The Noether Theorems: from Noether to Ševera

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Lectures 4 and 5
Lie Algebroids and Courant Algebroids in the Lagrangian and Hamiltonian Formalisms

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Motivation

Weinstein was motivated by the study of discrete Lagrangian and Hamiltonian systems. The natural framework for discrete systems is that of **Lie groupoids**.

For the case of continuous time, the natural framework is that of **Lie algebroids**.

For 1-dimensional variational problems – the case of mechanics – the framework of Lie algebroids extends that of tangent bundles.

For 2-dimensional problems, introduce **Courant algebroids** (Ševera).

Back to basics

Let M be a manifold, $TM \to M$ its **tangent bundle**.
Let $\Gamma(TM)$ be the $C^\infty(M)$-module of sections of TM (the space of vector fields on M).

Fact. The Lie bracket of vector fields makes $\Gamma(TM)$ a LIE ALGEBRA over $\mathbb{R}$:
skewsymmetry $[X, Y] = -[Y, X]$ and
Jacobi identity $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$

Fact. The Lie bracket of vector fields satisfies the LEIBNIZ IDENTITY:

$$\forall X, Y \in \Gamma(TM), \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (X \cdot f)Y$$

Lie algebroids are "generalized tangent bundles"

Definition of Lie algebroids

Definition

A *Lie algebroid* is a vector bundle $\tau : A \to M$ such that

• $\Gamma(A)$ is a LIE ALGEBRA over $\mathbb{R}$
(skewsymmetry + Jacobi identity)

• there exists a morphism of vector bundles $\rho : A \to TM$, called the *anchor*, such that the LEIBNIZ IDENTITY is satisfied,

$$\boxed{\forall X, Y \in \Gamma(A), \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (\rho(X) \cdot f)Y}$$

Proposition (yks and Franco Magri)

The mapping ρ induces a Lie algebra homomorphism $\Gamma(A) \to \Gamma(TM)$.

Examples

- TM
- foliations: integrable subbundles of TM
- cotangent bundle of a Poisson manifold, (M, π)
  there exists a unique Lie algebroid structure $[\ ,\ ]_\pi$ on T^*M such that
  the anchor is the map $\pi^\sharp : T^*M \to TM$ defined by π, and

$$[df, dg]_\pi = d\{f, g\}, \quad \forall f, g \in C^\infty(M).$$

- gauge Lie algebroids (also called Atiyah algebroids): TP/G, where P is a principal bundle with structure group G.
- action Lie algebroids : $\mathfrak{g} \times M$, where $\mathfrak{g}$ is a Lie algebra acting on M.

The differential of a Lie algebroid

For any Lie algebroid A, there is a differential d_A on $\Gamma(\wedge^\bullet A^*)$ defined by

$$(d_A\alpha)(X_0, X_1, \ldots, X_k) = \sum_{i=0}^k (-1)^i \rho(X_i)\cdot \alpha(X_0, \ldots, \widehat{X}_i, \ldots, X_k) \\ + \sum_{0\le i<j\le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),$$

for $\alpha \in \Gamma(\wedge^k A^*)$, $k \in \mathbb{N}$, $X_0, \ldots, X_k \in \Gamma A$.

$$\boxed{d_A^2 = 0}$$

The differential d_A turns $\Gamma(\wedge^\bullet A^*)$ into a complex

whose cohomology is called the *Lie algebroid cohomology* and is denoted by $H^\bullet(A)$.

The de Rham differential and the Chevalley–Eilenberg differential

When $A = TM$, the differential d_A acts on the differential forms $\Gamma(\wedge^\bullet(T^*M))$ on the manifold M, and $H^\bullet(A)$ is the de Rham cohomology of M.

When A is a Lie algebra considered as a Lie algebroid over a point, $H^\bullet(A)$ is the Chevalley–Eilenberg cohomology.

The *Lie derivative* of sections of $\Lambda^\bullet A^*$ with respect to $X \in \Gamma A$ is defined by $\mathcal{L}_X = [i_X, d_A]$,
where [,] is the graded commutator of derivations of $\Gamma(\wedge^\bullet A^*)$.

The Gerstenhaber algebra of a Lie algebroid

There is a unique extension to $\Gamma(\wedge^\bullet A)$ of the Lie bracket of sections of A, again denoted by $[\ ,\ ]_A$, as a (graded) bi-derivation satisfying $[X, f]_A = < X, d_A f >$, for all $X \in \Gamma A$, $f \in C^\infty(M)$. This bracket of degree -1 is called a *Gerstenhaber bracket*, and $(\Gamma(\wedge^\bullet A), \wedge, [\ ,\ ]_A)$ is a *Gerstenhaber algebra*.

To summarize, to any Lie algebroid are associated both
• a differential d_A on $\Gamma(\wedge^\bullet A^*)$,
and
• a Gerstenhaber bracket $[\ ,\ ]_A$ on $\Gamma(\wedge^\bullet A)$.

The Schouten–Nijenhuis bracket of multivector fields

In the special case where $A = TM$ with the Lie bracket of vector fields, the Gerstenhaber bracket on $\Gamma(\wedge^\bullet TM)$ is called the *Schouten–Nijenhuis bracket of multivector fields*.
It is a bracket, $[\ ,\ ]_{SN}$, of degree -1 on the space of multivector fields on M, the unique extension to $\Gamma(\wedge^\bullet TM)$ as a (graded) bi-derivation of the Lie bracket of vector fields satisfying $[X, f]_{SN} = X \cdot f$, for all $X \in \Gamma(TM)$, $f \in C^\infty(M)$.

Example: the Lichnerowicz–Poisson differential

If (M, π) is a Poisson manifold, the differential associated to the Lie algebroid $(T^*M, [\ ,\ ]_\pi, \pi^\sharp)$ is

$$\boxed{d_\pi = [\pi, \cdot]_{SN}}$$

acting on $\Gamma(\wedge^\bullet TM)$, where $[\ ,\ ]_{SN}$ is the Schouten–Nijenhuis bracket of multivector fields on M.

