FINITE DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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1. Introduction

In the case of semi-simple Lie algebras, the structure of the Grothendieck ring of finite dimensional representations of the quantum algebra is well understood : it is analogous to the classic case q=1 and we have a ring homomorphism of characters. For the general case of Kac-Moody algebras the picture is less clear. In the affine case, E. Frenkel and N. Reshetikhin have recently introduced an injective ring homomorphism of q-characters. It gives informations about the decomposition in Jordan subspaces for a class of commutative elements. The homomorphism of q-characters has a nice symmetry property analogous to the classic action of the Weyl group : the image is the intersection of the kernels of screening operators. In the ADE case, H. Nakajima, motivated by the geometry of quiver varieties, introduced t-analogs of q-characters. From representation theory point of view, it gives more informations about Jordan subspaces. We define t-analogs of screening operators and give a symmetry property of q,t-characters. Our construction is purely algebraic and is extended to the non-simply laced case : it could help to extend Nakajima's theory in this case.

2. Homomorphism of characters in finite case

$\mathfrak{g}$: simple Lie algebra of rank n, $I = \{1, ..., n\}$

$\mathfrak{h} \subset \mathfrak{g}$: Cartan subalgebra

$\Lambda \subset \mathfrak{h}^*$: lattice of weights of $\mathfrak{g}$

$\omega_i \in \Lambda$: fundamental weights

$\mathrm{Rep}(\mathcal{U}(\mathfrak{g}))$: Grothendieck ring of finite dimensional representations (with $\bigoplus$ and $\bigotimes$)

We have an injective homomorphism of rings :

$$\chi : \mathrm{Rep}(\mathcal{U}(\mathfrak{g})) \to \mathbb{Z}[y_i^{\pm}]_{i\in I}$$

$$\chi(V) = \sum_{\lambda = \sum_{i\in I} m_i\omega_i \in \Lambda} \dim(V_\lambda) \prod_{i\in I} y_i^{m_i}$$

where V_λ is the weight space of V :

$$V_\lambda = \{x \in V / \forall h \in \mathfrak{h}, h.x = \lambda(h)x\}$$

We have a symmetry property related to the Weyl groups W :

$$\mathrm{Im}(\chi) = \mathbb{Z}[y_i^{\pm}]_{i\in I}^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

The quantum case is analogeous :

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \simeq \mathrm{Rep}(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

where $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$: finite-dimensional representations of type 1.

In quantum affine case $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, the picture is less clear.

3. Quantum affine algebra

$\mathfrak{g}$: simple Lie algebra of rank n, $I = \{1, ..., n\}$

$(C_{ij})_{1\leq i,j\leq n} = (\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)})$: Cartan matrix of $\mathfrak{g}$

$\mathfrak{g} \subset \hat{\mathfrak{g}}$: affine Lie algebra, $(C_{i,j})_{0\leq i,j\leq n}$ generalized Cartan matrix of $\hat{\mathfrak{g}}$

$r_i = \frac{(\alpha_i,\alpha_i)}{2} \in \mathbb{Z}$, $B_{i,j} = r_i C_{i,j}$ symmetric matrix

$q \in \mathbb{C}^*$ not a root of unity, $q_i = q^{r_i}$

Definition 1. *The quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is the $\mathbb{C}(q)$-algebra generated by E_i, F_i, K_i, K_i^{-1} ($0 \leq i \leq n$) with relations :*

$$K_iK_i^{-1} = 1 = K_i^{-1}K_i$$
$$K_iE_j = q^{B_{ij}}E_jK_i$$
$$K_iF_j = q^{-B_{ij}}F_jK_i$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$
$$\sum_{s=0..1-C_{ij}} (-1)^s \begin{bmatrix} 1-C_{ij} \\ s \end{bmatrix}_{q_i} E_i^{1-C_{ij}-s}E_jE_i^s = 0 \text{ si } i \neq j$$
$$\sum_{s=0..1-C_{ij}} (-1)^s \begin{bmatrix} 1-C_{ij} \\ s \end{bmatrix}_{q_i} F_i^{1-C_{ij}-s}F_jF_i^s = 0 \text{ si } i \neq j$$

An Hopf algebra structure is given by formulas :

