Algébroïdes de Lie et mécanique lagrangienne

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Résumé

Algébroïdes de Lie et algébroïdes de Courant dans le formalisme lagrangien et hamiltonien

Nous introduirons les algébroïdes de Lie, donnerons des exemples et quelques propriétés. Nous montrerons ensuite comment la mécanique lagrangienne sur le fibré tangent à un espace de configuration se généralise aux algébroïdes de Lie. Nous donnerons des indications sur le rôle des algébroïdes de Courant dans ces théories.

Some motivation and references

In 1992, motivated by the study of discrete Lagrangian and Hamiltonian system, Alan Weinstein showed that the natural framework for discrete systems is that of **Lie groupoids**, and for the case of continuous time, that of **Lie algebroids**.
For 1-dimensional variational problems – the case of mechanics – the framework of Lie algebroids extends that of tangent bundles.
For 2-dimensional problems, Pavol Ševera showed that it was necessary to introduce **Courant algebroids**.

Weinstein, *Lagrangian mechanics and groupoids*,
Fields Inst. Commun. 7 (Waterloo, ON, 1992),
Amer. Math. Soc., Providence, RI, 1996, 207–231.

Ševera, *Some title containing the words "homotopy" and "symplectic", e.g. this one*,
Travaux mathématiques (Luxembourg) 16 (2005), 121–137.

A classical fact

Let M be a manifold, $TM \to M$ its **tangent bundle**.
Let $\Gamma(TM)$ be the $C^\infty(M)$-module of sections of TM (the space of vector fields on M).

Fact. The Lie bracket of vector fields makes $\Gamma(TM)$ a
LIE ALGEBRA over $\mathbb{R}$:
skewsymmetry $[X, Y] = -[Y, X]$ and
Jacobi identity $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

Remark: The Jacobi identity is usually written in the equivalent form,
$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Fact. The Lie bracket of vector fields satisfies the
LEIBNIZ IDENTITY:

$$\forall X, Y \in \Gamma(TM), \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (X \cdot f)Y$$

Lie algebroids are "generalized tangent bundles"

Definition of Lie algebroids

Definition

A *Lie algebroid* is a vector bundle $\tau : A \to M$ such that

• ΓA is a LIE ALGEBRA over $\mathbb{R}$
(skewsymmetry + Jacobi identity)

• there exists a morphism of vector bundles $\rho : A \to TM$, called the *anchor*, such that the LEIBNIZ IDENTITY is satisfied,

$$\boxed{\forall X, Y \in \Gamma A, \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (\rho(X) \cdot f)Y}$$

Proposition

The mapping ρ induces a Lie algebra homomorphism $\Gamma A \to \Gamma(TM)$.

Remark: Although this proposition is a consequence of the definition, it is often included in the list of axioms.

Examples

- TM
- foliations: integrable subbundles of TM
- gauge Lie algebroids (also called Atiyah algebroids): TP/G, where P is a principal bundle with structure group G.
- action Lie algebroids : $M \times \mathfrak{g} \to M$, where $\mathfrak{g}$ is a Lie algebra acting on M.

Example: action Lie algebroid

In an action Lie algebroid $M \times \mathfrak{g}$, the anchor is $(m, X) \in M \times \mathfrak{g} \mapsto X_M(m) \in TM$ and the Lie bracket of constant sections is their Lie bracket in $\mathfrak{g}$ considered as a constant map. Then use the Leibniz rule for the bracket of arbitrary sections.

Another example is the cotangent bundle of a Poisson manifold.

Definition of Poisson manifolds

[Moché Flato, André Lichnerowicz, Daniel Sternheimer, 1974]
A Poisson manifold is a smooth manifold equipped with a Poisson bracket on its ring of functions, i.e., a skew-symmetric, $\mathbb{R}$-bilinear, biderivation, $\{\ ,\ \}$, satisfying the Jacobi identity,
$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$,
for all smooth functions f, g, h.
Such a bracket is defined by a bivector (i.e., a field of skew-symmetric contravariant tensors) π and $\{f, g\} = \pi(df, dg)$, where π satisifes a condition that ensures the Jacobi identity for $\{\ ,\ \}$ ($[\pi, \pi]_{SN} = 0$, see *infra*).
Poisson manifolds generalize symplectic manifolds since, if π is non-degenerate, its inverse is a symplectic 2-form.
Example: The dual of a Lie algebra is a (linear) Poisson manifold (and not a symplectic manifold).

