QUESTIONS ON SELF MAPS OF ALGEBRAIC VARIETIES

NAJMUDDIN FAKHRUDDIN

1. Introduction

In this note we shall consider some arithmetic and geometric questions concerning projective varieties X with a self-map ϕ and an ample line bundle L such that $\phi^*(L) \cong L^{\otimes d}$ for some $d > 1$, all defined over some field k. Such a situation is interesting arithmetically, if k is a number field, because using a method of Tate (see for example [17, Chapter 3]) one can define a canonical height function $h_\phi : X(\bar{k}) \to \mathbb{R}$ which can be used to study the dynamics of ϕ on the set of rational points of X. One is then naturally led to ask which varieties admit such self maps and line bundles.

Basic examples of such varieties are projective spaces and abelian varieties. The results of Section 2 show that projective spaces are in a sense the "universal" such varieties, so several questions like the uniform boundedness of the torsion of abelian varieties over a number field or the generalisation of Bogomolov's conjecture due to Zhang [20, Conjecture 2.5], reduce to questions about self maps of projective spaces. (Unfortunately, the questions do not seem to become any easier!)

In the next section we raise some questions on the arithmetic of self maps of projective spaces, among them a generalisation of the Morton-Silverman Uniform Boundedness conjecture [10]. Most of these seem beyond reach at the moment, however, we do show (Proposition 3.3) that the weakest of these questions is implied by a conjecture of Lang, via the work of Caporaso, Harris and Mazur [3].

The remaining two sections are devoted to purely geometric questions. In Section 4 we obtain a partial classification (Theorem 4.2) of varieties admitting self maps and line bundles as above. This suggests that all étale self maps "come" from maps of families of abelian varieties. We also ask if projective spaces can be characterised in terms of self maps. Section 5 is devoted to showing that the self maps considered above have many periodic points; in fact the periodic points always form a Zariski dense set.

2.

Proposition 2.1. *Let X be a projective variety over an infinite field k, $\phi : X \to X$ a morphism and L a very ample line bundle on X such that $\phi^*(L) \cong L^{\otimes d}$ for some $d > 1$. Let $V = H^0(X, L)$ and suppose the following conditions are satisfied:*

(1) *The embedding $\iota : X \to \mathbb{P}(V)$ induced by L has the property that the maps $H^0(\mathbb{P}(V), \mathcal{O}(n)) \xrightarrow{\iota^*} H^0(X, L^{\otimes n})$ are surjective for all $n \geq 0$.*
(2) *$\iota(X)$ is cut out set-theoretically by homogenous forms of degree $\leq d$.*

Then there exists a morphism $\psi : \mathbb{P}(V) \to \mathbb{P}(V)$ such that $\psi \circ \iota = \iota \circ \phi$. .

Proof. Let $g = \dim(X)$, let $s_0, s_1, \ldots, s_n$ be a basis of V and assume that $s_0, s_1, \ldots, s_g$ have no common zeros on X. By (1), it follows that after choosing an isomorphism $\phi^*(L) \cong L^{\otimes d}$, each $\phi^*(s_j)$, $0 \leq j \leq n$, can be written as a homogenous polynomial of degree d (not necessarily uniquely) in the s_i's. Any choice of such polynomials f_j, $0 \leq j \leq n$, gives rise to a rational map $\psi : \mathbb{P}(V) \to \mathbb{P}(V)$ defined by $s_j \mapsto f_j$, such that $\psi \circ \iota = \iota \circ \phi$. We shall show that if the f_j's are chosen generally then ψ is actually a morphism.

Let $D(f_j)$ denote the hypersurface in $\mathbb{P}(V)$ which is the zero set of f_j. Using induction we will prove that the intersection of all the $D(f_j)$'s is empty. Firstly, each component of the intersection of the first $g+1$ $D(f_i)$'s is of codimension $g+1$, since otherwise it would intersect $\iota(X)$, and by choice of the s_j's this cannot happen. Now we prove that we can choose the remaining f_i's in such a way so that the codimension of the intersection goes down by 1 at each step, this clearly suffices to complete the proof.

Suppose $g \leq k < n$, we have chosen f_i for for all $i \leq k$ and we want to choose f_{k+1}. Let h_1 be any polynomial of degree d which restricts to $\phi^*(s_{k+1})$ on A. If $D(h_1)$ does not contain any of the components of the intersection of the $D(f_j)$'s, $j \leq k$, then let $f_{k+1} = h_1$. Otherwise, by (2) we can choose a polynomial h_2 of degree d in the ideal of X such that $D(h_2)$ does not contain any of the above components. Consider the family of hypersurfaces $D(ah_1 + bh_2)$, $[a, b] \in \mathbb{P}^1$. When $a = 0$, by construction the corresponding hypersurface does not contain any of the components. Since this is an open condition and k is infinite, there exists $b \in k$ such that the hypersurface $D(h_1 + bh_2)$ does not contain any of the components of the intersection of the $D(f_i)$'s, $i \leq k$. Let $f_{k+1} = h_1 + bh_2$. □