(Bhaskara and Viswanath, yks and Franco Magri [1990])

Example: deformation by a Nijenhuis tensor

If M is a manifold and $N : TM \to TM$ is a Nijenhuis tensor,

$$[NX, NY] - N([NX, Y] + [X, NY]) - N^2[X, Y] = 0,$$

then

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$$

is a Lie bracket on TM, and TM becomes a Lie algebroid with anchor $N : TM \to TM$.

The associated differential is

$$d_N = [i_N, d]$$

(graded commutator).

Alternate definition of Lie algebroids

Defining a Lie algebroid structure on A is equivalent to defining a differential d on $\Gamma(\wedge^\bullet A^*)$.

Set

$$\rho(X)f = < X, df >$$

and

$$< [X, Y], \alpha > = \rho(X) \cdot < Y, \alpha > - \rho(Y) \cdot < X, \alpha > - (d\alpha)(X, Y).$$

$d^2 = 0$ is equivalent to the Jacobi and Leibniz identities for [,].

Lie bialgebroids

A Lie bialgebroid (A, A^*) is a pair of Lie algebroids A and A^* in duality, satisfying the following compatibility assumption

$\boxed{\text{the differential } d_A \text{ is a derivation of the Gerstenhaber bracket } [\ ,\ ]_{A^*}.}$

(Mackenzie and Xu, Lie bialgebroids and Poisson groupoids [1994])
(yks, Exact Gerstenhaber algebras and Lie bialgebroids [1995])

More generally, one can define proto-Lie-bialgebroids.

Lie algebroids as supermanifolds (1)

Let $E \to M$ be a vector bundle.
We denote by ΠE the supermanifold obtained by "making the coordinates in the fibers odd."
Let (x^i, ξ^α) be adapted local coordinates on ΠE. Then

$$x^i x^j = x^j x^i, \qquad \xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha.$$

The smooth functions on ΠE are of the form

$$f(x, \xi) = f_0(x) + f_\alpha(x)\xi^\alpha + f_{\alpha\beta}(x)\xi^\alpha\xi^\beta + \cdots ,$$

where the highest term contains at most p factors ξ^α
where p is the rank of E (the dimension of the fibers)
[since the ξ^α anti-commute].

Lie algebroids as supermanifolds (2)

Fact. The smooth functions on the supermanifold ΠE are the sections of $\wedge^\bullet E^*$:

$$\boxed{C^\infty(\Pi E) = \Gamma(\wedge^\bullet E^*)}$$

Question. How can we interpret a Lie algebroid structure on E in terms of the supermanifold ΠE?

Answer. (Vaĭntrob [1997])

• If E is a vector bundle over M, a derivation of $\Gamma(\wedge^\bullet E^*) = C^\infty(\Pi E)$ is identified with a vector field on ΠE.

• If A is a Lie algebroid over M, the derivation d_A of square 0 is identified with a vector field Q_A on ΠA, satisfying

$$[Q_A, Q_A] = 0.$$

Such a vector field is called a homological vector field, or an integrable vector field.

Proto-Lie-bialgebroids

Lie bialgebroids and, more generally, proto-Lie-bialgebroids can also be defined in terms of supermanifolds equipped with an additional structure.

A proto-Lie-bialgebroid is defined by a function of degree 3 on $T^*\Pi A$ which has square 0 with respect to the canonical Poisson bracket of this cotangent bundle.

Mechanics on Lie algebroids: Lagrangian formalism

In 1992 (published 1996), Weinstein described a Lagrangian formalism on Lie algebroids as follows.
Let $A \to M$ be a Lie algebroid, with bracket $[\ ,\ ]_A$ and anchor ρ.
In local coordinates (x^i) on M and a local basis (e_α) of sections,

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \qquad [e_\alpha, e_\beta]_A = c^\gamma_{\alpha\beta} e_\gamma.$$

$\boxed{\text{Since } A \text{ is a Lie algebroid, } A^* \text{ is a Poisson manifold.}}$

In local coordinates, (x^i, ξ_α), on A^*,

$$\{x^i, \xi_\alpha\} = \rho^i_\alpha, \qquad \{\xi_\alpha, \xi_\beta] = c^\gamma_{\alpha\beta} \xi_\gamma.$$

Example. If $A = TM$, $\rho^i_j = \delta^i_j$ and $c^k_{ij} = 0$. The Poisson structure on T^*M is the canonical symplectic structure.

The Legendre transformation

Let L be a Lagrangian (real-valued function) on A. Define the Legendre transformation, $\mathcal{F}L : A \longrightarrow A^*$, as the fiber derivative of L.
Assume that L is regular, i.e., $\mathcal{F}L$ is a fiber-preserving local diffeomorphim.
A becomes a Poisson manifold with the pull-back of the Poisson structure of A^*. Denote this Poisson bracket by $\{\ ,\ \}_L$.
If $L = L(x, y)$,

$$\{x^i, x^j\}_L = 0, \quad \{x^i, \frac{\partial L}{\partial y^\alpha}\}_L = \rho^i_\alpha,$$

$$\{\frac{\partial L}{\partial y^\alpha}, \frac{\partial L}{\partial y^\beta}\}_L = c^\gamma_{\alpha\beta}\frac{\partial L}{\partial y^\gamma}.$$

The energy on a Lie algebroid

Define the action function $\mathcal{A}_L : A \to \mathbb{R}$ and the energy function $\mathcal{E}_L : A \to \mathbb{R}$ by

$$\mathcal{A}_L(a) = < (\mathcal{F}L)(a), a >$$

$$\mathcal{E}_L = \mathcal{A}_L - L.$$

In adapted local coordinates, if $a = (x, y) = (x^i, y^\alpha)$, then

$$\mathcal{A}_L(x, y) = \frac{\partial L}{\partial y^\alpha}(x, y) y^\alpha$$

$$\mathcal{E}_L(x, y) = \frac{\partial L}{\partial y^\alpha}(x, y) y^\alpha - L(x, y).$$

Question. What is the Hamiltonian vector field $X_{\mathcal{E}_L}$ on A with Hamiltonian $\mathcal{E}_L$ with respect to the Poisson bracket $\{\ ,\ \}_L$?

