The Noether Theorems: from Noether to Ševera

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Lecture 1
In Noether's Words: The Two Noether Theorems

Emmy Noether 1918
Invariant Variational Problems

Invariante Variationsprobleme *Göttinger Nachrichten* (1918), pp. 235–257.

"We consider variational problems which are invariant under a continuous group (in the sense of Lie). [...] What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups."

The two Noether Theorems

"In what follows we shall examine the following two theorems:

I. *If the integral I is invariant under a [group]* $\mathfrak{G}_\rho$*, then there are* ρ *linearly independent combinations among the Lagrangian expressions which become divergences–and conversely, that implies the invariance of I under a group* $\mathfrak{G}_\rho$*. The theorem remains valid in the limiting case of an infinite number of parameters.*

II. *If the integral I is invariant under a [group]* $\mathfrak{G}_{\infty\rho}$ *depending upon arbitrary functions and their derivatives up to order* σ*, then there are* ρ *identities among the Lagrangian expressions and their derivatives up to order* σ*. Here as well the converse is valid.*[1]"

[1] For some trivial exceptions, see §2, note 13.

Questions

- What variational problem is Noether considering? What is the integral I?
- What are "the Lagrangian expressions"?
- Is $\mathfrak{G}_\rho$ a Lie group of transformations of dimension ρ?
- In what sense is the integral I invariant?
- What is a $\mathfrak{G}_{\infty\rho}$?
- What is "the formal calculus of variations"?

What variational problem is Noether considering? What is the integral I?

Noether considers an n-**dimensional variational problem of order** κ **for an** $\mathbb{R}^\mu$**-valued function** (n, μ and κ arbitrary integers)

$$\boxed{I = \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) dx} \qquad (1)$$

"I omit the indices here, and in the summations as well whenever it is possible, and I write $\frac{\partial^2 u}{\partial x^2}$ for $\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\gamma}$, etc."

"I write dx for $dx_1 \dots dx_n$ for short."

$x = (x_1, \dots, x_n) = (x_\alpha)$ **independent variables**
$u = (u_1, \dots, u_\mu) = (u_i)$ **dependent variables**

Calculus of variations in a nutshell

Consider a variation of (x, u).
Compute the variation of L, hence that of I.
Use integration by parts to obtain the **Euler–Lagrange equation**.

- Define the $\boxed{\text{variational derivative}}$ of L also called $\boxed{\text{Euler–Lagrange derivative}}$ or $\boxed{\text{Euler–Lagrange differential}}$.
  Denote it by $\frac{\delta L}{\delta q}$ or EL.
- A necessary condition for a map $x \to u(x)$ with fixed values on the boundary of the domain of integration to minimize the integral

$$I = \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) dx$$

  is the **Euler–Lagrange equation**

$$\boxed{EL = 0}$$

Example: Elementary case $x = t$, $u = q = (q^i)$

- Lagrangian of order 1

$$(EL)_i = \frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}$$

The Euler–Lagrange equation in this case is

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0.$$

- Lagrangian of order k

$$(EL)_i = \frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^i} - \dots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial q^{i(k)}}$$

The Euler–Lagrange equation in this case is

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^i} - \dots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial q^{i(k)}} = 0.$$

What are "the Lagrangian expressions"?

Fact. When there are μ dependent variables, there are μ scalar Euler-Lagrange equations. Their left-hand sides are the components of the Euler–Lagrange derivative of L.

The "Lagrangian expressions" in Noether's article are the left-hand sides of the Euler–Lagrange equations.
In other words, the "Lagrangian expressions" are the components of the variational derivative of the Lagrangian, denoted by f, with respect to the dependent variables, denoted by $u = (u_i)$, and Noether denotes these "Lagrangian expressions" by ψ_i.

Recall that Noether considers the very general case of a multiple integral, u is a map from a domain in n-dimensional space to a μ-dimensional space, and the Lagrangian f depends on an arbitrary number of derivatives of u.