Corollary 2.2. *Let X be a projective variety over an infinite field k, $\phi : X \to X$ a morphism and L an ample line bundle on X such that $\phi^*(L) \cong L^{\otimes d}$ for some $d \geq 1$. Then there exists an embedding ι of X in $\mathbb{P}^N_k$ and a morphism $\psi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ such that $\psi \circ \iota = \iota \circ \phi$.*

Proof. If $d = 1$ then f must be an automorphism. Replacing L by a very ample tensor power $L' = L^{\otimes n}$, one sees that the natural embedding $\iota : X \to \mathbb{P}(H^0(X, L))$ and the automorphism ψ of $\mathbb{P}(H^0(X, L'))$ induced by $f^* : H^0(X, L') \to H^0(X, L')$ have the required properties.

If $d > 1$, we use the results of Mumford ([12], Theorems 1 and 3) to replace L by a tensor power $L' = L^{\otimes n}$ so that the conditions of Proposition 2.1 are satisfied by X, ϕ and L'. □

Our main application of Proposition 2.1 is to show that the Boundedness Conjecture of Morton and Silverman for $\mathbb{P}^N$ implies the uniform boundedness of torsion for abelian varieties, using an idea of Bjorn Poonen. Recall that for a map of sets $f : X \to X$, a point $x \in X$ is called *preperiodic* for f if $f^n(x) = f^m(x)$ for some $m > n \geq 0$; it is called *periodic* if we can take $n = 0$.

Conjecture 2.3 (Morton-Silverman [10]). *For all positive integers D, N, d with $d > 2$, there exists an integer $\kappa(D, N, d)$ such that for each number field k of degree D over $\mathbb{Q}$, and each morphism $\psi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ defined by homogenous polynomials of degree d over k, the number of preperiodic points of ψ in $\mathbb{P}^N(k)$ is less than or equal to $\kappa(D, N, d)$.*

For an abelian variety A and $n \in \mathbb{Z}$ we denote by $[n] : A \to A$ the multiplication by n map. If $n \neq 0$, then any point of $A(k)_{tors}$ is preperiodic for $[n]$.

Corollary 2.4. *The Boundedness Conjecture implies that there exists a constant $\eta(D, g)$ such that for any abelian variety A of dimension g defined over a number field k of degree D over $\mathbb{Q}$, $|A(k)_{tors}| \leq \eta(D, g)$.*

Proof. Let A be an abelian variety of dimension g defined over a number field k. By Zarhin's trick ([19, Lemma 2.5] or [9, p. 205]) $X = (A \times A^t)^4$ has a principal polarisation λ. By [13, p. 121], 2λ is represented by a canonical symmetric[1] ample line bundle M on X. Let $L = M^{\otimes 3}$; this is a symmetric and very ample line bundle [11, p. 163]. By a theorem of Kempf [8], the imbedding of X in $\mathbb{P}(H^0(X, L))$ is projectively normal and the homogenous ideal of the image of X is generated by its elements of degree two and three. Furthermore, since L is symmetric $[n]^*L \cong L^{\otimes n^2}$ for all $n \in \mathbb{Z}$.

Let $n = 2$, so $d = 2^2 = 4$. We apply Proposition 2.1 to X, L and $\phi = [2]$. By the Riemann-Roch theorem for abelian varieties ([11, p. 150]) $h^0(X, L) = (8g)!\ 6^{8g}$. It follows from the Boundedness Conjecture that $|X| = |(A \times A^t)^4(k)_{tors}| \leq \kappa(D, (8g)!\ 6^{8g} - 1, 4)$. Thus we may take $\eta(D, g) = \kappa(D, (8g)!\ 6^{8g} - 1, 4)^{1/4}$. $\square$

Remark 2.5. If we assume Conjecture 2.3 only for periodic points then Corollary 2.4 still holds, but we get a slightly weaker bound: A point in $A(k)_{tors}$ is periodic for $[n]$ iff its order is prime to n. Applying the method of proof of Corollary 2.4 to the maps [2] and [3], we get uniform bounds for the torsion of order prime to 2 and the torsion of order prime to 3. Then $|A(k)_{tors}|$ is bounded by the product of these bounds.

Remark 2.6. Let $\psi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ be a morphism. By the functorial property of restriction of scalars, there is an induced morphism $\psi' : \mathrm{Res}^k_{\mathbb{Q}}(\mathbb{P}^N_k) \to \mathrm{Res}^k_{\mathbb{Q}}(\mathbb{P}^N_k)$. Preperiodic points for ψ in $\mathbb{P}^N(k)$ then correspond bijectively to

[1] The symmetry follows from the construction in loc. cit.

preperiodic points for ψ' in $\mathrm{Res}^k_{\mathbb{Q}}(\mathbb{P}^N_k)(\mathbb{Q})$. One can apply Proposition 2.1 to ψ' and the line bundle L on $\mathrm{Res}^k_{\mathbb{Q}}(\mathbb{P}^N_k)$ induced by $\mathcal{O}(1)$, uniformly as ψ varies over all morphisms $\mathbb{P}^N_k \to \mathbb{P}^N_k$ with N, D, d fixed and $[k:\mathbb{Q}] = D$, to see that the special case of Conjecture 2.3 for $D = 1$ and all d, N, in fact implies the general case.

