives given by*

$$X_0 + \sum_{j=1}^{k/2} X_{2j-1} - X_{2j}$$

and

$$X_0 - uX_1 - \pi X_2 + \sum_{j=2}^{k/2} X_{2j-1} - X_{2j}.$$

(b) *If $k+1 = rank(\tilde{\mathfrak{g}}) \geq 3$ is even, then $\chi_0(G) = F^{*2}$ and the group G has 5 open orbits in V^+, with representatives given by*

$$\sum_{j=0}^{(k-1)/2} X_{2j} - X_{2j+1},$$

$$X_0 + vX_1 + \sum_{j=1}^{(k-1)/2} X_{2j} - X_{2j+1}, \text{ with } v \in \{-u, -\pi, u\pi\}.$$

and

$$X_0 - uX_1 - \pi X_2 + u\pi X_3 + \sum_{j=2}^{(k-1)/2} X_{2j} - X_{2j+1} .$$

(c) *For $m \in \{0, \dots, k\}$, two generic elements of V_m^+ are G-conjugated if and only if they are G_m-conjugated.*
If $k+1 = rank(\tilde{\mathfrak{g}}) = 2p+1$ with $p \geq 1$ then the group G has $7p$ non zero orbits and if $k+1 = rank(\tilde{\mathfrak{g}}) = 2p+2$ with $p \geq 1$ then G has $7p+5$ non zero orbits. The representatives of these orbits are the representatives of the G_m-orbits of the generic elements of V_m^+ where $m \in \{0, \dots, k\}$.

(d) *Let $X = x_0X_0 + \cdots + x_kX_k$ and $X' = x_0'X_0 + \cdots + x_k'X_k$ be two generic elements in V^+. Then X and X' are L -conjugated if and only if there exits $\mu \in F^*$ such that $\mu x_i x_i' \in F^{*2}$ for all $i \in \{0, \dots, k\}$.*

Proof. If $k+1 = 2$ the result is a consequence of Proposition 3.8.7.
If $k+1 \geq 3$ then the classification in Table 1 implies that $d = e = 1$. As we have noticed in Remark 1.8.8, one can suppose that $\tilde{\mathfrak{g}}$ is the symplectic algebra

$$\mathfrak{s}p(2n, F) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} ;\ A, B, C \in M(k+1, F), {}^tB = B, {}^tC = C \right\}, \text{ with } k+1 = n,$$

which is graded by the element $H_0 = \begin{pmatrix} I_{k+1} & 0 \\ 0 & -I_{k+1} \end{pmatrix}$. It follows that, V^+ (respectively V^-) can be identified with the space $Sym(k+1, F)$ of symmetric matrices of size $k+1$ on F through the map

$$B \in Sym(k+1, F) \mapsto X(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

(respectively through the map

$$B \in Sym(k+1, F) \mapsto Y(B) = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix})$$

.

The algebra $\mathfrak{a}$ is then the set of matrices $H(t_k, \dots, t_0) = \begin{pmatrix} diag(t_k, \dots, t_0) & 0 \\ 0 & -diag(t_k, \dots, t_0) \end{pmatrix}$ where $diag(t_k, \dots, t_0)$ stands for the diagonal matrix whose diagonal elements are respectively $t_k, \dots, t_0$. The set of strongly orthogonal roots associated to these choices is given by $\lambda_j(H(t_k, \dots, t_0)) = 2t_j$ for $j = 0, \dots, k+1$ and the space $\tilde{\mathfrak{g}}^{\lambda_j}$ is the space of matrices

$X(x_j\mathbf{E}_{k+1-j,k+1-j})$ with $x_j\in F$, where $\mathbf{E}_{i,j}$ is the square matrix of size $k+1$ whose coefficients are zero except the coefficient of index (i,j) which is equal to 1.

By ([4] Chap VIII, §13, $n^0 3$), we know that the group $\mathrm{Aut}_0(\tilde{\mathfrak{g}})$ is the group of conjugations by the similarities of the symplectic form defining $\tilde{\mathfrak{g}}$. This implies that G is the group of elements $[\mathbf{g},\mu]=\mathrm{Ad}\begin{pmatrix}\mathbf{g} & 0\\ 0 & \mu\,{}^t\mathbf{g}^{-1}\end{pmatrix}$ where $\mathbf{g}\in GL(k+1,F)$ and $\mu\in F^*$. Let us denote by $g=[\mathbf{g},\mu]$ such an element of G. Its action on V^+ is given by $[\mathbf{g},\mu]B=\mu^{-1}\mathbf{g}B\,{}^t\mathbf{g}$ for $B\in Sym(k+1,F)$. We normalize the relatively invariant polynomial Δ_0 by setting $\Delta_0(X(B))=\det(B)$ for $B\in Sym(k+1,F)$ and from above we see that

$$\chi_0([\mathbf{g},\mu])=\mu^{-(k+1)}\det(\mathbf{g})^2.$$

In particuliar , on a $\chi_0(G)=F^*$ if $k+1$ is odd and $\chi_0(G)=F^{*2}$ if $k+1$ is even.

Hence the orbits of G in V^+ are the classes of similar quadratic forms.
In order to give a set of representatives of these G-orbits, we will first normalize the elements X_j. Remember that the $X_j,\ j=0,\dots,k$ satisfy the conditions of Proposition 3.5.2. This means that for $i\neq j$, the quadratic form q_{X_i,X_j} represents 1. This quadratic form is defined on $E_{i,j}(-1,-1)$ by $q_{X_i,X_j}(Y)=-\frac{1}{2}b([X_i,Y],[X_j,Y])$. As $\dim V^+=\dfrac{(k+1)(k+2)}{2}$, the normalization of the Killing form given in Definition 1.10.1 is

$$b(X,Y)=-\frac{k+1}{2(k+1)(k+2)}\tilde{B}(X,Y)=\mathrm{Tr}(XY),\quad X,Y\in\tilde{\mathfrak{g}}.$$

Then, if we set $X_j=X(v_j\mathbf{E}_{k+1-j,k+1-j})$ with $v_j\in F^*$, a simple computation shows that for $Y=Y(y(\mathbf{E}_{k+1-i,k+1-j}+\mathbf{E}_{k+1-j,k+1-j}))\in E_{i,j}(-1,-1)$, we have

$$q_{X_i,X_j}(Y)=v_iv_jy^2.$$

Hence q_{X_i,X_j} represents 1 if and only if $v_iv_j\in F^{*2}$. Therefore for all $j\in\{0,\dots k\}$, there exists $a_j\in F^*$ such that $v_j=a_j^2v_0$. Any element $X=\sum_{j=0}^k x_jX_j$ is then conjugated to $X(\sum_{j=0}^k x_j\mathbf{E}_{k+1-j,k+1-j})$ by the element $g=[\mathbf{g},v_0^{-1}]$ where $\mathbf{g}$ is the diagonal matrix $diag(a_k,\dots,a_0)$.

For $X=X(B)\in V^+$ with $B\in Sym(k+1,F)$, let us denote by f_X the quadratic form on F^{k+1} defined by B (ie. $f_X(z)={}^tzBz$). From above we obtain that for $X=\sum_{j=0}^k x_jX_j$, the quadratic form f_X is similar to the form $T\in F^{k+1}\mapsto\sum_{j=0}^k x_jT_j^2$.

We describe now the similarity classes of quadratic forms. From Witt's Theorems ([13] Theorem I.4.1 and Theorem I.4.2, p.12), any quadratic form Q of rank r is the orthogonal sum of an unique (up to equivalence) anisotropic form Q_{an} of rank r_{an}, and a hyperbolic form $Q_{(m,m)}$ which is the sum of m hyperbolic planes with $2m+r_{an}=r$ (m is the so-called *Witt index* of Q). Moreover, two quadratic forms are similar if and only if they have the same Witt index and if their anisotropic parts are similar.

We recall the following classical results ([13] Chapter VI, Theorem 2.2 page 152, and Corollary 2.5 p. 153-154):
(1) Every quadratic form of rank ≥ 5 is isotropic.
(2) Up to equivalence, there exists a unique anisotropic form of rank 4, given by x^2-uy^2-

$\pi z^2 + u\pi z^2$.
(3) If Q is an anisotropic form of rank 3, then Q represents every class modulo F^{*2} except $-disc(Q)$ where $disc(Q)$ is the discriminant of Q.

If Q' is anisotropic of rank 3 with the same discriminant as Q then $Q + disc(Q)t^2$ and $Q' + disc(Q)t^2$ are anisotropic of rank 4, hence they are equivalent. Witt's cancellation Theorem ([13] Chapter I, Theorem 4.2, p.12) implies then that $Q \sim Q'$. Therefore there exist 4 equivalence classes of anisotropic quadratic forms of rank 3 characterised by the discriminant. Hence all the anisotropic quadratic forms of rank 3 are similar. Such a form is given by $x^2 - uy^2 - \pi z^2$.

We describe first the similarity classes of anisotropic quadratic forms of rank 2. We know from Lemma 3.8.6, that an anisotropic quadratic form Q of rank 2 represents exactly two classes of squares a and b in F^*/F^{*2} which characterize the equivalence class of Q. Moreover μQ is equivalent to Q if and only if $\mu = 1$ or ab modulo F^{*2}.
Let $a \neq b$ be two elements of F^*/F^{*2}. If $ab \neq -1$ the form $ax^2 + by^2$ is anisotropic and represents a and b and if $a = -b$ with $-1 \notin F^{*2}$ then -1 the sum of two squares ([13] Chapter VI, Corollary 2.6 page 154) and then $ax^2 + ay^2$ is anisotropic and represents $\pm a$. As there are four classes modulo F^{*2}, there $\binom{4}{2} = 6$ equivalence classes of anisotropic quadratic forms of rank 2.
Let $a \neq b \in F^*/F^{*2}$ defining the equivalence class of an anisotropic form Q of rank 2. Then, for $w \neq 1, ab$ modulo F^{*2}, one has $F^*/F^{*2} = \{1, ab, w, wab\}$. As $ab\,Q$ is equivalent to Q, and as wQ is not equivalent to Q, a form which is similar to Q is equivalent either to Q or to $ab\,Q$. Hence there are 3 similarity classes of anisotropic quadratic forms of rank 2. A set of representatives of these classes are given by $x^2 + vy^2$ with $v \in \{-u, -\pi, u\pi\}$.

Let Q be a quadratic form of rank $k+1 \geq 3$ and Witt index m: $Q = Q_{an} + Q_{(m,m)}$. Hence $\text{rank}(Q_{an}) \leq 4$ and $k+1 = \text{rank}(Q_{an}) + 2m$. By the classical results we recalled above, we get:

- If $k+1$ is odd, then $\text{rank}(Q_{an}) = 1$ or 3 there are two similarity classes of quadratic forms,
- and if $k+1$ is even, then $\text{rank}(Q_{an}) = 0, 2$ or 4 and hence there are 5 similarity classes of quadratic forms.

The statements $2(a)$ and $2(b)$ are consequences of the description of the anisotropic quadratic forms of rank ≤ 4 given above.

Let us prove statement *2.(c)*. We denote by ι the natural injection from $Sym(k+1-m, F)$ into $Sym(k+1, F)$ given by $M \mapsto \iota(M) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0_m \end{array}\right) \in Sym(k+1, F)$. Therefore the space V_m^+ is identified to the space $Sym(k+1-m, F)$ by the map $M \mapsto X(\iota(M))$. An element $X(\iota(M)) \in V_m^+$ is generic in V_m^+ if and only if $\det(M) \neq 0$. The group G_m is the group of elements $g_1 = [\mathbf{g}_1, \mu]$ with $\mathbf{g}_1 \in GL(k+1-m, F)$ and $\mu \in F^*$ acting on $Sym(k+1-m, F)$ by $[\mathbf{g}_1, \mu].M = \mu^{-1}\mathbf{g}_1 M\, {}^t\mathbf{g}_1$.

Let $Z = X(\iota(M))$ and $Z' = X(\iota(M'))$ be two generic elements in V_m^+. If Z and Z' are G-conjugated then there exist $\mathbf{g} \in GL(k+1, F)$ and $\mu \in F^*$ such that $\mathbf{g}\iota(M)\, {}^t\mathbf{g} = \mu\, \iota(M')$. Let $\mathbf{g}_1 \in GL(k+1-m, F)$ be the submatrix of $\mathbf{g}$ of the $k+1-m$ first rows and the $k+1-m$ first columns of $\mathbf{g}$. Then one sees easily that $\mathbf{g}_1 M\, {}^t\mathbf{g}_1 = \mu\, M'$. As Z and Z' are generic in V_m^+, the matrices M and M' are invertible and hence $\mathbf{g}_1 \in GL(k+1-m, F)$. This shows that Z and Z' are G_m-conjugated.
Conversely, if Z and Z' are G_m-conjugated then there exist $\mathbf{g}_1 \in GL(k+1-m, F)$ and $\mu \in F^*$ such that $\mathbf{g}_1 M\, {}^t\mathbf{g}_1 = \mu\, M'$. We set $\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 & 0 \\ 0 & I_m \end{pmatrix}$. The element $[\mathbf{g}, \mu]$ belongs to G and satisfies $Z' = [\mathbf{g}, \mu].Z$. Hence Z and Z' are G-conjugated.
This proves that two generic elements in V_m^+ are G-conjugated if and only if they are G_m-conjugated.
Let S be the number of non zero G-orbits in V^+ and let S_r be the number of G-orbits of rank r in V^+. By Proposition 3.6.1, we have $S = \sum_{r=1}^{k+1} S_r$. From above, S_r is exactly the number of open G_{k+1-r}- orbits in V_{k+1-r}. Using the statements 1, 2(a) and 2(b) we see that $S_1 = 1$, $S_2 = 4$ and $S_{2p+1} = 2$ for $p \geq 1$, $S_{2p} = 5$ for $p \geq 2$. For $k = 2$, we therefore have $S = S_1 + S_2 + S_3 = 7$. If $k = 2p \geq 4$ is even, we get $S = S_1 + S_2 + \sum_{j=1}^{p} S_{2j+1} + \sum_{j=2}^{p} S_{2j} = 5 + 2p + 5(p-1) = 7p$ and for $k = 2p+1 \geq 3$ odd, we get $S = S_1 + S_2 + \sum_{j=1}^{p} S_{2j+1} + \sum_{j=2}^{p+1} S_{2j} = 5 + 2p + 5p = 7p + 5$. This ends the proof of *2.(c)*.

Here the subalgebra $\mathfrak{a}^0$ is equal to $\mathfrak{a}$. Therefore the group L is the group of elements $[\mathbf{g}, \mu]$ where $\mu \in F^*$ and where $\mathbf{g}$ is a diagonal matrix of $GL((k+1), F)$. If $\mathbf{g}$ is the diagonal matrix in $GL(k+1, F)$ whose diagonal elements are $(a_k, \dots a_0)$, then we have $[\mathbf{g}, \mu]. \sum_{j=0}^{k} x_j X_j = \mu^{-1}(a_0^2 x_0 X_0 + \ldots + a_k^2 x_k X_k)$. This proves 2$(d)$ and ends the proof of the Theorem.

□

Theorem 3.8.10. *Suppose that $e = 2$.*

(1) *If $k+1 =$ rank$(\tilde{\mathfrak{g}}) = 2$, then $[F^* : \chi_0(G)] = 2$. This case is case* (9) *and case (2) with $n = 2$ in Table 1. The group G has* 3 *non zero orbits, for which* 2 *are open. A set of representatives of the open orbits is given by $X_0 + vX_1$ where $v \in F^*/\chi_0(G)$. Two generic elements of $\tilde{\mathfrak{g}}^{\lambda_0} \oplus \tilde{\mathfrak{g}}^{\lambda_1}$ are conjugated under the group $L = Z_G(\mathfrak{a}^0)$ if and only if they are G-conjugated.*

(2) *If $k+1 =$ rank$(\tilde{\mathfrak{g}}) \geq 3$ then $e = d = 2$. This case corresponds to case* (2) *in Table 1, namely $\tilde{\mathfrak{g}} = \mathfrak{u}(2n, E, H_n)$ where E is an unramified quadratic extension of F.*
   (a) *If $k+1 =$ rank$(\tilde{\mathfrak{g}})$ is odd then $\chi_0(G) = F^*$ and the group G has a unique open orbit in V^+.*
   (b) *If $k+1 =$ rank$(\tilde{\mathfrak{g}})$ is even then $\chi_0(G) = N_{E/F}(E^*)$ and the group G has two open orbits given by $\mathcal{O}_1 = \{X \in V^+; \Delta_0(X) = 1 \bmod N_{E/F}(E^*)\}$ and $\mathcal{O}_\pi = \{X \in V^+; \Delta_0(X) = \pi \bmod N_{E/F}(E^*)\}$, where π is a uniformizer in F.*
   (c) *For $m \in \{0, \dots, k\}$, two generic elements in V_m^+ are G-conjugated if and only if they are G_m-conjugated. If $k+1 = 2p$ is even then the group G has $3p$ non zero orbits and if $k+1 = 2p+1$ is odd, the group G has $3p+1$ non zero orbits. The*

representatives of these orbits are the representatives of the orbits under G_m of the generic elements of V_m^+ where $m \in \{0, \dots, k\}$.

(d) *Let $X = x_0X_0 + \cdots + x_kX_k$ and $X' = x'_0X_0 + \cdots + x'_kX_k$ be two generic elements of V^+. Then X and X' are L-conjugated if and only if there exists $\mu \in F^*$ such $\mu x_i x'_i \in N_{E/F}(E^*)$ for all $i \in \{0, \dots, k\}$.*

Proof. The case $k+1 = 2$ is a consequence of Proposition 3.8.7.
If $k+1 \geq 3$, then from Table 1, we have $d = e = 2$. Using Remark 1.8.8, we can suppose that $\tilde{\mathfrak{g}}$ is $\mathfrak{u}(2n, E, H_n)$. This algebra can be realized as follows. Let $E = F[\sqrt{u}]$ be a quadratic extension of F where u is a non square unit. We denote by $x \mapsto \bar{x}$ the natural conjugation of E and by $N_{E/F}$ the norm on E defined by associated to this extension $N_{E/F}(x) = x\bar{x}$. We also set $n = k+1$.
Define $S_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Consider the Lie algebra

$$\tilde{\mathfrak{g}} = \{Z \in \mathfrak{sl}(2n, E); ZS_n + S_n\, {}^t\bar{Z} = 0\},$$

which is graded by the element $H_0 = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. This implies that

$$\tilde{\mathfrak{g}} = \left\{ Z = \begin{pmatrix} A & B \\ C & -\,{}^t\bar{A} \end{pmatrix}; A, B, C \in M(n, E)\ \ Tr(A - {}^t\bar{A}) = 0,\ {}^t\bar{B} = -B \text{ and } {}^t\bar{C} = -C \right\}.$$

The subspace V^+ is then isomorphic to the space $\mathrm{Herm}(n, E)$ of hermitian matrices in $M(n, E)$ (ie. matrices B such that ${}^t\bar{B} = B$) through the map

$$B \in \mathrm{Herm}(n, E) \mapsto X(B) = \begin{pmatrix} 0 & \sqrt{u}B \\ 0 & 0 \end{pmatrix} \in V^+.$$

The space $\mathfrak{a}$ is then the space of matrices

$$H(t_0, \dots, t_{n-1}) = \begin{pmatrix} diag(t_{n-1}, \dots, t_0) & 0 \\ 0 & diag(-t_{n-1}, \dots, -t_0) \end{pmatrix},$$

with $(t_0, \dots, t_{n-1}) \in (F^*)^n$ and the roots λ_j are given by $\lambda_j(H(t_0, \dots, t_{n-1})) = 2t_j$.
We fix a basis of $\tilde{\mathfrak{g}}^{\lambda_j}$ by setting $X_j = X(B_j)$ where $B_j \in \mathrm{Herm}(n, E)$ is a diagonal matrix whose coefficient are zero except the coefficient of index $(n-j, n-j)$ which is equal to 1. As in the proof of Theorem 3.8.9, we see easily that $X_0, \dots, X_k$ satisfy conditions of Proposition 3.5.2.