Noether's computation (p. 238)

"[...] On the other hand, I calculate for an arbitrary integral I, that is not necessarily invariant, the first variation δI, and I transform it, according to the rules of the calculus of variations, by integration by parts. Once one assumes that δu and all the derivatives that occur vanish on the boundary, but remain arbitrary elsewhere, one obtains the well known result,

$$\delta I = \int \cdots \int \delta f \; dx = \int \cdots \int \left(\sum \psi_i(x, u, \frac{\partial u}{\partial x}, \cdots) \delta u_i \right) dx, \tag{2}$$

where the ψ represent the *Lagrangian expressions*; that is to say, the left-hand sides of the Lagrangian equations of the associated variational problem, $\delta I = 0$."

Noether's text (continued)

"To that integral relation there corresponds an *identity* without an integral in the δu and their derivatives that one obtains by adding the boundary terms. As an integration by parts shows, these boundary terms are integrals of divergences, that is to say, expressions,

$$\mathrm{Div}\ A = \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n},$$

where A is linear in δu and its derivatives."

So Noether writes the Euler–Lagrange equations:

$$\boxed{\sum \psi_i \delta u_i = \delta f + \mathrm{Div}\ A.} \tag{3}$$

In the modern literature, A is expressed in terms of the Legendre transformation associated to L.

Noether's explicit computations

"In particular, if f only contains the first derivatives of u, then, in the case of a simple integral, identity (3) is identical to Heun's 'central Lagrangian equation',

$$\sum \psi_i \delta u_i = \delta f - \frac{d}{dx}\left(\sum \frac{\partial f}{\partial u_i'} \delta u_i\right), \qquad \left(u_i' = \frac{du_i}{dx}\right), \tag{4}$$

while, for an n-uple integral, (3) becomes

$$\sum \psi_i \delta u_i = \delta f - \frac{\partial}{\partial x_1}\left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_1}} \delta u_i\right) - \cdots - \frac{\partial}{\partial x_n}\left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_n}} \delta u_i\right)." \tag{5}$$

Noether's explicit computations (continued)

"For the simple integral and κ derivatives of the u, (**??**) yields

$$\sum \psi_i \delta u_i = \delta f - \tag{6}$$

$$-\frac{d}{dx}\left\{\sum\left(\binom{1}{1}\frac{\partial f}{\partial u_i^{(1)}}\delta u_i + \binom{2}{1}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i^{(1)} + \cdots + \binom{\kappa}{1}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-1)}\right)\right\}$$

$$+\frac{d^2}{dx^2}\left\{\sum\left(\binom{2}{2}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i + \binom{3}{2}\frac{\partial f}{\partial u_i^{(3)}}\delta u_i^{(1)} + \cdots + \binom{\kappa}{2}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-2)}\right)\right\}$$

$$+ \cdots + (-1)^\kappa \frac{d^\kappa}{dx^\kappa}\left\{\sum \binom{\kappa}{\kappa}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i\right\},$$

and there is a corresponding identity for an n-uple integral;
in particular, A contains δu and its derivatives up to order $\kappa - 1$."

Noether's proof

"That the Lagrangian expressions ψ_i are actually defined by (**??**), (**??**) and (**??**) is a result of the fact that, by the combinations of the right-hand sides, all the higher derivatives of the δu are eliminated, while, on the other hand, relation (**??**), which one clearly obtains by an integration by parts, is satisfied."

Q.E.D

Then Noether states her two theorems.

The theorems of Noether

First theorem

If the integral I is invariant under a group $\mathfrak{G}_\rho$, then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences–and conversely, that implies the invariance of I under a group $\mathfrak{G}_\rho$. The theorem remains valid in the limiting case of an infinite number of parameters.

Second theorem

If the integral I is invariant under a group $\mathfrak{G}_{\infty\rho}$ depending upon arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid.

Is $\mathfrak{G}_\rho$ a Lie group of transformations of dimension ρ?

In some cases, yes, $\mathfrak{G}_\rho$ is a Lie group of transformations of dimension ρ.
Noether considers "the infinitesimal transformations contained in $\mathfrak{G}_\rho$", which she denotes by

$$y_\lambda = x_\lambda + \Delta x_\lambda; \qquad v_i(y) = u_i + \Delta u_i.$$

In modern terms, she considers the LIE ALGEBRA of the ρ-dimensional LIE GROUP, $\mathfrak{G}_\rho$.

