Sur les algébroïdes de Lie et de Courant

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Résumé

Algébroïdes de Lie et algébroïdes de Courant dans le formalisme lagrangien

Aprs̀ un exposé de diverses définitions, propriétés et exemples d'algébroïdes de Lie et d'algébroïdes de Courant, nous présenterons un formalisme lagrangien sur les algébroïdes de Lie.

Some motivation and references

In 1992, motivated by the study of discrete Lagrangian and Hamiltonian system, Alan Weinstein showed that the natural framework for discrete systems is that of **Lie groupoids**, and for the case of continuous time, that of **Lie algebroids**.
For 1-dimensional variational problems – the case of mechanics – the framework of Lie algebroids extends that of tangent bundles.
For 2-dimensional problems, Pavol Ševera showed that it was necessary to introduce **Courant algebroids**.

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What are Lie algebroids?

Charles Ehresmann introduced the “groupoïdes différentiels” which are now called “groupoïdes de Lie”.
Groupoids generalize groups, Lie groupoids generalize Lie groups.

Lie algebroids are the infinitesimals of Lie groupoids.
Therefore Lie algebroids are generalized Lie algebras.
But they are also generalized tangent bundles!

A classical fact

Let M be a manifold, $TM \to M$ its **tangent bundle**.
Let $\Gamma(TM)$ be the $C^\infty(M)$-module of sections of TM (the space of vector fields on M).

Fact. The Lie bracket of vector fields makes $\Gamma(TM)$ a LIE ALGEBRA over $\mathbb{R}$:
skewsymmetry $[X, Y] = -[Y, X]$ and
Jacobi identity $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$

Fact. The Lie bracket of vector fields satisfies the LEIBNIZ IDENTITY:

$$\forall X, Y \in \Gamma(TM), \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (X \cdot f)Y$$

Lie algebroids are "generalized tangent bundles"

Definition of Lie algebroids

Definition

A *Lie algebroid* is a vector bundle $\tau : A \to M$ such that

• ΓA is a LIE ALGEBRA over $\mathbb{R}$
(skewsymmetry + Jacobi identity)

• there exists a morphism of vector bundles $\rho : A \to TM$, called the *anchor*, such that the LEIBNIZ IDENTITY is satisfied,

$$\boxed{\forall X, Y \in \Gamma A, \forall f \in C^\infty(M), [X, fY] = f[X, Y] + (\rho(X) \cdot f)Y}$$

Proposition

The mapping ρ induces a Lie algebra homomorphism $\Gamma A \to \Gamma(TM)$.

Remark: Although this proposition is a consequence of the definition, it is often included in the list of axioms.

Examples

- TM

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- foliations: integrable subbundles of TM
- gauge Lie algebroids (also called Atiyah algebroids): TP/G, where P is a principal bundle with structure group G.
- action Lie algebroids : $M \times \mathfrak{g} \to M$, where $\mathfrak{g}$ is a Lie algebra acting on M.

Example: action Lie algebroid

In an action Lie algebroid $M \times \mathfrak{g}$, the anchor is $(m, X) \in M \times \mathfrak{g} \mapsto X_M(m) \in TM$ and the Lie bracket of constant sections is their Lie bracket in $\mathfrak{g}$ considered as a constant map. Then use the Leibniz rule for the bracket of arbitrary sections.

Another example is the cotangent bundle of a Poisson manifold.

Poisson manifolds

We recall the definition of Poisson manifolds:

A Poisson manifold is a smooth manifold equipped with a bivector (i.e., a field of skew-symmetric contravariant tensors) π such that the bracket of smooth functions, defined by $\{f, g\} = \pi(df, dg)$, satisfies the Jacobi identity,
$\{f, \{g, h\}\} = \{g, \{h, f\}\} + \{h, \{f, g\}\}$,
for all smooth functions f, g, h.

On a Poisson manifold, the Poisson bracket, $\{\ ,\ \}$, is a skew-symmetric, $\mathbb{R}$-bilinear, biderivation of the ring $C^\infty(M)$, and $(C^\infty(M), \{\ ,\ \})$ is an $\mathbb{R}$-Lie algebra.