3.

We now consider some uniformity properties of the images of the set of *all* rational points for varying morphisms of projective spaces. For the rest of this section k will be a number field and $d > 1, N$ will be positive integers. We shall also abuse convention and say that a self map of $\mathbb{P}^N_k$ is of degree d if it is defined by homogenous polynomials of degree d over k.

Given a morphism $f : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d over k, we may form the inverse system of sets

$$\cdots \xrightarrow{f} \mathbb{P}^N(k) \xrightarrow{f} \cdots \xrightarrow{f} \mathbb{P}^N(k) \xrightarrow{f} \mathbb{P}^N(k).$$

Using the theory of heights one can show that the set $\varprojlim \mathbb{P}^N(k)$ is finite, and its points correspond in a natural way with the periodic points of f. It is then natural to ask the following

Question 3.1. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of morphisms $f_i : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d over k. Consider the inverse system of sets

$$\cdots \xrightarrow{f_n} \mathbb{P}^N(k) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} \mathbb{P}^N(k) \xrightarrow{f_1} \mathbb{P}^N(k).$$

Is the set $\varprojlim \mathbb{P}^N(k)$ finite?

If true, the following would have a positive answer.

Question 3.2. Does there exist a finite subset $S(d, N, k)$ of $\mathbb{P}^N(k)$ such that for all morphisms $f : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d over k, $S(d, N, k)$ is not contained in $f(\mathbb{P}^N(k))$?

Proposition 3.3. *Lang's conjecture on rational points on varieties of general type implies that Question 3.2 has a positive answer.*

Proof. We shall consider the cases $N = 1$ and $N > 1$ separately.

$\underline{N = 1}$. $PGL(2, k)$ acts 3-transitively on $\mathbb{P}^1(k)$, so for a given rational function f we may find an automorphism h of $\mathbb{P}^1_k$ such that $hf : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is not ramified over 0 or ∞. Since rational functions f of a fixed degree d depend on a finite number of parameters, we may find a finite set of automorphisms $\{h_i\}$ such that for any rational function f of degree d, at least one of the rational functions $h_i f$ is not ramified over 0 or ∞.

Consider the smooth projective model X of the curve defined by the equation $y^3 = f(x)$, where f is a rational function of degree $d > 1$ not ramified over 0 or ∞. X is ramified over all the zeros and poles of f and the ramification degree at such points is 3. Applying the Riemann-Hurwitz formula, one sees that the genus of X is at least 2. The genera of all curves obtained in this way (for a fixed d) is also bounded above, since there are only finitely many possibilities for the ramification type. By the theorem of Caporaso-Harris-Mazur [3], Lang's conjecture implies that the number of k-rational points on all such curves is bounded by a constant C_k. Let r_d be the maximum number of points of $\mathbb{P}^1$ over which a rational function of degree d can be ramified, and let T be a subset of cubes in $\mathbb{Q}^*$ with $|T| > C_k + r_d$. Let $S = \cup_i h_i^{-1}(T)$. If $S \subset f(\mathbb{P}^1(k))$ for some f of degree d, the above considerations would imply that one of the curves constructed above has more than C_k k-rational points, a contradiction.

$\underline{N > 1}$. For a fixed f, Bertini's theorem implies that $f^{-1}(L)$ is smooth and irreducible for a general line L in $\mathbb{P}^N$. It follows that for a fixed degree d, there exists a finite set of lines $\{L_i\}_{i=1}^{r(d)}$ such that for any f of degree d at least one of the curves $\{f^{-1}(L_i)\}_{i=1}^{r(d)}$ is smooth and irreducible.

Suppose f is general. By the adjunction formula the degree of the branch locus of f is $(N+1)(d-1)$, so the discriminant locus is an irreducible (since f is general) hypersurface of degree $d^{N-1}(N+1)(d-1)$. It follows that $f|_{f^{-1}(L_i)} : f^{-1}(L_i) \to L_i$ is of degree d^N and branched over $d^{N-1}(N+1)(d-1)$ points of L_i. From the Riemann-Hurwitz formula we see that the genus of $f^{-1}(L_i)$ is at least 2 if

$$d^{N-1}(N+1)(d-1) \geq 2d^N + 2.$$

This holds for $d \geq 4$ if $N = 2$, $d \geq 3$ if $N = 3$ and for $d \geq 2$ if $N \geq 4$. Since the genus remains constant in families, it follows that this will hold for arbitrary f as long as $f^{-1}(L_i)$ is smooth.