Example of Lie algebroid: The cotangent bundle of a Poisson manifold

On the cotangent bundle T^*M of a Poisson manifold, (M, π),
there exists a unique Lie algebroid structure such that
the anchor is the map $\pi^\sharp : T^*M \to TM$ defined by $\pi^\sharp(\alpha) = i_\alpha \pi$,
for all 1-forms α, and
the bracket of 1-forms, denoted by $[\ ,\ ]_\pi$, satisfies

$$[df, dg]_\pi = d\{f, g\}, \quad \forall f, g \in C^\infty(M).$$

The general formula for the bracket of 1-forms on a Poisson manifold is

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp \alpha}\beta - \mathcal{L}_{\pi^\sharp \beta}\alpha - d(\pi(\alpha, \beta)).$$

The differential of a Lie algebroid

For any Lie algebroid A, there is a differential d_A on $\Gamma(\wedge^\bullet A^*)$ defined by

$$(d_A\alpha)(X_0, X_1, \dots, X_k) = \sum_{i=0}^{k}(-1)^i \rho(X_i) \cdot \alpha(X_0, \dots, \widehat{X_i}, \dots, X_k) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k),$$

for $\alpha \in \Gamma(\wedge^k A^*)$, $k \in \mathbb{N}$, $X_0, \dots, X_k \in \Gamma A$.

$$\boxed{d_A^2 = 0}$$

The differential d_A turns $\Gamma(\wedge^\bullet A^*)$ into a complex

whose cohomology is called the *Lie algebroid cohomology* and is denoted by $H^\bullet(A)$.

The de Rham differential and the Chevalley–Eilenberg differential

Lie algebroid cohomology unifies de Rham cohomology and Chevalley–Eilenberg cohomology.

When $A = TM$, the differential d_A acts on the differential forms $\Gamma(\wedge^\bullet(T^*M))$ on the manifold M, and $H^\bullet(A)$ is the de Rham cohomology of M.

When A is a Lie algebra considered as a Lie algebroid over a point, $H^\bullet(A)$ is the Chevalley–Eilenberg cohomology.

The *Lie derivative* of sections of $\Lambda^\bullet A^*$ with respect to $X \in \Gamma A$ is defined by $\mathcal{L}_X = [i_X, d_A]$,
where $[\ ,\ ]$ is the graded commutator of derivations of $\Gamma(\wedge^\bullet A^*)$.

The Gerstenhaber algebra of a Lie algebroid

There is a unique extension to $\Gamma(\wedge^\bullet A)$ of the Lie bracket of sections of A, again denoted by $[\ ,\ ]_A$, as a (graded) biderivation satisfying $[X, f]_A = \langle X, d_A f \rangle$, for all $X \in \Gamma A$, $f \in C^\infty(M)$. This bracket of degree -1 is called a *Gerstenhaber bracket*, and $(\Gamma(\wedge^\bullet A), \wedge, [\ ,\ ]_A)$ is a *Gerstenhaber algebra*.

To summarize, to any Lie algebroid are associated both
• a differential d_A on $\Gamma(\wedge^\bullet A^*)$,
and
• a Gerstenhaber bracket $[\ ,\ ]_A$ on $\Gamma(\wedge^\bullet A)$.

The Schouten–Nijenhuis bracket of multivector fields

In the special case where $A = TM$ with the Lie bracket of vector fields, the Gerstenhaber bracket on $\Gamma(\wedge^\bullet TM)$ is called the *Schouten–Nijenhuis bracket of multivector fields*.
It is a bracket, $[\ ,\ ]_{SN}$, of degree -1 on the space of multivector fields on M, the unique extension to $\Gamma(\wedge^\bullet TM)$ as a (graded) biderivation of the Lie bracket of vector fields satisfying $[X, f]_{SN} = X \cdot f$, for all $X \in \Gamma(TM)$, $f \in C^\infty(M)$.