Admissible vector fields

Definition

A tangent vector X at a point a on a Lie algebroid $\tau : A \to M$, with anchor $\rho : A \to TM$, is called admissible if the tangent vector to M obtained by projecting X under $T\tau : TA \to TM$ is equal to the tangent vector to M, $\rho(a)$.

In adapted local coordinates $(x, y) = (x^i, y^\alpha)$ on A,

$$X(x, y) = X^i(x, y)\frac{\partial}{\partial x^i} + \mathcal{X}^\alpha(x, y)\frac{\partial}{\partial y^\alpha}$$

is admissible if and only if

$$\boxed{X^i = \rho^i_\alpha y^\alpha.}$$

A smooth curve is called admissible if its tangent vector at each point is admissible.

Admissible vector fields as second-order differential equations

An admissible vector field is also called a second-order differential equation.

In fact, when $A = TM$, in local coordinates $(q, \dot{q}) = (q^i, \dot{q}^i)$, the integral curves $(q(t), \dot{q}(t))$ of $X(q, \dot{q}) = \dot{q}^i \frac{\partial}{\partial x^i} + \mathcal{X}^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}$ satisfy

$$\begin{cases} \frac{dq^i}{dt} & = \dot{q}^i \\ \frac{d\dot{q}^i}{dt} & = \mathcal{X}^i(q, \dot{q}) \end{cases}$$

which is

$$\boxed{\frac{d^2 q^i}{dt^2} = \mathcal{X}^i(q, \dot{q})},$$

a second-order differential equation.

Proposition

The Hamiltonian vector field $X_{\mathcal{E}_L}$ with Hamiltonian $\mathcal{E}_L$ with respect to the Poisson bracket $\{\ ,\ \}_L$ on A is admissible.

The proof will use adapted local coordinates on A, denoted by $(x, y) = (x^i, y^\alpha)$. Let

$$X_{\mathcal{E}_L} = X^i(x, y)\frac{\partial}{\partial x_i} + \mathcal{X}^\alpha(x, y)\frac{\partial}{\partial y^\alpha}.$$

We shall prove that $X^i(x, y) = \rho^i_\alpha y^\alpha$.

$X_{\mathcal{E}_L}$ is admissible (proof)

Proof. By definition, $X^i = \{x^i, \mathcal{E}_L\}_L$. By the derivation property of $\{\ ,\ \}_L$,

$$\{x^i, \mathcal{E}_L\}_L = \{x^i, y^\alpha\}_L \,\frac{\partial \mathcal{E}_L}{\partial y^\alpha}.$$

By the definition of $\mathcal{E}_L$,

$$\frac{\partial \mathcal{E}_L}{\partial y^\alpha} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} y^\beta,$$

therefore

$$X^i = \{x^i, y^\alpha\}_L \,\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\, y^\beta = \{x^i, \frac{\partial L}{\partial y^\beta}\}_L \, y^\beta.$$

From $\rho^i_\alpha = \{x^i, \frac{\partial L}{\partial y^\alpha}\}_L$, we obtain

$$\boxed{X^i = \rho^i_\alpha y^\alpha}$$

$X_{\mathcal{E}_L}$ is a second order differential equation

Therefore the integral curves $(x^i(t), y^\alpha(t))$ of $X_{\mathcal{E}_L}$ satisfy

$$\frac{dx^i}{dt} = \rho(y^\alpha e_\alpha) \cdot x^i.$$

When $A = TM$, set $x^i = q^i$ and $y^i = \dot{q}^i$. The equation $X^i = \rho^i_\alpha y^\alpha$ is written $X^i = \rho^i_j \dot{q}^j$, and the preceding equation reduces to

$$\frac{dq^i}{dt} = \dot{q}^i.$$

Since $\dot{q}^i$ satisfies a first-order differential equation, q^i satisfies a second-order differential equation.

The Euler–Lagrange equations on a Lie algebroid (2)

Another computation shows that

$$\{\frac{\partial L}{\partial y^\alpha}, \mathcal{E}_L\}_L = \rho^i_\alpha \frac{\partial L}{\partial x^i} + c^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}.$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} + c^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma}. \tag{1}$$

Definition

Equations (1) are the *Euler–Lagrange equations* for the Lagrangian L.

The case of TM

Proposition

When $A = TM$, the Euler–Lagrange equations are the usual ones.

Proof. In local coordinates , $x^i = q^i$ and $y^i = \dot{q}^i$, equations (1) become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0,$$

since $\rho^i_j = \delta^i_j$ and $c^k_{ij} = 0$.

Variational principles on Lie algebroids (Weinstein)

In order to relate the solutions of the Euler–Lagrange equations on a Lie algebroid A to the extremals of a variational principle with fixed boundary conditions, it is necessary to assume that the Lie algebroid A can be integrated to a Lie groupoid Γ.
The action integral is $\int L(\sigma(t))dt$, for admissible curves $t \in [a, b] \to \sigma(t) \in A$.
Boundary conditions are imposed on the curves $t \in [a, b] \to g(t) \in \Gamma$ that lift the integral curves of the admissible vector field $X_{\mathcal{E}_L}$.

Another approach to mechanics on Lie algebroids

For $A = TM$, this approach goes back to Joseph Klein,
Joseph Klein, *Espaces variationnels et mécanique*, Ann. Inst. Fourier (Grenoble) 12 (1962), 1-124.

For a review, see
Claude Godbillon, *Géométrie différentielle et mécanique analytique*, Hermann, Paris, 1969.

It was further developed by Włodzimierz Tulczyjew [1974], and many other authors: Demeter Krupka, Olga Krupkova, David Saunders, Michael Crampin, Manuel de León,
impossible to cite them all....

