Modular classes of Lie algebroids: recent results

Yvette Kosmann-Schwarzbach
(Ecole Polytechnique, Palaiseau)

We present recent results and work in progress on the modular classes and relative modular classes of Lie algebroids, a report mainly based on joint papers with Camille-Laurent Gengoux [7], Alan Weinstein [10], Milen Yakimov [11] and Franco Magri [9].

On a Poisson manifold, given a volume form, the map which associates to a function the divergence of the corresponding hamiltonian vector field is a derivation, i.e., a vector field, called a *modular vector field*. It is a 1-cocycle in the Lichnerowicz-Poisson cohomology, and its class, called the *modular class*, is independent of the volume form [12] [15]. If the manifold is not orientable, densities must be used instead of volume forms.

Evens, Lu and Weinstein [4] introduced the notion of a *modular class of a Lie algebroid*, and they observed that the modular class of a Poisson manifold is one-half that of its cotangent Lie algebroid.

It is straightforward [6] to extend the notion of modular class from the case of a Poisson manifold to that of a Lie algebroid A with a Poisson structure, i.e., a section π of $\wedge^2 A$ such that $[\pi,\pi]_A = 0$, where $[\ ,\ ]_A$ is the Schouten-Nijenhuis bracket on $\Gamma(\wedge^\bullet A)$ defining its Gerstenhaber structure. The question that then arises is how to determine what relation exists in general between the modular class $\theta(A,\pi)$ and the modular class $\mathrm{Mod}(A^*)$ of the dual A^* of A equipped with the Lie algebroid structure defined by π.

In order to solve this problem, the notion of *relative modular class*, which also appears in [5] under the name of *modular class of a morphism*, was introduced in [10]. If $\Phi : E \to F$ is a morphism of Lie algebroids over the same base, then $\wedge^\bullet \Phi^*$ is a chain map from the complex $\Gamma(\wedge^\bullet F^*)$ of the Lie algebroid F to the complex $\Gamma(\wedge^\bullet E^*)$ of the Lie algebroid E . Therefore $\mathrm{Mod}\, E - \Phi^*(\mathrm{Mod}\, F)$ is a cohomology class in the Poisson cohomology of E. This is the relative class, denoted $\mathrm{Mod}^\Phi(E,F)$. Then the relation

$$\theta(A,\pi) = \frac{1}{2}(\mathrm{Mod}(A^*) - (\pi^\sharp)^*\mathrm{Mod}A)$$

is valid in general and, since $\mathrm{Mod}(TM) = 0$, it reduces to the fact recalled above in the case of Poisson manifolds. The relative modular classes of general, not necessarily base-preserving, morphisms are treated in [8].

A *twisted Poisson structure*, also called Poisson structure with background [14], on a Lie algebroid A is a pair (π,ψ), where ψ is a 3-cocycle on A and

$$\frac{1}{2}[\pi,\pi]_A = (\wedge^3\pi^\sharp)\psi\ .$$

A representative of the modular class $\theta(A,\pi) = \frac{1}{2}(\mathrm{Mod}(A^*) - (\pi^\sharp)^*\mathrm{Mod}A)$ is $X+Y$, with $i_X\lambda = -d_A i_\pi \lambda$, where λ is a section of $\wedge^{\mathrm{top}} A^*$, and $Y = \pi^\sharp i_\pi \psi$ [7].

In the *spinor approach* to Poisson and Dirac structures [1] [13], the modular field appears as the obstruction to the existence of a pure spinor defining the graph of π which is closed in the Lie algebroid cohomology of A. This fact extends to the twisted case, replacing d_A by $d_A + \epsilon_\psi$, where ϵ_ψ is exterior product of forms by ψ.

In the case of a *regular* Poisson or twisted Poisson structure, the modular class can be computed in terms of the characteristic class of a representation of the image of $\pi^\sharp$ on the top exterior power of its kernel [11]. This result extends to Lie algebroid extensions with unimodular kernel [8].