Application: (M, π) is a Poisson manifold if and only if $[\pi . \pi]_{SN} = 0$.

Example: the Lichnerowicz–Poisson differential

If (M, π) is a Poisson manifold, the differential associated to the Lie algebroid $(T^*M, [\ ,\ ]_\pi, \pi^\sharp)$ is

$$\boxed{d_\pi = [\pi, \cdot]_{SN}}$$

acting on $\Gamma(\wedge^\bullet TM)$, where $[\ ,\ ]_{SN}$ is the Schouten–Nijenhuis bracket of multivector fields on M.

(Bhaskara and Viswanath [1988], yks and Franco Magri [1990])

Example: deformation by a Nijenhuis tensor

Let M be a manifold and $N : TM \to TM$ a Nijenhuis tensor, i.e., $\mathcal{T}(N) = 0$, where $\mathcal{T}(N)$ is the Nijenhuis torsion of N defined by

$$\mathcal{T}(N)(X, Y) = [NX, NY] - N([NX, Y] + [X, NY]) + N^2[X, Y],$$

for all vector fields X, Y. Then

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$$

is a Lie bracket on TM, and TM becomes a Lie algebroid with anchor $N : TM \to TM$ and bracket $[\ ,\ ]_N$.

The associated differential is

$$d_N = [i_N, d]$$

(graded commutator).

Lie bialgebroids

A Lie bialgebroid (A, A^*) is a pair of Lie algebroids A and A^* in duality, satisfying the following compatibility assumption

the differential d_A is a derivation of the Gerstenhaber bracket $[\ ,\ ]_{A^*}$.

Proposition (A, A^*) is a Lie bialgebroid if and only if (A^*, A) is a Lie bialgebroid.
(Kirill Mackenzie and Ping Xu [1994]; yks [1995])

Example: If (M, π) is a Poisson manifold, then (TM, T^*M) is a Lie bialgebroid.

Lagrangian mechanics

Let L be a first-order Lagrangian on configuration space M.
Then L is a function on J^1M, the first-order jets of functions from $\mathbb{R}$ to M.
Its Euler-Lagrange differential EL is a 1-form on J^2M.
In local coordinates, $(EL)_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}$.

$(EL)_i = 0$ are the Euler-Lagrange equations of L.

Assume that L is time-independent,
then L is just a function on TM, $L = L(q^i, \dot{q}^i)$.

The Legendre transform, defined by the fiber derivative of L, is a bundle map $\mathcal{F}L : TM \to T^*M$.
L is called regular if $\mathcal{F}L$ is an isomorphism.
Assume L is regular. Then the pull-back of the canonical symplectic structure of the cotangent bundle T^*M is a symplectic structure ω_L on TM.

Define the action function A_L as the function on TM such that $A_L(X) = \langle \mathcal{F}L(X), X \rangle$, for $X \in TM$, and the energy,

$$E_L = A_L - L.$$

With respect to the symplectic structure ω_L , the Hamiltonian vector field, $\mathcal{X}_L$, with Hamiltonian E_L is defined by $i_{\mathcal{X}_L}\omega_L = dE_L$.
It is a vector field on TM, i.e., a section of $T(TM) \to TM$.

$\mathcal{X}_L$ is admissible, i.e., it is written locally as $\mathcal{X}_L = \dot{q}^i \frac{\partial}{\partial q^i} + X^i(q^j, \dot{q}^j) \frac{\partial}{\partial \dot{q}^i}$. It is therefore associated to a system of second-order differential equations. These equations are the Euler–Lagrange equations for L.

Liouville vector field

On any vector bundle $E \to M$, the Liouville vector field Z
(also called the Euler vector field)
is the generator of the group of dilations of E.
In local coordinates (x^i, y^α),

$$Z = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

The Liouville vector field is vertical (tangent to the fibers).