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Example: The dual of a Lie algebra is a (linear) Poisson manifold (and not a symplectic manifold).

Example of Lie algebroid: The cotangent bundle of a Poisson manifold

On the cotangent bundle T^*M of a Poisson manifold, (M, π), there exists a unique Lie algebroid structure such that
the anchor is the map $\pi^\sharp : T^*M \to TM$ defined by $\pi^\sharp(\alpha) = i_\alpha \pi$, for all 1-forms α, and
the bracket of 1-forms, denoted by $[\ ,\ ]_\pi$, satisfies

$$[df, dg]_\pi = d\{f, g\}, \quad \forall f, g \in C^\infty(M).$$

The general formula for the bracket of 1-forms on a Poisson manifold is

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp \alpha}\beta - \mathcal{L}_{\pi^\sharp \beta}\alpha - d(\pi(\alpha, \beta)).$$

The differential of a Lie algebroid

For any Lie algebroid A, there is a differential d_A on $\Gamma(\wedge^\bullet A^*)$ defined by

$$(d_A\alpha)(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i \rho(X_i)\cdot\alpha(X_0, \ldots, \widehat{X}_i, \ldots, X_k) + \sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),$$

for $\alpha \in \Gamma(\wedge^k A^*)$, $k \in \mathbb{N}$, $X_0, \ldots, X_k \in \Gamma A$.

$$\boxed{d_A^2 = 0}$$

The differential d_A turns $\Gamma(\wedge^\bullet A^*)$ into a complex

whose cohomology is called the *Lie algebroid cohomology* and is denoted by $H^\bullet(A)$.

The de Rham differential and the Chevalley–Eilenberg differential

When $A = TM$, the differential d_A acts on the differential forms $\Gamma(\wedge^\bullet(T^*M))$ on the manifold M, and $H^\bullet(A)$ is the de Rham cohomology of M.

When A is a Lie algebra considered as a Lie algebroid over a point, $H^\bullet(A)$ is the Chevalley–Eilenberg cohomology.

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The *Lie derivative* of sections of $\Lambda^\bullet A^*$ with respect to $X \in \Gamma A$ is defined by $\mathcal{L}_X = [i_X, d_A]$,
where [,] is the graded commutator of derivations of $\Gamma(\wedge^\bullet A^*)$.

Alternate definition of Lie algebroids

Defining a Lie algebroid structure on A is equivalent to defining a differential d on $\Gamma(\wedge^\bullet A^*)$.

Set

$$\rho(X)f = < X, df >$$

and

$$< [X, Y], \alpha > = \rho(X) \cdot < Y, \alpha > - \rho(Y) \cdot < X, \alpha > - (d\alpha)(X, Y).$$

$d^2 = 0$ is equivalent to the Jacobi and Leibniz identities for [,].

The Gerstenhaber algebra of a Lie algebroid

There is a unique extension to $\Gamma(\wedge^\bullet A)$ of the Lie bracket of sections of A, again denoted by $[\ ,\ ]_A$, as a (graded) bi-derivation satisfying $[X, f]_A = < X, d_A f >$, for all $X \in \Gamma A$, $f \in C^\infty(M)$. This bracket of degree -1 is called a *Gerstenhaber bracket*, and $(\Gamma(\wedge^\bullet A), \wedge, [\ ,\ ]_A)$ is a *Gerstenhaber algebra*.

To summarize, to any Lie algebroid are associated both
• a differential d_A on $\Gamma(\wedge^\bullet A^*)$,
and
• a Gerstenhaber bracket $[\ ,\ ]_A$ on $\Gamma(\wedge^\bullet A)$.