For d, N satisfying the above conditions, we can apply the Caporaso-Harris-Mazur theorem to obtain subsets S_i of $L_i(k)$, $i = 1, \ldots, r(d)$, such that $\bigcup S_i$ is not contained in $f(\mathbb{P}^N(k))$ for any f of degree d. For the exceptional pairs, we repeat the above arguments using,instead of lines, smooth degree 2 curves which are rational over k.

$\square$

It would be very interesting to find a complete proof i.e. not relying on any conjectures.

Question 3.2 is evidently equivalent to the following which we state in order to motivate the succeeding questions.

Question 3.4. Let $h : \mathbb{P}^N(k) \to \mathbb{R}$ be a logarithmic Weil height. Does there exist a constant $C(d, N, k) \in \mathbb{R}$ such that $\{x \in \mathbb{P}^N(k) \mid h(x) < C(d, N, k)\}$ is not contained in $f(\mathbb{P}^N(k))$ for all f as above?

One could also ask for the constant to depend only on the degree of k over $\mathbb{Q}$.

In fact, some computer calculations for the case $k = \mathbb{Q}$, $N = 1$ and morphisms of the form $f(x) = x^2 + c, c \in \mathbb{Q}$ suggest that the following might be true.

Question 3.5. For a morphism $f : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d over k and $c \in \mathbb{R}$, let $N(f,c) = \#\{x \in \mathbb{P}^N(k) \mid h(f(x)) \leq c\ \}$ and let $N(c) = \sup_{\deg(f)=d} N(f,c)$. Let $M(c) = \#\{x \in \mathbb{P}^N(k) \mid h(x) \leq c\}$. Then is

$$\lim_{c\to\infty} \frac{log(M(c))}{log(N(c))} = d\ ?$$

A generalisation of the Morton-Silverman Boundedness Conjecture (Conjecture 2.3) is the following:

Question 3.6. For a morphism $f : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d over k, let $h_f : \mathbb{P}^N(k) \to \mathbb{R}$ denote the corresponding canonical height [17, p. 30]. For $c \in \mathbb{R}$, let $R(c) = \sup_{\deg(f)=d} \#\{x \in \mathbb{P}^N(k) \mid h_f(x) \leq c\ \}$. Then is $R(c) < \infty$ for all c? If so, then with $M(c)$ as above, is

$$\lim_{c\to\infty} \frac{log(R(c))}{log(M(c))} = 1\ ?$$

It would be desirable to understand the precise relation between the previous two questions. For example, does either of the questions imply the other?

If the previous question has a positive answer, then using the methods of Section 2 one would get a positive answer to the following:

Question 3.7. For each $c \in \mathbb{R}$, does there exist an integer $C(k,g,D,c)$ such that for any abelian variety A of dimension g over k and an ample symmetric line bundle L of degree D on A,

$$\#\{x \in A(k) \mid h_L(x) \leq c\ \} < C(k,g,D,c)\ ?$$

Here h_L denotes the canonical height on $A(k)$ associated to the line bundle L.

4.

The existence of a self map of a variety satisfying the conditions of Corollary 2.2 with $d > 1$ imposes strong restrictions on the geometry of such varieties. The main result of this section, Theorem 4.2, gives a complete classification in the case of smooth varieties of non-negative Kodaira dimension. We then state some questions concerning the general case.

Lemma 4.1. *Let X, L and ϕ be as in Corollary 2.2 with $d > 1$. Assume further that ϕ is separable and the Kodaira dimension of X is ≥ 0 i.e. $H^0(X, \omega_X^{\otimes n}) \neq 0$ for some $n > 0$, where ω_X is the canonical line bundle of X. Then ϕ is étale*[2]*, the class of ω_X is torsion in $Pic(X)$ and X does not contain any positive dimensional subvarieties with finite algebraic fundamental group.*

[2] This is probably well-known.

Proof. Let $n > 0$ be such that $H^0(X, \omega_X^{\otimes n}) \neq 0$. Since ϕ is separable, $\omega_X = \phi^*\omega_X \otimes \mathcal{O}_X(R)$ where R is the ramification divisor. Since R is an effective divisor, it follows that any section of $H^0(X, \omega_X^{\otimes n})$ is in the image of $H^0(X, \omega_X^{\otimes n} \otimes \mathcal{O}_X(-nR))$. By iterating ϕ it follows that

$$H^0(X, \omega_X^{\otimes n}) \subset \bigcap_{i \geq 0} H^0(X, \omega_X^{\otimes n} \otimes \mathcal{O}_X(-n(R + \phi^*(R) + \cdots + (\phi^{i-1})^*(R)))).$$

The right hand side is zero unless $R = 0$, so ϕ must be étale.