Consider the double tangent bundle $T(TM)$.
In local coordinates $(q^i, \dot{q}^i)$,

$$Z_{TM} = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

Vertical endomorphism

Let $\mathcal{V} : T(TM) \to T(TM)$ be the vertical endomorphism of TM.
Let $\mathcal{V}(TM)$ be the vector bundle of vertical vectors over TM.
In local coordinates $(q^i, \dot{q}^i)$, for $\mathcal{X} \in \mathcal{V}(TM)$,

$$\boxed{\mathcal{X} = X^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}}$$

The image of $\mathcal{V}$ is contained in $V(TM)$.
In local coordinates $(q^i, \dot{q}^i)$,

$$\mathcal{V} : \mathcal{X} = X^i(q, \dot{q}) \frac{\partial}{\partial q^i} + Y^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} \longmapsto \mathcal{V}(X) = X^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}.$$

A vector field $\mathcal{X}$ on TM is admissible if and only if $\mathcal{V}(\mathcal{X}) = Z_{TM}$.

An approach to mechanics on Lie algebroids

For $A = TM$, this approach goes back to Joseph Klein,
Joseph Klein, *Espaces variationnels et mécanique*, Ann. Inst. Fourier (Grenoble) 12 (1962), 1-124.

The Euler–Lagrange equations for L are equivalent to

$$\mathcal{L}_{\mathcal{X}_L}\theta_L = dL,$$

where θ_L is the 1-form on TM which is the image of dL under the dual of the vertical endomorphism.
Question. On a Lie algebroid?

Recent contributions include

• Eduardo Martínez, Lagrangian mechanics on Lie algebroids, *Acta Appl. Math.* 67 (2001), no. 3, 295–320,

• Katarzyna Grabowska, Pawel Urbanski, and Janusz Grabowski Geometrical mechanics on algebroids,
Int. J. Geom. Methods Mod. Phys. 3 (2006), no. 3, 559-575.

The vector bundle $\mathcal{L}A$

Let $A \xrightarrow{\tau} M$ be a Lie algebroid with base M.
Define $\boxed{\mathcal{L}A = TA \oplus_{TM} A}$ as the subbundle of $TA \oplus A$ that completes the diagram:

$$\begin{array}{ccc} TA \oplus_{TM} A & \longrightarrow & A \\ \downarrow & & \downarrow \quad \rho \\ TA & \xrightarrow{T\tau} & TM \end{array}$$

The elements of $\mathcal{L}A$ are pairs $(X, a) \in TA \times A$ such that $\boxed{(T\tau)(X) = \rho(a)}$.

Fact. If $A = TM$, then $\mathcal{L}A = \mathcal{L}(TM)$ reduces to $TA = T(TM)$.
In fact, in this case, elements of $\mathcal{L}(TM)$ are pairs $(v, (T\tau)v)$, for $v \in T(TM)$. Such pairs are in 1-to-1 correspondence with elements $v \in T(TM)$.

The Lie algebroid $\mathcal{L}A$

$\mathcal{L}A$ can be viewed as a Lie algebroid over A called the pull-back Lie algebroid of A over τ.

The projection onto the base manifold, $p : \mathcal{L}A \to A$, maps a pair (X, a), where X is tangent to A at u, to u. It is the composition of the projection p_1 from $\mathcal{L}A$ onto TA with the projection of the vector bundle $TA \to A$.

The anchor is the projection p_1 onto TA.

The bracket is such that for $X, Y \in \Gamma(TA)$ and for sections u, v of A which satisfy $T\tau \circ X = \rho \circ u \circ \tau$ and $T\tau \circ Y = \rho \circ v \circ \tau$,

$$[(X, \tau^* u), (Y, \tau^* v)]_{\mathcal{L}A} = ([X, Y]_{TA}, \tau^*[u, v]_A) \ .$$

A reference

The Lie algebroid $\mathcal{L}A$ is a special case of the construction by Higgins and Mackenzie [1990] of the pull-back of a Lie algebroid by a submersion. See

Kirill Mackenzie,
General Theory of Lie Groupoids and Lie Algebroids,
London Mathematical Society Lecture Note Series, 213,
Cambridge University Press, Cambridge, 2005, p. 256.