The Schouten–Nijenhuis bracket of multivector fields

In the special case where $A = TM$ with the Lie bracket of vector fields, the Gerstenhaber bracket on $\Gamma(\wedge^\bullet TM)$ is called the *Schouten–Nijenhuis bracket of multivector fields*.
It is a bracket, $[\ ,\ ]_{SN}$, of degree -1 on the space of multivector fields on M, the unique extension to $\Gamma(\wedge^\bullet TM)$ as a (graded) bi-derivation of the Lie bracket of vector fields satisfying $[X, f]_{SN} = X \cdot f$, for all $X \in \Gamma(TM)$, $f \in C^\infty(M)$.

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Application: (M, π) is a Poisson manifold if and only if $[\pi . \pi]_{SN} = 0$.

Example: the Lichnerowicz–Poisson differential

If (M, π) is a Poisson manifold, the differential associated to the Lie algebroid $(T^*M, [\ ,\ ]_\pi, \pi^\sharp)$ is

$$\boxed{d_\pi = [\pi, \cdot]_{SN}}$$

acting on $\Gamma(\wedge^\bullet TM)$, where $[\ ,\ ]_{SN}$ is the Schouten–Nijenhuis bracket of multivector fields on M.

(Bhaskara and Viswanath [1988], yks and Franco Magri [1990])

Example: deformation by a Nijenhuis tensor

If M is a manifold and $N : TM \to TM$ is a Nijenhuis tensor,

$$[NX, NY] - N([NX, Y] + [X, NY]) - N^2[X, Y] = 0,$$

then

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$$

is a Lie bracket on TM, and TM becomes a Lie algebroid with anchor $N : TM \to TM$.

The associated differential is

$$d_N = [i_N, d]$$

(graded commutator).

Lie bialgebroids

A Lie bialgebroid (A, A^*) is a pair of Lie algebroids A and A^* in duality, satisfying the following compatibility assumption

the differential d_A is a derivation of the Gerstenhaber bracket $[\ ,\ ]_{A^*}$.

Proposition (A, A^*) is a Lie bialgebroid if and only if (A^*, A) is a Lie bialgebroid.
(Kirill Mackenzie and Ping Xu [1994]; yks [1995])

Example: If (M, π) is a Poisson manifold, then (TM, T^*M) is a Lie bialgebroid.

Lie algebroids as supermanifolds (1)

Let $E \to M$ be a vector bundle.
We denote by ΠE the supermanifold obtained by "making the coordinates in the fibers odd."
Let (x^i, ξ^α) be adapted local coordinates on ΠE. Then

$$x^i x^j = x^j x^i, \qquad \xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha.$$

The smooth functions on ΠE are of the form

$$f(x,\xi) = f_0(x) + f_\alpha(x)\xi^\alpha + f_{\alpha\beta}(x)\xi^\alpha\xi^\beta + \cdots ,$$

where the highest term contains at most p factors ξ^α
where p is the rank of E (the dimension of the fibers)
[since the ξ^α anti-commute].

Lie algebroids as supermanifolds (2)

Fact. The smooth functions on the supermanifold ΠE are the sections of $\wedge^\bullet E^*$:

$$\boxed{C^\infty(\Pi E) = \Gamma(\wedge^\bullet E^*)}$$

Question. How can we interpret a Lie algebroid structure on E in terms of the supermanifold ΠE?

Answer. (Vaĭntrob [1997])

• If E is a vector bundle over M, a derivation of $\Gamma(\wedge^\bullet E^*) = C^\infty(\Pi E)$ is identified with a vector field on ΠE.

• If A is a Lie algebroid over M, the derivation d_A of square 0 is identified with a vector field Q_A on ΠA, satisfying

$$[Q_A, Q_A] = 0.$$

Such a vector field is called a homological vector field.

Why Courant algebroids?

The tangent bundle framework, more generally the Lie algebroid framework, are for 1-dimensional variational problems: one independent variable, denoted t, "time".
What is the proper framework for 2-dimensional variational problems?
Answer (Pavol Ševera): Courant algebroids.

What is the "double" of a Lie algebroid? of a Lie bialgebroid?
Answer: a Courant algebroid.

How can one unify symplectic and complex geoemtry?
Answer: on the standard Courant algebroid.