Let $P(L, x)$ denote the Hilbert polynomial of L. We apply the Grothendieck-Riemann-Roch theorem (see for example [6, Theorem 15.2]) to the morphism ϕ. Since ϕ is étale, $\phi^*(T_X) = T_X$, so the relative tangent bundle is trivial. Thus, for any integer x we get

$$ch(\phi_*(L^{\otimes dx})) = \phi_*(ch(L^{\otimes dx})). \tag{1}$$

Since $\phi^*(L) \cong L^{\otimes d}$, the projection formula implies that $\phi_*(L^{\otimes dx}) \cong L^{\otimes x} \otimes \phi_*(\mathcal{O}_X)$. The Hirzebruch-Riemann-Roch theorem ([6, Corollary 15.2.1]) then gives:

$$\begin{gathered} P(L, dx) = \int_X ch(L^{\otimes dx}) \cdot td(T_X) = \int_X \phi_*(ch(L^{\otimes dx}) \cdot td(T_X)) = \\ \int_X \phi_*(ch(L^{\otimes dx})) \cdot td(T_X) = \int_X ch(\phi_*(L^{\otimes dx})) \cdot td(T_X) = \\ \int_X ch(L^{\otimes x}) \cdot ch(\phi_*(\mathcal{O}_X)) \cdot td(T_X) = d^g \int_X ch(L^{\otimes x}) \cdot td(T_X) = d^g P(L, x) \end{gathered}$$

with $g = \dim(X)$. Here the third equality follows from the projection formula since $\phi^*(T_X) = T_X$, the fourth from (1), the fifth from multiplicativity of ch, the sixth from $ch(\phi_*(\mathcal{O}_X)) = deg(\phi) = d^{\dim(X)}$ in $A^*(X)_{\mathbb{Q}}$, which holds since ϕ is étale (and the first and last follow from HRR).

Since $P(L, x)$ is a polynomial of degree g, it follows that $P(L, x) = P(L, 1)\, x^g$. Examining the terms in the Hirzebruch-Riemann-Roch formula we see that $\int_X (c_1(L)^{g-1} \cdot c_1(\omega_X)$, which is the coefficient of x^{g-1} in $P(L, x)$, must be 0. Since L is ample and some multiple of ω_X has a section, this can only happen if the zero set of the section is empty. Thus ω_X is torsion.

Let $Y \subset X$ be a subvariety of dimension e, $0 < e \leq g$, with finite algebraic fundamental group, say of order f. Let $c = \int_X (c_1(L)^e \cdot Y) > 0$ and choose $m > 0$ so that $cf/d^{em} < 1$. Let Y' be an irreducible component of $(\phi^m)^{-1}(Y)$. It follows that $deg\ \phi^m|_{Y'} \leq f$. By the projection formula

$$d^{e \cdot m} \int_X (c_1(L)^e \cdot Y') = \int_X (c_1(L^{\otimes d^m})^e \cdot Y') \leq \int_X (c_1(L)^e \cdot Y)$$

Thus $\int_X (c_1(L)^e \cdot Y') \leq cf/d^{em} < 1$. This is a contradiction because L is ample so the left hand side must be a positive integer. $\square$

The classification result that we obtain is the following

Theorem 4.2. *Let X be a smooth projective variety of non-negative Kodaira dimension, L an ample line bundle on X and ϕ a self map of X such that $\phi^*(L) \cong L^{\otimes d}$ for some $d > 1$, all over an algebraically closed field k of characteristic zero*[3]. *Then X is isomorphic to a quotient A/G of an abelian variety A by a finite group G acting fixed point freely. Furthermore, all such quotients, over an algebraically closed field of arbitrary characteristic, have line bundles L and self maps ϕ as above.*

Proof. By the previous lemma we have that ω_X is torsion. By a theorem of Beauville [1, Théorme 2] it follows that there exists an abelian variety A, a simply connected smooth projective variety Z and a finite group G acting on $A \times Z$ without fixed points such that $X \cong (A \times Z)/G$. We need to show that Z is a point. Since Z is simply connected, there are no nonconstant morphisms from Z to A. It follows that if p_1 denotes the projection from $A \times Z$ to the first factor, then $p_1(g(\{a\} \times Z))$ is a point for all $g \in G$. Thus G acts on A (not necessarily faithfully) and we get a surjective morphism $\bar{p_1} : (A \times Z)/G \to A/G$ such that the diagram

$$\begin{tikzcd} A \times Z \arrow[r] \arrow[d, "p_1"'] & (A \times Z)/G \arrow[d, "\bar{p_1}"'] \\ A \arrow[r] & A/G \end{tikzcd}$$

commutes. For all $a \in A$, $\bar{p_1}^{-1}(a)$ is isomorphic to a quotient of Z by a finite group and hence has a finite fundamental group. Lemma 1 then implies that Z must be a point.