Comutiplication :

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \ , \ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \ , \ \Delta(K_i) = K_i \otimes K_i$$

Counit :

$$\epsilon(E_i) = \epsilon(F_i) = 0 \ , \ \epsilon(K_i) = 1$$

Antipode :

$$S(E_i) = -K_i^{-1}E_i \ , \ S(F_i) = -F_iK_i \ , \ S(K_i) = K_i^{-1}$$

4. Drinfeld new realization

Theorem 1. (Drinfeld new realization)

$\mathcal{U}_q(\hat{\mathfrak{g}})$ is generated by $x_{i,m}^{\pm}$ ($1 \leq i \leq n$, $m \in \mathbb{Z}$), $k_i^{\pm}$ ($1 \leq i \leq n$), $h_{i,m}$ ($1 \leq i \leq n$, $m \in \mathbb{Z}^*$), $c^{\pm\frac{1}{2}}$, with relations :

$$k_ik_j = k_jk_i$$
$$k_ih_{j,m} = h_{j,m}k_i$$
$$k_ix_{j,m}^{\pm}k_i^{-1} = q^{\pm B_{ij}}x_{j,m}^{\pm}$$
$$[h_{i,m}, x_{j,m'}^{\pm}] = \pm\frac{1}{m}[mB_{ij}]_q c^{\mp\frac{|m|}{2}}x_{j,m+m'}^{\pm}$$
$$x_{i,m+1}^{\pm}x_{j,m'}^{\pm} - q^{\pm B_{ij}}x_{j,m'}^{\pm}x_{i,m+1}^{\pm} = q^{\pm B_{ij}}x_{i,m}^{\pm}x_{j,m'+1}^{\pm} - x_{j,m'+1}^{\pm}x_{i,m}^{\pm}$$
$$[h_{i,m}, h_{j,m'}] = \delta_{m,-m'}\frac{1}{m}[mB_{ij}]_q\frac{c^m - c^{-m}}{q_j - q_j^{-1}}$$
$$[x_{i,m}^{+}, x_{j,m'}^{-}] = \delta_{ij}\frac{c^{\frac{m-m'}{2}}\phi_{i,m+m'}^{+} - c^{-\frac{m-m'}{2}}\phi_{i,m+m'}^{-}}{q_i - q_i^{-1}}$$

$$\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}x^{\pm}_{i,m_{\pi(1)}}...x^{\pm}_{i,m_{\pi(k)}}x^{\pm}_{j,m'}x^{\pm}_{i,m_{\pi(k+1)}}...x^{\pm}_{i,m_{\pi(s)}}=0$$

$\phi^{\pm}_{i,m}\in\mathcal{U}_q(\hat{\mathfrak{g}})$ are defined by :

$$\Phi_i^{\pm}(u)=\sum_{m=0..\infty}\phi^{\pm}_{i,\pm m}u^{\pm m}=k_i^{\pm 1}e^{\pm(q-q^{-1})\sum\limits_{m'=1..\infty}h_{i,\pm m'}u^{\pm m'}}\in\mathcal{U}_q(\hat{\mathfrak{g}})[[\pm u]]$$

Remark: $\{\phi^{\pm}_{i,\pm m},i\in I,m\in\mathbb{Z}\}$ is commutative

Every finite dimensional represention is a sum

$$V=\bigoplus_{\gamma}V_\gamma$$

where V_γ is a simultaneous generalized eigenspace :

$$V_\gamma=\{x\in V/\exists p\in\mathbb{N},\forall i\in\{1,...,n\},\forall m\in\mathbb{Z},(\phi^{\pm}_{i,m}-\gamma^{\pm}_{i,m})^p.x=0\}$$

5. Finite dimensional representation of $\mathcal{U}_q(\mathfrak{g})$

Theorem 2. (Chari-Pressley) *Every simple $V\in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is a highest weight representation V, that is to say $\exists v_0\in V$ (highest weight vector) :*

$$x^{+}_{i,r}.v_0=0\ ,\ V=\mathcal{U}_q(\hat{\mathfrak{g}}).v_0\ ,\ c^{\frac{1}{2}}.v_0=v_0\ ,\ \phi^{\pm}_{i,m}.v_0=\gamma^{\pm}_{i,m}v_0$$

Moreover we have an i-uplet $(P_i(u))_{i\in I}$ of (Drinfeld-)polynomials, $P_i(0)=1$, such that :

$$\gamma_i^{\pm}(u)=\sum_{m\in\mathbb{N}}\gamma^{\pm}_{i,\pm m}u^{\pm}=q_i^{\deg(P_i)}\frac{P_i(uq_i^{-1})}{P_i(uq_i)}\in\mathbb{C}[[u^{\pm}]]$$

and $(P_i)_{i\in I}$ parametrizes simple modules in $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$.