In particular, see the book
Paulette Libermann and Charles-Michel Marle,
Symplectic Geometry and Analytical Mechanics,
Reidel, Dordrecht, 1987.

Recent contributions

The recent contributions are mainly (but not only!) due to

Eduardo Martínez and his collaborators beginning with:
• Eduardo Martínez, Lagrangian mechanics on Lie algebroids,
Acta Appl. Math. 67 (2001), no. 3, 295–320.

and to Janusz Grabowski and his collaborators:
• Katarzyna Grabowska, Pawel Urbanski, and Janusz Grabowski
Geometrical mechanics on algebroids,
Int. J. Geom. Methods Mod. Phys. 3 (2006), no. 3, 559-575.
[and earlier references]

See also

Jorge Cortés, Manuel de León, Juan C. Marrero, D. Martín de Diego, and Eduardo Martínez,
A survey of Lagrangian mechanics and control on Lie algebroids and groupoids,
Int. J. Geom. Methods Mod. Phys. 3 (2006), no. 3, 509-558.

Katarzyna Grabowska and Janusz Grabowski,
Variational calculus with constraints on general algebroids,
J. Phys. A 41 (2008), no. 17, 175204, 25 pp.

Liouville vector field

On any vector bundle $E \to M$, the Liouville vector field Z
(also called the Euler vector field)
is the generator of the group of dilations of E.
In local coordinates (x^i, y^α),

$$Z = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

The Liouville vector field is vertical (tangent to the fibers).

Consider the double tangent bundle $T(TM)$.
In local coordinates $(q^i, \dot{q}^i)$,

$$Z = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

Vertical endomorphism

Let $V(TM)$ be the vector bundle of vertical vectors over TM.
In local coordinates $(q^i, \dot{q}^i)$, for $X \in V(TM)$,

$$\boxed{X = X^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}}$$

On $T(TM)$, there is a vertical endomorphism,

$$\mathcal{V} : T(TM) \to T(TM).$$

The image of $\mathcal{V}$ is contained in $V(TM)$.
In local coordinates $(q^i, \dot{q}^i)$,

$$\mathcal{V} : X = X^i(q, \dot{q}) \frac{\partial}{\partial q^i} + Y^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} \longmapsto \mathcal{V}(X) = X^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}.$$

Question. On a Lie algebroid?

The vector bundle $\mathcal{L}A$

Define $\boxed{\mathcal{L}A = TA \oplus_{TM} A}$ as the subbundle of $TA \oplus A$ that appears in the diagram:

$$\begin{array}{ccc} TA \oplus_{TM} A & \rightarrow & A \\ \downarrow & & \downarrow \quad \rho \\ TA & \overset{T\tau}{\rightarrow} & TM \end{array}$$

The elements of $\mathcal{L}A$ are pairs $(X, a) \in TA \times A$ such that $\boxed{(T\tau)(X) = \rho(a)}$.

Fact. If $A = TM$, then $\mathcal{L}A = \mathcal{L}(TM)$ reduces to $TA = T(TM)$. In fact, in this case, elements of $\mathcal{L}(TM)$ are pairs $(v, (T\tau)v)$, for $v \in T(TM)$. Such pairs are in 1-to-1 correspondence with elements $v \in T(TM)$.

The Lie algebroid $\mathcal{L}A$

$\mathcal{L}A$ can be viewed as a Lie algebroid over A called the pull-back Lie algebroid of A over τ.

The projection onto the base manifold, $p : \mathcal{L}A \to A$, maps a pair (X, a), where X is tangent to A at u, to u. It is the composition of the projection p_1 from $\mathcal{L}A$ onto TA with the projection of the vector bundle $TA \to A$.

The anchor is the projection p_1 onto TA.

The bracket is such that for $X, Y \in \Gamma(TA)$ and for section u, v of A which satisfy $T\tau \circ X = \rho \circ u \circ \tau$ and $T\tau \circ Y = \rho \circ v \circ \tau$,

$$[(X, \tau^* u), (Y, \tau^* v)]_{\mathcal{L}A} = ([X, Y]_{TA}, \tau^*[u, v]_A) \ .$$

A reference

The Lie algebroid $\mathcal{L}A$ is a special case of a construction of Higgins and Mackenzie [1990]. See

Kirill Mackenzie,
General Theory of Lie Groupoids and Lie Algebroids,
London Mathematical Society Lecture Note Series, 213,
Cambridge University Press, Cambridge, 2005, p. 256.

Pull-backs of Lie algebroids

For $\tau : A \to M$, a Lie algebroid, and $\phi : N \to M$, a submersion, the general construction of Higgins and Mackenzie starts from the diagram,

$$\begin{array}{ccc} \phi^! A & \to & A \\ \downarrow & & \downarrow \quad \tau \\ N & \xrightarrow{\phi} & M \end{array}$$

The special case of $\mathcal{L}A$ starts from the particular case of the preceding diagram,

$$\begin{array}{ccc} \phi^! A & \to & A \\ \downarrow & & \downarrow \quad \tau \\ A & \xrightarrow{\tau} & M \end{array}$$

Pull-back by the projection

To the general case corresponds the Lie algebroid $\phi^{!!}A$ with base A that completes the diagram

$$\begin{array}{ccc} \phi^{!!}A = TN \oplus_{TM} A & \rightarrow & A \\ \downarrow & & \downarrow \quad \rho \\ TN & \stackrel{T\phi}{\rightarrow} & TM \\ \downarrow & & \\ N & & \end{array}$$

To the special case where $N = A$ and $\phi = \tau$, there corresponds the Lie algebroid $\tau^{!!}A = \mathcal{L}A$ with base A that completes the diagram

$$\begin{array}{ccc} \mathcal{L}A = TA \oplus_{TM} A & \rightarrow & A \\ \downarrow & & \downarrow \quad \rho \\ TA & \stackrel{T\tau}{\rightarrow} & TM \\ \downarrow & & \\ A & & \end{array}$$

Properties of $\mathcal{L}A \to A$

On $\mathcal{L}A \to A$ there is

• a Liouville section Z: the value of Z at (X, a) is the vertical lift of a,

• a vertical endomorphism $\mathcal{V}$: the vertical endomorphism maps a pair (X, a) to the vertical lift of X.
The vertical endomorphism is a Nijenhuis tensor of square 0.