Pull-back by the projection

To the general case corresponds the Lie algebroid $\phi^{!!}A$ with base A that completes the diagram

$$\begin{array}{ccc} \phi^{!!}A = TN \oplus_{TM} A & \longrightarrow & A \\ \downarrow & & \downarrow \quad \rho \\ TN & \stackrel{T\phi}{\longrightarrow} & TM \\ \downarrow & & \downarrow \\ N & \stackrel{\phi}{\longrightarrow} & M \end{array}$$

To the special case where $N = A$ and $\phi = \tau$, there corresponds the Lie algebroid $\tau^{!!}A = \mathcal{L}A$ with base A that completes the diagram

$$\begin{array}{ccc} \mathcal{L}A = TA \oplus_{TM} A & \longrightarrow & A \\ \downarrow & & \downarrow \quad \rho \\ TA & \stackrel{T\tau}{\longrightarrow} & TM \\ \downarrow & & \downarrow \\ A & \stackrel{\tau}{\longrightarrow} & M \end{array}$$

Properties of $\mathcal{L}A \to A$

On $\mathcal{L}A \to A$ there is

• a Liouville section Z: the value of Z at a is the vertical lift of a,

• a vertical endomorphism $\mathcal{V}$: the vertical endomorphism maps a pair (X, a) to the vertical lift of a.
The vertical endomorphism is a Nijenhuis tensor of square 0.

An admissible element of $\mathcal{L}A$ is a pair $(X, a) \in TA \times A$ such that $(T\tau)(X) = \rho(a)$ and X is tangent to A at $a \in A$.
The anchor $p_1 : \mathcal{L}A \to TA$ maps an admissible section of $\mathcal{L}A \to A$ to an admissible vector field on A (satisfying $T\tau(X_a) = \rho(a)$).
Admissible sections play the rôle of second-order differential equations.

Joseph Klein's formalism extended

By duality, the vertical endomorphism of $\mathcal{L}A$ acts on the sections of the dual bundle $(\mathcal{L}A)^* \to A$.

Let L be a Lagrangian on A. Applying $\mathcal{V}$ to dL yields a section θ_L of $(\mathcal{L}A)^*$, the Cartan section.
Since L is a function, $(\mathcal{V} \circ d)(L) = i_{\mathcal{V}} dL = \mathcal{L}_{\mathcal{V}} L$, so $\boxed{\theta_L = \mathcal{L}_{\mathcal{V}} L}$.

θ_L defines a map from A to A^*, the Legendre transformation.

Klein's formalism extended (continued)

Set $\omega_L = -d\theta_L$.
The Lagrangian L is called regular if the differential 2-form ω_L is non-degenerate.
Define the action function of L,

$$\mathcal{A}_L = i_Z dL,$$

and the energy

$$\mathcal{E}_L = \mathcal{A}_L - L.$$

Klein's formalism extended (continued)

If L is regular, define a section $\mathcal{X}_L$ of $\mathcal{L}A$ by the equation

$$i_{\mathcal{X}_L}\omega_L = d\mathcal{E}_L.$$

Then (Martínez, pp. 313–314),

- the section $\mathcal{X}_L$ is admissible,
- $\mathcal{X}_L$ satisfies the Euler–Lagrange equation which takes the form,

$$\mathcal{L}_{\mathcal{X}_L}\theta_L = dL.$$

In local coordinates,

$$\mathcal{L}_{\mathcal{X}_L}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} + c^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}.$$

The case of a regular Lagrangian

Fact. X is an admissible section of $\mathcal{L}A \to A$ if and only if $\mathcal{V}(X) = Z$.
Therefore, if X is admissible, then for any section Λ of $(\mathcal{L}A)^* \to A$,

$$i_X(\mathcal{V}\Lambda) = i_Z\Lambda.$$

Proposition

When the Lagrangian L is regular, an equivalent form of the Euler-Lagrange equations is

$$\mathcal{L}_{\mathcal{X}_L}\theta_L = dL.$$

Here the differential d is that of the Lie algebroid $\mathcal{L}A$, and both sides of the equation are sections of $(\mathcal{L}A)^* \to A$.
Proof. $i_{\mathcal{X}_L}\omega_L - d\mathcal{E}_L = -\mathcal{L}_{\mathcal{X}_L}\theta_L + (di_{\mathcal{X}_L}\mathcal{V}(dL) - di_Z dL) + dL$.
It follows from $i_{\mathcal{X}_L} \circ \mathcal{V} = i_Z$ that the middle term vanishes.