Theorem 3. (Frenkel-Reshetikhin) *The eigenvalues $\gamma_i(u)^{\pm}\in\mathbb{C}[[u]]$ of $V\in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ have the form :*

$$\gamma_i^{\pm}(u)=q_i^{deg(Q_i)-deg(R_i)}\frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}$$

where $Q_i(u),R_i(u)\in\mathbb{C}[u]$ and $Q_i(0)=R_i(0)=1$.

Example: $\mathfrak{g}=sl_2$, $I=\{1\}$. Explicit calculations with Jimbo's evaluation homomorphism ev_a : $\mathcal{U}_q(\hat{sl_2})\rightarrow\mathcal{U}_q(sl_2)$.

For $P(u)=1-ua$ we have $M_a=\mathbb{C}v_0\bigoplus_{\mathbb{C}}\mathbb{C}v_1$:

v_0 highest weight vector : $\phi^{\pm}(u).v_0=q\frac{1-uaq^{-1}}{1-uaq}v_0$, $P(u)=1-ua$

v_1 simultaneous eigenvector :

$$\phi^{\pm}(u).v_1=q^{-1}\frac{1-uaq^3}{1-uaq}v_1\ ,\ Q(u)=(1-ua)\ ,\ R(u)=(1-uaq^2)(1-ua)$$

6. q-characters

Definition 2. (Frenkel-Reshetikhin's q-characters)

$$\chi_q(V)=\sum_{\gamma}dim(V_{(\gamma)})\prod_{i\in I}\prod_{r=1..k_{\gamma_i}}Y_{i,a_{\gamma ir}}\prod_{s=1..l_{\gamma_i}}Y^{-1}_{i,b_{\gamma ir}}\in\mathbb{Z}[Y^{\pm}_{i,a}]_{i\in I,a\in\mathbb{C}^*}$$

where

$$Q_{\gamma i}(z)=\prod_{r=1..k_{\gamma i}}(1-za_{\gamma ir})\ ,\ R_{\gamma i}(z)=\prod_{r=1..l_{\gamma i}}(1-zb_{\gamma ir})$$

Example:

$$\chi_q(M_a) = Y_a + Y_{aq^2}^{-1}$$

Theorem 4. (Frenkel-Reshetikhin) *The map*

$$\chi_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*} = \mathcal{Y}$$

is an injective ring homomorphism and the diagram is commutative :

$$\begin{array}{ccc} Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \\ \downarrow res & & \downarrow \beta \\ Rep(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i\in I} \end{array}$$

Corollary 1. *The ring $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$.*

Example:

$$\chi_q(M_{aq^2}\otimes M_a) = (Y_{aq^3} + Y_{aq}^{-1})(Y_{aq} + Y_{aq^{-1}}^{-1})$$

$$= Y_{aq^3}Y_{aq} + Y_{aq^3}Y_{aq^{-1}}^{-1} + Y_{aq}^{-1}Y_{aq^{-1}}^{-1} + 1 = \chi_q(N) + \chi_q(\text{trivial module})$$

where N simple module with $dim(N) = 3$ and $P(u) = (1 - aq^3u)(1 - aqu)$.

Note that $M_{aq^2}\otimes M_a \neq N \oplus$ triv. is not semi-simple. In fact we have an exact sequence :

$$0 \to N \to M_{aq^2}\otimes M_a \to \text{trivial module} \to 0$$

Example:

$$\chi_q(M_a\otimes M_a) = Y_a^2 + 2Y_aY_{aq^2}^{-1} + Y_{aq^2}^{-1}$$

$M_a\otimes M_a$ simple module with $P(u) = (1 - au)^2$.