An admissible element of $\mathcal{L}A$ is a pair $(X, a) \in TA \times A$ such that $(T\tau)(X) = \rho(a)$ and X is tangent to A at $a \in A$.
The anchor $p_1 : \mathcal{L}A \to TA$ maps an admissible section of $\mathcal{L}A \to A$ to an admissible vector field on A.
Admissible sections play the rôle of second-order differential equations.

Joseph Klein's formalism extended

By duality, the vertical endomorphism of $\mathcal{L}A$ acts on the sections of the dual bundle $(\mathcal{L}A)^* \to A$.

Let L be a Lagrangian on A. Applying $\mathcal{V}$ to dL yields a section θ_L of $(\mathcal{L}A)^*$, the Cartan section.
Since L is a function, $(\mathcal{V} \circ d)(L) = i_v dL = \mathcal{L}_{\mathcal{V}} L$, so $\boxed{\theta_L = \mathcal{L}_{\mathcal{V}} L}$.

θ_L defines a map from A to A^*, the Legendre transformation.

Klein's formalism extended

Set $\omega_L = -d\theta_L$.
The Lagrangian L is called regular if the differential 2-form ω_L is non-degenerate.
Define the action function of L,

$$\mathcal{A}_L = i_Z dL,$$

and the energy

$$\mathcal{E}_L = \mathcal{A}_L - L.$$

Klein's formalism extended (continued)

If L is regular, define a section $\mathcal{X}_L$ of $\mathcal{L}A$ by the equation

$$i_{\mathcal{X}_L}\omega_L = d\mathcal{A}_L.$$

Then (Martínez, pp. 313–314),

- the section $\mathcal{X}_L$ is admissible,
- $\mathcal{X}_L$ satisfies the Euler–Lagrange equation which takes the form,

$$\mathcal{L}_{\mathcal{X}_L}\theta_L = dL.$$

In local coordinates,

$$\mathcal{L}_{\mathcal{X}_L}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \frac{\partial L}{\partial x^i} + c^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma},$$

as in Weinstein's formulation.

The case of a regular Lagrangian

Fact. X is an admissible section of $\mathcal{L}A \to A$ if and only if $\mathcal{V}(X) = Z$.
Therefore, if X is admissible, then for any section λ of $(\mathcal{L}A)^* \to A$,

$$i_X(\mathcal{V}\lambda) = i_Z\lambda.$$

Proposition

When the Lagrangian L is regular, an equivalent form of the Euler-Lagrange equation is

$$\mathcal{L}_{\mathcal{X}_L}\theta_L = dL.$$

Here the differential d is that of the Lie algebroid $\mathcal{L}A$, and both sides of the equation are sections of $(\mathcal{L}A)^* \to A$.
Proof. $i_{\mathcal{X}_L}\omega_L - d\mathcal{E}_L = -\mathcal{L}_{\mathcal{X}_L}\theta_L + (di_{\mathcal{X}_L}\mathcal{V}(dL) - di_Z dL) + dL$.
It follows from $i_{\mathcal{X}_L} \circ \mathcal{V} = i_Z$ that the middle term vanishes.

Symmetries

The complete lift of a section X of A is the projectable section X^c of $\mathcal{L}A \to A$ such that $p_1(X^c)\lambda = \mathcal{L}_X\lambda$, for each section λ of A^*, where λ and $\mathcal{L}_X\lambda$ are considered as functions on A.

The case of TM. If $A = TM$, then $\mathcal{L}A = T(TM)$ and the complete lift of a section $X = X^i \frac{\partial}{\partial q^i}$ of TM (i.e., a vector field) is the usual lift to $T(TM)$, $\check{X} = X^i \frac{\partial}{\partial q^i} + \frac{\partial X^i}{\partial q^j}\dot{q}^j \frac{\partial}{\partial \dot{q}^i}$.

An (infinitesimal) symmetry of L is defined as a section X of A such that $\mathcal{L}_{X^c}L = 0$.

It is clear that this definition generalizes the definition of a symmetry of a time-independent first-order Lagrangian on M,

Noether currents on Lie algebroids

Let $\nu_X = < \theta_L, X^c >$. The function ν_X on $\mathcal{L}A$ is called the Noether current associated with the section X of A.

Since the Cartan section generalizes the 1-form $\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i$ on TM, and since X^c generalizes $\check{X}$, ν_X generalizes the Noether current associated to a symmetry, X, of a time-independent first-order Lagrangian, L, on M.

Finally, the (elementary) Noether theorem in the Lie algebroid case

Theorem

If a section X of A is a symmetry of the regular Lagrangian L, then the Noether current associated with X is invariant under the flow of $\mathcal{X}_L$.

Proof. It follows from the properties of the vertical endomorphism and the complete lifts that
$\mathcal{V}([X^c, \mathcal{X}_L]) = [X^c, \mathcal{V}\mathcal{X}_L] = [X^c, Z] = 0$, for each section X of A.
Therefore the section $[X^c, \mathcal{X}_L]$ of $p : \mathcal{L}A \to A$ is vertical and $< \theta_L, [X^c, \mathcal{X}_L] >= 0$.
It follows that $\mathcal{L}_{\mathcal{X}_L} < \theta_L, X^c >=< \mathcal{L}_{\mathcal{X}_L}\theta_L, X^c >$.
The section $\mathcal{X}_L$ satisifies the Euler-Lagrange equation, $\mathcal{L}_{\mathcal{X}_L}\theta_L = dL$.
Therefore, $\mathcal{L}_{\mathcal{X}_L} < \theta_L, X^c >=< dL, X^c >$.
Thus, if X is a symmetry of L, then $\mathcal{L}_{\mathcal{X}_L}\nu_X = 0$.