In modern notation

The ρ infinitesimal generators of the Lie group are linearly independent VECTOR FIELDS $X_{(1)}, \ldots, X_{(\rho)}$, each a vector field on $\mathbb{R}^n \times \mathbb{R}^\mu$ of the form

$$X = \sum_{\alpha=1}^{n} X^\alpha(x) \frac{\partial}{\partial x_\alpha} + + \sum_{i=1}^{\mu} Y^i(x,u) \frac{\partial}{\partial u_i}$$

[infinitesimal automorphism of the trivial vector bundle $F \longrightarrow M$, $F = \mathbb{R}^n \times \mathbb{R}^\mu$, $M = \mathbb{R}^n$]
[projectable vector field on $F \longrightarrow M$]

Generalized vector fields

But $\mathfrak{G}_\rho$ can be much more general. In fact, for Noether, a "transformation" is a GENERALIZED VECTOR FIELD:

$$\boxed{X = \sum_{\alpha=1}^{n} X^\alpha(x)\frac{\partial}{\partial x_\alpha} + \sum_{i=1}^{\mu} Y^i\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right)\frac{\partial}{\partial u_i}}$$

[NOT a vector field on the vector bundle $F \to M$]
[introduce JET BUNDLES]

Generalized vector fields will be re-discovered much, much later under many names: "a new type of vector fields", "Lie-Bäcklund transformations".

In what sense is the integral I invariant?

Noether defines invariance of the action integral $\int f dx$:
"An integral I is an invariant of the group if it satisfies the relation,

$$\begin{aligned} I &= \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) dx \\ &= \int \cdots \int f\left(y, v, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \cdots\right) dy \end{aligned} \tag{7}$$

integrated upon an *arbitrary* real domain in x, and upon the corresponding domain in y."

Infinitesimal invariance

Then she seeks a criterion in terms of the infinitesimal generators of the invariance group.

"Now let the integral I be invariant under $\mathfrak{G}$, then relation (**??**) is satisfied. In particular, I is also invariant under the infinitesimal transformations contained in $\mathfrak{G}$,

$$y_i = x_i + \Delta x_i; \qquad v_i(y) = u_i + \Delta u_i,$$

and therefore relation (**??**) becomes

$$\begin{aligned} 0 = \quad \Delta I \quad &= \int \cdots \int f\left(y, v(y), \frac{\partial v}{\partial y}, \cdots\right) dy \\ &- \quad \int \cdots \int f\left(x, u(x), \frac{\partial u}{\partial x}, \cdots\right) dx, \end{aligned} \tag{8}$$

where the first integral is defined upon a domain in $x + \Delta x$ corresponding to the domain in x."

The variation $\bar{\delta} u_i$

"But this integration can be replaced by an integration on the domain in x by means of the transformation that is valid for infinitesimal Δx,

$$\int \cdots \int f\left(y, v(y), \frac{\partial v}{\partial y}, \cdots\right) dy \tag{9}$$

$$= \int \cdots \int f\left(x, v(x), \frac{\partial v}{\partial x}, \cdots\right) dx + \int \cdots \int \mathrm{Div}(f.\ \Delta x)\ dx.''$$

Noether then introduces the variation (*in modern terms*, the vertical generalized vector field)

$$\bar{\delta} u_i = v_i(x) - u_i(x) = \Delta u_i - \sum \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda,$$

so that she obtains the condition

$$0 = \int \cdots \int \{\bar{\delta} f + \mathrm{Div}(f.\ \Delta x)\} dx. \tag{10}$$

In her own words

"The right-hand side is the classical formula for the simultaneous variation of the dependent and independent variables. Since relation (**??**) is satisfied by integration on an *arbitrary* domain, the integrand must vanish identically; Lie's differential equations for the invariance of I thus become the relation

$$\bar{\delta} f + \mathrm{Div}(f.\ \Delta x) = 0.'' \tag{11}$$

In modern terms, Lie's differential equations express the infinitesimal invariance of the integral by means of the Lie derivative of f with respect to the given infinitesimal transformation.