These definitions and properties can be applied to Lie algebras, considered as Lie algebroids over a point, whence the notion of *twisted triangular r-matrix*. In [11], we obtained a formula for the modular class of a Lie algebra equipped with a twisted triangular r-matrix in terms of the infinitesimal character of the adjoint representation of $\mathfrak{p}$ in $\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{p}$ is the carrier of the r-matrix, i.e., its image in the Lie algebra. When the carrier of the r-matrix is a Frobenius Lie algebra with respect to a 1-form ξ, the modular class is the unique element X in $\mathfrak{p}$ such that $\mathrm{ad}^*_X \xi$ is equal to the above character. This method is applied to the computation of the class defined by the Gerstenhaber-Giaquinto r-matrix on $\mathfrak{sl}(n, \mathbb{R})$.

Other examples of modular classes appear in the theory of *Poisson-Nijenhuis manifolds* [3] [9]. When a Poisson tensor π and a Nijenhuis tensor N on a manifold are *compatible*, there is a hierarchy of vector fields, $NX^{k-1} - X^k$, $k \geq 1$, where X^k is a modular vector field for the k-th Poisson structure $N^k\pi$, which are cocycles in the Poisson cohomology defined by $N^k\pi$, and independent of the choice of a volume form. Up to a factor of one-half, these modular vector fields coincide with the well-known hierarchy of commuting hamiltonian vector fields defined on a Poisson-Nijenhuis manifold. This construction has been generalized to Lie algebroids with a Poisson-Nijenhuis structure [2].

References

[1] A. Alekseev, P. Xu, Courant algebroids and derived brackets, preprint 2002.
[2] R. Caseiro, Modular classes of Poisson-Nijenhuis Lie algebroids, preprint arXiv:math/0701476.
[3] P. A. Damianou, R. Fernandes, Integrable hierarchies and the modular class, preprint arXiv:math/0607784.
[4] S. Evens, J.-H. Lu, A. Weinstein, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*, Quart. J. Math. Ser.2 **50** (1999), 417–436.
[5] J. Grabowski, G. Marmo, P. W. Michor, *Homology and modular classes of Lie algebroids*, Ann. Inst. Fourier (Grenoble) **56** (2006), 69–83.
[6] Y. Kosmann-Schwarzbach, *Modular vector fields and Batalin-Vilkovisky algebras*, in Poisson Geometry, J. Grabowski, P. Urbanski, eds., Banach Center Publ. **51** (2000), 109–129.
[7] Y. Kosmann-Schwarzbach, C. Laurent-Gengoux, *The modular class of a twisted Poisson structure*, Travaux mathématiques (Luxembourg) (2005) **16**, 315–339.
[8] Y. Kosmann-Schwarzbach, C. Laurent-Gengoux, A. Weinstein, in preparation.
[9] Y. Kosmann-Schwarzbach, F. Magri, On the modular classes of Poisson-Nijenhuis manifolds, preprint arXiv:math/0611202.
[10] Y. Kosmann-Schwarzbach, A. Weinstein, *Relative modular classes of Lie algebroids*, C. R. Acad. Sci. Paris, Ser. I **341** (2005), 509–514.

[11] Y. Kosmann-Schwarzbach, M. Yakimov, *Modular classes of regular twisted Poisson structures on Lie algebroids*, Lett. Math. Phys. **80** (2007), 183–197.

[12] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, in Élie Cartan et les mathématiques d'aujourd'hui, Astérisque (1985), Numéro hors série, 257–271.

[13] E. Meinrenken, Lectures on pure spinors and moment maps, preprint arXiv:math/0609319.

[14] P. Ševera, A. Weinstein, *Poisson geometry with a 3-form background*, in Noncommutative Geometry and String Theory, Progr. Theoret. Phys. Suppl. no. **144** (2001), 145–154.

[15] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. **23** (1997), 379–394.