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(relation with variational problem)
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Also applications to optimal control problems.

Graded manifolds

If, in the supermanifold ΠE,
– the coordinates on the base are given the degree 0, and
– the coordinates on the fibers are given the degree 1,
ΠE becomes a graded manifold and it is denoted by $E[1]$.

[The degree is also called weight.]

More generally, $E[N]$ is the graded manifold where the coordinates on the base are given the degree 0 and the coordinates on the fibers are given the degree N (N a positive integer).

A tower of fibrations $E_2 \longrightarrow E_1 \longrightarrow M$ can also be made into a graded manifold by assigning increasing positive integer degrees to the coordinates on the fibers.

Graded manifolds of degree 1

A graded manifold of degree 1 is a vector bundle $E \to M$.

- A graded manifold of degree 1 with a homological vector field of degree 1 is a Lie algebroid $A \to M$.
  The correspondence is $(A, d_A) \longleftrightarrow (A[1], Q_A)$.
  Notation: $\boxed{Q\text{-manifold}}$.
- Manifolds are in one-to-one correspondence with graded manifolds of degree 1 with a symplectic structure of degree 1.
  The correspondence is $M \longleftrightarrow (T^*[1]M, \omega_0)$, where ω_0 is the canonical symplectic structure of the cotangent bundle.
  Notation: $\boxed{P\text{-manifold}}$.
  In coordinates, $\omega_0 = dq^i \wedge dp_i$.
  The symplectic structure of degree 1 of $T^*[1]M$ defines a Poisson bracket of degree -1 on $C^\infty(T^*[1]M) = \Gamma(\wedge^\bullet TM)$ which is the Schouten-Nijenhuis bracket of multivector fields, $[\ ,\ ]_{SN}$.

Graded manifolds of degree 1 (continued)

- Poisson manifolds are in one-to-one correspondence with graded manifolds of degree 1 with a homological vector field of degree 1 and an invariant symplectic structure of degree 1. The correspondence is $(M, \pi) \longleftrightarrow (T^*[1]M, Q_\pi, \omega_0)$.
  Notation: $\boxed{QP\text{-manifold}}$.
  Here Q_π is the Lichnerowicz–Poisson differential on $\Gamma(\wedge^\bullet TM)$, also denoted by $d_\pi = [\pi, \cdot]_{SN}$, seen as a homological vector field on $T^*[1]M$.
  There is a compatibility condition between the Q-structure and the P-structure: invariance of ω_0 with respect to the vector field Q_π.
  This condition is satisfied because d_π is a derivation of the Schouten–Nijenhuis bracket (it is an interior derivation).

Example

More generally, let $A \to M$ be a Lie algebroid.
Let the coordinates in the fiber of $T^*A \to A$ have degree 1. This graded manifold is denoted by $T^*[1]A[1]$.
As a cotangent bundle, the graded manifold $T^*[1]A[1]$ has a canonical symplectic structure. In this case the associated Poisson bracket is of degree -1. This bracket is the Gerstenhaber bracket of sections of $\wedge^\bullet A$. It generalizes the Schouten-Nijenhuis bracket of multivectors.

In local coordinates $(x^i, z^\alpha, p_i, \zeta_\alpha)$ of degrees $(0, 1, 1, 0)$
(the degree of ζ_α in $T^*A[1]$ is -1, so it is $-1 + 1 = 0$ in $T^*[1]A[1]$)

the Poisson brackets of coordinates are

$$\{x^i, p_j\} = \delta^i_j, \quad \{z^\alpha, \zeta_\beta\} = \delta^\alpha_\beta.$$

Courant algebroids. The standard Courant algebroid

Definition (standard Courant algebroid)

- vector bundle $E = TM \oplus T^*M$,
- fiberwise non-degenerate bilinear symmetric form

$$< X + \xi, Y + \eta > = < X, \eta > + < Y, \xi >,$$

- non-skewsymmetric bracket on sections

$$\boxed{[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi}$$

$\forall X, Y \in \Gamma(TM), \xi, \eta \in \Gamma(T^*M)$.

In particular, $[X, \eta] = \mathcal{L}_X \eta$ and $[\eta, X] = -i_X d\eta$.

This bracket is called the Dorfman bracket.

The Dorfman bracket is a derived bracket

Derived bracket (yks [1996], [2004])

$$[a, b]_d = [[a, d], b]$$

(graded Lie algebra, d odd element, $[d, d] = 0$)
The Dorfman bracket is a *derived bracket*.
Consider the graded Lie algebra of graded endomorphisms of $\Gamma(\wedge^\bullet T^*M)$. Set $\epsilon_\xi(\eta) = \xi \wedge \eta$.

$$[i_X, i_Y]_d = [[i_X, d], i_Y] = [\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

$$[i_X, \epsilon_\eta]_d = [[i_X, d], \epsilon_\eta] = [\mathcal{L}_X, \epsilon_\eta] = \epsilon_{\mathcal{L}_X \eta}$$

$$[\epsilon_\xi, i_Y]_d = [[\epsilon_\xi, d], i_Y] = [\epsilon_{d\xi}, i_Y] = -\epsilon_{i_Y d\xi}$$

$$[\epsilon_\xi, \epsilon_\eta] = 0.$$

Therefore,
when $X \leftrightarrow i_X$ and $\xi \leftrightarrow \epsilon_\xi$, $[X, \eta]_d = \mathcal{L}_X \eta$ and $[\xi, Y]_d = -i_Y d\xi$.

Loday algebras

Definition

A *Loday algebra* is a (graded) vector space equipped with a bracket (bilinear operation, in general non-skewsymmetric) satisfying the Jacobi identity

$$[a,[b,c]] = [[a,b],c] + [a,[b,c]],$$

$\forall a, b, c$.

Loday algebras are also called *Leibniz algebras*.