In her own words (continued)

"If, using (**??**), one expresses $\bar{\delta} f$ here in terms of the Lagrangian expressions, one obtains

$$\sum \psi_i \bar{\delta} u_i = \mathrm{Div}\ B \qquad (B = A - f.\ \Delta x), \tag{12}$$

and that relation thus represents, for each *invariant* integral I, an identity in all the arguments which occur; that is the form of Lie's differential equations for I that was sought."

Noether's first theorem is proved in all generality!

The equations $\boxed{\mathrm{Div}\ B = 0}$ are the conservation laws that are satisfied when the Euler–Lagrange equations $\boxed{\psi_i = 0}$ are satisfied.

Conservation laws

Is the condition $\mathrm{Div} B = 0$ a conservation law in the usual sense?

In mechanics, a conservation law is a quantity that depends upon the configuration variables and their derivatives, and which remains constant during the motion of the system.

In field theory, a conservation law is a relation of the form $\dfrac{\partial B_1}{\partial t} + \displaystyle\sum_{\lambda=2}^{n} \dfrac{\partial B_\lambda}{\partial x_\lambda} = 0$, where $x_1 = t$ is time and the x_λ, $\lambda = 2, \ldots, n$, are the space variables, and $B_1, \ldots, B_n$ are functions of the field variables and their derivatives, which relation is satisfied when the field equations are satisfied.

If the conditions for the vanishing of the quantities being considered at the boundary of a domain of the space variables, $x_2, \ldots, x_n$, are satisfied, then, by Stokes's theorem, the integral of B_1 over this domain is constant in time.

In physics a conservation law is also called a continuity equation.

The second theorem. What is a $\mathfrak{G}_{\infty\rho}$?

Noether assumes the existence of ρ symmetries of the Lagrangian, each of which depends linearly upon an arbitrary function $p^{(\lambda)}$ $(\lambda = 1, 2, \ldots, \rho)$ of the variables $x_1, x_2, \ldots, x_n$, and its derivatives up to order σ. In Noether's notation, each symmetry is written

$$a_i^{(\lambda)}(x, u,\ldots)p^{(\lambda)}(x) + b_i^{(\lambda)}(x, u,\ldots)\frac{\partial p^{(\lambda)}}{\partial x} + \cdots + c_i^{(\lambda)}(x, u,\ldots)\frac{\partial^\sigma p^{(\lambda)}}{\partial x^\sigma}.$$

In modern terms, such a symmetry is defined by a vector-valued linear differential operator of order σ acting on the arbitrary function $p^{(\lambda)}$.

Towards the proof of the second theorem

Noether introduces the adjoint operator of each of these differential operators.
But she does not propose a name or a notation for them.
She writes
"Now, by the following identity which is analogous to the formula for integration by parts,

$$\varphi(x, u, \ldots)\frac{\partial^\tau p(x)}{\partial x^\tau} = (-1)^\tau \cdot \frac{\partial^\tau \varphi}{\partial x^\tau} \cdot p(x) \quad \text{mod divergences,}$$

the derivatives of the p are replaced by p itself and by divergences that are linear in p and its derivatives."
In modern terms, call the operators $\mathcal{D}_i^{(\lambda)}$, $i = 1, 2, \ldots, \mu$, and denote their adjoints by $(\mathcal{D}_i^{(\lambda)})^*$. The above identity implies

$$\psi_i \, \mathcal{D}_i^{(\lambda)}(p^{(\lambda)}) = (\mathcal{D}_i^{(\lambda)})^*(\psi_i)p^{(\lambda)} \quad \text{modulo divergences} \quad \text{Div}\, \Gamma_i^{(\lambda)}.$$

Expressing the invariance of the Lagrangian

Now the preceding equation

$$\sum \psi_i \bar{\delta} u_i = \mathrm{Div}\ B \qquad (B = A - f.\ \Delta x),$$

is written

$$\sum_{i=1}^{\mu} \psi_i\, \mathcal{D}_i^{(\lambda)}(p^{(\lambda)}) = \mathrm{Div}\, B^{(\lambda)} \qquad (\lambda = 1, 2, \ldots, \rho)\ .$$

These relations imply

$$\sum_{i=1}^{\mu} (\mathcal{D}_i^{(\lambda)})^*(\psi_i)\ p^{(\lambda)} = \mathrm{Div}(B^{(\lambda)} - \Gamma^{(\lambda)})\ ,$$

where $\Gamma^{(\lambda)} = \sum_{i=1}^{\mu} \Gamma_i^{(\lambda)}$. Since the $p^{(\lambda)}$ are arbitrary,

$$\sum_{i=1}^{\mu} (\mathcal{D}_i^{(\lambda)})^*(\psi_i) = 0, \quad \text{for} \quad \lambda = 1, 2, \ldots, \rho.$$

Differential identities

These are the ρ differential relations among the components ψ_i of the Euler-Lagrange derivative of the Lagrangian f that are identically satisfied.