We will now describe the group G.
As $\tilde{\mathfrak{g}} \otimes_F E = \mathfrak{sl}(n, E)$, the group $\mathrm{Aut}_0(\tilde{\mathfrak{g}})$ is the subgroup of $\mathrm{Aut}_0(\mathfrak{sl}(2n, E))$ which stabilizes $\tilde{\mathfrak{g}}$, and hence (using [4] Chap VIII, §13, $n^0 1$, VII, p.189) it is the group of the automorphisms $\mathrm{Ad}(g)$ (conjugation by g) where $g \in GL(2n, E)$ such there exists $\mu \in E^*$ vérifiant ${}^t\bar{g}S_n g = \mu S_n$.
The group G is the subgroup of $\mathrm{Aut}_0(\tilde{\mathfrak{g}})$ of elements which commute with H_0. Therefore an element of G corresponds to the action of $\mathrm{Ad}(g)$ where $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in GL(2n, E)$ is such

that there exists $\mu \in E^*$ satisfying $g_2 = \mu\ {}^t\bar{g_1}^{-1}$ and $g_1 = \mu^t\bar{g_2}^{-1}$. This implies that $\bar{\mu} = \mu$, hence $\mu \in F^*$, and $g_2 = \mu\ {}^t\bar{g_1}^{-1}$. Finally:

$$G = \left\{ \mathrm{Ad}(g);\ g = \begin{pmatrix} \mathbf{g} & 0 \\ 0 & \mu\ {}^t\bar{\mathbf{g}}^{-1} \end{pmatrix}, \mu \in F^*\ \mathbf{g} \in GL(n,E) \right\}.$$

We denote by $[\mathbf{g},\mu]$ such an element of G. The action of $[\mathbf{g},\mu]$ on V^+ corresponds to the action $[\mathbf{g},\mu].B = \mu^{-1}\mathbf{g}B\ {}^t\bar{\mathbf{g}}$ on $\mathrm{Herm}(n,E)$.

The polynomial $B \in \mathrm{Herm}(n,E) \mapsto \det(B) \in F^*$ is relatively invariant under the action of G and we normalize the polynomial Δ_0 onV^+ by setting $\Delta_0(X(B)) = \det(B),\ B \in \mathrm{Herm}(n,E)$. This implies that

$$\Delta_0(x_0X_0 + \ldots x_{n-1}X_{n-1}) = x_0 \ldots x_{n-1}.$$

Therefore

$$\chi_0([\mathbf{g},\mu]) = \mu^{-n}N_{E/F}(\det(\mathbf{g})) \qquad (*)$$

and hence if n is even, we have $\chi_0(G) = N_{E/F}(E^*)$ and if n is odd , we have $\chi_0(G) = F^*$. This is a part of statements $(2)(a)$ and $(2)(b)$

We describe now the G-orbits in V^+. We will prove the results by induction on $n =\ \mathrm{rank}(\tilde{\mathfrak{g}})$.

In what follows, we identify V^+ with $Herm(n,E)$ and we recall that the action of G is given by $[\mathbf{g},\mu].B = \mu^{-1}\mathbf{g}B\ {}^t\bar{\mathbf{g}}$. By Proposition 3.6.1, any generic element of V^+ is G-conjugated to an element of the form $x_0X_0 + \ldots + x_{n-1}X_{n-1}$. Therefore it suffices to study the G-orbits in the space of diagonal matrices (with coefficients in F) under this action. We set $I_\pi = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \pi \end{pmatrix}$.
If $n = 2$ (ie. $k = 1$), one has $\det(I_\pi) = \pi$ and $\det(I_2) = 1$. As $\chi_0(G) = N_{E/F}(E^*)$, we see from relation $(*)$, that the elements I_π and I_2 are not conjugated .

Let $X = \begin{pmatrix} x_1 & 0 \\ 0 & x_0 \end{pmatrix}$ with $x_0x_1 \neq 0$. As $N_{E/F}(E^*) = F^{*2} \cup uF^{*2}$, we obtain that if $x_0x_1 \notin N_{E/F}(E^*)$ then $x_0 = \pi a\bar{a}x_1$ for an element $a \in E^*$ and hence $X = x_1 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} I_\pi \begin{pmatrix} 1 & 0 \\ 0 & \bar{a} \end{pmatrix} = [\mathbf{g}, x_1^{-1}].I_\pi$ where $\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, and therefore X is G-conjugated to I_π.

If $x_0x_1 \in N_{E/F}(E^*)$, then:
- either x_0 and x_1 belong to $N_{E/F}(E^*)$, then $x_0 = a\bar{a}$ and $x_1 = b\bar{b}$ and hence X is conjugated to I_2 by the matrix $\mathbf{g} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$.
- or x_0 and x_1 belong to $\pi N_{E/F}(E^*)$, then $x_0 = \pi a\bar{a}$ and $x_1 = \pi b\bar{b}$ and then $X = [\mathbf{g}, \pi^{-1}].I_2$.
This ends the proof for $n = 2$.

We will need the following result for the induction:

$$\text{there exists } \mathbf{g} \in GL(2,E) \text{ tel que } \pi I_2 = \mathbf{g}\ {}^t\bar{\mathbf{g}}. \qquad (**)$$

(remember that from above πI_2 is either conjugated to I_2 or to I_π, this proves that it is actually conjugated to I_2.)

Suppose that $-1 \in F^{*2}$, then $-1 = \alpha_0^2$ with $\alpha_0 \in F^*$ and we set $\mathbf{g} = \begin{pmatrix} \dfrac{1+\pi}{2} & \dfrac{1-\pi}{2\alpha_0} \\ -\dfrac{1-\pi}{2\alpha_0} & \dfrac{1+\pi}{2} \end{pmatrix}$.

Then

$$\det(\mathbf{g}) = \frac{(1+\pi)^2}{4} - \frac{(1-\pi)^2}{4} = \pi$$

hence $\mathbf{g} \in GL(2,E)$ and

$$\mathbf{g}\,{}^t\bar{\mathbf{g}} = \begin{pmatrix} \dfrac{1+\pi}{2} & \dfrac{1-\pi}{2\alpha_0} \\ -\dfrac{1-\pi}{2\alpha_0} & \dfrac{1+\pi}{2} \end{pmatrix} \begin{pmatrix} \dfrac{1+\pi}{2} & -\dfrac{1-\pi}{2\alpha_0} \\ \dfrac{1-\pi}{2\alpha_0} & \dfrac{1+\pi}{2} \end{pmatrix} = \pi I_2.$$

If $-1 \notin F^{*2}$ then we can take $u = -1$. The quadratic form on F^4 defined by $q(x,y,z,t) = x^2 + y^2 + z^2 + t^2$ is isotropic (because -1 is a sum of two squares by [13], Chapter VI, Corollary 2.6. p. 154) and hence it represents all elements of F^*. Therefore there exist $a, b, c, d \in F$ such that $\pi = a^2 + b^2 + c^2 + d^2$. We set $\alpha = a + \sqrt{u}b$, $\beta = c + \sqrt{u}d$ and $\mathbf{g} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. One has $\det(\mathbf{g}) = |\alpha|^2 + |\beta|^2 = a^2 - ub^2 + c^2 - ud^2 = \pi$ (as $-u = 1$). Finally $\mathbf{g} \in GL(2,E)$ and $\mathbf{g}\,{}^t\bar{\mathbf{g}} = (|\alpha|^2| + |\beta|^2)I_2 = \pi I_2$.

This ends the proof of $(**)$.

We suppose now that the statements 2.(a) and 2.(b) in the Theorem are true if $\operatorname{rank}(\tilde{\mathfrak{g}}) = p \leq n$ and we will prove that they remain true for $n+1$.

Let $X = \begin{pmatrix} x_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_0 \end{pmatrix}$ whis $x_j \in F^*$. We denote by $n(X)$ the cardinality of set $\{j \in \{0, \ldots, , n\}, x_j \in \pi N_{E/F}(E^*)\}$.

Suppose first that $n(X)$ is even.
As there exist automorphisms $\gamma_{i,j}$ interverting X_i and X_j (see Proposition 3.4.3), we can suppose that $x_j \in \pi N_{E/F}(E^*)$ if and only if $j = 0, \ldots, n(X) - 1$ and then $x_j = \pi a_j \bar{a}_j$, and if $j \geq n(X)$, we have $x_j \in N_{E/F}(E^*)$ and hence $x_j = a_j \bar{a}_j$. We denote by $\mathbf{g}_1$ a matrix in $GL(2,E)$ satisfying $(**)$.
Let $\mathbf{g}_\pi$ the block diagonal matrix whose $n - n(X)$ first blocks are just scalars equal to 1 and whose $\frac{1}{2}n(X)$ last blocks are equal to $\mathbf{g}_1$. Let $diag(a_n, \ldots, a_0)$ be the diagonal matrix whose diagonal coefficients are $a_n, \ldots, a_0$. Then $\mathbf{g} = diag(a_n, \ldots, a_0)\mathbf{g}_\pi$ satisfies $\mathbf{g}\,{}^t\bar{\mathbf{g}} = X$, in other words X is G-conjugated to I_{n+1}.

Suppose now that $n(X)$ is odd.

If n is even then $n+1$ is odd, and hence $n(\pi X)$ is even. From above, there exists $\mathbf{g} \in GL(n+1, E)$ such that $\pi X = \mathbf{g}\,{}^t\bar{\mathbf{g}}$. This means that X is G-conjugated to I_{n+1} by the element $[\mathbf{g}, \pi]$.

We have proved that if n is even , any generic element is G-conjugated to I_{n+1}.

If n is odd, as $\det(I_{n+1}) = 1$, $\det(I_\pi) = \pi$ and $\chi_0(G) \subset N_{E/F}(E^*)$, the relation $(*)$ implies that I_{n+1} and I_π are not G-conjugated .

Using again the automorphisms $\gamma_{i,j}$,we can suppose that $x_0 = \pi a\bar{a}$ with $a \in E^*$. We can write $X = \begin{pmatrix} X_1 & 0 \\ 0 & x_0 \end{pmatrix}$ where $X_1 \in M(n, E)$ is the diagonal matrix $diag(x_n, \dots, x_1)$. Then $n(X_1)$ is even, and hence there exists $\mathbf{g}_1 \in GL(n, E)$ tel que $X_1 = \mathbf{g}_1\, {}^t\bar{\mathbf{g}}_1$. If we set $\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 & 0 \\ 0 & a \end{pmatrix} \in GL(n, E)$, we get $X = \mathbf{g} I_\pi\, {}^t\bar{\mathbf{g}}$ and hence X is G-conjugated to I_π.

This ends the prove of statement (2) (a) and (2) (b) of the Theorem.

Let us now prove the statement 2 (c). Let ι be the natural injection of $\mathrm{Herm}(n-m, E)$ into $\mathrm{Herm}(n, E)$ which associates to $M \in \mathrm{Herm}(n-m, E)$ the matrix $\iota(M) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0_m \end{array}\right) \in M(n, E)$. This way we identify the V_m^+ to $\mathrm{Herm}(n-m, E)$ by the map $M \mapsto X(\iota(M))$. An element $X(\iota(M)) \in V_m^+$ is generic in V_m^+ if and only if $\det(M) \neq 0$. The group G_m is the group of elements $g_1 = [\mathbf{g}_1, \mu]$ where $\mathbf{g}_1 \in GL(n-m, E)$ and $\mu \in F^*$, acting on $\mathrm{Herm}(n-m, E)$ by $[\mathbf{g}_1, \mu].M = \mu^{-1}\mathbf{g}_1 M\, {}^t\bar{\mathbf{g}}_1$.

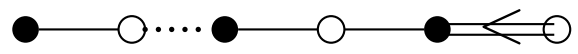
Let $Z = X(\iota(M))$ and $Z' = X(\iota(M'))$ be two generic elements in V_m^+. If Z and Z' are G-conjugated then there exist $\mathbf{g} \in GL(n, E)$ and $\mu \in F^*$ such that $\mathbf{g}\iota(M)\, {}^t\bar{\mathbf{g}} = \mu\,\iota(M')$. Let $\mathbf{g}_1 \in GL(n-m, E)$ be the submatrix of $\mathbf{g}$ given by the first $m-n$ rows and the first $m-n$ first columns of $\mathbf{g}$. An easy computations shows that $\mathbf{g}_1 M\, {}^t\bar{\mathbf{g}}_1 = \mu M'$. As Z and Z' are generic in V_m^+, the matrices M and M' are invertible and therefore $\mathbf{g}_1 \in Gl(n-m, E)$. The preceding relation shows then that Z and Z' are G_m-conjugated.
Conversely, if Z and Z' are G_m-conjugated then there exist $\mathbf{g}_1 \in GL(n-m, E)$ and $\mu \in F^*$ such that $\mathbf{g}_1 M\, {}^t\bar{\mathbf{g}}_1 = \mu M'$. We set b $\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 & 0 \\ 0 & I_m \end{pmatrix}$. The element $[\mathbf{g}, \mu]$ belongs to G and we have $Z' = [\mathbf{g}, \mu].Z$. Hence Z and Z' are G-conjugated. The assertion concerning the number of orbits can be easily proved the same way as the corresponding assertion in Theorem 3.8.9. The statement (2) (c) is now proved.

The group L is the group of elements $l = [\mathbf{g}, \mu]$ where $\mathbf{g} = diag(a_{n-1}, \dots, a_0)$ is a diagonal matrix and where $\mu \in F^*$. For such an element l, we have $l.\sum_j x_j X_j = \sum_j (\mu^{-1} x_j a_j \bar{a}_j) X_j$. The last statement is then easy.
This ends the proof.

□

3.9. **G-orbits in the case $\ell = 3$.**
In this subsection we suppose $\ell = 3$.

By Remark 1.8.8, we can assume that $\widetilde{\mathfrak{g}}$ is simple. If $\ell = 3$ (and rank($\widetilde{\mathfrak{g}}$)$= k+1 > 1$) then it corresponds to case (7) in Table 1 and its Satake-Tits diagram is of type $C_{2(k+1)}$, ($k \in \mathbb{N}$) and is given by



Note that case (5) in Table 1 is a particular case of case (7).
By ([27] Proposition 5.4.5.), $\widetilde{\mathfrak{g}}$ splits over any quadratic extension E of F, and (up to isomorphism) $\widetilde{\mathfrak{g}} \otimes_F E \simeq \mathfrak{sp}(4(k+1), E)$. We will use the following classical realization of $\widetilde{\mathfrak{g}}$ (also used in [17]):

Let F^{*2} the set of squares in F^*. Let u be a unit of F which is not a square and let π be a uniformizer of F. According to [13] (Theorem VI. 2.2, p.152), the set

$$\{1, u, \pi, u\pi\}$$

is a set of representatives of F^*/F^{*2}. We set $E = F[\sqrt{u}]$ and we note $\mapsto \bar{x}$ the conjugation in E. For $n \in \mathbb{N}$, we denote by I_n the identity matrix of size n. Consider the symplectic form Ψ on $E^{4(k+1)}$ defined by

$$\Psi(X,Y) = {}^tXK_{2(k+1)}Y, \quad \text{where} \quad K_{2(k+1)} = \begin{pmatrix} 0 & I_{2(k+1)} \\ -I_{2(k+1)} & 0 \end{pmatrix}.$$

Then

$$\widetilde{\mathfrak{g}} \otimes_F E = \mathfrak{sp}(4(k+1), E) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix};\ A, B, C \in M(2(k+1), E), {}^tB = B, {}^tC = C \right\}.$$

We set:

$$J_\pi = \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} \in M(2, E),\ J = \begin{pmatrix} J_\pi & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_\pi \end{pmatrix} \in M(2(k+1), E),\ T = \begin{pmatrix} J & 0 \\ 0 & {}^tJ \end{pmatrix} \in M(4(k+1), E).$$

The subalgebra $\widetilde{\mathfrak{g}}$ is then the subalgebra of $\widetilde{\mathfrak{g}} \otimes_F E$ whose elements are the matrices X satisfying $T\overline{X} = XT$.

Let us make precise the different objects which have been introduced in earlier section in relation with the Lie algebra $\widetilde{\mathfrak{g}}$.

The algebra $\widetilde{\mathfrak{g}}$ is graded by the element $H_0 = \begin{pmatrix} I_{2(k+1)} & 0 \\ 0 & -I_{2(k+1)} \end{pmatrix}$. More precisely we have

$$V^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix};\ C \in M(2(k+1), E), {}^tC = C,\ {}^tJ\overline{C} = CJ \right\},$$

$$V^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix};\ B \in M(2(k+1), E), {}^tB = B, J\overline{B} = B\,{}^tJ \right\},$$

and

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix};\ A \in M(2(k+1), E), J\overline{A} = AJ \right\}.$$

The subspace $\mathfrak{a}$ defined by

$$\mathfrak{a}=\left\{H(t_0,\dots,t_k)=\begin{pmatrix}\mathbf{H} & 0\\ 0 & -\mathbf{H}\end{pmatrix};\ \text{with } \mathbf{H}=\begin{pmatrix}t_kI_2 & & & \\ & t_{k-1}I_2 & & \\ & & \ddots & \\ & & & t_0I_2\end{pmatrix}\in M(2(k+1),F)\right\}$$

is a maximal split abelian subalgebra of $\widetilde{\mathfrak{g}}$. If the linear forms η_j on $\mathfrak{a}$ are defined by

$$\eta_j(H(t_0,\dots,t_k))=t_j,$$

then the root system of the pair $(\widetilde{\mathfrak{g}},\mathfrak{a})$ is given by

$$\widetilde{\Sigma}=\{\pm\eta_i\pm\eta_j,\ \text{for}\quad 0\le i<j\le k\}\cup\{2\eta_j;\ \text{for}\quad 0\le j\le k\}$$

and the set of strongly orthogonal roots given in Theorem 1.6.1 is the set

$$\{\lambda_0,\dots,\lambda_k\}\quad\text{where}\quad\lambda_j=2\eta_j.$$

We set also

$$\mathbb{S}^+=\left\{X\in M(2,E);\ {}^tX=X, J_\pi\overline{X}=X\,{}^tJ_\pi\right\}=\left\{\begin{pmatrix}\pi\bar{x} & \mu\\ \mu & x\end{pmatrix};x\in E,\mu\in F\right\},$$

$$\mathbb{S}^-=\left\{Y\in M(2,E);\ {}^tY=Y,\ {}^tJ_\pi\overline{Y}=YJ_\pi\right\}=\left\{\begin{pmatrix}y & \mu\\ \mu & \pi\bar{y}\end{pmatrix};y\in E,\mu\in F\right\},$$

and

$$\mathbb{L}=\left\{A\in M(2,E), J_\pi\overline{A}=AJ_\pi\right\}=\left\{\begin{pmatrix}x & \pi y\\ \bar{y} & \bar{x}\end{pmatrix};x,y\in E\right\}.$$

For $M\in M(2,E)$, the matrix $E_{i,i}(M)\in M(2(k+1),E)$ is block diagonal, the $(k+1-i)$-th block being equal to M and all other 2×2 blocks being equal to 0. In other words:

$$E_{i,i}(M)=\begin{pmatrix}X_k & & \\ & \ddots & \\ & & X_0\end{pmatrix}\text{ where } X_j=0 \text{ if }\ i\neq j \text{ and } X_i=M.$$

All the algebras $\widetilde{l}_j$ are isomorphic and given by

$$\widetilde{l}_j=\left\{\begin{pmatrix}E_{j,j}(A) & E_{j,j}(X)\\ E_{j,j}(Y) & E_{j,j}(-\,{}^tA)\end{pmatrix},A\in\mathbb{L},X\in\mathbb{S}^+,Y\in\mathbb{S}^-\right\}.$$

The algebra $\widetilde{\mathfrak{g}}_j$ is the centralizer of $\widetilde{\mathfrak{l}}_0\oplus\ldots\oplus\widetilde{\mathfrak{l}}_{j-1}$. It is given by

$$\widetilde{\mathfrak{g}}_j=\left\{\begin{pmatrix}\mathbf{A}_j & 0 & \mathbf{X}_j & 0\\ 0 & 0 & 0 & 0\\ \mathbf{Y}_j & 0 & -{}^t\mathbf{A}_j & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\in\widetilde{\mathfrak{g}}\right\},$$

where the matrices $\mathbf{A}_j,\mathbf{X}_j$ and $\mathbf{Y}_j$ are square matrices of size $2(k+1-j)$.

Let us now describe the group $G=\mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0)$.
By ([4] Chap VIII, §13, $n^0 3$), we know that the group $\mathrm{Aut}_0(\widetilde{\mathfrak{g}}\otimes_F E)=\mathrm{Aut}(\widetilde{\mathfrak{g}}\otimes_F E)$ is the

group of automorphisms of the form $\varphi_s : X \mapsto sXs^{-1}$ where s is a symplectic similarity of the form Ψ. This means that $s \in GL(4(k+1), E)$ and there exists a scalar $\mu(s) \in E^*$ such that ${}^t s K_{2(k+1)} s = \mu(s) K_{2(k+1)}$. The scalar $\mu(s)$ is called the ratio of the similarity s. It is easily seen that any similarity s with ratio μ such that $\varphi_s(H_0) = H_0$ can be written $s = s(\mathbf{g}, \mu) := \begin{pmatrix} \mathbf{g} & 0 \\ 0 & \mu {}^t\mathbf{g}^{-1} \end{pmatrix}$ with $\mathbf{g} \in GL(2(k+1), E)$ and $\mu \in E^*$. As $s(\mu I_{2(k+1)}, \mu^2) = \mu I_{4(k+1)}$, the elements $s(\mu I_{2(k+1)}, \mu^2)$ with $\mu \in E^*$ act trivially on $\widetilde{\mathfrak{g}} \otimes_F E$. More precisely one can easily show that $\varphi_{s(g,\lambda)} = \mathrm{Id} \iff \exists \mu$ such that $g = \mu I_{2(k+1)}$, and $\lambda = \mu^2$.