Symmetries

The complete lift of a section X of A is the projectable section X^c of $\mathcal{L}A \to A$ such that $p_1(X^c)\lambda = \mathcal{L}_X\lambda$, for each section λ of A^*, where λ and $\mathcal{L}_X\lambda$ are considered as functions on A.

The case of TM. If $A = TM$, then $\mathcal{L}A = T(TM)$ and the complete lift of a section $X = X^i \frac{\partial}{\partial q^i}$ of TM (i.e., a vector field) is the usual lift to $T(TM)$, $\check{X} = X^i \frac{\partial}{\partial q^i} + \frac{\partial X^i}{\partial q^i} \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$.

An (infinitesimal) symmetry of L is defined as a section X of A such that $\mathcal{L}_{X^c} L = 0$.

It is clear that this definition generalizes the definition of a symmetry of a time-independent first-order Lagrangian on M,

Noether currents on Lie algebroids

Let $\nu_X = \langle \theta_L, X^c \rangle$. The function ν_X on $\mathcal{L}A$ is called the Noether current associated with the section X of A.

Since the Cartan section generalizes the 1-form $\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i$ on TM, and since X^c generalizes $\check{X}$, ν_X generalizes the Noether current associated to a symmetry, X, of a time-independent first-order Lagrangian, L, on M.

The elementary Noether theorem in the Lie algebroid case

Theorem

If a section X of A is a symmetry of the regular Lagrangian L, then the Noether current associated with X is invariant under the flow of $\mathcal{X}_L$.

Proof. It follows from the properties of the vertical endomorphism and the complete lifts that $\mathcal{V}(X^c, \mathcal{X}_L]) = [X^c, \mathcal{V}\mathcal{X}_L] = [X^c, Z] = 0$, for each section X of A. Therefore the section $[X^c, \mathcal{X}_L]$ of $p : \mathcal{L}A \to A$ is vertical and $\langle \theta_L, [X^c, \mathcal{X}_L] \rangle = 0$.
It follows that $\mathcal{L}_{\mathcal{X}_L} \langle \theta_L, X^c \rangle = \langle \mathcal{L}_{\mathcal{X}_L} \theta_L, X^c \rangle$.
The section $\mathcal{X}_L$ satisifies the Euler-Lagrange equations, $\mathcal{L}_{\mathcal{X}_L} \theta_L = dL$.
Therefore, $\mathcal{L}_{\mathcal{X}_L} \langle \theta_L, X^c \rangle = \langle dL, X^c \rangle$.
Thus, if X is a symmetry of L, then $\mathcal{L}_{\mathcal{X}_L} \nu_X = 0$.

2-dimensional variational problems

Question. Why Courant algebroids?
The tangent bundle framework, more generally the Lie algebroid framework, are for 1-dimensional variational problems: one independent variable, denoted t, "time".
What is the proper framework for 2-dimensional variational problems?

Courant algebroids. The standard Courant algebroid

Definition (standard Courant algebroid)

- vector bundle $E = TM \oplus T^*M$,
- fiberwise non-degenerate bilinear symmetric form

$$\langle X + \xi, Y + \eta \rangle = \langle X, \eta \rangle + \langle Y, \xi \rangle,$$

- non-skewsymmetric bracket on sections

$$\boxed{[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi}$$

$\forall X, Y \in \Gamma(TM), \xi, \eta \in \Gamma(T^*M)$.

In particular, $[X, \eta] = \mathcal{L}_X \eta$ and $[\eta, X] = -i_X d\eta$.