Noether writes these ρ identities as

$$\sum \left\{ (a_i^{(\lambda)} \psi_i) - \frac{\partial}{\partial x}(b_i^{(\lambda)} \psi_i) + \cdots + (-1)^\sigma \frac{\partial^\sigma}{\partial x^\sigma}(c_i^{(\lambda)} \psi_i) \right\} = 0$$

$$(\lambda = 1, 2, \ldots, \rho).$$

"*These are the identities that were sought among the Lagrangian expressions and their derivatives when I is invariant under* $\mathfrak{G}_{\infty\rho}$."

Improper conservation laws

Noether observes that her identities may be written

$$\sum_{i=1}^{\mu} a_i^{(\lambda)} \psi_i = \mathrm{Div}\, \chi^{(\lambda)},$$

where each $\chi^{(\lambda)}$ is defined by a linear differential operator acting upon the Lagrangian expressions ψ_i. She then deduces that each $B^{(\lambda)}$ can be considered as the sum of two terms,

$$B^{(\lambda)} = C^{(\lambda)} + D^{(\lambda)},$$

where

- the quantity $C^{(\lambda)}$ and not only its divergence vanishes on $\psi_i = 0$,
- the divergence of $D^{(\lambda)}$ vanishes identically, i.e., whether $\psi_i = 0$ or not.

Noether calls these conservation laws "improper".

What is “the formal calculus of variations”?

In the “formal calculus of variations”,

• the aim is to determine necessary conditions for a map $x \mapsto u(x)$ to realize a minimum of the variational integral, and the second variation is not considered,

• the boundary conditions are assumed to be such that the integrals of functions that differ by a divergence are equal.

In modern terms, the formal calculus of variations is an algebraic formulation of the calculus of variations where functionals defined by integrals are replaced by equivalence classes of functions modulo a total differential.
See Gelfand-Dickey [1976], Gelfand-Dorfman [1979], Manin [1978].

A question in the general theory of relativity

Noether's research was prompted by a quesiton in the general theory of relativity (Einstein, 1915) concerning the law of energy conservation in general relativity.

She showed that each invariance transformation implies a conservation law.

• Invariance under a group of transformations depending upon a finite or denumerable number of parameters implies proper conservation laws.

• Invariance under transformations depending upon arbitrary functions (a continuous set of parameters) yields improper conservation laws.

Later, physicists working in general relativity called the improper conservation laws of the second type "strong laws".

Important remarks

- Noether carefully proves the converse of the first and the second theorem.
- Noether is aware of the problem of defining equivalent infintesimal invariance transformations and equivalent conservation laws in order to make the correspondence 1-to-1.
- Symmetries up to divergence were introduced by Erich Bessel-Hagen in 1921,
  Über die Erhaltungssätze der Elektrodynamik,
  Mathematische Annalen, 84 (1921), pp. 258–276.
  He writes that he will formulate Noether's theorems slightly more generally than they were formulated in the article he cites, but that he is "in debt for that to an oral communication by Miss Emmy Noether herself."

Emmy Noether versus William Hamilton

Noether's article deals with the Lagrangian formalism.
There is NO Hamiltonian formalism in Noether's work.

In the Hamiltonian formalism, it follows immediately from the skew-symmetry of the Poisson bracket that,
if X_H is a Hamiltonian vector field, then for any Hamiltonian vector field X_K that commutes with X_H, the quantity K is conserved under the flow of X_H.

The name "Noether theorem" applied to this result is a misnomer.