Let H_E be the subgroup of $GL(2(k+1), E) \times E^*$ whose elements are of the form $(\mu I_{2(k+1)}, \mu^2)$ with $\mu \in E^*$. Then, from above, the map $(\mathbf{g}, \lambda) \mapsto \varphi_{s(\mathbf{g},\lambda)}$ induces an isomorphism from $GL(2(k+1), E) \times E^*/H_E$ onto the centralizer of H_0 in $\mathrm{Aut}_0(\widetilde{\mathfrak{g}} \otimes_F E)$, which we will denote by G_E. If $(\mathbf{g}, \mu) \in GL(2(k+1), E) \times E^*$, we will denote by $[\mathbf{g}, \mu]$ its class in G_E. The action of G_E on $\widetilde{\mathfrak{g}} \otimes_F E$ stabilizes $V^+ \otimes_F E$ and $V^- \otimes_F E$. More precisely the action on $V^+ \otimes_F E$ is as follows:

$$[\mathbf{g}, \mu] \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu^{-1}\mathbf{g}\mathbf{X}\,{}^t\mathbf{g} \\ 0 & 0 \end{pmatrix}.$$

The group we are interested in, namely $G = \mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{g}})}(H_0)$, is the subgroup of G_E of all elements which normalize $\widetilde{\mathfrak{g}}$. As V^+ and V^- generate $\widetilde{\mathfrak{g}}$, G is also the subgroup of G_E of all elements normalizing V^+ and V^-.

A result in the spirit of the following is indicated without proof in [17] (p. 112).

Proposition 3.9.1. *Let $G^0(2(k+1))$ be the subgroup of elements $\mathbf{g} \in GL(2(k+1), E)$ such that $J\bar{\mathbf{g}} = \mathbf{g}J$. Then*

$$G = \{g = [\mathbf{g}, \mu], \mathbf{g} \in G^0(2(k+1)) \cup \sqrt{u}G^0(2(k+1)), \mu \in F^*\}$$

Proof. We identify V^+ with the space $Sym_J(2(k+1))$ of symmetric matrices $\mathbf{X} \in M(2(k+1), E)$ such that $J\overline{\mathbf{X}} = \mathbf{X}\,{}^tJ$ through the map $\mathbf{X} \mapsto X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix}$.

As the extension $E = F(\sqrt{u})$ is unramified ([13], Chap. VI, Remark 2.7 p. 154), the uniformizer π of F is still a uniformizer in E. We will now define a unit e of E which is not a square (and hence, as before for F, the set $\{1, e, \pi, e\pi\}$ will be a set of representatives of the classes in E^*/E^{*2}).

If $-1 \in F^{*2}$ then one can easily see that $\sqrt{u}$ is not a square in E^*. In this case we set $e = \sqrt{u}$ (of course e is still a unit in E). If $-1 \notin F^{*2}$, we can suppose that $u = -1$. Then by ([13] Corollary V.2.6, p.154) -1 is a sum of two squares in F^*. That is $-1 = x_0^2 + y_0^2$ $(x_0, y_0 \in F^*)$ and in this case we set $e = x_0 + y_0\sqrt{u} \notin E^{*2}$ (again one verifies easily that e is not a square in E^*).

Let $g \in G_E$. From the definition of the group G_E, one sees that there exist $\mathbf{g} \in GL(2(k+1), E)$ and $\mu \in \{1, e, \pi, e\pi\}$ such that $g = [\mathbf{g}, \mu]$. We will now fix such a pair $(\mathbf{g}, \mu)$. If $g \in G$ then $[\mathbf{g}, \mu]V^+ \subset V^+$, and this implies that for all $\mathbf{X} \in Sym_J(2(k+1))$, one has

$$J\overline{\mu^{-1}\mathbf{g}\mathbf{X}\,{}^t\mathbf{g}} = \mu^{-1}\mathbf{g}\mathbf{X}\,{}^t\mathbf{g}\,{}^tJ.$$

As $J^2=\pi I_{4(k+1)}$, we obtain that for all $\mathbf{X}\in Sym_J(2(k+1))$, on has

$$(\mathbf{g}^{-1}J\overline{\mathbf{g}}J^{-1})\mathbf{X}\,{}^t(\mathbf{g}^{-1}J\overline{\mathbf{g}}J^{-1})=\mu^{-1}\overline{\mu}\,\mathbf{X}. \qquad (*)$$

The proposition will then be a consequence of the following Lemma:

Lemma 3.9.2. *Let $M\in M(2(k+1),E)$ and $\nu\in E^*$ such that $M\mathbf{X}\,{}^tM=\nu\mathbf{X}$ for all $\mathbf{X}\in Sym_J(2(k+1))$, then there exists $a\in E^*$ such that $\nu=a^2$ and $M=aI_{2(k+1)}$.*

Proof of Proposition 3.9.1 (Lemma 3.9.2 being assumed)
Remember that we have fixed a pair $(\mathbf{g},\mu)$ in $GL(2(k+1),E)\times\{1,e,\pi,e\pi\}$ such that $g=[\mathbf{g},\mu]$.
• If $\mu=1$ or $\mu=\pi$ then $\mu^{-1}\overline{\mu}=1$. The preceding Lemma and the relation $(*)$ imply $\mathbf{g}^{-1}J\overline{\mathbf{g}}J^{-1}=\pm I_{2(k+1)}$ and hence $J\overline{\mathbf{g}}=\pm\mathbf{g}J$. If $J\overline{\mathbf{g}}=\mathbf{g}J$ then $\mathbf{g}\in G^0(2(k+1))$. If $J\overline{\mathbf{g}}=-\mathbf{g}J$ then $\sqrt{u}\mathbf{g}$ satisfies $J\overline{\sqrt{u}\mathbf{g}}=\sqrt{u}\mathbf{g}J$ and hence $\mathbf{g}\in\sqrt{u}G^0(2(k+1))$. Conversely, it is easy to see that if $\mathbf{g}$ verifies $J\overline{\mathbf{g}}=\pm\mathbf{g}J$ and if μ belongs to F^* then $[\mathbf{g},\mu]$ stabilizes V^+ and V^-.

•• We show now that if $\mu=e$ or $\mu=e\pi$ then the element $[\mathbf{g},\mu]$ of G_E does not belong to G. Suppose that $\mu=e$ or $\mu=e\pi$ and $[\mathbf{g},\mu]\in G$.
If $-1\in F^{*2}$ then $-1=\alpha_0^2$ with $\alpha_0\in F^*$ and we have set $e=\sqrt{u}\notin E^{*2}$. Then $\mu^{-1}\overline{\mu}=-1=\alpha_0^2$. The preceding lemma implies $J\overline{\mathbf{g}}=\epsilon\alpha_0\mathbf{g}J$ with $\epsilon=\pm1$. Taking the conjugate of this equality, one obtains $J\mathbf{g}=\epsilon\alpha_0\overline{\mathbf{g}}J$. And therefore

$$\overline{\mathbf{g}}=\epsilon\alpha_0J^{-1}\mathbf{g}J=\alpha_0^2J^{-2}\overline{\mathbf{g}}J^2=-\overline{\mathbf{g}},$$

and this is impossible as $\mathbf{g}\neq0$.
If $-1\notin F^{*2}$, we have set $u=-1=x_0^2+y_0^2$, with x_0,y_0 in F^*, and $e=x_0+y_0\sqrt{u}$. Then $\mu^{-1}\overline{\mu}=\dfrac{\bar{e}^2}{e\bar{e}}=-\bar{e}^2=(\bar{e}\sqrt{u})^2$. Hence the preceding Lemma gives $J\overline{\mathbf{g}}=\epsilon\bar{e}\sqrt{u}\mathbf{g}J$ with $\epsilon^2=1$, and this implies that $J\mathbf{g}=-\epsilon e\sqrt{u}\overline{\mathbf{g}}J$ and therefore

$$\overline{\mathbf{g}}=\epsilon\bar{e}\sqrt{u}J^{-1}\mathbf{g}J=-u\bar{e}eJ^{-2}\overline{\mathbf{g}}J^2=-\overline{\mathbf{g}}.$$

Again this is impossible as $\mathbf{g}\neq0$.
This proves the proposition. □

Proof of Lemma 3.9.2:
For $k=0$, we have $Sym_J(2,E)=\mathbb{S}^+=\left\{\begin{pmatrix}\pi\overline{x}&\lambda\\\lambda&x\end{pmatrix};x\in E,\lambda\in F\right\}$. Let us set $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL(2,E)$. The relation which is satisfied by M and ν implies that for all $x\in E$ and $\lambda\in F$, one has

$$\left\{\begin{array}{rcl}a^2\pi\overline{x}+b^2x+2ab\lambda&=&\nu\pi\overline{x}\\c^2\pi\overline{x}+d^2x+2dc\lambda&=&\nu x\\ac\pi\overline{x}+bdx+(ad+bc)\lambda&=&\nu\lambda\end{array}\right.$$

The first two equations imply $b=c=0$ and $a^2=d^2=\nu$ and the third equation implies then $a=d$. Therefore it exists $a\in E^*$ such that

$$M=aI_2,\quad\text{and}\quad\nu=a^2\in E^{*2}.$$

On the other hand, a similar computation shows that if $M \in M(2,E)$ satisfies $M\mathbf{X}\,{}^tM = 0$ for all $\mathbf{X} \in \mathbb{S}^+$ then $M = 0$.

By induction we suppose that the expected result is true up to the rank $k-1$. Let $(M,\nu) \in M(2(k+1),E) \times E^*$ satisfying the hypothesis of Lemma 3.9.2. Let us write

$$M = \begin{pmatrix} & M' & & M_k \\ & & & \vdots \\ L_k & \dots & L_1 & M_0 \end{pmatrix},$$

with $M' \in M(2k,E)$ and $M_j, L_j \in M_2(E)$. The equality $M\mathbf{X}\,{}^tM = \nu\mathbf{X}$ for all matrices $\mathbf{X} \in Sym_J(2(k+1),E)$ of the form

$$\mathbf{X} = \begin{pmatrix} & & & 0 \\ & \mathbf{X}' & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & \mathbf{X}_0 \end{pmatrix},$$

(with $\mathbf{X}' \in Sym_J(2k,E)$ and $\mathbf{X}_0 \in Sym_J(2,E)$) implies then that for all $\mathbf{X}' \in Sym_J(2k,E)$ and all $\mathbf{X}_0 \in Sym_J(2,E)$, one has

$$\begin{cases} M'\mathbf{X}'\,{}^tM' = \nu\mathbf{X}' \\ M_0\mathbf{X}_0\,{}^tM_0 = \nu\mathbf{X}_0 \\ M_j\mathbf{X}_0\,{}^tM_j = 0 \quad \text{for} \quad j = 1,\dots k. \end{cases}$$

By induction , there exists $a \in E^*$ such that $\nu = a^2$, $M_k = \dots = M_1 = 0$ and $M' = \epsilon' a I_{2k}$, $M_0 = \epsilon_0 a I_2$ with $\epsilon', \epsilon_0 \in \{\pm 1\}$.

Taking $\mathbf{X}_0 = 0$ and $\mathbf{X}'$ diagonal by blocks, each block being equal to $\mathbf{Y} \in Sym_J(2,E)$, one obtains

$$L_j\mathbf{Y}\,{}^tL_j = 0 \quad \text{for} \quad j = 1,\dots k, \qquad \text{for all} \;\; \mathbf{Y} \in Sym_J(2,E)$$

and hence $L_k = \dots = L_1 = 0$ from the case $k = 0$.

The equality $M\mathbf{X}\,{}^tM = a^2\mathbf{X}$ for all matrices $\mathbf{X} \in Sym_J(2(k+1),E)$ of the form

$$\mathbf{X} = \begin{pmatrix} & & & \mathbf{Y} \\ & 0 & & 0 \\ & & & \vdots \\ {}^t\mathbf{Y} & 0 & \dots & 0 \end{pmatrix},$$

where $\mathbf{Y} \in M(2,E)$ and $J_\pi\overline{\mathbf{Y}} = \mathbf{Y}\,{}^tJ_\pi$ implies then that $\epsilon'\epsilon_0 = 1$.

This ends the proof of Lemma 3.9.2. □

Definition 3.9.3. *The subgroup G^0 of G is defined to be the subgroup of elements $[\mathbf{g},1]$ of G, with $\mathbf{g} \in G^0(2(k+1))$. Hence we have:*

$$\mathrm{Aut}_e(\mathfrak{g}) \subset G^0 \subset G.$$

Remark 3.9.4. Recall that for $j \in \{0, \dots, k\}$, the group G_j (with $G_0 = G$) is the analogue of the group G for the Lie algebra $\tilde{\mathfrak{g}}_j$ and that $G_j \subset \overline{G}$. From the preceding Definition, one has an injection $G_j^0 \hookrightarrow G^0$ given by $[\mathbf{g}_j, 1] \mapsto [\mathbf{g}, 1]$ where $\mathbf{g}_j \in G^0(2(k+1-j))$ and where $\mathbf{g} = \begin{pmatrix} \mathbf{g}_j & 0 \\ 0 & I_j \end{pmatrix}$. In what follows, we will identify G_j^0 with a subgroup of G^0 under this injection.
In particular, an element $g_j \in G_j^0$ will have a trivial action on $\oplus_{s=0}^{j-1} \tilde{\mathfrak{g}}^{\lambda_j}$.
In the same manner, we will denote by L_i^0 the corresponding subgroup for the Lie algebra $\tilde{\mathfrak{l}}_i$.

We will now normalize the relative invariants Δ_j on V_j^+ for $j = 0, \dots, k$. The determinant is an irreducible polynomial on the space of symmetric matrices which satisfies:

$$\det(\mu^{-1}\mathbf{g}\mathbf{X}\,{}^t\mathbf{g}) = \mu^{-2(k+1)}\det(\mathbf{g})^2\det(\mathbf{X}), \quad \mathbf{X} \in Sym_J(2(k+1), E), [\mathbf{g}, \mu] \in G.$$

Therefore one can normalize the fundamental relative invariant Δ_0 by setting

$$\Delta_0 \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} = (-1)^{k+1}\det(\mathbf{X}).$$

The character χ_0 is given by

$$\chi_0(g) = \mu^{-2(k+1)}\det(\mathbf{g})^2$$

for $g = [\mathbf{g}, \mu] \in G$. This implies that

$$\chi_0(G) \subset F^{*2}.$$

Similarly, the fundamental relative invariant δ_0 of $\widetilde{\mathfrak{g}}^{\lambda_0}$ (cf. Definition 1.14.1) is given by

$$\delta_0(X) = -\det(\mathbf{X}); \quad \text{for } X = \begin{pmatrix} 0 & E_{0,0}(\mathbf{X}) \\ 0 & 0 \end{pmatrix}, \ \mathbf{X} \in \mathbb{S}^+.$$

Let us now fix elements $\gamma_{i,j} \in \mathrm{Aut}_e(\mathfrak{g}) \subset G^0$ satisfying the properties of Proposition 3.4.3. Then

$$\gamma_{i,j} \begin{pmatrix} 0 & E_{i,i}(\mathbf{X}) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_{j,j}(\mathbf{X}) \\ 0 & 0 \end{pmatrix}, \text{ for } \mathbf{X} \in \mathbb{S}^+.$$

We normalize the relative invariant polynomials δ_j of $\widetilde{\mathfrak{g}}^{\lambda_j}$ by setting

$$\delta_j(X) = \delta_0(\gamma_{j,0}(X)), \quad X \in \widetilde{\mathfrak{g}}^{\lambda_j}.$$

From Theorem 1.12.4, the L_j-orbits de $\widetilde{\mathfrak{g}}^{\lambda_j} \backslash \{0\}$ are given by

$$\{X \in \widetilde{\mathfrak{g}}^{\lambda_j}; \delta_j(X) \equiv v \bmod F^{*2}\}, \quad v \in (F^*/F^{*2}) \backslash \{-disc(\delta_j)\}\}.$$

Then, we normalize the polynomials Δ_i (defined by their restriction to V_i^+) (cf. Definition 1.14.1) by setting

$$\Delta_i(X_i + \ldots + X_k) = \prod_{i=j}^{k} \delta_j(X_j), \quad X_i \in \widetilde{\mathfrak{g}}^{\lambda_i}.$$

Lemma 3.9.5. *Let* $\mathbf{X}$ *and* $\mathbf{X}'$ *be two elements of* $\mathbb{S}^+$ *such that* $\det(\mathbf{X}) \equiv \det(\mathbf{X}') \bmod F^{*2}$. *Then there exists* $\mathbf{g} \in G^0(2)$ *such that* $\mathbf{X} = \mathbf{g}\mathbf{X}'\,{}^t\mathbf{g}$.

Proof. We consider here the case $k = 0$, that is the algebra $\widetilde{\mathfrak{l}}_0 = \widetilde{\mathfrak{g}}^{-\lambda_0} \oplus \mathfrak{l}_0 \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$ with $\mathfrak{l}_0 = [\widetilde{\mathfrak{g}}^{-\lambda_0}, \widetilde{\mathfrak{g}}^{\lambda_0}] \simeq \mathbb{L}$, $\widetilde{\mathfrak{g}}^{\lambda_0} \simeq \mathbb{S}^+ \simeq F^3$ and $\widetilde{\mathfrak{g}}^{-\lambda_0} \simeq \mathbb{S}^-$. Let Q be the quadratic form on $\mathbb{S}^+$ defined by $Q(\mathbf{Y}) = -\det(\mathbf{Y}) = -\pi a^2 + u\pi b^2 + c^2$ for $\mathbf{Y} = \begin{pmatrix} \pi(a+\sqrt{u}b) & c \\ c & a-\sqrt{u}b \end{pmatrix} \in \mathbb{S}^+$ (where $a, b, c \in F$). Remember also that this form is anisotropic.

Let U be the connected algebraic subgroup of $G^0(2) \subset GL(2,E)$ whose Lie algebra is $\mathfrak{u} = [\mathbb{L}, \mathbb{L}] \subset \mathfrak{sl}(2,E)$. We will first give a surjection from U onto $SO(Q)$. By [24] (Corollary 24.4.5 and Remark 24.2.6) the group U is the intersection of the algebraic subgroups of $GL(2,E)$ whose Lie algebra contains $\mathfrak{u}$. As the Lie algebra of $SL(2,E)$ is $\mathfrak{sl}(2,E)$ (see for example [2], Chap. I, 3.9 (d)), we get that $U \subset SL(2,E)$ and hence the elements in U have determinant 1. We denote by $\Psi_{\mathbf{g}}$ the action of an element $\mathbf{g}$ of U on $\mathbb{S}^+$, in other words $\Psi_{\mathbf{g}}(\mathbf{Y}) = \mathbf{g}\mathbf{Y}\,{}^t\mathbf{g}$. As $U \subset SL(2,E)$ one has $\Psi_{\mathbf{g}} \in O(Q)$. From Lemma 3.9.2 one has $\Psi_{\mathbf{g}} = Id_{\mathbb{S}^+}$ if and only if $\mathbf{g} = \pm I_2$. Therefore the map

$$\begin{array}{rcl} U_1 = U/\{\pm I_2\} & \longrightarrow & O(Q) \\ \mathbf{g} & \longmapsto & \Psi_{\mathbf{g}} \end{array}$$

(defined up to a abuse of notation) is injective.
Let $\overline{F}$ be an algebraic closure of F. It is well known that $SO(Q, \overline{F})$ is connected. As U is connected, the same is true for U_1. Therefore $\Psi(U_1(\overline{F})) \subset SO(Q, \overline{F})$, and Ψ is an isomorphism from U_1 on its image.
From [24] (Theorem 24.4.1), the differential of Ψ is injective. As the Lie algebra of U_1 (which is equal to $\mathfrak{u}$) has the same dimension (3) as $\mathfrak{o}(Q)$, the map Ψ is also a submersion. Hence $\Psi(U_1(\overline{F}))$ is open in $SO(Q, \overline{F})$. By [2] Chap. I, Corollary 1.4, the group $\Psi(U_1(\overline{F}))$ is also closed in $SO(Q, \overline{F})$. Therefore $\Psi(U_1(\overline{F})) = SO(Q, \overline{F})$.

Let $\varphi \in SO(Q)$ and $\widetilde{\mathbf{g}} \in U(\overline{F})$ be such that $\Psi_{\widetilde{\mathbf{g}}} = \varphi$. The element $g = \begin{pmatrix} \widetilde{\mathbf{g}} & 0 \\ 0 & {}^t\widetilde{\mathbf{g}}^{-1} \end{pmatrix}$ normalizes $\widetilde{\mathfrak{g}}^{\lambda_0}$, hence by duality g normalizes $\widetilde{\mathfrak{g}}^{-\lambda_0}$ and therefore it normalizes $\mathfrak{l}_0$ and $[\mathfrak{l}_0, \mathfrak{l}_0]$. This implies that $\widetilde{\mathbf{g}} \in U$. Finally the map $\mathbf{g} \mapsto \Psi_{\mathbf{g}}$ is surjective from U onto $SO(Q)$.