Definition (standard Courant algebroid)

- vector bundle $E = TM \oplus T^*M$,
- fiberwise non-degenerate bilinear symmetric form

$$< X + \xi, Y + \eta > = < X, \eta > + < Y, \xi >,$$

- non-skewsymmetric bracket on sections

$$\boxed{[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi}$$

$\forall X, Y \in \Gamma(TM), \xi, \eta \in \Gamma(T^*M)$.

In particular, $[X, \eta] = \mathcal{L}_X \eta$ and $[\eta, X] = -i_X d\eta$.

This bracket is called the Dorfman bracket.

The Dorfman bracket is a derived bracket

Derived bracket (yks [1996], [2004])

$$[u, v]_d = [[u, d], v]$$

(graded Lie algebra, d odd derivation, $[d, d] = 0$)
The Dorfman bracket is a *derived bracket*.
Set $\epsilon_\xi(\eta) = \xi \wedge \eta$.

$$[i_X, i_Y]_d = [[i_X, d], i_Y] = [\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

$$[i_X, \epsilon_\eta]_d = [[i_X, d], \epsilon_\eta] = [\mathcal{L}_X, \epsilon_\eta] = \epsilon_{\mathcal{L}_X \eta}$$

$$[\epsilon_\xi, i_Y]_d = [[\epsilon_\xi, d], i_Y] = [\epsilon_{d\xi}, i_Y] = -\epsilon_{i_Y d\xi}$$

$$[\epsilon_\xi, \epsilon_\eta] = 0.$$

Therefore,
when $X \leftrightarrow i_X$ and $\xi \leftrightarrow \epsilon_\xi$, $[X, \eta]_d = \mathcal{L}_X \eta$ and $[\xi, Y]_d = -i_Y d\xi$.

Loday algebras

Definition

A *Loday algebra* is a (graded) vector space equipped with a bracket (bilinear operation, in general non-skewsymmetric) satisfying the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [a, [b, c]],$$

$\forall a, b, c$.

Loday algebras are also called *Leibniz algebras*.

Examples.

Derived brackets.

The vector space of sections of the standard Courant algebroid.

Properties of the Dorfman bracket

Fact. If $[\ ,\ ]$ is a Lie bracket, then the derived bracket $[\ ,\ ]_d$ is a Loday bracket.

Theorem

The Dorfman bracket satisfies the Jacobi identity

$$[a,[b,c]] = [[a,b],c] + [a,[b,c]]$$

$\forall a,b,c \in \Gamma(TM \oplus T^*M)$.

Remark. The Dorfman bracket does NOT satisfy the identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

Fact. $\Gamma(TM \oplus T^*M)$ with the Dorfman bracket is a LODAY ALGEBRA, and NOT, in general, a Lie algebra.

The Courant bracket

The skewsymmetrized Dorfman bracket is called the Courant bracket,

$$[X+\xi, Y+\eta]^C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\xi - i_Y\eta).$$

The Courant bracket does NOT satify the Jacobi identity.

A vector bundle $E \to M$ is called a pseudo-orthogonal vector bundle if it is equipped with a fiberwise non-degenerate symmetric bilinear form, $< \ , \ >$.

Definition

A Courant algebroid is a pseudo-orthogonal vector bundle $(E, < \ , \ >)$ equipped with a Loday bracket $[\ , \ ]$ and an anchor $\rho : E \to TM$ satisfying

$$\rho(a) < b, c > = < a, [b, c] + [c, b] >,$$

$$\rho(a) < b, c > = < [a, b], c > + < b, [a, c] > .$$

Examples of Courant algebroids

• Case $M = \{pt\}$, E a Lie algebra with an invariant non-degenerate symmetric bilinear form (for example a semi-simple Lie algebra with its Killing form).

• $E = TM \oplus T^*M$ with the non-degenerate bilinear symmetric form

$$< X + \xi, Y + \eta > = < X, \eta > + < Y, \xi >$$

and the Dorfman bracket.