7. Symmetry property of q-characters

Let be $A_{i,a}^{\pm}\in\mathcal{Y}$:

$$A_{i,a} = Y_{i,aq_i}Y_{i,aq_i^{-1}}\Big(\prod_{j/C_{j,i}=-1} Y_{j,a}^{-1}\Big)\Big(\prod_{l/C_{l,i}=-2} Y_{l,aq}^{-1}Y_{l,aq^{-1}}^{-1}\Big)\Big(\prod_{m/C_{m,i}=-3} Y_{m,aq^2}^{-1}Y_{m,a}^{-1}Y_{m,aq^{-2}}^{-1}\Big)$$

Definition 3. (Frenkel-Reshetikhin's screening operators)

$$S_i : \mathcal{Y} \to \bigoplus_{a\in\mathbb{C}^*}\mathcal{Y}.S_{i,a}/\sum_{a\in\mathbb{C}^*}\mathcal{Y}.(S_{i,aq_i^2} - A_{i,aq_i}.S_{i,a})$$

$$S_i(Y_{i,a}) = Y_{i,a}.S_{i,a}$$

$$\forall U, V\in\mathcal{Y}, S_{i,a}(UV) = US_{i,a}(V) + VS_{i,a}(U)$$

Theorem 5. (Frenkel-Reshetikhin-Mukhin)

$$Im(\chi_q) = \bigcap_{i\in I} Ker(S_i)$$

and

$$Ker(S_i) = \mathbb{Z}[Y_{j,a}^{\pm}]_{j\neq i, a\in\mathbb{C}^*}\otimes\mathbb{Z}[Y_{i,b}(1 + A_{i,bq_i}^{-1})]_{b\in\mathbb{C}^*} = \mathfrak{K}_i$$

Example:

$$\chi_q(M_a) = Y_a(1 + A_{aq}^{-1})$$

$$\chi_q(N) = Y_{aq^3}Y_{aq}(1 + A_a^{-1} + A_{aq^2}^{-1}A_a^{-1})$$

$$= Y_{aq^3}Y_{aq}(1 + A_a^{-1} + A_{aq^2}^{-1}A_a^{-1})$$

$$= Y_{aq^3}Y_{aq}(1 + A_a^{-1})(1 + A_{aq^2}^{-1}) - 1$$

Remark: It is analogeous to the classic symmetry property of χ : $\text{Im}(\chi) = \mathbb{Z}[y_i^{\pm}]_{i\in I}^W$

It gives an algorithm to construct in purely combinatorial way the q-characters.

8. t-analogs of q-characters (ADE case)

We assume $\mathfrak{g}$ is of type ADE. Motivated by the geometry of quiver varieties, Nakajima defines : homomorphisms of additive group :

$$\hat{\chi}_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \hat{\mathcal{Y}}_t = \mathbb{Z}[W_{i,a}, V_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

$$\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}_t = \mathbb{Z}[Y_{i,a}^{\pm}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

a new twisted multiplication $*$ on $\hat{\mathcal{Y}}_t$

$$m_1 * m_2 = t^{2d(m_1,m_2)} m_1 m_2$$

t-analogs of $\mathfrak{K}_i$:

$$\mathfrak{K}_{t,i} = \subset \mathcal{Y}_t \ , \ \hat{\mathfrak{K}}_{t,i} \subset \hat{\mathcal{Y}}_t$$

Theorem 6. **(Nakajima)** *The maps $\chi_{q,t}, \hat{\chi}_{q,t}$ are injective, the diagram :*

$$\begin{array}{ccc} Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\hat{\chi}_{q,t}} & \hat{\mathcal{Y}}_t \\ \downarrow & & \downarrow \ \hat{\Pi}_t \\ Rep(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\chi_{q,t}} & \mathcal{Y}_t \\ \downarrow & & \downarrow \ \Pi_t \\ Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathcal{Y} \end{array}$$

is commutative. We have :

$$Im(\hat{\chi}_{q,t}) \subset \bigcap_{i\in I} \hat{\mathfrak{K}}_{t,i} = \hat{\mathfrak{K}}_t$$

$$Im(\chi_{q,t}) = \bigcap_{i\in I} \mathfrak{K}_{t,i} = \mathfrak{K}_t$$

and $\hat{\chi}_{q,t}$ is "almost" a ring homomorphism for the new multiplication $*$.