Let us now prove the Lemma. From the assumption, there exists $x \in F^*$ such that $Q(\mathbf{X}) = Q(x\mathbf{X}')$. From Witt's Theorem $SO(Q)$ acts transitively on the set $\{\mathbf{Y} \in \mathbb{S}^+; Q(\mathbf{Y}) = t\}$, hence there exists $\mathbf{g} \in U \subset G^0(2)$ such that $\mathbf{X} = x\mathbf{g}\mathbf{X}'\,{}^t\mathbf{g}$.
In order to prove the Lemma, it is now enough to prove that for all $\mathbf{Y} \in \mathbb{S}^+ \setminus \{0\}$, the elements $u\mathbf{Y}$, $\pi\mathbf{Y}$ and $\mathbf{Y}$ are conjugated under $G^0(2)$, as $F^*/F^{*2} = \{1, u, \pi, u\pi\}$.
Let $\mathbf{Y} \in \mathbb{S}^+ \setminus \{0\}$. As $J_\pi = \pi J_\pi^{-1}$, one has $\pi\mathbf{Y} = J_\pi \overline{\mathbf{Y}}\,{}^tJ_\pi$. And as $\det(\mathbf{Y}) = \det(\overline{\mathbf{Y}})$, the preceding discussion implies that there exists $\mathbf{g} \in U \subset G^0(2)$ such that $\mathbf{g}\mathbf{Y}\,{}^t\mathbf{g} = \overline{\mathbf{Y}}$. As $J_\pi \in G^0(2)$, it follows that $\pi\mathbf{Y}$ and $\mathbf{Y}$ are conjugated by the element $J_\pi\mathbf{g}$ of $G^0(2)$.

Let us prove that $u\mathbf{Y}$ is $G^0(2)$-conjugate to $\mathbf{Y}$. As $-disc(Q)=u$, a system of representatives of the L_0 - orbits in $\tilde{\mathfrak{g}}^{\lambda_0}\simeq \mathbb{S}^+$ is given by

$$X_{1,0}=\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix},\quad X_{0,1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \text{and}\ \ X_{\alpha,0}=\begin{pmatrix}\pi\overline{\alpha} & 0\\ 0 & \alpha\end{pmatrix},$$

where $\alpha\overline{\alpha}=u$ (if $-1=\alpha_0^2\in F^{*2}$ then $\alpha=\alpha_0\sqrt{u}$, and if $-1\notin F^{*2}$ then $u=-1=x_0^2+y_0^2$ with $x_0,y_0\in F^*$ and $\alpha=x_0+y_0\sqrt{u}$) (see Theorem 1.12.4 2)).
By proposition 3.9.1, L_0 is the group of elements $[\mathbf{g},\mu]$ with $\mathbf{g}\in G^0(2)\cup\sqrt{u}G^0(2)$ and $\mu\in F^*$. Hence, there exist $y\in F^*$ and $\mathbf{g}\in G^0(2)$ such that $y\mathbf{g}\mathbf{Y}\,{}^t\mathbf{g}$ equals to $X_{1,0},X_{0,1}$ or $X_{\alpha,0}$. Thus it is enough to prove the result for this set of representatives.

Let $\mathbf{g}_u=\begin{pmatrix}\sqrt{u} & 0\\ 0 & -\sqrt{u}\end{pmatrix}\in G^0(2)$, then $\mathbf{g}_uX_{1,0}\,{}^t\mathbf{g}_u=uX_{1,0}$ and $\mathbf{g}_uX_{\alpha,0}\,{}^t\mathbf{g}_u=uX_{\alpha,0}$. If $\mathbf{g}_\alpha=\begin{pmatrix}\alpha & 0\\ 0 & \overline{\alpha}\end{pmatrix}\in G^0(2)$, then $\mathbf{g}_\alpha X_{0,1}\,{}^t\mathbf{g}_\alpha=uX_{0,1}$.
The Lemma is proved.

□

Corollary 3.9.6. *Let $k\in\mathbb{N}$. Let $X=X_0+\ldots+X_k$ be a generic element of $\oplus_{i=0}^k\tilde{\mathfrak{g}}^{\lambda_i}$. Let $j\in\{0,\ldots,,k\}$ and $X_j'\in\tilde{\mathfrak{g}}^{\lambda_j}$ such that $\delta_j(X_j)=\delta_j(X_j')$ mod F^{*2}. Then $X_0+\ldots X_{j-1}+X_j'+X_{j+1}+\ldots X_k$ is L_j^0-conjugated to X.*

Proof. Let $X_j=\begin{pmatrix}0 & E_{i,i}(\mathbf{X}_j)\\ 0 & 0\end{pmatrix}$ and $X_j'=\begin{pmatrix}0 & E_{i,i}(\mathbf{X}_j')\\ 0 & 0\end{pmatrix}$, where $\mathbf{X}_j$ and $\mathbf{X}_j'$ belong to $\mathbb{S}^+$. Then $\delta_j(X_j)=\delta_j(X_j')$ mod F^{*2} if and only if $\det(\mathbf{X}_j)=\det(\mathbf{X}_j')$ mod F^{*2}. By Lemma 3.9.5, the elements $\mathbf{X}_j$ and $\mathbf{X}'_j$ are $G^0(2)$-conjugated. The result is then a consequence of the definition of L_j^0 (see Remark 3.9.4).

□

Lemma 3.9.7. *Suppose that $k=1$. Let $X=X_0+X_1$ and $Y=Y_0+Y_1$ be two elements of V^+ such that $X_j,Y_j\in\widetilde{\mathfrak{g}}^{\lambda_j}\backslash\{0\}$ and $\delta_0(X_0)\equiv\delta_1(X_1)$ mod F^{*2} and $\delta_0(Y_0)\equiv\delta_1(Y_1)$ mod F^{*2}. Then X and Y are in the same G^0-orbit.*

Proof. The normalization of δ_1, the hypothesis, and Corollary 3.9.6 imply that the elements X_1 and $\gamma_{0,1}(X_0)$(respectively Y_1 and $\gamma_{0,1}(Y_0)$) are in the same L_1^0-orbit. As L_1^0 acts trivially on $\widetilde{\mathfrak{g}}^{\lambda_0}$, one can suppose that $X_1=\gamma_{0,1}(X_0)$ and $Y_1=\gamma_{0,1}(Y_0)$.

If $\delta_0(X_0)\equiv\delta_0(Y_0)$ mod F^{*2} then Corollary 3.9.6 implies that X_j and Y_j are conjugated by an element l_j in L_j^0 for $j=0,1$ and then X and Y are conjugated by $l_0l_1\in G^0$.

We assume now that $\delta_0(X_0)\not\equiv\delta_0(Y_0)$ mod F^{*2}. We can then write

$$X=\begin{pmatrix}0 & \begin{matrix}\mathbf{X} & 0\\ 0 & \mathbf{X}\end{matrix}\\ 0 & 0\end{pmatrix}\quad\text{and}\quad Y=\begin{pmatrix}0 & \begin{matrix}\mathbf{Y} & 0\\ 0 & \mathbf{Y}\end{matrix}\\ 0 & 0\end{pmatrix},$$

with $\mathbf{X}, \mathbf{Y} \in \mathbb{S}^+\backslash\{0\}$, and $\det(\mathbf{X}) \not\equiv \det(\mathbf{Y}) \bmod F^{*2}$.
Consider the polynomial $Q(a,b)$ defined on F^2 by

$$Q(a,b) = \det(a\mathbf{X} - \mathbf{Y}) - b^2\det(\mathbf{X}).$$

Then $Q(a,b) = \det(\mathbf{X})(a^2 - b^2) + 2a\, C_0(\mathbf{X},\mathbf{Y}) + \det(\mathbf{Y})$ where C_0 is a polynomial function in the variables $(\mathbf{X},\mathbf{Y})$. We set $C_1(\mathbf{X},\mathbf{Y}) = (\det(\mathbf{X}))^{-1}C_0(\mathbf{X},\mathbf{Y})$, and hence

$$Q(a,b) = \det(\mathbf{X})(a^2 - b^2) + 2a \det(\mathbf{X})C_1(\mathbf{X},\mathbf{Y}) + \det(\mathbf{Y}).$$

We obtain then

$$Q(a,b) = \det(\mathbf{X})\Big[\big(a + C_1(\mathbf{X},\mathbf{Y})\big)^2 - b^2\Big] + \det(\mathbf{Y}) - \det(\mathbf{X})C_1(\mathbf{X},\mathbf{Y})^2.$$

As the quadratic form $(A,B) \mapsto A^2 - B^2$ is isotropic, it represents all elements of F ([13] Theorem I.3.4 p.10) and hence it exists (A_0, B_0) such that

$$\det(\mathbf{X})(A_0^2 - B_0^2) + \det(\mathbf{Y}) - \det(\mathbf{X})C_1(\mathbf{X},\mathbf{Y})^2 = 0.$$

It follows that the pair $(a_0, b_0) = (A_0 - C_1(\mathbf{X},\mathbf{Y}), B_0)$ satisfies $Q(a_0,b_0) = 0$ or equivalently, $\det(a_0\mathbf{X} - \mathbf{Y}) = b_0^2\det(\mathbf{X})$.
As $\mathbf{X}$ and $\mathbf{Y}$ belong to $\mathbb{S}^+$, we have $\det(a_0\mathbf{X} - \mathbf{Y}) = 0$ if and only if $a_0\mathbf{X} - \mathbf{Y} = 0$. As $\det(\mathbf{X}) \not\equiv \det(\mathbf{Y}) \bmod F^{*2}$, we deduce that $a_0 \neq 0$ and $b_0 \neq 0$. Therefore $a_0\mathbf{X} - \mathbf{Y}$ and $-a_0\mathbf{X}$ are two non zero elements of $\mathbb{S}^+$ such that $\det(a_0\mathbf{X} - \mathbf{Y}) \equiv \det(-a_0\mathbf{X}) \bmod F^{*2}$. By Lemma 3.9.5, it exists $\mathbf{g}_1 \in G^0(2)$ such that $a_0\mathbf{X} - \mathbf{Y} = -a_0\mathbf{g}_1\mathbf{X}\,{}^t\mathbf{g}_1$, which is equivalent to

$$\mathbf{Y} = a_0\mathbf{X} + a_0\mathbf{g}_1\mathbf{X}({}^t\mathbf{g}_1).$$

Let us set

$$\mathbf{g} = \begin{pmatrix} I_2 & \mathbf{g}_1 \\ -\mathbf{X}({}^t\mathbf{g}_1)\mathbf{X}^{-1} & I_2 \end{pmatrix}.$$

As ${}^t\mathbf{X} = \mathbf{X}$, an easy computation shows that:

$$\mathbf{g}\begin{pmatrix} a_0\mathbf{X} & 0 \\ 0 & a_0\mathbf{X} \end{pmatrix}({}^t\mathbf{g}) = \begin{pmatrix} \mathbf{Y} & 0 \\ 0 & \mathbf{Y}' \end{pmatrix}, \text{ with } \mathbf{Y}' = a_0\mathbf{X} + a_0\mathbf{X}({}^t\mathbf{g}_1)\mathbf{X}^{-1}\mathbf{g}_1\mathbf{X} \in \mathbb{S}^+. \qquad (*)$$

As $\mathbf{Y} = a_0\mathbf{X} + a_0\mathbf{g}_1\mathbf{X}({}^t\mathbf{g}_1) \neq 0$, one has $\mathbf{X} \neq -\mathbf{g}_1\mathbf{X}({}^t\mathbf{g}_1)$ and this is equivalent to $\mathbf{X}({}^t\mathbf{g}_1)\mathbf{X}^{-1}\mathbf{g}_1 \neq -I_2$. This implies that $\mathbf{Y}' \neq 0$. Then, by computing the determinant in $(*)$ above, we get $\det(\mathbf{g}) \neq 0$ and $\det(\mathbf{Y}) = \det(\mathbf{Y}')$ modulo F^{*2}.
As $J_\pi\overline{\mathbf{g}_1} = \mathbf{g}_1J_\pi$ and $J_\pi\overline{\mathbf{X}} = \mathbf{X}\,{}^tJ_\pi$, a simple computation shows that $J\overline{\mathbf{g}} = \mathbf{g}J$ and hence $\mathbf{g} \in G^0(4)$.
The relation $\det(\mathbf{Y}) = \det(\mathbf{Y}')$ modulo F^{*2} and Lemma 3.9.5 imply that there exists $\mathbf{g}_0 \in G^0(2)$ such that

$$\mathbf{g}_0\mathbf{Y}'\,{}^t\mathbf{g}_0 = \mathbf{Y}.$$

We set

$$\mathbf{l}_0 := \begin{pmatrix} I_2 & 0 \\ 0 & \mathbf{g}_0 \end{pmatrix} \in G^0(4).$$

Then $g=[\mathbf{g},1]$ and $l_0=[\mathbf{l}_0,1]$ belong to G^0 and satisfy $l_0g(a_0X)=Y$. Using Lemma 3.9.5 again, there exists $\mathbf{g}_2\in G^0(2)$ such that $a_0\mathbf{X}=\mathbf{g}_2\mathbf{X}\ {}^t\mathbf{g}_2$. Taking $\mathbf{g}_2'=\begin{pmatrix}\mathbf{g}_2 & 0\\ 0 & \mathbf{g}_2\end{pmatrix}\in G^0(4)$ and $g_2=[\mathbf{g}_2',1]\in G^0$, we deduce that $l_0gg_2X=Y$ and $l_0gg_2\in G^0$. The Lemma is proved.

□

We are now able to describe the G-orbits in V^+ in the case $\ell=3$.
Let us set:

$$e_1=1,\ e_2=\pi,\ e_3=u\pi$$

and remember that $\{1,\pi,u\pi\}=F^*/F^{*2}\backslash\{-disc(\delta_0)\}$.

For $l\in\{1,2,3\}$ and $X=X_m+\ldots+X_k\in V^+$ with $X_j\in\widetilde{\mathfrak{g}}^{\lambda_j}\backslash\{0\}$, we define

$$n_l(X)=\#\{j\in\{m,\ldots,k\}\text{ such that }\delta_j(X_j)=e_l\bmod F^{*2}\}$$

.

Theorem 3.9.8.
We suppose that $\ell=3$
1) *Let $X=X_0+\ldots+X_k$ and $X'=X_0'+\ldots+X_k'$ be two elements of V^+ such that $X_i\in\widetilde{\mathfrak{g}}^{\lambda_i}\backslash\{0\}$ (resp. $X_j'\in\widetilde{\mathfrak{g}}^{\lambda_j}\backslash\{0\}$). Then the following assertions are equivalent:*
(a) X and X' are in the same G-orbit,
(b) $n_l(X)\equiv n_l(X')$ mod 2 for $l=1,2$ and 3,
(c) X and X' are in the same G^0-orbit.

2) *Suppose that the rank of $\widetilde{\mathfrak{g}}$ is $k+1$. Then the number of G-orbits in V^+ is $4(k+1)$ with 3 open orbits if the rank is 1 (i.e. $k=0$) and 4 open orbits if the rank is ≥ 2 (i.e. $k\geq 1$).*

3) *For $v\in\{1,\pi,u\pi\}$, let us fix a representative $X_0(v)\in\widetilde{\mathfrak{g}}^{\lambda_0}$ of the orbit $\{Y\in\widetilde{\mathfrak{g}}^{\lambda_0};\delta_0(Y)\equiv v\bmod F^{*2}\}$ and, if $k\geq 1$, we set $X_j(v)=\gamma_{0,j}(X_0(v))$, for $j=1,\ldots k$. A set of representatives of the non zero orbits is then:*

$$X_0(1),\quad X_0(\pi),\quad X_0(u\pi),$$

and, if $k\geq 1$, for $m\in\{0,\ldots,k\}$,

$$\begin{aligned}&X_m(1)+\ldots+X_{k-1}(1)+X_k(1),\\&X_m(1)+\ldots+X_{k-1}(1)+X_k(\pi),\\&X_m(1)+\ldots+X_{k-1}(1)+X_k(u\pi),\\&X_m(1)+\ldots+X_{k-2}(1)+X_{k-1}(\pi)+X_k(u\pi).\end{aligned}\quad .$$

(where we assume that if $k=1$ then $X_{k-2}(1)=0$).
For $k\geq 1$, the 4 open orbits are those of the preceding elements where $m=0$. For $k=0$, the 3 open orbits are those of the elements $X_0(1),X_0(\pi)$ and $X_0(u\pi)$.

Proof.
1) Clearly, (c) implies (a).

If X and X' are in the same G-orbit then $\Delta_0(X)=\Delta_0(X')$ mod F^{*2} (because $\chi_0(G)\subset F^{*2}$), and hence

$$\prod_{j=0}^{k}\delta_j(X_j)\equiv\prod_{j=0}^{k}\delta_j(X'_j) \text{ mod } F^{*2}.$$

This implies that

$$\pi^{n_2(X)}(u\pi)^{n_3(X)}=\pi^{n_2(X')}(u\pi)^{n_3(X')} \text{ mod } F^{*2}.$$

Which is the same as:

$$\pi^{n_2(X)+n_3(X)}(u)^{n_3(X)}=\pi^{n_2(X')+n_3(X')}(u)^{n_3(X')} \text{ mod } F^{*2}.$$

And as $F^*/F^{*2}\simeq(\mathbb{Z}/2\mathbb{Z})^2$ this last equality implies that

$$n_2(X)+n_3(X)=n_2(X')+n_3(X') \mod 2 \text{ and } n_3(X)=n_3(X') \mod 2,$$

and therefore $n_2(X)=n_2(X') \mod 2$.
As $n_1(X)+n_2(X)+n_3(X)=n_1(X')+n_2(X')+n_3(X')=k+1$, we obtain also $n_1(X)\equiv n_1(X')$ mod 2. Finally

$$n_\ell(X)=n_\ell(X') \mod 2, \text{for } \ell=1,2,3.$$

Thus (a) implies (b).

Suppose that $n_l(X)=n_l(X')$ mod 2 for $l=1,2$ and 3. We will show by induction on k that X and X' are G^0-conjugated. By Corollary 3.9.6 the result is true for $k=0$.

Suppose now that $k\geq 1$.
• If there exists an $l\in\{1,2,3\}$ such that $n_l(X)\equiv n_l(X')\equiv 1$ mod 2. Then, applying eventually the elements $\gamma_{i,j}\in \mathrm{Aut}_e(\mathfrak{g})\subset G^0$, we can suppose that $\delta_0(X_0)=\delta_0(X'_0)=e_l$. From the case $k=0$ there exists $g_0\in L_0^0$ such that $g_0X'_0=X_0$. As L_0^0 stabilizes the space $\oplus_{j=1}^k\widetilde{\mathfrak{g}}^{\lambda_j}$, we get $g_0X'=X_0+X'_1+\dots X'_k$.
By induction on the elements $X_1+\dots+X_k$ and $X'_1\dots+X'_k$ of V_1^+ there exists $g_1\in G_1^0$ such that $g_1(X'_1+\dots+X'_k)=X_1+\dots+X_k$. As g_1 stabilizes $\widetilde{\mathfrak{g}}^{\lambda_0}$, we obtain $g_1g_0X'=X$ and hence X and X' are G^0-conjugated.

• Suppose that $n_l(X)\equiv n_l(X')\equiv 0$ mod 2 for all $l\in\{1,2,3\}$. This implies that $k\geq 1$ is odd. The case $k=1$ is a consequence of Lemma 3.9.7. Hence we assume $k\geq 3$.
As $n_1(X)+n_2(X)+n_3(X)=k+1$, we cannot have $n_\ell(X)=0$ (or $n_\ell(X')=0$) for all $\ell=1,2,3$. Therefore there exist $r\neq s$ and $r'\neq s'$ such that $\delta_r(X_r)\equiv\delta_s(X_s)$ mod F^{*2} and $\delta_{r'}(X'_{r'})=\delta_{s'}(X'_{s'})$ mod F^{*2} . Using the elements $\gamma_{i,j}\in G^0$ if necessary, we can suppose that $r=r'=k-1$ and $s=s'=k$.
Then $X_{k-1}+X_k$ and $X'_{k-1}+X'_k$ are two elements of V_{k-1}^+ such that

$$\delta_{k-1}(X_{k-1})=\delta(X_k)) \text{ mod } F^{*2} \text{ and } \delta_{k-1}(X'_{k-1})=\delta(X'_k) \text{ mod } F^{*2}.$$

Then by Lemma 3.9.7 there exists $g_{k-1}\in G_{k-1}^0$ such that $g_{k-1}(X'_{k-1}+X'_k)=(X_{k-1}+X_k)$ and hence $g_{k-1}(X')=X'_0+\dots+X'_{k-2}+X_{k-1}+X_k$.