Fact. If $[\ ,\ ]$ is a Lie bracket, then the derived bracket $[\ ,\ ]_d$ is a Loday bracket.
Therefore a Lie algebra with a derived bracket is a Loday algebra.
Example.
The vector space of sections of the standard Courant algebroid with the Dorfman bracket is a Loday algebra.

Properties of the Dorfman bracket

Theorem

The Dorfman bracket satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [a, [b, c]]$$

$\forall a, b, c \in \Gamma(TM \oplus T^*M)$.

Remark. The Dorfman bracket does NOT satisfy the identity $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Fact. $\Gamma(TM \oplus T^*M)$ with the Dorfman bracket is a LODAY ALGEBRA, and NOT, in general, a Lie algebra.

The Courant bracket

The skewsymmetrized Dorfman bracket is called the Courant bracket,

$$[X+\xi, Y+\eta]^C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\xi - i_Y\eta).$$

The Courant bracket does NOT satify the Jacobi identity.

Courant Algebroids (general case)

A vector bundle $E \to M$ is called a pseudo-Euclidean vector bundle if it is equipped with a fiberwise non-degenerate symmetric bilinear form, $< \ , \ >$.

Definition

A Courant algebroid is a pseudo-Euclidean vector bundle $(E, < \ , \ >)$ equipped with a Loday bracket $[\ ,\ ]$ and an anchor $\rho : E \to TM$ satisfying the Leibniz rule and the identities,

$$\rho(a) < b, c > = < a, [b, c] + [c, b] >,$$

$$\rho(a) < b, c > = < [a, b], c > + < b, [a, c] > .$$

Examples of Courant algebroids

• Case $M = \{pt\}$, E a Lie algebra with an invariant non-degenerate symmetric bilinear form (for example a semi-simple Lie algebra with its Killing form).
• $E = TM \oplus T^*M$ with the non-degenerate bilinear symmetric form

$$< X + \xi, Y + \eta > = < X, \eta > + < Y, \xi >$$

and the Dorfman bracket.
• more generally, $E = A \oplus A^*$, when A is a Lie algebroid,
• more generally, $E = A \oplus A^*$ when (A, A^*) is a Lie bialgebroid
• more generally, $E = A \oplus A^*$ when (A, A^*) is a Lie proto-bialgebroid.

Theorem

Let $A \to M$ be a vector bundle. A Courant algebroid structure on the vector bundle $A \oplus A^*$ is equivalent to a Lie proto-bialgebroid structure on (A, A^*).

Courant algebroids $V \oplus V^*$ as graded manifolds

Question. How can we define a Courant algebroid structure on $V \oplus V^*$ using graded manifolds?

Answer. (Pavol Ševera, Dmitry Roytenberg, Theodore Voronov)
Let $V \to M$ be a vector bundle. Then $(E = V \oplus V^*, < \ , \ >)$ is a pseudo-Euclidean vector bundle,

Consider the graded manifold $V[1]$.
The cotangent bundle $T^*V[1]$ of $V[1]$ can be made into a graded manifold as follows.

Courant algebroids $V \oplus V^*$ as graded manifolds (continued)

Let the coordinates in the fiber of $T^*V \to V$ have degree 2. This graded manifold is denoted by $T^*[2]V[1]$.
As a cotangent bundle, the graded manifold $T^*[2]V[1]$ has a canonical symplectic structure. It is of degree 2 and the associated Poisson bracket is of degree -2.

Local coordinates $(x^i, z^\alpha, , p_i, \zeta_\alpha)$ of degrees $(0, 1, 2, 1)$
(the degree of ζ_α in $T^*V[1]$ is -1, so it is $-1 + 2 = 1$ in $T^*[2]V[1]$).

The local coordinates have bi-degrees: $(0, 0), (1, 0), (1, 1), (0, 1)$.

The Poisson bracket is of bi-degree $(-1, -1)$. The brackets of coordinates are

$$\{x^i, p_j\} = \delta^i_j, \quad \{z^\alpha, \zeta_\beta\} = \delta^\alpha_\beta.$$

Courant algebroids $V \oplus V^*$ as graded manifolds (continued)

Consider a function Θ of degree 3 on $T^*[2]V[1]$,

$$\Theta = \rho^i_\alpha(x) p_i z^\alpha + C_{\alpha\beta\gamma}(x) z^\alpha z^\beta z^\gamma .$$

The function Θ defines a Hamiltonian vector field on $T^*[2]V[1]$,

$$\boxed{d_\Theta = \{\Theta, \cdot\}}$$

Since d_Θ is a Hamiltonian vector field, it leaves the symplectic form invariant.

Courant algebroids $V \oplus V^*$ as graded manifolds (continued)

Consider $d_\Theta = \{\Theta, \cdot\}$, a derivation of the algebra of functions on $T^*[2]V[1]$.

Fact. The following properties are equivalent
• $\{\Theta, \Theta\} = 0$ (Poisson bracket of functions of degree 3 on a graded manifold equipped with a Poisson bracket of degree -2)
• $d_\Theta^2 = 0$ (composition of derivations)
• $[d_\Theta, d_\Theta] = 0$ (Lie bracket of odd vector fields)

In field theory, the equation $\boxed{\{\Theta, \Theta\} = 0}$ is called the classical master equation. It is also called the Maurer-Cartan equation. Θ satisfies the Maurer-Cartan equation if and only if the vector field d_Θ is homological.

Theorem

Θ defines a Courant algebroid structure on $V \oplus V^*$ if and only if $\{\Theta, \Theta\} = 0$.

Method of **proof.** Given Θ, define an anchor and a bracket on $V \oplus V^*$ by

$$\boxed{\rho(a)f = \{\{a, \Theta\}, f\}}$$

$$\boxed{[a, b] = \{\{a, \Theta\}, b\}}$$

$\forall a, b \in \Gamma(V \oplus V^*)$, $f \in C^\infty(M)$.
Then show that the condition $\{\Theta, \Theta\} = 0$ is equivalent to the axioms of a Courant algebroid.
Then prove the converse.