Now suppose that $X \cong A/G$, with A, G as above. Let $V = A(k)/A(k)_{tors}$; this is a vector space over $\mathbb{Q}$ and the action of G on A induces an action of G on V which factors through the semi-direct product of $GL(V)$ and V. Since G is finite, this action must have a fixed point: if $\bar{x} \in V$ is any point, the point $|G|^{-1} \sum_{g \in G} g\bar{x}$ is fixed by G. By translation, we may assume that $\bar{0} \in V$ is a fixed point of the action of G on V i.e. for all $g \in G$, $g(0) \in A(k)_{tors}$. Let n be an integer such that $n \equiv 1 \mod ord(g(0))$ for all $g \in G$. Then the multiplication by n map on A, $[n]$, commutes with the action of G, hence descends to a morphism $\phi_n : X \to X$. Let M_1 be an ample, symmetric line bundle on A. Let m be the $g.c.d.$ of all $ord(g(0))$, $g \in G$, and let $M = \bigotimes_{g \in G} g^*(M_1)^{\otimes m}$. Then $[n]^*(M) \cong M^{\otimes n^2}$ and moreover $g^*(M) = M$ for all $g \in G$. Since the kernel of the pullback map $Pic(X) \to Pic(A)$ is finite, it follows that a suitable positive tensor power of M descends to an ample line bundle L on X such that $\phi_n^*(L) \cong L^{\otimes n^2}$. $\square$

[3] It seems likely that this also holds in positive characteristic if we assume that the map ϕ is separable.

We thus have a reasonably complete description of varieties of non-negative Kodaira dimension which have self maps and line bundles as above. However, the general case seems much more difficult. As a beginning, one may ask the following:

Question 4.3. Let X be an N-dimensional smooth projective variety over an algebraically closed field of characteristic zero. Suppose $Pic(X) \cong \mathbb{Z}$, X has a a self map of degree > 1 and the anti-canonical bundle of X is ample. Then is X isomorphic to $\mathbb{P}^N$?

The question has a positive answer for $N = 1, 2$ as can be easily checked from the classification of smooth projective curves and surfaces. It follows from results of Paranjape-Srinivas [15] that a variety of the form G/P, where G is a simple linear algebraic group and P is a maximal parabolic subgroup (such varieties have $Pic(G/P) \cong \mathbb{Z}$) has a self map of degree > 1 iff $G/P \cong \mathbb{P}^N$ for some N. The same holds for smooth projective hypersurfaces H with $\dim(H) > 2$ (these also have $Pic(H) \cong \mathbb{Z}$) by results of Beauville [2].

One may ask the same question for separable morphisms of varieties over algebraically closed fields of arbitrary characteristic. If one also allows singular varieties, there are other examples. But perhaps all such, at least those which are normal, are toric varieties. (It is well known that projective toric varieties do have self maps and line bundles verifying the conditions of Corollary 2.2).

The following is a possible generalisation.

Question 4.4. Let X be a smooth projective rationally connected variety over an algebraically closed field of characteristic zero. Suppose X has a self map ϕ and an ample line bundle L such that $\phi^* L \cong L^{\otimes d}$, for some $d > 1$. Then is X a toric variety?

It is not difficult to check this (using classification) if $\dim(X) \leq 2$. A more general result has been proved by N. Nakayama [14].

One may also try to classify étale maps of varieties of arbitrary Kodaira dimension.

Question 4.5. Let X be a smooth projective variety over an algebraically closed field and $f : X \to X$ an étale map. Does there exist a finite étale cover Y of X, a self map $g : Y \to Y$, a smooth projective morphism $\pi : Y \to S$ whose fibres are abelian varieties, and an *automorphism* $h : S \to S$ such that the following diagrams commute?

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ S \arrow[r, "h"] & S \end{tikzcd}$$

Again, this can be easily checked for curves and surfaces. Note that this is also true for X of the from A/G considered above: for such X, among all étale maps $B \to X$ with B an abelian variety, there is a unique (upto isomorphism) map of minimal degree. Some general results for X of dimension 3 have been obtained by Y. Fujimoto [5].

5.

Let X be a projective variety over an algebraically closed field k, $\phi : X \to X$ a morphism and L an ample line bundle on X such that $\phi^*(L) \cong L^{\otimes d}$ for some $d > 1$. Using the Lefschetz trace formula and a result of Serre [16, Théorème 1], one can show that if X is smooth and k is of characteristic zero then ϕ has infinitely many periodic points in $X(k)$. However, we shall show below that under more general conditions the set of periodic points is always Zariski dense in X. The main ingredient is a generalisation of the Lang-Weil estimates due to E. Hrushovski [7].

Theorem 5.1. *Let X be a projective variety over an algebraically closed field k, $\phi : X \to X$ a dominant morphism and L a line bundle on X such that $\phi^* L \otimes L^{-1}$ is ample. Then the subset of $X(k)$ consisting of periodic points of ϕ is Zariski dense in X.*

Remark 5.2. From the arguments below one may extract an elementary proof of the fact that under the conditions of the theorem the set of all preperiodic points is Zariski dense—this does not need Hrushovski's theorem.