The elements $\widetilde{X}' = \gamma_{0,k-1}\gamma_{1,k}(X'_0 + \ldots + X'_{k-2})$ and $\widetilde{X} = \gamma_{0,k-1}\gamma_{1,k}(X_0 + \ldots + X_{k-2})$ of V_2^+ satisfy the condition $n_l(\widetilde{X}) = n_l(\widetilde{X}') \equiv 0 \bmod 2$ for all $l \in \{1,2,3\}$. By induction (applied to V_2^+) there exists $g'_2 \in G_2^0$ such that $g'_2\widetilde{X}' = \widetilde{X}$.

The element $\gamma_{0,k-1}\gamma_{1,k}(X_{k-1} + X_k) \in \widetilde{\mathfrak{g}}^{\lambda_0} + \widetilde{\mathfrak{g}}^{\lambda_1}$ is fixed by g'_2. We obtain:

$$g'_2\gamma_{0,k-1}\gamma_{1,k}g_{k-1}(X') = g'_2\gamma_{0,k-1}\gamma_{1,k}(X'_0 + \ldots + X'_{k-2} + X_{k-1} + X_k)$$
$$= g'_2(\widetilde{X}') + \gamma_{0,k-1}\gamma_{1,k}(X_{k-1} + X_k) = \widetilde{X} + \gamma_{0,k-1}\gamma_{1,k}(X_{k-1} + X_k) = \gamma_{0,k-1}\gamma_{1,k}(X),$$

and this proves the first assertion.

2) Let $Z \in V^+ \setminus \{0\}$. We know from Theorem 3.2.2 that the element Z is G-conjugated to an element of the form $Z_0 + \ldots + Z_k$ with $Z_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$. Let m be the number of indices j such that $Z_j = 0$. Using the elements $\gamma_{i,j} \in G$ of Proposition 3.4.3, we see that Z is G-conjugated to an element of the form $X = X_m + \ldots + X_k$ with $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}\setminus\{0\}$ for $j = m, \ldots, k$.

Z belongs to an open orbit if and only if $m = 0$ (if not we would have $\Delta_0(Z) = 0$).
If $k = 0$ we have already seen (in Theorem 1.12.4 2)) that the number of open orbits is 3. If $k \geq 1$, the number of open orbits is equal, according to the first assertion, to the number of classes modulo 2 of triples $(n_1(X), n_2(X), n_3(X))$ such that $n_1(X) + n_2(X) + n_3(X) = k + 1$. This number of classes is 4.

3) Suppose $m \neq m'$. Then, according to Theorem 3.5.1, two elements $X = X_m + \ldots + X_k$ with $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j}\setminus\{0\}$ and $Y = Y_{m'} + \ldots + Y_k$ with $Y_i \in \widetilde{\mathfrak{g}}^{\lambda_i}\setminus\{0\}$ are not in the same G-orbit because rang $Q_X \neq$ rang Q_Y.
Finally, to conclude the proof, it will be enough to show that two generic elements $Y = Y_m + \ldots Y_k$ and $Y' = Y'_m + \ldots Y'_k$ of V_m^+ are G-conjugated if and only if they are G_m-conjugated. Let $g = [\mathbf{g}, \mu] \in G$ such that $gY = Y'$. Denote by $\mathbf{g}_1 \in M(2(k-m+1), E)$ the submatrix of $\mathbf{g}$ of the coefficients in the first $k-m+1$ rows and columns. Set

$$\mathbf{g}' = \begin{pmatrix} \mathbf{g}_1 & 0 \\ 0 & I_m \end{pmatrix}$$

A simple block by block computation shows that $[\mathbf{g}', \mu] \cdot Y = Y'$. This implies that $\Delta_m(Y) = \mu^{-2}\det(\mathbf{g}_1)^2\Delta_m(Y')$. As Y and Y' are generic in V_m^+, it follows that $\det(\mathbf{g}_1) \neq 0$. Hence the element $[\mathbf{g}_1, \mu]$ belongs to G_m (see Proposition 3.9.1).
Conversely, if Y and Y' generic in V_m^+ are G_m - conjugate then, by the first assertion, they are G_m^0-conjugate. Since $G_m^0 \subset G^0$, this achieves the proof of the Theorem.

□

4. The symmetric spaces G/H

4.1. **The involutions.**

Let I^+ be a generic element of V^+. By Proposition 1.7.12, there exists $I^- \in V^-$ such that $\{I^-, H_0, I^+\}$ is an $\mathfrak{sl}_2$-triple. The action on $\widetilde{\mathfrak{g}}$ of the non trivial element of the Weyl group of this $\mathfrak{s}l_2$-triple is given by the element $w \in \widetilde{G}$ defined by

$$w = e^{\text{ad } I^+} e^{\text{ad } I^-} e^{\text{ad } I^+} = e^{\text{ad } I^-} e^{\text{ad } I^+} e^{\text{ad } I^-}.$$

We denote by σ the corresponding isomorphism of $\widetilde{\mathfrak{g}}$:

$$\sigma(X) = w.X, \quad X \in \widetilde{\mathfrak{g}}.$$

We denote also by σ the automorphism of $\widetilde{G}$ induced by σ:

$$\sigma(g) = wgw^{-1}, \quad \text{for } g \in \widetilde{G}.$$

If $X \in \widetilde{\mathfrak{g}}$ is nilpotent, then $\sigma(e^{\text{ad } X}) = e^{\text{ad } \sigma(X)}$.

Theorem 4.1.1.
The automorphism σ is an involution of $\widetilde{\mathfrak{g}}$ which satisfies the following properties:

(1) *Define $\mathfrak{h} = \{X \in \mathfrak{g}; \sigma(X) = X\}$. Then $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(I^+) = \mathfrak{z}_{\mathfrak{g}}(I^-)$.*
(2) *Define $\mathfrak{q} = \{X \in \mathfrak{g}; \sigma(X) = -X\}$.Then $\text{ad } I^+$ is an isomorphism from $\mathfrak{q}$ onto V^+ and $\text{ad } I^-$ is an isomorphism from $\mathfrak{q}$ onto V^-.*
(3) *$\sigma(I^+) = I^-$ and $\sigma(V^+) = V^-$. Moreover one has $\sigma(X) = \dfrac{1}{2}\big(\text{ad } I^-\big)^2 X$, for $X \in V^+$ and $\sigma(X) = \dfrac{1}{2}\big(\text{ad } I^+\big)^2 X$, for $X \in V^-$.*
(4) *$\sigma(H_0) = -H_0$ and $\sigma(\mathfrak{g}) = \mathfrak{g}$. Moreover, for $X \in \mathfrak{g}$, one has $\sigma(X) = X + (\text{ad } I^- \text{ ad } I^+)X$.*

Proof. For the convenience of the reader we give the proof although it is the same as for the real case (See [7]). It is just elementary representation theory of the $\mathfrak{sl}_2$-triple $\{I^-, H_0, I^+\}$. The irreducible components of $\widetilde{\mathfrak{g}}$ under the action of this $\mathfrak{s}l_2$-triple are of dimension 1 or 3 (because the weights of the primitive elements are 0 or 2). The action of w^2 is trivial on each of these components as they have odd dimension (see section 3.1). Hence σ is an involution of $\widetilde{\mathfrak{g}}$.

The subalgebra $\mathfrak{g}$ is the sum of the 0-weight spaces of theses irreducible components. If the dimension of the component is 1 (respectively 3) then the action of w is trivial (respectively multiplication by -1). Therefore $\mathfrak{h}$ is the sum of the irreducible components of dimension 1, and this proves the assertion (1), and $\mathfrak{q}$ is the sum of the 0-weight spaces of the irreducible components of dimension 3, and this proves the assertion (2).

The space V^+ is the sum of the sum of the subspaces of primitive elements of the irreducible components of dimension 3. Hence the action of w on V^+ is given by $\frac{1}{2}(\text{ad } I^-)^2$ (see section 3.1). This implies that $\sigma(I^+) = I^-$ and $\sigma(V^+) = V^-$.
If $X \in V^-$ then $Y = (\text{ad } I^+)^2 X$ belongs to V^+ and $\sigma(Y) = (\text{ad } \sigma(I^+))^2 \sigma(X) = (\text{ad } I^-)^2 \sigma(X)$. From the preceding discussion, we obtain $\sigma(Y) = \frac{1}{2}(\text{ad } I^-)^2(Y)$. As $(\text{ad } I^-)^2$ is injective on V^+, we get

$$\sigma(X) = \frac{1}{2}(\text{ad } I^+)^2 X.$$

The assertion (3) is now proved.

As $H_0 = [I^-, I^+]$, we have $\sigma(H_0) = -H_0$ and therefore $\sigma(\mathfrak{g}) = \mathfrak{g}$. If $X \in \mathfrak{h}$, we have $\operatorname{ad} I^- \operatorname{ad} I^+ X = 0$ and this means that $\sigma(X) = X = X + \operatorname{ad} I^- \operatorname{ad} I^+ X$.
If $X \in \mathfrak{q} \setminus \{0\}$, then $\operatorname{ad} I^+ X$ is a non zero element of V^+. Hence it is a primitive element of an irreducible component of dimension 3. Therefore $\operatorname{ad} I^+ \operatorname{ad} I^- \operatorname{ad} I^+ X = -2 \operatorname{ad} I^+ X$. As $\operatorname{ad} I^+$ is injective on $\mathfrak{q}$, we obtain $\operatorname{ad} I^- \operatorname{ad} I^+ X = -2X$. And hence

$$w.X = -X = X + \operatorname{ad} I^- \operatorname{ad} I^+ X.$$

This proves (4). □

Definition 4.1.2. *An $\mathfrak{sl}_2$-triple $\{I^-, H_0, I^+\}$ is called a diagonal $\mathfrak{sl}_2$-triple if $I^+ = X_0 + \ldots X_k$ is a generic element of V^+ such that $X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and if $I^- = Y_0 + \ldots + Y_k$ where $Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j} \setminus \{0\}$ and $\{Y_j, H_{\lambda_j} X_j\}$ is an $\mathfrak{sl}_2$-triple for all $j \in \{0, \ldots, k\}$.*

Remark 4.1.3. Any open G-orbit in V^+ contains an element I^+ which can be put in a diagonal $\mathfrak{sl}_2$-triple $\{I^-, H_0, I^+\}$ (this is a consequence of Theorem 3.2.2).

For the rest of this section, we fix a diagonal $\mathfrak{sl}_2$-triple $\{I^-, H_0, I^+\}$ and we will denote by σ the corresponding involution of $\widetilde{\mathfrak{g}}$.
Recall also that $\mathfrak{a}$ is a maximal split abelian subalgebra of $\mathfrak{g}$ containing H_0 and that $\mathfrak{a}^0$ is the subspace of $\mathfrak{a}$ defined by

$$\mathfrak{a}^0 = \oplus_{j=0}^k F H_{\lambda_j}.$$

Definition 4.1.4. *A maximal split abelian subalgebra of $\mathfrak{q}$ is called a Cartan subspace of $\mathfrak{q}$.*

Lemma 4.1.5.
The maximal split abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ satisfies $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} \oplus \mathfrak{a}^0$. Moreover $\mathfrak{a} \cap \mathfrak{h} = \{H \in \mathfrak{a}; \lambda_j(H) = 0 \text{ for } j = 0, \ldots, k\}$.
The subalgebra $\mathfrak{a}^0$ is a Cartan subspace of $\mathfrak{q}$.

Proof. From Theorem 4.1.1 (4), for $H \in \mathfrak{a}$, we get $\sigma(H) = H + (\operatorname{ad} I^- \operatorname{ad} I^+) H = H - \sum_{j=0}^k \lambda_j(H) H_{\lambda_j}$. This proves that $\mathfrak{a}$ is σ-stable and also the given decomposition of $\mathfrak{a}$.

Of course $\mathfrak{a}^0$ is a split abelian subalgebra of $\mathfrak{q}$. It remains to show that $\mathfrak{a}^0$ is maximal among such subalgebras. Let X be an element of $\mathfrak{q}$ such that $\mathfrak{a}^0 + FX$ is split abelian in $\mathfrak{q}$. From the root space decomposition of $\mathfrak{g}$ relatively to Σ, we get

$$X = U + \sum_{\lambda \in \Sigma} X_\lambda, \qquad \text{where } U \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \text{ and } X_\lambda \in \mathfrak{g}^\lambda.$$

As X centralizes $\mathfrak{a}^0$, if $X_\lambda \neq 0$ for $\lambda \in \Sigma$, we obtain that $\lambda_{|_{\mathfrak{a}^0}} = 0$. Corollary 1.8.2 implies now that λ is strongly orthogonal to all roots λ_j and hence $\operatorname{ad} I^+ X_\lambda = 0$. Therefore $X_\lambda \in \mathfrak{h}$. As $\sigma(U)$ belongs to $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ and $\sigma(X) = -X$, we have $X = U$. This implies (maximality of $\mathfrak{a}$) that $X \in \mathfrak{a}$ and hence $X \in \mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}^0$. This proves that $\mathfrak{a}^0$ is a Cartan subspace of $\mathfrak{q}$.

□

Lemma 4.1.6. *Let $\underline{\sigma}$ be the involution of $\widetilde{\mathfrak{g}}$ defined by $\underline{\sigma}(X) = \sigma(X)$ for $X \in \mathfrak{g}$ and by $\underline{\sigma}(X) = -\sigma(X)$ for $X \in V^- \oplus V^+$. Let $\widetilde{\mathfrak{q}} = \{X \in \widetilde{\mathfrak{g}}, \underline{\sigma}(X) = -X\}$. Then $\mathfrak{a}^0$ is a Cartan subspace of $\widetilde{\mathfrak{q}}$.*

Proof. As $\sigma = \underline{\sigma}$ on $\mathfrak{g}$, the space $\mathfrak{a}^0$ is a split abelian subspace of $\widetilde{\mathfrak{q}}$. It remains to prove the maximality. Let $X \in \widetilde{\mathfrak{q}}$ such that $\mathfrak{a}^0 + FX$ is abelian split. Then X commutes with $\mathfrak{a}^0$ and hence with H_0. Therefore $X \in \mathfrak{g}$. Then $X \in \mathfrak{a}^0$ by Lemma 4.1.5. □

Remark 4.1.7. From [10] (Proposition 5.9), the set of roots $\Sigma(\widetilde{\mathfrak{g}}, \mathfrak{a}^0)$ of $\widetilde{\mathfrak{g}}$ with respect to $\mathfrak{a}^0$ is a root system which will be denoted by $\widetilde{\Sigma}^0$. The decomposition $\widetilde{\mathfrak{g}}$ given in Theorem 1.8.1:

$$\widetilde{\mathfrak{g}} = \mathfrak{z}_{\widetilde{\mathfrak{g}}}(\mathfrak{a}^0) \oplus \Big(\oplus_{0\leq i<j\leq k} E_{i,j}(\pm 1, \pm 1)\Big) \oplus \Big(\oplus_{j=0}^{k} \widetilde{\mathfrak{g}}^{\lambda_j}\Big),$$

is in fact the root space decomposition associated to the root system $\widetilde{\Sigma}^0$. Setting

$$\eta_j(H_{\lambda_i}) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

we obtain

$$\widetilde{\Sigma}^0 = \{\pm\eta_i \pm \eta_j (0 \leq i < j \leq k), \pm 2\eta_j (0 \leq j \leq k)\}.$$

And this shows that $\widetilde{\Sigma}^0$ is a root system of type C_{k+1}. In fact, as seen in the next Proposition, this root system is the root system of a subalgebra $\tilde{\mathfrak{g}}_{C_{k+1}} \subset \tilde{\mathfrak{g}}$, which is isomorphic to $\mathfrak{sp}(2(k+1), F)$ and which contains $\mathfrak{a}^0$ as a maximal split abelian subalgebra.

Proposition 4.1.8.
For $0 \leq i \leq k-1$, consider a family of $\mathfrak{sl}_2$-triples $(B_i, H_{\lambda_i} - H_{\lambda_{i+1}}, A_i)$ where $B_i \in E_{i,i+1}(-1, 1)$ and $A \in E_{i,i+1}(1, -1)$ (such triples exist by Lemma 3.4.2). Consider also an $\mathfrak{sl}_2$-triple of the form $(B_k, H_{\lambda_k}, A_k)$, where $B_k \in \tilde{\mathfrak{g}}^{-\lambda_k}$ and $A_k \in \tilde{\mathfrak{g}}^{\lambda_k}$. Then these $k+1$ $\mathfrak{sl}_2$-triples generate a subalgebra $\tilde{\mathfrak{g}}_{C_{k+1}} \subset \tilde{\mathfrak{g}}$ which is isomorphic to $\mathfrak{sp}(2(k+1), F)$ and which contains $\mathfrak{a}^0$ as a maximal split abelian subalgebra.

Proof. The linear forms $\eta_0 - \eta_1, \eta_1 - \eta_2, \ldots, \eta_{k-1} - \eta_k, 2\eta_k$ form a basis of the root system $\widetilde{\Sigma}^0$ which is of type C_{k+1} as seen in the preceding Remark. As the η_i's form the dual basis of the H_{λ_i}'s, it is well known that the set of elements $\{H_{\lambda_0} - H_{\lambda_1}, H_{\lambda_1} - H_{\lambda_2}, \ldots, H_{\lambda_{k-1}} - H_{\lambda_k}, H_{\lambda_k}\}$ is a basis of the dual root system in $\mathfrak{a}^0$, which is of course of type B_{k+1}. Define $H_i = H_{\lambda_i} - H_{\lambda_{i+1}}$ with $0 \leq i \leq k-1$, and $H_k = H_{\lambda_k}$. As usual we define also $n(\alpha, \beta) = \alpha(H_\beta)$, for $\alpha, \beta \in \widetilde{\Sigma}^0$ and where $H_\beta \in \mathfrak{a}^0$ is the coroot of β. It is also convenient to set $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i = 0, \ldots, k-1$, and $\alpha_k = \lambda_k$.
Then the generators satisfy the following relations, for $i, j \in \{0, \ldots, k\}$:
(1) $[H_i, H_j] = 0$,
(2) $[B_i, A_j] = \delta_{i,j} H_i$,
(3) $[H_i, A_j] = n(\alpha_j, \alpha_i) A_j$,
(3′) $[H_i, B_j] = -n(\alpha_j, \alpha_i) B_j$,
(4) $(\operatorname{ad} B_j)^{-n(\alpha_i,\alpha_j)+1} B_i = 0$ if $i \neq j$,
(5) $(\operatorname{ad} A_j)^{-n(\alpha_i,\alpha_j)+1} A_i = 0$ if $i \neq j$.
The relations (1), (2), (3), (3′) are obvious. Let us show relation (4). The $\mathfrak{sl}_2$-triple (B_j, H_j, A_j) defines a structure of finite dimensional $\mathfrak{sl}_2$-module on $\tilde{\mathfrak{g}}$. We have $[A_j, B_i] = 0$ and $[H_j, B_i] = -n(\alpha_i, \alpha_j) B_i$ by relations (2) and (3′). This means that B_i is a primitive vector of weight $-n(\alpha_i, \alpha_j)$. Therefore B_i generates an $\mathfrak{sl}_2$-module of dimension $-n(\alpha_i, \alpha_j) + 1$. And this implies (4). The same argument proves (5).

The above relations are the well known Serre relations for $\mathfrak{sp}(2(k+1), F)$. Hence the algebra generated by these elements is isomorphic to $\mathfrak{sp}(2(k+1), F)$.

□

Remark 4.1.9. The preceding construction of the subalgebra $\tilde{\mathfrak{g}}_{C_{k+1}}$ uses the same argument as the construction of the so-called "admissible" subalgebras (see [22], Théorème 3.1 p.273).

4.2. The minimal σ-split parabolic subgroup P of G.

Definition 4.2.1. ([10]) *A parabolic subgroup R of G (resp. a parabolic subalgebra $\mathfrak{r}$ of $\mathfrak{g}$) is called a σ-split parabolic subgroup of G (resp. a σ-split parabolic subalgebra of $\mathfrak{g}$) if $\sigma(R)$ (resp. $\sigma(\mathfrak{r})$) is the opposite parabolic subgroup (resp. parabolic subalgebra) of R (resp. of $\mathfrak{r}$).*

Let $\{I^-, H_0, I^+\}$ be a diagonal $\mathfrak{sl}_2$-triple (see Definition 4.1.2). As above we denote by σ the involution of $\widetilde{\mathfrak{g}}$ associated to this triple. Hence we have the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(I^+) = \mathfrak{z}_{\mathfrak{g}}(I^-)$. We also denote by σ the involution of $\widetilde{G} = \mathrm{Aut}_0(\widetilde{\mathfrak{g}})$ given by the conjugation by the element

$$w = e^{\operatorname{ad} I^+} e^{\operatorname{ad} I^-} e^{\operatorname{ad} I^+} = e^{\operatorname{ad} I^-} e^{\operatorname{ad} I^+} e^{\operatorname{ad} I^-}.$$

As $\sigma(H_0) = -H_0$, the group G is invariant under the action of σ. Let $G^\sigma \subset G$ be the fixed point group under σ. The Lie algebra of G^σ is equal to $\mathfrak{h}$. Define $H = Z_G(I^+)$. Then the Lie algebra of H is $\mathfrak{h}$ and hence H is an open subgroup of G^σ.