Some historical facts: Einstein, Klein, Hilbert and the general theory of relativity

Noether's two theorems in pure mathematics can hardly be understood outside their historical context, the inception of the general theory of relativity in the period of great intellectual effervescence in Germany and especially in Göttingen that coincided with the war and the first years of the Weimar republic. Einstein had published his article "The Field equations of gravitation" in 1915, where he first wrote "the Einstein equations" of general relativity.
She wrote quite explicitly in her article that questions arising from the general theory of relativity were the inspiration for her research, and that her article clarifies what should be the nature of the law of conservation of energy in that new theory.

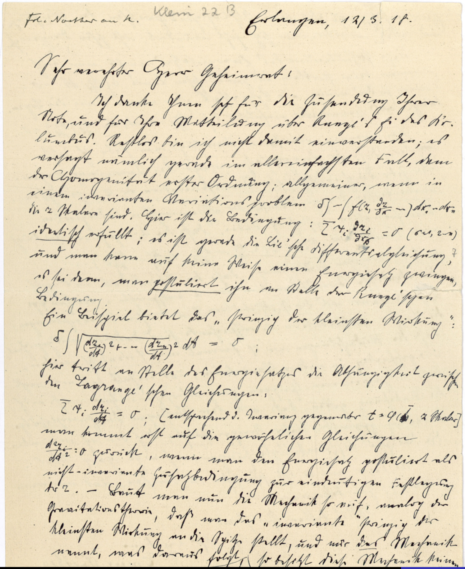
Noether's first attempts

In their articles of 1917 and 1918 on the fundamental principles of physics, Felix Klein and David Hilbert, who were attempting to understand Einstein's work, said clearly that they had solicited Noether's assistance to resolve these questions and that she proved a result which had been asserted by Hilbert without proof.

- 1915, notes written for Hilbert
- February 1918, a postcard to Klein

The fundamental identity can be read on the verso of her postcard, and she announced the result of her second theorem, but only for a very special case: the variation of the u_i in the direction of the coordinate line x_κ.

March 1918, letter to Felix Klein


Erlangen, 12/3. 18.

Sehr verehrter Herr Geheimrat!

Noether's letter to Klein, March 1918

In this letter she formulated the fundamental idea that the lack of a theorem concerning energy in general relativity is due to the fact that the invariance groups that are considered are in fact subgroups of an infinite group depending upon arbitrary functions, and therefore lead to identities that are satisfied by the Lagrangian expressions:

"By my additional research, I have now established that the [conservation] law for energy is not valid in the case of invariance *under any extended group generated by the transformation induced by the z's*."

The end of her letter is a preliminary formulation of the conclusion of her article.

Noether's conclusion: the discussion of Hilbert's assertion

Hilbert asserted (without proof) in early 1918 that, in the case of general relativity and in that case only, there are no proper conservation laws.

Here Noether shows that the situation is better understood "in the more general setting of group theory."

She explains (p. 255) the apparent paradox that arises from the consideration of the finite-dimensional subgroups of groups that depend upon arbitrary functions.

"Given I invariant under the group of translations, then the energy relations are improper if and only if I is invariant under an infinite group which contains the group of translations as a subgroup."

Noether's conclusion

"Hilbert asserts that the lack of a proper law of [conservation of] energy constitutes a characteristic of the 'general theory of relativity'. For that assertion to be literally valid, it is necessary to understand the term 'general relativity' in a wider sense than is usual, and to extend it to the afore-mentioned groups that depend upon n arbitrary functions."

In her final footnote, Noether remarks the relevance of Klein's observation [1910] in the spirit of his Erlangen program [1872]. In Noether's striking formulation, Klein's remark becomes: "The term *relativity* that is used in physics should be replaced by *invariance with respect to a group*."

Before "The Dawning of Gauge Theory"

Noether extrapolates from the problems arising from
(1) the invariance group of the equations of mechanics and
(2) the invariance group of the general theory of relativity,
to a general theory of invariance groups of variational problems.
She made the essential distinction between the case (1) of invariance groups that are finite-dimensional Lie groups and that (2) of groups of transformations that depend upon arbitrary functions.
This latter case would become, in the work of Hermann Weyl and, much later, Chen Ning Yang and Robert L. Mills, gauge theory.
The question of the geometric nature of the Noether theorems could not even be formulated in 1918. It remained open - until the 1970's.