• more generally, $E = A \oplus A^*$, when A is a Lie algebroid,

• more generally, $E = A \oplus A^*$ when (A, A^*) is a Lie bialgebroid

• more generally, $E = A \oplus A^*$ when (A, A^*) is a Lie proto-bialgebroid.

Theorem

Let $V \to M$ be a vector bundle. A Courant algebroid structure on the vector bundle $V \oplus V^*$ is equivalent to a Lie proto-bialgebroid structure on (V, V^*).

Graded manifolds

If, in the supermanifold ΠE,
– the coordinates on the base are given the degree 0, and
– the coordinates on the fibers are given the degree 1,
ΠE becomes a graded manifold and it is denoted by $E[1]$.

[The degree is also called weight.]

More generally, $E[N]$ is the graded manifold where
the coordinates on the base are given the degree 0 and
the coordinates on the fibers are given the degree N
(N a positive integer).

A tower of fibrations $E_2 \to E_1 \to M$ can also be made into a graded manifold by assigning increasing positive integer degrees to the coordinates on the fibers.

Graded manifolds of degree 1

A graded manifold of degree 1 is a vector bundle $E \to M$.

- A graded manifold of degree 1 with a homological vector field of degree 1 is a Lie algebroid $A \to M$.
  The correspondence is $(A, d_A) \longleftrightarrow (A[1], Q_A)$.
  Notation: $\boxed{Q\text{-manifold}}$.

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  The correspondence is $(A, d_A) \longleftrightarrow (A[1], Q_A)$.
  Notation: $\boxed{Q\text{-manifold}}$.
- Manifolds are in one-to-one correspondence with graded manifolds of degree 1 with a symplectic structure of degree 1.
  The correspondence is $M \longleftrightarrow (T^*[1]M, \omega_0)$, where ω_0 is the canonical symplectic structure of the cotangent bundle.
  Notation: $\boxed{P\text{-manifold}}$.
  In coordinates, $\omega_0 = dq^i \wedge dp_i$.
  The symplectic structure of degree 1 of $T^*[1]M$ defines a Poisson bracket of degree -1 on $C^\infty(T^*[1]M) = \Gamma(\wedge^\bullet TM)$ which is the Schouten-Nijenhuis bracket of multivector fields, $[\ ,\ ]_{SN}$.

Graded manifolds of degree 1 (continued)

- Poisson manifolds are in one-to-one correspondence with graded manifolds of degree 1 with a homological vector field of degree 1 and an invariant symplectic structure of degree 1.
  The correspondence is $(M, \pi) \longleftrightarrow (T^*[1]M, Q_\pi, \omega_0)$.
  Notation: $\boxed{QP\text{-manifold}}$.
  Here Q_π is the Lichnerowicz–Poisson differential on $\Gamma(\wedge^\bullet TM)$, also denoted by $d_\pi = [\pi, \cdot]_{SN}$, seen as a homological vector field on $T^*[1]M$.
  There is a compatibility condition between the Q-structure and the P-structure: invariance of ω_0 with respect to the vector field Q_π.
  This condition is satisfied because d_π is a derivation of the Schouten–Nijenhuis bracket (it is an interior derivation).

Courant algebroids $V \oplus V^*$ as graded manifolds

Question. How can we define a Courant algebroid structure on $V \oplus V^*$ using graded manifolds?

Answer. (Pavol Ševera, Dmitry Roytenberg, Theodore Voronov)
Let $V \to M$ be a vector bundle. Then $(E = V \oplus V^*, < \ , \ >)$ is a pseudo-orthogonal vector bundle,

Consider the graded manifold $V[1]$.
The cotangent bundle $T^*V[1]$ of $V[1]$ can be made into a graded manifold as follows.

Courant algebroids $V \oplus V^*$ as graded manifolds (continued)

Let the coordinates in the fiber of $T^*V \to V$ have degree 2. This graded manifold is denoted by $T^*[2]V[1]$.
As a cotangent bundle, the graded manifold $T^*[2]V[1]$ has a canonical symplectic structure. It is of degree 2 and the associated Poisson bracket is of degree -2.