Consider the subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ defined by

$$\mathfrak{p} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \oplus \Big(\oplus_{0 \le i < j \le k} E_{i,j}(1, -1) \Big).$$

Proposition 4.2.2.
The subalgebra $\mathfrak{p}$ is a minimal σ-split parabolic subalgebra of $\mathfrak{g}$. Its Langlands decomposition is given by $\mathfrak{p} = \mathfrak{l} + \mathfrak{n} = \mathfrak{m}_1 \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}$ where

$$\begin{cases} \mathfrak{n} = \oplus_{0 \le i < j \le k} E_{i,j}(1, -1) \\ \mathfrak{l} = \mathfrak{m}_1 \oplus \mathfrak{a}_{\mathfrak{p}} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) \\ \mathfrak{a}_{\mathfrak{p}} = \{H \in \mathfrak{a}; (\lambda \in \Sigma, \lambda(\mathfrak{a}^0) = 0) \Longrightarrow \lambda(H) = 0\} \\ \mathfrak{m}_1 \textit{ is the orthogonal of } \mathfrak{a}_{\mathfrak{p}} \text{ in } \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0) = \mathfrak{m}_1 \oplus \mathfrak{a}_{\mathfrak{p}} \end{cases}.$$

Proof. The Theorem 1.8.1 and the Proposition 1.9.1 imply that $\mathfrak{p}$ contains all root spaces corresponding to the negative roots in Σ. Hence $\mathfrak{p}$ contains a minimal parabolic subalgebra of $\mathfrak{g}$. Therefore $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$.

Let Γ be the set of roots $\lambda \in \Sigma$ such that $\mathfrak{g}^\lambda \subset \mathfrak{p}$. From the definition of $\mathfrak{p}$, one has

$$\Gamma = \Sigma^- \cup \{\alpha \in \Sigma^+; \lambda(\mathfrak{a}^0) = 0\} \text{ and } \mathfrak{p} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \big(\oplus_{\lambda \in \Gamma} \mathfrak{g}^\lambda \big).$$

It follows that $\Gamma \cap -\Gamma = \{\lambda \in \Sigma; \lambda(\mathfrak{a}^0) = 0\}$. And then $\mathfrak{n} = \oplus_{\lambda \in \Gamma \setminus (\Gamma \cap -\Gamma)} \mathfrak{g}^\lambda = \oplus_{0 \le i < j \le k} E_{i,j}(1, -1)$ is the nilradical of $\mathfrak{p}$. If $H \in \mathfrak{a}^0$, then $\sigma(H) = -H$. Therefore if $X \in E_{i,j}(1, -1)$ then $\sigma(X) \in E_{i,j}(-1, 1)$. Then $\mathfrak{l} = \sigma(\mathfrak{p}) \cap \mathfrak{p} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$ is a σ-stable Levi component of $\mathfrak{p}$ and $\mathfrak{p}$ is a σ-split parabolic subalgebra. Let $\mathfrak{a}_p$ be the maximal split abelian subalgebra of the center of

$\mathfrak{l}$. Then $\mathfrak{a}_p = \cap_{\lambda\in\Gamma\cap-\Gamma}\ker(\lambda)$. Hence the Langlands decomposition is given by $\mathfrak{p} = \mathfrak{m}_1 \oplus \mathfrak{a}_p \oplus \mathfrak{n}$ where $\mathfrak{m}_1$ is the orthogonal of $\mathfrak{a}_p$ in $\mathfrak{l}$ for the Killing form.
As $\mathfrak{a}^0$ is a Cartan subspace of $\mathfrak{q}$ and as $\sigma(\mathfrak{p})\cap\mathfrak{p} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$, Proposition 4.7 (iv) of [10] (see also Proposition 1.13 of [9]) implies that $\mathfrak{p}$ is a minimal σ-split parabolic subalgebra of $\mathfrak{g}$.

□

Let $N = \exp^{\mathrm{ad}\,\mathfrak{n}} \subset G$ and $L = Z_G(\mathfrak{a}^0)$. Then $P = LN$ is a parabolic subgroup of G. From the above discussion P is in fact a minimal σ-split parabolic subgroup of G with σ-stable Levi component $L = P\cap\sigma(P)$ and with nilradical N.

We denote by A and A^0 the split tori of G whose Lie algebras are respectively $\mathfrak{a}$ and $\mathfrak{a}^0$. A is a maximal split torus of G and the set of weights of A in $\mathfrak{g}$ is a root system $\Phi(G,A)$ isomorphic to Σ. More precisely, any root in Σ is the differential of a unique root in $\Phi(G,A)$. In the sequel of the paper we will always identify these two root systems. For $\lambda\in\Sigma$ and $a\in A$, we will denote by a^λ the eigenvalue of the action of a on $\mathfrak{g}^\lambda$.

4.3. **The prehomogeneous vector space (P, V^+).**

For the convenience of the reader, and although it will be a consequence of the proof of Theorem 4.3.2, let us give a simple proof of the prehomogeneity of (P, V^+).

Proposition 4.3.1.
The representation (P, V^+) is prehomogeneous.

Proof. Remember that prehomegeneity is an infinitesimal condition. Therefore it is enough to prove that $(\overline{\mathfrak{p}}, \overline{V}^+)$ is prehomogeneous. But the parabolic subalgebra $\overline{\mathfrak{p}}$ of $\overline{\mathfrak{g}}$ will contain a Borel subalgebra $\overline{\mathfrak{b}}$. The space $(\overline{\mathfrak{b}}, \overline{V}^+)$ is prehomogeneous by [18], Prop. 3.8 p. 112 (the proof is the same over $\overline{F}$ as over $\mathbb{C}$).

□

We will now show that the polynomial Δ_j (see Definition 1.14.1) are the fundamental relative invariants of this prehomogeneous vector space .

Recall ℓ is the common dimension of the $\widetilde{\mathfrak{g}}^{\lambda_j}$'s and that ℓ is either a square or equal to 3 (cf. Theorem 1.12.4).

Theorem 4.3.2.
The polynomials Δ_j are irreducible, and relatively invariant under the action of P: there exists a rational character χ_j of P such that

$$\Delta_j(p.X) = \chi_j(p)\Delta_j(X), \quad X\in V^+, p\in P$$

More precisely:

- $\chi_j(n) = 1, \quad n\in N,$
- $\chi_j(a) = a^{\kappa(\lambda_j+\ldots+\lambda_k)}, \quad a\in A,$
- $\chi_j(m) = 1, \quad m\in L\cap H. \quad (H = Z_G(I^+))$

Proof. The fact that the Δ_j's are irreducible, and their invariance under N, have already been obtained in Theorem 1.14.2 and its proof.
We have $A \subset G_j$ and $a \in A$ acts by a^{λ_j} on $\widetilde{\mathfrak{g}}^{\lambda_j}$. Therefore by Theorem 1.14.2 (3) we get

$$\Delta_j(a(X_0 + \ldots + X_k)) = \Delta_j(\sum_{s=0}^{k} a^{\lambda_s} X_s) = \prod_{s=j}^{k} a^{\kappa\lambda_s} \Delta_j(X_0 + \ldots + X_k),$$

for $X_s \in \widetilde{\mathfrak{g}}^{\lambda_s} \setminus \{0\}$. (This gives the value of χ_j on A).
Let us show now that Δ_j is relatively invariant under L (this is not given by Theorem 1.14.2 because L is in general not a subgroup of G_j).
Let $Z = X + Y \in V^+$ where $X \in V_j^+$ and $Y \in V_j^\perp$ ($V_j^\perp$ was defined at the beginning of section 1.14). By Theorem 1.14.2 (4), there exists a constant c such that $\Delta_j(X + Y) = \Delta_j(X) = c\Delta_0(X_0 + \ldots + X_{j-1} + X)$. An element of L normalizes V_j^+, $V_j^\perp \otimes \overline{F}$ and each root space $\widetilde{\mathfrak{g}}^{\lambda_j}$. As Δ_0 is relatively invariant under $\overline{G}$ and $L \subset \overline{G}$, we get for $m \in L$:

$$\Delta_j(m(X + Y)) = c\Delta_0(X_0 + \ldots + X_{j-1} + mX) = c\chi_0(m)\Delta_0(m^{-1}X_0 + \ldots + m^{-1}X_{j-1} + X).$$

Again by Theorem 1.14.2 (4), there exists a constant $c_j(m)$ such that $\Delta_0(m^{-1}X_0 + \ldots + m^{-1}X_{j-1}+X) = c_j(m)\Delta_j(X) = c_j(m)\Delta_j(X+Y)$. Therefore $\Delta_j(m(X+Y)) = c\, c_j(m)\chi_0(m)\Delta_j(X+Y)$, and hence Δ_j is relatively invariant under L.
This proves that the Δ_j's are relatively invariant under the parabolic subgroup P.
Let $m \in L \cap H$. Then $\Delta_j(mI^+) = \Delta_j(I^+) = \chi_j(m)\Delta_j(I^+)$. Hence $\chi_j(m) = 1$.

□

We define the dense open subset of V^+ as follows:

$$\mathcal{O}^+ := \{X \in V^+; \Delta_0(X)\Delta_1(X)\ldots\Delta_k(X) \neq 0\}.$$

We will now prove that $\mathcal{O}^+$ is the union of the open P-orbits of V^+.

Lemma 4.3.3. *Any element of $\mathcal{O}^+$ is conjugated under N to an element of $\oplus_{j=0}^k(\widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\})$.*

Proof. Let $X \in \mathcal{O}^+$. As $\Delta_1(X) \neq 0$, we know from the proof of Proposition 3.2.1, that there exists $Z \in \mathfrak{g}$ such that $[H_{\lambda_1}+\ldots H_{\lambda_k}, Z] = -Z$ (and hence $[H_{\lambda_0}, Z] = Z$,) and $e^{\operatorname{ad} Z}X \in V_1^+ \oplus \widetilde{\mathfrak{g}}^{\lambda_0}$. Therefore $Z \in \oplus_{j=1}^k E_{0,j}(1, -1) \subset \mathfrak{n}$. Let $X^1 \in V_1^+$ and $X^0 \in \widetilde{\mathfrak{g}}^{\lambda_0}$ such that $e^{\operatorname{ad} Z}X = X^0 + X^1$. Then, for $j \geq 1$, $\Delta_j(X) = \Delta_j(X^0 + X^1) = \Delta_j(X^1)$ as $X^0 \in V_1^\perp$.
As $X \in \mathcal{O}^+$, we have $\Delta_j(X^1) \neq 0$ for $j \geq 1$. Then, by induction, we obtain that X is N-conjugated to an element of $\oplus_{j=0}^k(\widetilde{\mathfrak{g}}^{\lambda_j})$. And as X is generic for the G action, we see that in fact X in N-conjugated to an element of $\oplus_{j=0}^k(\widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\})$.

□

Remark 4.3.4. Applying this Lemma to $\widetilde{\mathfrak{g}}_j$, we see that if $X \in V_j^+$ such $\Delta_s(X) \neq 0$ for $s \geq j$, then X is N-conjugated to an element of the form $Y_j + X^{j+1}$ with $X^{j+1} \in V_{j+1}^+$ and $Y_j \in \widetilde{\mathfrak{g}}^{\lambda_j}$.

Remark 4.3.5. (Normalization)

Suppose $\ell=3$. Recall that L_j is the analogue of the group G for the graded Lie algebra $\widetilde{\mathfrak{l}}_j$, that is $L_j=\mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{l}}_j)}(H_{\lambda_j})$. In the following Theorem we denote by δ_j a choice of a relative invariant of the prehomogeneous space $(L_j,\widetilde{\mathfrak{g}}^{\lambda_j})$. And then we normalize the Δ_j's in such a way that

$$\Delta_j(X_j+X_{j+1}+\ldots+X_k)=\delta_j(X_j)\Delta_{j+1}(X_{j+1}+\ldots+X_k)=\delta_j(X_j)\ldots\delta_k(X_k)$$

for $X_j\in\widetilde{\mathfrak{g}}^{\lambda_j}\setminus\{0\}$ and for $j=0,\ldots,k$. Conversely one could also choose arbitrarily the Δ_j's, and this choice defines uniquely the δ_j's, according to the above formulas.

For $k\geq 1$, recall that G_{k-1} is the analogue of the group G associated to the graded Lie algebra $\tilde{\mathfrak{g}}_{k-1}$ which is of rank 2. From Proposition 3.8.7 and Theorem 3.8.10 one has $\chi_{k-1}(G_{k-1})=F^{*2}$ if $e=1$ or 3 and $\chi_{k-1}(G_{k-1})=N_{E/F}(E)^*$ if $e=2$, where E is a quadratic extension of F.

Theorem 4.3.6.
In all cases, the dense open set $\mathcal{O}^+$ is the union of the open P-orbits in V^+.

1. *If $\tilde{\mathfrak{g}}$ is of Type I (that is, if ℓ is a square and $e=0$ or 4), then $\mathcal{O}^+$ is the unique open P-orbit in V^+.*
2. *Let $\tilde{\mathfrak{g}}$ be of of Type II (that is $\ell=1$ and $e\in\{1,2,3\}$) and let $S=\chi_{k-1}(G_{k-1})$. Then the subgroup P has $|F^*/S|^k$ open orbits in V^+ given for $k\geq 1$ by*
$$\mathcal{O}_u=\{X\in V^+;\frac{\Delta_j(X)}{\Delta_k(X)^{k+1-j}}u_j\ldots u_{k-1}\in S\text{ for }j=0,\ldots k-1\},$$
where $u=(u_0,\ldots,u_{k-1})\in(F^*/S)^k$. (i.e. P has 4^k open orbits in V^+ if $e=1$ or 3, and 2^k open orbits if $e=2$).
3. *If $\tilde{\mathfrak{g}}$ is of Type III (that is if $\ell=3$), then the subgroup P has 3^{k+1} open orbits in V^+ given by*
$$\mathcal{O}_u=\{X\in V^+;\Delta_j(X)u_j\ldots u_k\in F^{*2}\text{ for }j=0,\ldots k\},$$
where $u=(u_0,\ldots,u_k)\in\prod_{i=0}^k\left(\delta_i(\widetilde{\mathfrak{g}}^{\lambda_i}\setminus\{0\})/F^{*2}\right)$.

Proof. As the Δ_j's are relatively invariant under P, the union of the open P-orbits is a subset of $\mathcal{O}^+$.
(1) Suppose first that ℓ is a square and $e=0$ or 4 (in other words $\tilde{\mathfrak{g}}$ is of Type I). Let $X\in\mathcal{O}^+$. By Lemma 4.3.3, X is N-conjugated to an element $\sum_{j=0}^k Z_j$ with $Z_j\in\widetilde{\mathfrak{g}}^{\lambda_j}\setminus\{0\}$. This element is of course generic for G.

From Theorem 3.7.1 and 3.8.8, two generic elements of the"diagonal" $\oplus_{j=0}^k\tilde{\mathfrak{g}}^{\lambda_j}$ are L-conjugated. Hence all the elements of $\mathcal{O}^+$are P-conjugated.

(2) We suppose now that $\ell=1$ and $e=1$, 2 or 3.
Let $k\geq 1$. Let $S=\chi_{k-1}(G_{k-1})$. For $j\in\{0,\ldots,,k\}$ we fix a non zero element X_j of $\tilde{\mathfrak{g}}^{\lambda_j}$ such that for $I^+=X_0+\ldots+X_k$ one has $\Delta_j(I^+)=1$ for all j.
Let $Z\in\mathcal{O}^+$. As before, Z is N-conjugated to an element $X\in\oplus_{j=0}^k(\tilde{\mathfrak{g}}^{\lambda_j}\setminus\{0\})$. As $\ell=1$, we can write $X=\sum_{j=0}^k x_jX_j$ with $x_j\neq 0$.

By Theorem 1.14.2, the polynomials Δ_j are N-invariant. If we set $u_s = \dfrac{x_s}{x_k}$ modulo S for $s = 0, \dots, k-1$, we get

$$\frac{\Delta_j(Z)}{\Delta_k(Z)^{k+1-j}} = \frac{\Delta_j(X)}{\Delta_k(X)^{k+1-j}} = \prod_{s=j}^{k-1} u_j \text{ modulo } S,$$

and this implies $Z \in \mathcal{O}_u$.
Conversely, let $u = (u_0, \dots, u_{k-1}) \in (F^*/S)^k$ and let Z and Z' be two elements of $\mathcal{O}_u$. These elements are respectively N-conjugated to diagonal elements $X = \sum_{j=0}^k x_j X_j$ and $X' = \sum_{j=0}^k x'_j X_j$. From the definition of $\mathcal{O}_u$, we have $\dfrac{x_j}{x_k} \dots \dfrac{x_{k-1}}{x_k} = \dfrac{x'_j}{x'_k} \dots \dfrac{x'_{k-1}}{x'_k}$ modulo S for all $j \in \{0, \dots k-1\}$. Therefore $\dfrac{x_j}{x_k} = \dfrac{x'_j}{x'_k}$ modulo S for all $j \in \{0, \dots k-1\}$. This implies that $\frac{1}{x_k x'_k} x_i x'_i \in S$ for all $i \in \{0, 1, \dots, k\}$ (because $F^{*2} \subset S$ by Lemma 3.8.2). By Theorem 3.8.9 (d) (in the case $e = 1$ or 3) and Theorem 3.8.10 (d) (in the case $e = 2$), the elements X and X' are L-conjugated. It follows that two elements in $\mathcal{O}_u$ are P-conjugated.

If u and v are two elements in $(F^*/S)^{k-1}$ such that $\mathcal{O}_u \cap \mathcal{O}_v \neq \emptyset$, then $u_j \dots u_{k-1} = v_j \dots v_{k-1}$ modulo S for $j = 0, \dots k-1$. Therefore $u_j = v_j$ modulo S for all j and hence $\mathcal{O}_u = \mathcal{O}_v$. The statement (2) is now proved.

(3) Consider finally the case $\ell = 3$. Remember (see Theorem 1.12.4, 2)) that in this case, δ_j is a quadratic form which represents three classes modulo F^{*2} (all classes in $(F^*/F^*)^2$ distinct from $-disc(\delta_j)$). Let $X \in V^+$ be a generic element of (P, V^+). We will show that X belongs to $\mathcal{O}_u$ for some $u = (u_0, \dots, u_k) \in \prod_{i=0}^k \big(\delta_i(\widetilde{\mathfrak{g}}^{\lambda_i} \setminus \{0\})/F^{*2}\big)$.

As before, the element X is N-conjugated to an element $Z = \sum_{j=0}^k Z_j$ where $Z_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$.
From the normalization made in Remark 4.3.5, we have $\Delta_j(Z) = \prod_{s=j}^k \delta_s(Z_s)$ where δ_s is a fundamental relative invariant of $(L_s, \widetilde{\mathfrak{g}}^{\lambda_s})$.
If we define $u_s = \delta_s(Z_s)$ modulo F^{*2}, we get $\Delta_j(X) u_j \dots u_k = \Delta_j(Z) u_j \dots u_k = \prod_{s=j}^k \delta_s(Z_s) u_s \in F^{*2}$, and therefore X belong to $\mathcal{O}_u$ with $u = (u_0, \dots, u_k)$.

Conversely, let $X, X' \in \mathcal{O}_u$ These elements are N-conjugated to (respectively) two "diagonal" elements $Z = Z_0 + \dots + Z_k$ and $Z' = Z'_0 + \dots + Z'_k$ (Lemma 4.3.3). From the definitions we have $\delta_j(Z_j) = \delta_j(Z'_j)$ modulo F^{*2} for $j = 0, \dots, k$. By Corollary 3.9.6, there exists $l_j \in L_j^0$ such that $l_j Z_j = Z'_j$ (Recall that L_i^0 is the subgroup of L_i defined in definition 3.9.3). As L_j^0 centralizes $\oplus_{s \neq j} \widetilde{\mathfrak{g}}^{\lambda_s}$, we get $l_0 \dots l_k . Z = Z'$. Moreover $l_0 \dots l_k \in P$. Hence two elements in $\mathcal{O}_u$ are P-conjugated.

If u and v are two elements of $\big(\Delta_k(\widetilde{\mathfrak{g}}^{\lambda_k} \setminus \{0\})/F^{*2}\big)^{k+1}$ such that $\mathcal{O}_u \cap \mathcal{O}_v \neq \emptyset$ then $u_j \dots u_k = v_j \dots v_k$ modulo F^{*2} for $j = 0, \dots k$. And hence $v_j = u_j$ modulo F^{*2} for all j, and therefore $\mathcal{O}_u = \mathcal{O}_v$.
Assertion (3) is proved.
The fact that $\mathcal{O}^+$ is the union of the open P-orbits is now clear.