Exercise

Let $\mathfrak{g}$ be a Lie algebra with an invariant non-degenerate symmetric bilinear form $< \ , \ >$. Define the Cartan 3-form $\Theta \in \wedge^3\mathfrak{g}^*$ by

$$\Theta(X, Y, Z) = < [X, Y], Z > .$$

Equip $\wedge^\bullet\mathfrak{g}^*$ with the Poisson bracket of degree -2 extending $< \ , \ >$, considered as a bilinear form on $\mathfrak{g}^*$.
Show that

1. Θ is invariant.
2. $\{\Theta, \Theta\} = 0$.
3. $\{\Theta, \cdot\}$ is the Chevalley–Eilenberg differential on $\wedge^\bullet\mathfrak{g}^*$.
4. $\boxed{[X, Y] = \{\{X, \Theta\}, Y\}}$, $\forall X, Y \in \mathfrak{g}$.

Courant algebroids as graded manifolds. General case

More generally, a Courant algebroid structure on an arbitrary pseudo-Euclidian vector bundle (not necessarily of the form $V \oplus V^*$) can be defined by a function of degree 3 of Poisson square 0 on a suitably defined graded manifold of degree 2 with canonical symplectic structure of degre 2, called the minimal symplectic realization of $E[1]$. The canonical Poisson bracket is then of degree -2.

Given a pseudo-Euclidean vector bundle E, the minimal symplectic realization of $E[1]$ is the pull-back of the vector bundle $T^*[2]E[1] \to (E \oplus E^*)[1]$ by the embedding $E[1] \to (E \oplus E^*)[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, \cdot \rangle)$.

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2-dimensional variational problems

Question. Why Courant algebroids?
The tangent bundle framework, more generally the Lie algebroid framework, are for 1-dimensional variational problems: one independent variable, denoted t, "time".
What is the proper framework for 2-dimensional variational problems?

Remarks from Ševera's article and notes

In a 2-dimensional variational problem one considers integrals of 2-forms (depending on elements of a jet bundle) on a manifold M. Ševera shows that the condition "symmetry up to divergence" is expressed naturally in the framework of Courant algebroids.

Let α be a 2-form on M.
A vector field X is an (infinitesimal) symmetry of α if $\mathcal{L}_X\alpha = 0$.
X is an (infinitesimal) symmetry up to divergence if there exists a 1-form ξ on M such that $\mathcal{L}_X\alpha = d\xi$. So we must consider the pair $(X, \xi) \in \Gamma(TM \oplus T^*M)$.

Proposition

Let α be a 2-form. Consider the pair $(X, \xi) \in \Gamma(TM \oplus T^*M)$. The following conditions are equivalent.

- (X, ξ) *leaves the graph of α invariant (with respect to the Dorfman bracket).*
- (X, ξ) *defines a symmetry of α up to divergence.*

Proof

Proof Let

$$\mathrm{Gr}(\alpha) = \{(Y, i_Y\alpha) \in TM \oplus T^*M \mid Y \in TM\}$$

be the graph of α.
The condition for (X, ξ) to preserve $\mathrm{Gr}\alpha$ infinitesimally is
$[(X, \xi), (Y, i_Y\alpha)] \in \mathrm{Gr}(\alpha)$, for all $Y \in \Gamma(TM)$, i.e.,
$([X, Y], \mathcal{L}_X(i_Y\alpha) - i_Y d\xi) \in \mathrm{Gr}(\alpha)$, for all $Y \in \Gamma(TM)$, i.e.,
$\mathcal{L}_X(i_Y\alpha) - i_Y d\xi) = i_{[X,Y]}\alpha$, for all $Y \in \Gamma(TM)$.
Replacing $i_{[X,Y]}$ by $\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$, we obtain the condition
$i_Y\mathcal{L}_X\alpha - i_Y d\xi = 0$, for all $Y \in \Gamma(TM)$.
Therefore the graph of α is infinitesimally invariant under (X, ξ) if and only if

$$\boxed{\mathcal{L}_X\alpha = d\xi}$$

Exact Courant algebroids

In addition, since α can be replaced by a 2-form, $\alpha + B$, where B is a closed 2-form, one should replace the graph of α in $TM \oplus T^*M$ by an isotropic splitting in an "exact Courant algebroid".

A Courant algebroid $E \to M$ is called "exact" if the following sequence is exact,

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0,$$

where $\rho^* : T^*M \to E^*$ is the dual of the anchor ρ, and E^* is identified with E by means of the non-degenerate symmetric bilinear form.

Ševera's "nonabelian Noether theorem"

Ševera introduces the action of a Poisson-Lie group G (i.e., a Lie group with a Poisson structure compatible with the multiplication) on a manifold M equipped with a 2-form α (corresponding to the Lagrangian). He defines an action of G on the Courant algebroid $TM \oplus T^*M$ by means of the Poisson tensor of G. He states that when the graph of α is invariant under this action, there is a $\mathfrak{g}^*$-valued 1-form on M, ν, satisfying an equation $d\nu \pm \frac{1}{2}[\nu, \nu] = 0$ on surfaces in M that are extremal for α. Thus, there are "nonabelian conservation laws" associated to the action of G on M, and this result constitutes a "nonabelian Noether theorem". Indeed, when the Lie group G is Abelian, the Maurer-Cartan type equation satisfied by ν reduces to $d(\nu_X) = 0$, for each $X \in \mathfrak{g}$, i.e., to a conservation law associated to each infinitesimal symmetry of the 2-form α.

What I left out

Question. What did I leave out? Answer. Nearly everything and in particular,

the infinite jet bundle,
contact geometry,
the geometry of partial differential equations,
the exact sequence of the calculus of variations,
the inverse problem of the calculus of variations,
constrained systems,
a modern formulation of Noether's second theorem,
non-variational equations,
Noether identities in gauge theory,
non-commutative conservation laws,
the first Noether theorem for dicrete systems
and numerical analysis, etc., etc.

A few essential references

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