Local coordinates $(x^i, z^\alpha, p_i, \zeta_\alpha)$ of degrees $(0, 1, 2, 1)$
(the degree of ζ_α in $T^*V[1]$ is -1, so it is $-1+2=1$ in $T^*[2]V[1]$).

The local coordinates have bi-degrees: $(0,0), (1,0), (1,1), (0,1)$.

The Poisson bracket is of bi-degree $(-1,-1)$. The brackets of coordinates are

$$\{x^i, p_j\} = \delta^i_j, \quad \{z^\alpha, \zeta_\beta\} = \delta^\alpha_\beta.$$

Courant algebroids $V \oplus V^*$ as graded manifolds (continued)

Consider a function Θ of degree 3 on $T^*[2]V[1]$,

$$\Theta = \rho^i_\alpha(x) p_i z^\alpha + C_{\alpha\beta\gamma}(x) z^\alpha z^\beta z^\gamma .$$

The function Θ defines a Hamiltonian vector field on $T^*[2]V[1]$,

$$\boxed{d_\Theta = \{\Theta, \cdot\}}$$

Since d_Θ is a Hamiltonian vector field, it leaves the symplectic form invariant.
This vector field is homological if and only if $\{\Theta, \Theta\} = 0$.

Courant algebroids $V \oplus V^*$ as graded manifolds (continued)

Consider $d_\Theta = \{\Theta, \cdot\}$, a derivation of the algebra of functions on $T^*[2]V[1]$.

Fact. The following properties are equivalent

• $\{\Theta, \Theta\} = 0$ (Poisson bracket of functions of degree 3 on a graded manifold equipped with a Poisson bracket of degree -2)

• $d_\Theta^2 = 0$ (composition of derivations)

• $[d_\Theta, d_\Theta] = 0$ (Lie bracket of odd vector fields)

In field theory, the equation $\boxed{\{\Theta, \Theta\} = 0}$ is called the classical master equation. It is also called the Maurer-Cartan equation.

Theorem

Θ defines a Courant algebroid structure on $V \oplus V^*$ if and only if $\{\Theta, \Theta\} = 0$.

Method of **proof.** Given Θ, define an anchor and a bracket on $V \oplus V^*$ by

$$\boxed{\rho(a)f = \{\{a, \Theta\}, f\}}$$

$$\boxed{[a, b] = \{\{a, \Theta\}, b\}}$$

$\forall a, b \in \Gamma(V \oplus V^*)$, $f \in C^\infty(M)$.
Then show that the condition $\{\Theta, \Theta\} = 0$ is equivalent to the axioms of a Courant algebroid.
Then prove the converse.

Exercise

Let $\mathfrak{g}$ be a Lie algebra with an invariant non-degenerate symmetric bilinear form $< \ , \ >$. Define the Cartan 3-form $\Theta \in \wedge^3 \mathfrak{g}^*$ by

$$\Theta(X, Y, Z) = < [X, Y], Z > .$$

Equip $\wedge^\bullet \mathfrak{g}^*$ with the Poisson bracket of degree -2 extending $< \ , \ >$, considered as a bilinear form on $\mathfrak{g}^*$.

Show that

1. Θ is invariant.
2. $\{\Theta, \Theta\} = 0$.
3. $\{\Theta, \cdot\}$ is the Chevalley–Eilenberg differential on $\wedge^\bullet \mathfrak{g}^*$.
4. $\boxed{[X, Y] = \{\{X, \Theta\}, Y\}}$, $\forall X, Y \in \mathfrak{g}$.

Courant algebroids as graded manifolds. General case

More generally, a Courant algebroid structure on an arbitrary pseudo-euclidian vector bundle E (not necessarily of the form $V \oplus V^*$) can be defined by a function of degree 3 of Poisson square 0 on a suitably defined graded manifold of degree 2 with canonical symplectic structure of degre 2, called the minimal symplectic realization of $E[1]$.
The canonical Poisson bracket is then of degree -2.

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