$\square$

4.4. **The involution γ.**
From Remark 4.1.7 we know that the root system of $(\widetilde{\mathfrak{g}}, \mathfrak{a}^0)$ is always of type C_{k+1} and consists of the linear forms $\pm\eta_j \pm \eta_i$ for $i \neq j$ and $\pm 2\eta_j$, $1 \leq i, j \leq k$ where

$$\eta_j(H_{\lambda_i}) = \delta_{ij}.$$

We know also ([3]) that then, there exists an element w of the Weyl group of C_{k+1} such that

$$(*) \qquad w.\eta_i = -\eta_{k-i} \text{ for } i = 0, \ldots, k.$$

As this Weyl group is isomorphic to $N_{\widetilde{G}}(\mathfrak{a}^0)/Z_{\widetilde{G}}(\mathfrak{a}^0)$, there exists an element $\gamma \in N_{\widetilde{G}}(\mathfrak{a}^0)$ such that $w = \mathrm{Ad}(\gamma)_{|_{\mathfrak{a}^0}}$. The property $(*)$ implies that γ normalizes $\mathfrak{g}$, exchanges V^+ and V^- and normalizes also P (ie. $\gamma P \gamma^{-1} = P$).

In the Theorem below, we will give explicitly such an element γ, which moreover, will be an involution of $\widetilde{\mathfrak{g}}$.

We choose a diagonal $\mathfrak{s}l_2$-triple $\{I^-, H_0, I^+\}$ that is such that $I^+ = X_0 + \ldots + X_k$ $(X_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\})$, $I^- = Y_0 + \ldots + Y_k$ $(Y_j \in \widetilde{\mathfrak{g}}^{-\lambda_j} \setminus \{0\})$, where each $(Y_j, H_{\lambda_j}, X_j)$ is an $\mathfrak{s}l_2$-triple.
Let $\{Y, H_{\lambda_i} - H_{\lambda_j}, X\}$ be an $\mathfrak{s}l_2$-triple such that $X \in E_{i,j}(1,-1)$ and $Y \in E_{i,j}(-1,1)$ (see Lemma 3.4.2). Remember the elements

$$\gamma_{i,j} = e^{\mathrm{ad}\, X} e^{\mathrm{ad}\, Y} e^{\mathrm{ad}\, X} = e^{\mathrm{ad}\, Y} e^{\mathrm{ad}\, X} e^{\mathrm{ad}\, Y}.$$

which have been introduced in Proposition 3.4.3. We suppose moreover that the sequence X_i is such that $\gamma_{i,k-i}(X_i) = X_{k-i}$, for $0 \leq i \leq n$, where n is the integer defined by $k = 2n + 2$ if k is even and $k = 2n + 1$ if k is odd. It is always possible to choose such a sequence.
Once we have chosen such an $\mathfrak{s}l_2$-triple, we normalize the polynomials Δ_j by the condition:

$$\Delta_j(I^+) = 1, \quad \text{for } j = 0, \ldots, k.$$

Theorem 4.4.1.
Suppose that $\{I^-, H_0, I^+\}$ is a diagonal $\mathfrak{s}l_2$-triple satisfying the preceding conditions.
There exists an element $\gamma \in N_{\widetilde{G}}(\mathfrak{a}^0)$ such that

(1) $\gamma.H_{\lambda_j} = -H_{\lambda_{k-j}}$ for $j = 0, \ldots, k$;
(2) $\gamma.X_j = Y_{k-j}$ for $j = 0, \ldots, k$;
(3) $\gamma^2 = \mathrm{Id}_{\widetilde{\mathfrak{g}}}$.

Such an element normalizes $\mathfrak{g}$, exchanges V^+ and V^- and normalizes G, P, M, A_0 and N.

Proof. We will first show the existence of an involution $\widetilde{\gamma}$ of $\widetilde{G}$ such that $\widetilde{\gamma}(H_{\lambda_j}) = H_{\lambda_{k-j}}$ and $\widetilde{\gamma}(X_{\lambda_j}) = X_{\lambda_{k-j}}$ for $j = 0, \ldots, k$. For $k = 0$, then the trivial involution satisfies this property.

We suppose that $k > 0$. Let w_i be the non trivial element of the Weyl group associated to the $\mathfrak{s}l_2$-triple $\{Y_i H_{\lambda_i}, X_i\}$. Recall (Proposition 3.4.3 and Proposition 3.4.4) that the elements $\widetilde{\gamma}_{i,j} = \gamma_{i,j} \circ w_i^2 \in N_{\widetilde{G}}(\mathfrak{a}^0)$ satisfy the following properties:

$$\widetilde{\gamma}_{i,j}^2 = \mathrm{Id}_{\widetilde{\mathfrak{g}}} \quad \text{and} \quad \widetilde{\gamma}_{i,j}(H_{\lambda_s}) = \begin{cases} H_{\lambda_i} & \text{for} \quad s = j \\ H_{\lambda_j} & \text{for} \quad s = i \\ H_{\lambda_s} & \text{for} \quad s \notin \{i, j\} \end{cases}$$

Consider again the integer n defined by $k = 2n+2$ if k is even and $k = 2n+1$ if k is odd and set:

$$\widetilde{\gamma} := \widetilde{\gamma_{0,k}} \circ \widetilde{\gamma_{1,k-1}} \circ \ldots \circ \widetilde{\gamma_{n,k-n}}.$$

Then

$$\widetilde{\gamma}(H_{\lambda_j}) = H_{\lambda_{k-j}} \text{ for } j = 0, \ldots, k$$

As the pairs of roots $(\lambda_i, \lambda_{k-i})$ are mutually strongly orthogonal, the involutions $\widetilde{\gamma_{i,k-i}}$ (see Proposition 3.4.4) commute, and hence $\widetilde{\gamma}$ is an involution of $\widetilde{\mathfrak{g}}$.

From our choice of the sequence X_j, and as the action of w_i^2 on $\oplus_{s=0}^{k}\widetilde{\mathfrak{g}}^{\lambda_s}$ is trivial, we obtain that $\widetilde{\gamma}(X_j) = X_{k-j}$.

The involution $\widetilde{\gamma}$ centralizes I^+ and H_0, and hence it centralizes I^-. Therefore $\widetilde{\gamma}$ commutes with $w = e^{\operatorname{ad} I^+}e^{\operatorname{ad} I^-}e^{\operatorname{ad} I^+}$ which is the element of $\widetilde{G}$ which defines the involution σ associated to the $\mathfrak{s}l_2$-triple $\{I^-, H_0, I^+\}$. Define

$$\gamma = \widetilde{\gamma} w = w \widetilde{\gamma}.$$

The automorphism γ commutes with w, and hence γ is an involution of $\widetilde{\mathfrak{g}}$. Moreover, using Theorem 4.1.1, we get

$$\begin{array}{lll} \gamma(H_{\lambda_j}) = \sigma(H_{\lambda_{k-j}}) = -H_{\lambda_{k-j}} & \text{for} & j = 0, \ldots, k \\ \gamma(X_j) = \sigma(X_{k-j}) = Y_{k-j} & \text{for} & j = 0, \ldots, k \end{array}$$

This implies that $\gamma(H_0) = -H_0$ and hence γ stabilizes $\mathfrak{g}$, normalizes G and exchanges V^+ and V^-.

As $\gamma(H_{\lambda_j}) = -H_{\lambda_{k-j}}$ for $j = 0, \ldots, k$, the element γ stabilizes $\mathfrak{a}^0$ and exchanges $E_{i,j}(1,-1)$ and $E_{k-i,k-j}(-1,1)$. Therefore γ stabilizes $\mathfrak{n}$ and $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^0)$, and hence it stabilizes $\mathfrak{p}$. It follows that γ normalizes A^0, L, N and P.

□

4.5. The P-orbits in V^- and the polynomials ∇_j.

In this section we fix an $\mathfrak{sl}_2$-triple (I^-, H_0, I^+) satisfying the same conditions as for Theorem 4.4.1 where the involution γ is defined. We set $H = Z_G(I^+)$

Definition 4.5.1. *For $j = 0, \ldots, k$, we denote by ∇_j the polynomial on V^- defined by*

$$\nabla_j(Y) = \Delta_j(\gamma(Y)), \quad \text{for } Y \in V^-.$$

Theorem 4.5.2.
The polynomials ∇_j are irreducible of degree $\kappa(k+1-j)$.

(1) *∇_0 is the relative invariant of V^- under the action of G.*
(2) *For $j = 0, \ldots, k$, the polynomial ∇_j is a relatively invariant polynomial on V^- under the action of the parabolic subgroup P. More precisely we have*

$$\nabla_j(p.Y) = \chi_j^-(p)\nabla_j(Y), \quad \text{for } p \in P,$$

where χ_j^- is a character of P with the following properties:

•$\chi_j^-(n) = 1, \quad$ for $n \in N$,

$\bullet \chi_j^-(a) = a^{-\kappa(\lambda_0+\ldots+\lambda_{k-j})}$, for $a \in A$,
$\bullet \chi_j^-(l) = 1$, for $l \in L \cap H$.

Proof. As γ is linear, ∇_j is effectively an irreducible polynomial of the same degree as Δ_j, that is $\kappa(k+1-j)$ (∇_j is non zero since $\gamma(I^-) = I^+$). As γ normalizes G and P, the fact that the ∇_j's are relatively invariant is direct consequence of the same property for the Δ_j's (Theorem 1.14.2).

For $p \in P$ we have:

$$\nabla_j(p.I^-) = \chi_j^-(p)\nabla_j(I^-) = \Delta_j(\gamma p \gamma^{-1}\gamma I^+) = \chi_j(\gamma p \gamma^{-1})\nabla_j(I^-),$$

and therefore

$$\chi_j^-(p) = \chi_j(\gamma p \gamma^{-1}).$$

As γ normalizes N, L and A and commutes with σ, the assertion concerning the values of χ_j^- on N, A, and $L \cap H$ is a consequence of the same properties for the χ_j's (cf. Theorem 4.3.2). □

Let $\mathcal{O}^-$ be the dense open subset of V^- defined by

$$\mathcal{O}^- = \{Y \in V^-; \nabla_0(Y)\nabla_1(Y)\ldots\nabla_k(Y) \neq 0\}.$$

Suppose $\ell = 3$. Recall that L_j is the analogue of the group G for the graded Lie algebra $\widetilde{\mathfrak{l}}_j$, that is $L_j = \mathcal{Z}_{\mathrm{Aut}_0(\widetilde{\mathfrak{l}}_j)}(H_{\lambda_j})$. In the following Theorem we denote by δ_j^- the relative invariant of the prehomogeneous space $(L_j, \widetilde{\mathfrak{g}}^{-\lambda_j})$ defined by the identity

$$\nabla_j(Y_j + Y_{j+1} + \ldots + Y_k) = \delta_j^-(Y_j)\nabla_{j+1}(Y_{j+1} + \ldots + Y_k) = \delta_j^-(Y_j)\ldots\delta_k^-(Y_k)$$

for $Y_j \in \widetilde{\mathfrak{g}}^{\lambda_j} \setminus \{0\}$ and for $j = 0, \ldots, k$.
Using the involution γ, the following description of the open P-orbits in V^- is an easy consequence of Theorem 4.3.6.

Theorem 4.5.3.
The dense open subset $\mathcal{O}^-$ is the union of the open P-orbits in V^-.

(1) *If $\tilde{\mathfrak{g}}$ is of Type I (that is if ℓ is a square and $e \in \{0,4\}$), then $\mathcal{O}^-$ is the unique open P-orbit in V^-.*
(2) *Let $\tilde{\mathfrak{g}}$ be of Type II (that is if $\ell = 1$ and $e \in \{1,2,3\}$) and let $S = \chi_{k-1}(G_{k-1})$. Then the subgroup P has $|F^*/S|^k$ open orbits in V^- given for $k \geq 1$ by*

$$\mathcal{O}_u^- = \{Y \in V^-; \frac{\nabla_j(Y)}{\nabla_k(Y)^{k+1-j}} u_j \ldots u_{k-1} \in S, \text{ for } j = 0, \ldots k-1\},$$

where $u = (u_0, \ldots, u_{k-1}) \in (F^*/S)^k$. (i.e. P has 4^k open orbits in V^- if $e = 1$ or 3, and 2^k open orbits if $e = 2$)
(3) *If $\tilde{\mathfrak{g}}$ is of Type III (that is if $\ell = 3$), then P has 3^{k+1} open orbits in V^- given by*

$$\mathcal{O}_u^- = \{Y \in V^-; \nabla_j(Y)u_j \ldots u_k \in F^{*2} \text{ for } j = 0, \ldots k\},$$

where $u = (u_0, \ldots, u_k) \in \prod_{i=0}^k \left(\delta_i^-(\widetilde{\mathfrak{g}}^{-\lambda_i} \setminus \{0\})/F^{*2}\right)$.

The following Lemma gives the relationship between the characters of the ∇_j's and those of the Δ_j's.

Lemma 4.5.4. *For $g \in G$ and $p \in P$, we have:*

$$\chi_0^-(g) = \frac{1}{\chi_0(g)}$$

and

$$\chi_j^-(p) = \frac{\chi_{k-j+1}(p)}{\chi_0(p)}, \quad j \in \{1, \dots k\}.$$

Proof. As the Killing forms induces a G-invariant duality between V^+ and V^-, we have:

$$\det_{V^+}\mathrm{Ad}(g) = \det_{V^-}\mathrm{Ad}(g^{-1}), \quad g \in G.$$

On the other hand, for $X \in V^+$, let us consider the determinant $P(X) = \det_{(V^-,V^+)}(\mathrm{ad}\ X)^2$ of the map $(\mathrm{ad}\ X)^2 : V^- \to V^+$ (for any choice of basis). This polynomial P is relatively invariant under the action of G because $P(g.X) = \det_{(V^-,V^+)}(g(\mathrm{ad}\ X)^2 g^{-1}) = (\det_{V^+}\mathrm{Ad}(g))^2 P(X)$. Hence it is, up to a multiplicative constant, a power of Δ_0. Therefore there exists $m \in \mathbb{N}$ such that

$$(\det_{V^+}\mathrm{Ad}(g))^2 = \chi_0(g)^m.$$

If we take $g \in G$ such that $g_{|_{V^+}} = t\ Id_{V^+} \in G_{|_{V^+}}$, $t \in F^*$ (cf. Lemma 1.11.3), then Theorem 1.13.2 implies that $m = \dfrac{2\mathrm{dim} V^+}{\kappa(k+1)}$.

The same argument for the dual space (G, V^-) implies that $(\det_{V^-}\mathrm{Ad}(g))^2 = \chi_0^-(g)^m$. Therefore we get $\chi_0^-(g)^m = \chi_0(g)^{-m}$ for all $g \in G$. As the group $X^*(G)$ of rational characters of G is a lattice (see for example [20], p.121), it has no torsion. Therefore $\chi_0^-(g) = \frac{1}{\chi_0(g)}$, and the first assertion is proved.

Let us show the second assertion. As all the characters we consider are trivial on N, it is enough to prove the relation for $m \in L$. We consider the subgroup $L' = L_0 \dots L_k$ of $\bar{L} = L(\overline{F})$ (keep in mind that the groups L_j are in general not included in G). The polynomials Δ_j are the restrictions to V^+ of polynomials defined on $\overline{V}^+$, which are relatively invariant under the action of $\bar{G}$. Therefore the polynomials ∇_j also are restrictions to V^- of polynomials of $\overline{V}^-$ which are relatively invariant under $\bar{G}$.

We first show that for $m \in L$, there exists $l \in L'$ such that $l^{-1}m \in \overline{L} \cap \overline{H}$ (*).

(Here $\overline{L} = L(\overline{F}) = Z_{\overline{G}}(\mathfrak{a}^0)$ and $\overline{H} = H(\overline{F}) = Z_{\overline{G}}(I^+)$).
If ℓ is a square, as $m.X_j \in \tilde{\mathfrak{g}}^{\lambda_j}$, it follows from Theorem 1.12.4 that $m.X_j$ is L_j-conjugated to X_j and hence there exists $l_j \in L_j$ such that $m.X_j = l_j.X_j$. Then the element $l = l_0 \dots l_k \in L'$ is such that $l^{-1}m.I^+ = I^+$ and therefore $l^{-1}m \in \overline{L} \cap \overline{H}$.
If $\ell = 3$, we will use the description of G given in Proposition 3.9.1. Let $E = F[\sqrt{u}]$ be a unramified quadratic extension of F and let π be a uniformizer of F. The group G is the group of automorphisms of $\tilde{\mathfrak{g}}$ given by conjugation by matrices of the form $\begin{pmatrix} \mathbf{g} & 0 \\ 0 & \mu\, {}^t\mathbf{g}^{-1} \end{pmatrix}$ where $\mathbf{g} \in G^0(2(k+1)) \cup \sqrt{u}G^0(2(k+1))$ and $\mu \in F^*$. We denote by $[\mathbf{g}, \mu]$ such an element of G.

The space $\mathfrak{a}^0$ is the space of matrices

$$\begin{pmatrix} \mathbf{H}(t_k,\dots,t_0) & 0 \\ 0 & -\mathbf{H}(t_k,\dots,t_0) \end{pmatrix}, \text{ where } \mathbf{H}(t_k,\dots,t_0) = \begin{pmatrix} t_kI_2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_0I_2 \end{pmatrix},$$

and $(t_k,\dots,t_0) \in F^{k+1}$. Therefore the centralizer of $\mathfrak{a}^0$ in G, that is L, is the subgroup of elements $m = [\mathbf{g}, \mu]$ where $\mathbf{g} = diag(\mathbf{g}_k,\dots,\mathbf{g}_0)$ is a 2×2 block diagonal matrix whose diagonal elements $\mathbf{g}_j$ belong to $G^0(2) \cup \sqrt{u}G^0(2)$. Let $l_j \in GL(2(k+1), E)$ be the 2×2 block diagonal matrix whose all diagonal blocks are the identity in $GL(2, E)$ except the $(k-j+1)$-th bock which is equal to $\mathbf{g}_j$ and the $(2(k+1)-j)$-th block which is equal to $\mu\ {}^t\mathbf{g}_j^{-1}$. Then from the definition of L_j, we have $\mathrm{Ad} l_j \in L_j$ and $m = \mathrm{Ad}(l_k \dots l_0)$ belongs to L'.

Hence we have proved that in all cases, for all $m \in L$, there exists $l \in L'$ such that $l^{-1}m \in \overline{L} \cap \overline{H}$. As the characters χ_j and χ_j^- are trivial on $\overline{L} \cap \overline{H}$, we have $\chi_j(m) = \chi_j(l)$ and $\chi_j^-(m) = \chi_j^-(l)$ for all $j \in \{0,\dots,k\}$. It suffices therefore to prove the result for $l \in L' = L_0L_1\dots L_k \subset \overline{G}$.

Let $l \in L'$ and $j \in \{1,\dots,k\}$. Consider the decomposition of V^+ into weight spaces under the action of $H_{\lambda_j} + \dots H_{\lambda_k}$:

$$V^+ = V_j^+ \oplus U_j^+ \oplus W_j^+,$$

where V_j^+, U_j^+ and W_j^+ are the spaces of weight 2, 1 and 0 under $H_{\lambda_j} + \dots H_{\lambda_k}$, respectively. More precisely:

$$\begin{aligned} V_j^+ &= \oplus_{s=j}^k \tilde{\mathfrak{g}}^{\lambda_s} \oplus \oplus_{j\le r<s} E_{r,s}(1,1) \\ U_j^+ &= \oplus_{r<j\le s} E_{r,s}(1,1) \\ W_j^+ &= \oplus_{s=0}^{j-1} \tilde{\mathfrak{g}}^{\lambda_s} \oplus \oplus_{r<s\le j-1} E_{r,s}(1,1). \end{aligned}$$

Similarly we denote by $V^- = V_j^- \oplus U_j^- \oplus W_j^-$ the decomposition of V^- into weight spaces of weight $-2, -1$ and 0 under $H_{\lambda_j} + \dots H_{\lambda_k}$. As the eigenspace, in V^+, for the eigenvalue r of $H_{\lambda_j} + \dots H_{\lambda_k}$, is the same as the eigenspace, for the eigenvalue $2-r$ of $H_{\lambda_1} + \dots H_{\lambda_{j-1}}$, it is easy to see that $W_j^- = \gamma(V_{k+1-j}^+)$. Therefore $(\gamma(G_{k+1-j}), W_j^-)$ is a regular irreducible prehomogeneous vector space whose fundamental relative invariant is the restriction of ∇_{k-j+1} to W_j^-.
Let us write $l = l_1l_2$ where $l_1 \in L_j \dots L_k \subset \overline{G_j}$ and $l_2 \in L_0 \dots L_{j-1} = \gamma(L_{k+1-j}\dots L_k) \subset \gamma(\overline{G_{k+1-j}})$.
As l_1 acts trivially on $\overline{W_j}^+$, and as l_2 acts trivially on $\overline{V_j^+}$, we have

$$\chi_0(l_1) = \Delta_0(X_0 + \dots + X_{j-1} + l_1(X_j + \dots + X_k))$$

$$= \Delta_j(l_1(X_j + \dots + X_k)) = \Delta_j(l(X_j + \dots + X_k)) = \chi_j(l).$$

Define $l_2' := \gamma l_2 \gamma^{-1} \in L_{k+1-j}\dots L_k \subset \overline{G_{k+1-j}}$.
If we consider the decomposition $V^+ = V_{k+1-j}^+ \oplus U_{k+1-j}^+ \oplus W_{k+1-j}^+$ and if we apply the same argument as before to $\gamma l \gamma^{-1}$ we get $\chi_0(l_2') = \chi_{k+1-j}(\gamma l \gamma^{-1})$, and this is equivalent to

$$\chi_0^-(l_2) = \chi_{k+1-j}^-(l).$$

Applying the first assertion of the Lemma, we obtain $\chi_0(l_2) = \chi^-_{k+1-j}(l)^{-1}$, and hence finally

$$\chi_0(l) = \chi_0(l_1)\chi_0(l_2) = \chi_j(l)\chi^-_{k+1-j}(l)^{-1},$$

and this proves the second assertion.