Corollary 5.3. *Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d > 1$ over an algebraically closed field k. Then the set of periodic points of f is Zariski dense in $\mathbb{P}^N$.*

Remark 5.4. Simple examples (e.g. $N = 1$, $\phi(x) = x^2$) show that if $k = \mathbb{C}$, the set of periodic points need not be dense in the analytic topology.

Theorem 5.1 will be deduced from the following result, whose proof is due to Bjorn Poonen.

Proposition 5.5. *Let X be an algebraic variety over $\mathbb{F}$, the algebraic closure of a finite field and $\phi : X \to X$ a finite surjective morphism. Then the subset of $X(\mathbb{F})$ consisting of periodic points of ϕ is Zariski dense in X.*

Proof. Let Y be the Zariski closure of the set of periodic points of ϕ in $X(\mathbb{F})$ and suppose $Y \neq X$. Let $q = p^n$, $p = char(\mathbb{F})$, be such that X as well as ϕ are defined over the subfield $\mathbb{F}_q$ of $\mathbb{F}$ consisting of q elements. Let σ denote the Frobenius morphism of X (which raises the coordinates of a point $x \in X$ to their q'th powers) and let Γ_ϕ (resp. Γ_m) denote the graph of ϕ (resp. σ^m) in $X \times X$. Let U be an irreducible affine open subset of $X - Y$ also defined over $\mathbb{F}_q$ and let $V = \Gamma_\phi \cap (U \times U)$. By a theorem of Hrushovski [7] (see [4, Conjecture 2][4] for the statement) there exists $m > 0$ such that

[4]Z. Chatzidakis has informed the author that in the statement of [4, Conjecture 2], hence in Hrushovski's theorem, the projections only need to be generically onto, not necessarily onto.

$(V \cap \Gamma_m)(\mathbb{F}) \neq \emptyset$ i.e. there exists $u \in U(\mathbb{F})$ such that $\phi(u) = \sigma^m(u)$. Since ϕ is defined over $\mathbb{F}_q$, it follows that u is a periodic point of ϕ. This contradicts the definition of Y and U, so the proof is complete.

□

Lemma 5.6. *Let X be a variety over a field k and $\phi : X \to X$ a dominant projective morphism. Then ϕ is a finite morphism.*

Proof. We may assume that k is algebraically closed.

Let $H_*(X)$ denote the étale homology of X with cofficients in $\mathbb{Q}_l$, l a prime with $(char\ k, l) = 1$ (i.e. the linear dual of $H^*_{et}(X, \mathbb{Q}_l)$) and let $P_*(X)$ denote the subspace of $H_*(X)$ generated by classes of complete subvarieties of X. ϕ induces a map $\phi_* : H_*(X) \to H_*(X)$ which preserves $P_*(X)$ i.e. $\phi_*(P_*(X)) \subset P_*(X)$. Suppose ϕ is not finite, so there is an irreducible positive dimensional subvariety Y contained in a fibre of ϕ. The class of Y in $P_*(X)$, $[Y]$, is non-zero because ϕ is projective: $c_1(L)^{\dim(Y)} \cap [Y] \in H_0(X)$ is non-zero for any relatively ample line bundle L on X. However, $\phi_*([Y]) = 0$ because $\dim(Y) > 0$ and $\dim(\phi(Y)) = 0$. Moreover, $\phi_* : P_*(X) \to P_*(X)$ is surjective since ϕ is dominant and proper, so any complete irreducible subvariety Z of X is of the form $\phi(Z')$ for a complete subvariety of X with $\dim(Z') = \dim(Z)$. This is a contradiction because $H_*(X)$, hence $P_*(X)$, is a finite dimensional $\mathbb{Q}_l$ vector space. □

Proof of Theorem 5.1. Since X is projective and ϕ is dominant, it follows from Lemma 5.6 that ϕ is a finite morphism. By Proposition 5.5, the theorem holds if k is the algebraic closure of a finite field. For a general k, X, ϕ and L will be defined over a finitely generated subfield k' of k, so we may assume that the transcendence degree of k over its prime field is finite. It is easy to see that we may also assume that X is irreducible.

Now consider the following situation: $\mathbf{X}$ is a projective scheme, flat over a d.v.r. R with algebraically closed residue field K, Φ is a dominant selfmap of $\mathbf{X}$ over R and $\mathbf{L}$ a line bundle on $\mathbf{X}$ such that $\Phi^*\mathbf{L} \otimes \mathbf{L}^{-1}$ is relatively ample. Let X be the special fibre of $\mathbf{X}$ and ϕ the restriction of Φ to X. The set of periodic points of ϕ of period dividing a positive integer n can be viewed as the set of points in $(\Delta_X \cap \Gamma_{\phi^n})(K)$, where Δ_X is the diagonal and Γ_{ϕ^n} is the graph of ϕ^n in $X \times X$. The hypothesis on the line bundle $\mathbf{L}$ implies that this set is always finite: The line bundle $p_1^*L^{-1} \otimes p_2^*L$ on $X \times X$, where $L = \mathbf{L}|_X$, restricts to a line bundle on $\Delta_X \cap \Gamma_{\phi^n}$ which is both ample and trivial.