This bracket is called the Dorfman bracket.

The Dorfman bracket is a derived bracket

Derived bracket (yks, Ann. Fourier [1996], LMP [2004])

$$[a, b]_d = [[a, d], b]$$

(graded Lie algebra, d odd element, $[d, d] = 0$)
The Dorfman bracket is a *derived bracket*.
Set $\epsilon_\xi(\eta) = \xi \wedge \eta$.

$$[i_X, i_Y]_d = [[i_X, d], i_Y] = [\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

$$[i_X, \epsilon_\eta]_d = [[i_X, d], \epsilon_\eta] = [\mathcal{L}_X, \epsilon_\eta] = \epsilon_{\mathcal{L}_X \eta}$$

$$[\epsilon_\xi, i_Y]_d = [[\epsilon_\xi, d], i_Y] = [\epsilon_{d\xi}, i_Y] = -\epsilon_{i_Y d\xi}$$

$$[\epsilon_\xi, \epsilon_\eta] = 0.$$

Therefore,
when $X \leftrightarrow i_X$ and $\xi \leftrightarrow \epsilon_\xi$, $[X, \eta]_d = \mathcal{L}_X \eta$ and $[\xi, Y]_d = -i_Y d\xi$.

Loday algebras

Definition

A *Loday algebra* is a (graded) vector space equipped with a bracket (bilinear operation, in general non-skewsymmetric) satisfying the Jacobi identity

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]],$$

$\forall a, b, c$.

Loday algebras are also called *Leibniz algebras*.

Fact. If $[\ ,\ ]$ is a Lie bracket, then the derived bracket $[\ ,\ ]_d$ is a Loday bracket.
Therefore a Lie algebra with a derived bracket is a Loday algebra.
Example.
The vector space of sections of the standard Courant algebroid with the Dorfman bracket is a Loday algebra.

Properties of the Dorfman bracket

Theorem

The Dorfman bracket satisfies the Jacobi identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]],$$

$\forall a,b,c \in \Gamma(TM \oplus T^*M)$.

Remark. The Dorfman bracket does NOT satisfy the identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

Fact. $\Gamma(TM \oplus T^*M)$ with the Dorfman bracket is a LODAY ALGEBRA, and NOT, in general, a Lie algebra.

The Courant bracket

The skewsymmetrized Dorfman bracket is called the Courant bracket,

$$[X+\xi, Y+\eta]^C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\xi - i_Y\eta).$$

The Courant bracket does NOT satify the Jacobi identity.

Courant Algebroids (general case)

A vector bundle $E \to M$ is called a pseudo-orthogonal vector bundle if it is equipped with a fiberwise non-degenerate symmetric bilinear form, $\langle\ ,\ \rangle$.

Definition

A Courant algebroid is a pseudo-orthogonal vector bundle $(E, \langle\ ,\ \rangle)$ equipped with a Loday bracket $[\ ,\ ]$ and an anchor $\rho : E \to TM$ satisfying the Leibniz rule, and the identities,

$$\rho(a)\langle b, c\rangle = \langle a, [b, c] + [c, b]\rangle,$$

$$\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle.$$

Examples of Courant algebroids

• Case $M = \{pt\}$, E a Lie algebra with an invariant non-degenerate symmetric bilinear form (for example a semi-simple Lie algebra with its Killing form).

• $E = TM \oplus T^*M$ with the non-degenerate bilinear symmetric form

$$\langle X + \xi, Y + \eta \rangle = \langle X, \eta \rangle + \langle Y, \xi \rangle$$

and the Dorfman bracket.

• more generally, $E = A \oplus A^*$ when (A, A^*) is a Lie bialgebroid.

Remarks from Ševera's article and notes

In a 2-dimensional variational problem one considers integrals of 2-forms (depending on elements of a jet bundle) on a manifold M. Ševera shows that the condition "symmetry up to divergence" is expressed naturally in the framework of Courant algebroids.