□

Let Ω^+ and Ω^- be the set of generic elements in V^+ and V^-, respectively. In other words:

$$\Omega^+ = \{x \in V^+, \Delta_0(X) \neq 0\}, \quad \Omega^- = \{x \in V^-, \nabla_0(X) \neq 0\}.$$

Definition 4.5.5. *Let $\psi : \Omega^+ \longrightarrow \Omega^-$ be the map which sends $X \in \Omega^+$ to the unique element $Y \in \Omega^-$ such that $\{Y, H_0, X\}$ is an $\mathfrak{s}l_2$-triple.*

If $\{Y, H_0, X\}$ is an $\mathfrak{s}l_2$-triple, then for each $g \in G$, $\{g.Y, H_0, g.X\}$ is again an $\mathfrak{s}l_2$-triple. Therefore the map ψ is G-equivariant.

Proposition 4.5.6. *For $X \in \Omega^+$, we have*

$$\nabla_0(\psi(X)) = \frac{1}{\Delta_0(X)}, \quad \text{and} \quad \nabla_j(\psi(X)) = \frac{\Delta_{k+1-j}(X)}{\Delta_0(X)}, \quad j = 1, \dots, k.$$

Proof. Fix a diagonal $\mathfrak{s}l_2$-triple $\{I^-, H_0, I^+\}$ which satisfies the condition of Theorem 4.4.1. Then $I^+ = X_0 + \ldots + X_k$ and $I^- = Y_0 + \ldots + Y_k$ where $\{Y_i, H_{\lambda_i}, X_i\}$ are $\mathfrak{s}l_2$-triples.
As $\mathcal{O}^+$ is open dense in Ω^+ and as the function we consider here are continuous on Ω^+, it suffices to prove the result for $X \in \mathcal{O}^+$. The proof depends on the Type of $\tilde{\mathfrak{g}}$.

If $\tilde{\mathfrak{g}}$ is of Type I (ie. ℓ is a square and $e = 0$ or 4) then any $X \in \mathcal{O}^+$ is P-conjugated to I^+ (see Theorem 4.3.6). Therefore it suffices to prove that for all $p \in P$, one has

$$\nabla_j(p.I^-) = \frac{\Delta_{k+1-j}(p.I^+)}{\Delta_0(p.I^+)}.$$

From the normalization of the polynomials ∇_j and Δ_j, the Lemma 4.5.4 implies

$$\nabla_j(p.I^-) = \chi^-_j(p) = \frac{\chi_{k+1-j}(p)}{\chi_0(p)} = \frac{\Delta_{k+1-j}(p.I^+)}{\Delta_0(p.I^+)},$$

and this proves the statement in this case.

Suppose now that $\tilde{\mathfrak{g}}$ is of Type II, that is that $\ell = 1$ and $e = 1$ or 2. By Lemma 4.3.3, X is N-conjugated to an element $Z = \sum_{j=0}^k z_j X_j$ with $z_0, \ldots, z_k \in F^*$. By the hypothesis on the X_j's and from the definition of the involution γ, we have $\psi(Z) = z_0^{-1} Y_0 + \ldots + z_k^{-1} Y_k$ and $\gamma(\psi(Z)) = z_k^{-1} X_0 + \ldots + z_0^{-1} X_k$. By Theorem 4.5.2 and 4.3.2, the polynomials ∇_j and Δ_j are N-invariant and hence we have

$$\nabla_0(\psi(X)) = \nabla_0(\psi(Z)) = \Delta_0(\gamma.\psi(Z)) = \prod_{s=0}^{k} z_s^{-1} = \frac{1}{\Delta_0(Z)} = \frac{1}{\Delta_0(X)},$$

and

$$\nabla_j(\psi(X)) = \nabla_j(\psi(Z)) = \Delta_j(\gamma.\psi(Z)) = \prod_{s=0}^{k-j} z_s^{-1} = \frac{\Delta_{k+1-j}(Z)}{\Delta_0(Z)} = \frac{\Delta_{k+1-j}(X)}{\Delta_0(X)},$$

and this again proves the statement.

Suppose now that $\tilde{\mathfrak{g}}$ is of Type III, this means that $\ell = 3$. We use the notations and the material developed in §3.9. In particuliar we realize the algebra $\tilde{\mathfrak{g}}$ as a subalgebra of $\mathfrak{sp}(4(k+1), E)$ where $E = F[\sqrt{u}]$ and where $u \in F^* \setminus F^{*2}$ is a unit. We will first describe precisely the involution γ and the polynomials ∇_j.

We define $I^+ = \begin{pmatrix} 0 & \mathbf{I}^+ \\ 0 & 0 \end{pmatrix}$ where $\mathbf{I}^+ \in M(2(k+1), E)$ is the 2×2 block diagonal matrix whose diagonal blocks are all equal to $J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $I^- = \begin{pmatrix} 0 & 0 \\ -\mathbf{I}^+ & 0 \end{pmatrix}$. We normalize the polynomials Δ_j in such a way that $\Delta_j(I^+) = 1$ for all $j \in \{0, \dots, k\}$.

We set $\Gamma = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$. We show now that $\gamma = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}$ satisfies the properties of 4.4.1. As γ centralizes the matrix $K_{2(k+1)} = \begin{pmatrix} 0 & I_{2(k+1)} \\ -I_{2(k+1)} & 0 \end{pmatrix}$, the element γ (or to be more precise, the conjugation by γ) belongs to $\mathrm{Aut}_0(\tilde{\mathfrak{g}} \otimes_F E)$. In order to verify that $\gamma \in \tilde{G}$, it suffices to verify that γ normalizes $\tilde{\mathfrak{g}}$. Recall that $\tilde{\mathfrak{g}}$ is the set of matrices $Z \in \tilde{\mathfrak{g}} \otimes_F E$ such that $T\overline{Z} = ZT$ where $T = \begin{pmatrix} J & 0 \\ 0 & {}^tJ \end{pmatrix}$ with $J = \begin{pmatrix} J_\pi & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_\pi \end{pmatrix}$ and $J_\pi = \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix}$. As $\gamma^{-1}T\gamma = \begin{pmatrix} \Gamma\, {}^tJ\Gamma & 0 \\ 0 & \Gamma J \Gamma \end{pmatrix}$ and $\Gamma J \Gamma = {}^tJ$, we obtain $\gamma^{-1}T\gamma = T$. It follows that γ normalizes $\tilde{\mathfrak{g}}$ and hence $\gamma \in \tilde{G}$.

For $Z = \begin{pmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{Y} & -{}^t\mathbf{A} \end{pmatrix} \in \tilde{\mathfrak{g}}$, we have

$$\gamma.Z = \begin{pmatrix} -\Gamma\, {}^t\mathbf{A}\Gamma & -\Gamma\mathbf{Y}\Gamma \\ -\Gamma\mathbf{X}\Gamma & \Gamma\mathbf{A}\Gamma \end{pmatrix}.$$

If $\mathbf{Z} \in M(2(k+1, E)$ can be written as $\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_{0,0} & \dots & \mathbf{Z}_{0,k} \\ \vdots & & \vdots \\ \mathbf{Z}_{k,0} & \dots & \mathbf{Z}_{k,k} \end{pmatrix}$ with $\mathbf{Z}_{r,s} \in M(2, E)$, then

$$\Gamma\mathbf{Z}\Gamma = \begin{pmatrix} J_1\mathbf{Z}_{k,k}J_1 & \dots & J_1\mathbf{Z}_{k,0}J_1 \\ \vdots & & \vdots \\ J_1\mathbf{Z}_{0,k}J_1 & \dots & J_1\mathbf{Z}_{k,0}J_1 \end{pmatrix}.$$

It is now easy to verify that γ normalizes $\mathfrak{a}^0$ and satisfies the properties of Theorem 4.4.1 .
From the normalization made in section 3.9, we have

$$\Delta_j \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} = (-1)^{k-j+1}\det(\tilde{\mathbf{X}}_j)$$

where $\tilde{\mathbf{X}}_j$ is the square matrix of size $2(k+1-j)$ defined by the $2(k+1-j)$ first rows and columns of $\mathbf{X}$. Explicitly, if $\mathbf{X} = (\mathbf{X}_{r,s})_{r,s=0,\dots,k}$ where $\mathbf{X}_{r,s} \in M(2,E)$, we have

$$\Delta_j(X) = (-1)^{k-j+1} \begin{vmatrix} \mathbf{X}_{0,0} & \dots & \mathbf{X}_{0,k-j} \\ \vdots & & \vdots \\ \mathbf{X}_{k-j,0} & \dots & \mathbf{X}_{k-j,k-j} \end{vmatrix}$$

From the definition of ∇_j, we get

$$\nabla_j \begin{pmatrix} 0 & 0 \\ \mathbf{Y} & 0 \end{pmatrix} = \Delta_j \begin{pmatrix} 0 & -\Gamma\mathbf{Y}\Gamma \\ 0 & 0 \end{pmatrix}.$$

Therefore if $\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_{0,0} & \dots & \mathbf{Y}_{0,k} \\ \vdots & & \vdots \\ \mathbf{Y}_{k,0} & \dots & \mathbf{Y}_{k,k} \end{pmatrix}$, we have

$$\nabla_j(Y) = (-1)^{k+1-j} \begin{vmatrix} -J_1\mathbf{Y}_{k,k}J_1 & \dots & -J_1\mathbf{Y}_{k,j}J_1 \\ \vdots & & \vdots \\ -J_1\mathbf{Y}_{j,k}J_1 & \dots & -J_1\mathbf{Y}_{j,j}J_1 \end{vmatrix} = \begin{vmatrix} J_1\mathbf{Y}_{k,k}J_1 & \dots & J_1\mathbf{Y}_{k,j}J_1 \\ \vdots & & \vdots \\ J_1\mathbf{Y}_{j,k}J_1 & \dots & J_1\mathbf{Y}_{j,j}J_1 \end{vmatrix}.$$

As $\det(J_1)^2 = 1$, we obtain:

$$\nabla_j(Y) = \begin{vmatrix} \mathbf{Y}_{k,k} & \dots & \mathbf{Y}_{k,j} \\ \vdots & & \vdots \\ \mathbf{Y}_{j,k} & \dots & \mathbf{Y}_{j,j} \end{vmatrix} = \begin{vmatrix} \mathbf{Y}_{j,j} & \dots & \mathbf{Y}_{j,k} \\ \vdots & & \vdots \\ \mathbf{Y}_{k,j} & \dots & \mathbf{Y}_{k,k} \end{vmatrix}.$$

Let $X = \begin{pmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{pmatrix} \in \Omega^+$. A simple computation shows that $\psi(X) = \begin{pmatrix} 0 & 0 \\ -\mathbf{X}^{-1} & 0 \end{pmatrix}$. If $X \in \mathcal{O}^+$, then by Lemma 4.3.3 , there exists $n \in N$ such that $n.X = \begin{pmatrix} 0 & \mathbf{Z} \\ 0 & 0 \end{pmatrix}$, where $\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbf{Z}_0 \end{pmatrix}$ and $\mathbf{Z}_j \in M(2,E)$. We set $Z = n.X$. From above we get:

$$\nabla_j(\psi(Z)) = (-1)^{k-j+1} \prod_{s=0}^{k-j} \frac{1}{\det(\mathbf{Z}_s)} = (-1)^{k-j+1} \frac{\prod_{s=k-j+1}^{k} \det(\mathbf{Z}_s)}{\prod_{s=0}^{k} \det(\mathbf{Z}_s)} = \frac{\Delta_{k+1-j}(Z)}{\Delta_0(Z)}.$$

As the Δ_j's and the ∇_j's are invariant under N, we have

$$\nabla_j(\psi(X)) = \frac{\Delta_{k+1-j}(X)}{\Delta_0(X)}$$

for all $X \in \mathcal{O}^+$and hence for all $X \in \Omega^+$. □

Definition 4.5.7. *Let $s = (s_0, \dots, s_k) \in \mathbb{C}^{k+1}$. We denote by $|\nabla|^s$ and $|\Delta^s|$ the functions respectively defined on $\mathcal{O}^+$ and $\mathcal{O}^-$ by*

$$|\Delta|^s(X) = |\Delta_0(X)|^{s_0} \dots |\Delta_k(X)|^{s_k}, \quad \text{for} X \in \mathcal{O}^+,$$

$$|\nabla|^s(Y) = |\nabla_0(Y)|^{s_0} \dots |\nabla_k(Y)|^{s_k}, \quad \text{for} Y \in \mathcal{O}^-.$$

Definition 4.5.8. *We denote by t the involution on $\mathbb{C}^{k+1}$ defined by*

$$t(s) = (-s_0 - s_1 - \ldots - s_k, s_k, s_{k-1}, \ldots, s_1),$$

for $s = (s_0, \ldots, s_k) \in \mathbb{C}^{k+1}$.

Corollary 4.5.9. *Let $X \in \Omega^+$. For $s \in \mathbb{C}^{k+1}$, we have*

$$|\nabla|^s(\psi(X)) = |\Delta|^{t(s)}(X).$$

In particular, the polynomials $|\nabla|^s$ and $|\Delta|^{s'}$ have the same A^0-character if and only if $s' = t(s)$.

Proof. The first statement is a straightforward consequence of Proposition 4.5.6. The second assertion follows then from Theorem 4.3.2 and 4.5.2 $\square$

Acknowledgments

During the writing of this paper we have greatly benefitted from help trough discussions or mails from several colleagues. We would like to thank Henri Carayol, Rutger Noot, Guy Rousseau, Torsten Schoeneberg and Marcus Slupinski.

References

[1] A. Blanchard — *Les corps non commutatifs* , Collection Sup: Le Mathématicien, $n°$ 9. Presses Universitaires de France, Vendôme, 1972. 135 pp. – cited on p. 33, 34

[2] A. Borel — *Linear algebraic groups*, Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.xii+288 pp. – cited on p. 85

[3] N. Bourbaki — *Groupes et Algèbres de Lie, Chapitres 4,5 et 6*, (1981) Masson, Paris. – cited on p. 6, 7, 12, 13, 16, 17, 20, 21, 44, 99

[4] N. Bourbaki — *Groupes et Algèbres de Lie, Chapitres 7 et 8*, (1975) Hermann, Paris. – cited on p. 5, 7, 10, 14, 19, 22, 27, 28, 33, 41, 51, 52, 61, 62, 65, 72, 75, 80

[5] N. Bourbaki — *Groupes et Algèbres de Lie, Chapitres 1*, (1971) Diffusion CCLS, Paris. – cited on p. 10, 27, 36

[6] N. Bourbaki — *Algèbre, Chapitre 8*, (2012) Springer, Berlin. – cited on p. 32

[7] N. Bopp-H. Rubenthaler — *Local Zeta functions attached to the minimal spherical series for a class of symmetric spaces*, Memoirs of the AMS (2005), Volume 174, $n°$ 821. – cited on p. 1, 4, 5, 47, 59, 91

[8] J. Faraut-A. Koranyi — *Analysis on Symmetric Cones*, (1994) *Oxford Sciences Publ.*

– cited on p. 4

[9] A. G., Helminck and G. F. Helminck, A class of parabolic k-subgroups associated with symmetric k-varieties. Trans. Amer. Soc. 350 (1998), 4669-4691. – cited on p. 95

[10] A. G. Helminck and S. P. Wang, On rationality properties of involutions of reductive groups, Adv. Math. **99** (1993) 26–96. – cited on p. 93, 94, 95

[11] J. E. Humphreys — *Linear Algebraic Groups*, Graduate Texts in Mathematics, v. 21, Springer-Verlag, New-York. – cited on p. 15

[12] J.I..Igusa — *On Functional equations of complex powers*, Invent. math. 85, 1986, pp 1-29. – cited on p. 15

[13] T.Y. Lam — *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, 67. American Mathematical Society, Providence, RI, 2005. xxii+550 pp. – cited on p. 34, 38, 66, 67, 68, 69, 70, 72, 73, 77, 79, 81, 87

[14] I.Muller — *Classification d'orbites pour une classe d'espaces préhomogènes*, Nagoya Math. J. Vol. 151 (1998), 161-197. – cited on p. 4, 14, 17, 59, 62, 63

[15] I.Muller — *Racines orthogonales et orbites d'algèbres de Lie semi-simples graduées*, J.of Algebra, 193 (1997), 41-74. – cited on p. 3, 4, 14

[16] I.Muller — *Racines orthogonales, formes quadratiques et orbites d'algèbres de Lie simples graduées*, J.of Algebra, 250 (2002), 288-334. – cited on p. 4

[17] I.Muller — *Une classe d'espaces préhomogènes de type parabolique faiblement sphériques*, preprint arXiv:0810.3148 [math.RT]. – cited on p. 4, 59, 79, 81

[18] I.Muller, H.Rubenthaler, G.Schiffmann — *Structure des espaces préhomogènes associés à certaines algèbres de Lie graduées*, Math. Ann. 274 (1986), $n°$ 1, 95-123.

– cited on p. 17, 39, 40, 42, 95

[19] R. S. Pierce — *Associative algebras*, Graduate Texts in Mathematics, 88. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982. xii+436 pp. – cited on p. 33

[20] D. Renard — *Représentations des groupes réductifs p-adiques. (French) [Representations of p-adic reductive groups]*, Cours Spécialisés [Specialized Courses], 17. Société Mathématique de France, Paris, 2010. vi+332 pp. – cited on p. 102

[21] H. Rubenthaler — *Formes réelles des espaces préhomogènes irréductibles de type parabolique*, (French) [Real forms of irreducible prehomogeneous spaces of parabolic type] Ann. Inst. Fourier (Grenoble) 36 (1986), $n°$. 1, 1-38.

– cited on p. 43

[22] H. Rubenthaler — *Construction de certaines sous-algèbres remarquables dans les algèbres de Lie semi-simples*(French) [Construction of certain remarkable subalgebras in semisimple Lie algebras] J. Algebra 81 (1983), no. 1, 268-278. – cited on p. 94

[23] D. J. Saltman — *Norm polynomials and algebras*, J. Algebra 62 (1980), no. 2, 333Ð345. – cited on p. 33

[24] P. Tauvel, R. W. Yu, — *Lie algebras and algebraic groups*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. xvi+653 pp.

– cited on p. 85

[25] F. Sato — *On Functional Equations of Zeta Distributions*, Advanced Studies in Pure Mathematics 15, 1989. Automorphic Forms and Geometry of Arithmetic Varieties, pp. 465-508. – cited on p. 15

[26] M. Sato – T. Kimura — *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155. – cited on p. 29

[27] T. Schoeneberg — *Semi-simple Lie algebras and thier classification over* $\mathfrak{p}$*-adic fields*, Mémoires de la S.M.F., $n°$ 151, Nouvelle série (2017). – cited on p. 5, 15, 18, 30, 31, 32, 34, 43, 45, 46, 52, 79

[28] I. B. Seligman — *Rational methods in Lie algebras*, Lecture Notes in Pure and Applied Mathematics, Vol. 17. Marcel Dekker, Inc., New York-Basel, 1976. viii+346 pp. – cited on p. 5, 15, 18, 30, 52

[29] J. Tits — *Classification of Algebraic Semisimple Lie Groups*, In: Algebraic Groups and Discontinuous Subgroups. Proceedings of symposia in pure mathematics, **IX** (Boulder, 1965), pp.33-62. – cited on p. 30, 31, 43

[30] E. B. Vinberg — *On the classification of the nilpotent elements of graded algebras*, Soviet. Math. Dokl. 16 (1975), 1517-1520. (Les démonstrations se trouvent dans *Classification of homogeneous nilpotent elements of a semi-simple graded Lie algebra*, Selecta Math. Sovietica, Vol. 6, $n°$1 (1987), 15-35). – cited on p. 15

[31] A. Weil — *Basic Number Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 144 Springer-Verlag New York, Inc., New York 1967 xviii+294 pp.

– cited on p. 47, 61

Pascale. Harinck, CMLS, CNRS, École polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France., E-mail: pascale.harinck@polytechnique.edu

Hubert Rubenthaler, Institut de Recherche Mathématique Avancée, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France, E-mail: rubenth@math.unistra.fr