Remark: Nakajima gives an altenative conjectural definition, for M standard module (tensor product of fumdamental representations) :

$$\chi_q(M) = \sum_{\gamma} dim(M_\gamma) m_\gamma \Rightarrow \chi_{q,t}(M) = \sum_{\gamma,r} t^{2r} dim(M_\gamma^{r+1}/M_\gamma^r) m_\gamma$$

where a filtration of generalized eigenspace is used :

$$M^r_{(\gamma^\pm)} = \{x \in V / \forall i \in \{1,...,n\}, \forall m \in \mathbb{Z}, (\phi^{\pm}_{i,m} - \gamma^{\pm}_{i,m})^r.x = 0\}$$

Example:

$$\chi_{q,t}(M_{aq^2} \otimes M_a) = Y_{aq^3} Y_{aq} + Y_{aq^3} Y^{-1}_{aq^{-1}} + Y^{-1}_{aq} Y^{-1}_{aq^{-1}} + 1$$

$$\chi_{q,t}(M_a \otimes M_a) = Y_a^2 + (1+t^2) Y_a Y^{-1}_{aq^2} + Y^{-1}_{aq^2} = Y_a^2(1 + (1+t^2)A^{-1}_{aq} + A^{-2}_{aq})$$

$$\hat{\chi}_{q,t}(M_a \otimes M_a) = W_a^2(1 + (1+t^2)V_{aq} + V_{aq})$$

$$= (W_a(1+V_{aq})) * (W_a(1+V_{aq})) = \hat{\chi}_{q,t}(M_a) * \hat{\chi}_{q,t}(M_a)$$

9. t-analogs of screening operators (general case)

Goal : Symmetry property for q, t-characters.

We define a bimodule structure on $\bigoplus_{a\in\mathbb{C}^*} \hat{\mathcal{Y}}_t.S_{i,a}$:

$$S_{i,a}.m = t^{2u_{i,a}(m)}.S_{i,a}$$

Then

$$\sum_{a\in\mathbb{C}^*} \hat{\mathcal{Y}}(V_{i,aq_i}S_{i,aq_i^2} - t^2 S_{i,a})$$

is a subbimodule and we note the quotient $\hat{\mathcal{Y}}_{t,i}$.

Definition 4. (t-analogs of screening operators)

$$\hat{S}_{t,i}(m) = m(\sum_{a\in\mathbb{C}^*}(1_{u_{i,a}(m)\geq 0}(1+t^2+...+t^{2(u_{i,a}(m)-1)}) - 1_{u_{i,a}(m)<0}(t^{-2}+...+t^{2u_{i,a}(m)}))S_{i,a}$$

Theorem 7. (H.) *The diagram :*

$$\begin{array}{ccccc} & \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}_{t,i}} & \hat{\mathcal{Y}}_{t,i} & \\ \hat{\Pi}_t \downarrow & & & \downarrow & \hat{\Pi}_{t,i} \\ & \mathcal{Y}_t & \xrightarrow{S_{t,i}} & \mathcal{Y}_{t,i} & \\ \Pi_t \downarrow & & & \downarrow & \Pi_{t,i} \\ & \mathcal{Y} & \xrightarrow{S_i} & \mathcal{Y}_i & \end{array}$$

is commutative and we have :

$$\bigcap_{i\in I} Ker(S_{i,t}) = \mathfrak{K}_t$$

$$\bigcap_{i\in I} Ker(\hat{S}_{i,t}) = \hat{\mathfrak{K}}_t$$

More over in the ADE case, $\hat{S}_{t,i}$ is a derivation for $*$:

$$\forall X, Y \in \hat{\mathcal{Y}}_t, \hat{S}_{t,i}(X * Y) = \hat{S}_{t,i}(X) * Y + X * \hat{S}_{t,i}(Y)$$

Remark: The construction is purely algebraic and is extended to the non-simply laced case. Next step : to extend Nakajima's theory...

We hope to understand these operators as quantum differential operators.

10. References

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