Suppose that X is reduced and the set of periodic points of ϕ is Zariski dense in X, so the set of periodic points of ϕ which are smooth points on X is also Zariski dense in X. For a positive integer n, consider the subscheme $\Delta_{\mathbf{X}} \cap \Gamma_{\Phi^n}$ of $X \times_R X$, where $\Delta_{\mathbf{X}}$ is the diagonal and Γ_{Φ^n} is the graph of Φ in $X \times_R X$. If $x \in X$ is a periodic point of ϕ which is also a smooth point on X, it follows from [18, Theorem 3, p.110] that (x, x) is contained in a closed subscheme of $\Delta_{\mathbf{X}} \cap \Gamma_{\Phi^n}$ of dimension at least 1. Since $(\Delta_X \cap \Gamma_{\phi^n})(K)$

is finite, it follows that this subscheme must intersect the generic fibre of $X \times_R X$. So x can be lifted to a periodic point of Φ on any geometric generic fibre of $X_{/R}$. It follows that the set of periodic points of Φ on any geometric generic fibre are also Zariski dense.

The proof is completed by induction on the transcendence degree of k, using models over discrete valuation rings as above to increase the transcendence degree by one at each step. For k of characteristic zero, we also need to use one such model to go from the algebraic closure of a finite field to $\overline{\mathbb{Q}}$. □

Acknowledgements. The results of Section 2 were motivated by a conversation with Bjorn Poonen who observed that the Morton-Silverman conjecture for degree 4 maps of $\mathbb{P}^1$ implies the uniform boundedness of torsion for elliptic curves, and asked if that could be generalised. I also thank him for informing me of Hrushovski's theorem [7], its application to the proof of Proposition 5.5, and for his permission to include it here. This note has benefited from his numerous comments on a preliminary version, in particular the first part of the proof of Proposition 3.3 is a simpler variant, due to him, of my original proof. Finally, I thank Zoé Chatzidakis for some clarifications concerning [4, Conjecture 2].

References

[1] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom., 18 (1983), pp. 755–782 (1984).

[2] ———, *Endomorphisms of hypersurfaces and other manifolds*, Internat. Math. Res. Notices, (2001), pp. 53–58.

[3] L. Caporaso, J. Harris, and B. Mazur, *Uniformity of rational points*, J. Amer. Math. Soc., 10 (1997), pp. 1–35.

[4] Z. Chatzidakis and E. Hrushovski, *Model theory of difference fields*, Trans. Amer. Math. Soc., 351 (1999), pp. 2997–3071.

[5] Y. Fujimoto, *Endomorphisms of smooth projective 3-folds with non-negative Kodaira dimension*, Publ. Res. Inst. Math. Sci., 38 (2002), pp. 33–92.

[6] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, second ed., 1998.

[7] E. Hrushovski, *The first order theory of the Frobenius automorphisms, Preprint.*

[8] G. R. Kempf, *Projective coordinate rings of abelian varieties*, in Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 225–235.

[9] L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque, (1985), p. 266.

[10] P. Morton and J. H. Silverman, *Rational periodic points of rational functions*, Internat. Math. Res. Notices, (1994), pp. 97–110.

[11] D. Mumford, *Abelian varieties*, Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[12] ———, *Varieties defined by quadratic equations*, in Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), Edizioni Cremonese, Rome, 1970, pp. 29–100.

[13] D. Mumford and J. Fogarty, *Geometric invariant theory*, Springer-Verlag, Berlin, second ed., 1982.

[14] N. Nakayama, *Ruled surfaces with non-trivial surjective endomorphisms, RIMS Preprint, 1286*, 2000.

[15] K. H. Paranjape and V. Srinivas, *Self-maps of homogeneous spaces*, Invent. Math., 98 (1989), pp. 425–444.

[16] J.-P. Serre, *Analogues kählériens de certaines conjectures de Weil*, Ann. of Math. (2), 71 (1960), pp. 392–394.

[17] ———, *Lectures on the Mordell-Weil theorem*, Friedr. Vieweg & Sohn, Braunschweig, 1989. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt.

[18] ———, *Local algebra*, Springer-Verlag, Berlin, 2000. Translated from the French by CheeWhye Chin and revised by the author.

[19] J. G. Zarhin, *Endomorphisms of abelian varieties and points of finite order in characteristic P*, Mat. Zametki, 21 (1977), pp. 737–744.

[20] S. Zhang, *Small points and adelic metrics*, J. Algebraic Geom., 4 (1995), pp. 281–300.

School of Mathematics, Tata Institue of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: `naf@math.tifr.res.in`