Let α be a 2-form on M.
A vector field X is an (infinitesimal) symmetry of α if $\mathcal{L}_X \alpha = 0$.
X is an (infinitesimal) symmetry up to divergence if there exists a 1-form ξ on M such that $\mathcal{L}_X \alpha = d\xi$. So we must consider the pair $(X, \xi) \in \Gamma(TM \oplus T^*M)$.

Remarks from Ševera's article and notes (continued)

Proposition

Let α be a 2-form. Consider the pair $(X, \xi) \in \Gamma(TM \oplus T^*M)$. The following conditions are equivalent.

- (X, ξ) *leaves the graph of α invariant (with respect to the Dorfman bracket).*
- (X, ξ) *defines a symmetry of α up to divergence.*

Proof

Proof Let

$$\mathrm{Gr}(\alpha) = \{(Y, i_Y\alpha) \in TM \oplus T^*M \mid Y \in TM\}$$

be the graph of α.
The condition for (X, ξ) to preserve $\mathrm{Gr}\alpha$ infinitesimally is
$[(X, \xi), (Y, i_Y\alpha)] \in \mathrm{Gr}(\alpha)$, for all $Y \in \Gamma(TM)$, i.e.,
$([X, Y], \mathcal{L}_X(i_Y\alpha) - i_Y d\xi) \in \mathrm{Gr}(\alpha)$, for all $Y \in \Gamma(TM)$, i.e.,
$\mathcal{L}_X(i_Y\alpha) - i_Y d\xi) = i_{[X,Y]}\alpha$, for all $Y \in \Gamma(TM)$.
Replacing $i_{[X,Y]}$ by $\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$, we obtain the condition
$i_Y\mathcal{L}_X\alpha - i_Y d\xi = 0$, for all $Y \in \Gamma(TM)$.
Therefore the graph of α is infinitesimally invariant under (X, ξ) if and only if

$$\boxed{\mathcal{L}_X\alpha = d\xi}$$

Exact Courant algebroids

In addition, since α can be replaced by a 2-form, $\alpha + B$, where B is a closed 2-form, one should replace the graph of α in $TM \oplus T^*M$ by an isotropic splitting in an "exact Courant algebroid".

A Courant algebroid $E \to M$ is called "exact" if the following sequence is exact,

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0,$$

where $\rho^* : T^*M \to E^*$ is the dual of the anchor ρ, and E^* is identified with E by means of the non-degenerate symmetric bilinear form.

Ševera's "nonabelian Noether theorem"

Ševera introduces the action of a Poisson-Lie group G (i.e., a Lie group with a Poisson structure compatible with the multiplication) on a manifold M equipped with a 2-form α (corresponding to the Lagrangian). He defines an action of G on the Courant algebroid $TM \oplus T^*M$ by means of the Poisson tensor of G. He states that when the graph of α is invariant under this action, there is a $\mathfrak{g}^*$-valued 1-form on M, ν, satisfying an equation $d\nu \pm \frac{1}{2}[\nu, \nu] = 0$ on surfaces in M that are extremal for α. Thus, there are "nonabelian conservation laws" associated to the action of G on M, and this result constitutes a "nonabelian Noether theorem". Indeed, when the Lie group G is Abelian, the Maurer-Cartan type equation satisfied by ν reduces to $d(\nu_X) = 0$, for each $X \in \mathfrak{g}$, i.e., to a conservation law associated to each infinitesimal symmetry of the 2-form α.

Applications

Applications to the reduction of Lagrangian systems and to systems with nonholonomic constraints can be found in recent articles:

• J.F. Cariñena, Miguel Rodríguez-Olmos,
Gauge equivalence and conserved quantities for Lagrangian systems on Lie algebroids, J. Phys. A. : Math. Theor. 42 (2009).

• J. Grabowski, M. de León, J. C. Marrero, D. Martín de Diego,
Nonholonomic constraints: a new viewpoint, J. Math. Phys. 50 (2009), no. 1, 013520, 17 pp.

• J. Cortés, M. de León, J.C. Marrero, E. Martínez,
Non-holonomic Lagrangian systems on Lie algebroids, Discrete Contin. Dyn. Syst. 24 (2009), no. 2, 213-271.