Finiteness theorems for étale cohomology of excellent schemes

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The goal is to prove

<u>Theorem 0.1</u> Let $f: X \to Y$ be a morphism of finite type between quasi-excellent noetherian $\mathbb{Z}[\frac{1}{n}]$ schemes and F a constructible sheaf of $\mathbb{Z}/n$-modules on X_{et}. Then the $R^i f_* F$ are constructible and 0 for $i \gg 0$.

We also prove other expected properties (affine cohomological dimension, existence of dualizing complexes). We use a weak form of resolution of singularities.

Epp's theorem
Popescu's theorem
Absolute cohomological purity

We give a proof of ACP without using AKTEC (Thomason)

In SGA4 XIX Artin proves Th. 0.1 for equichar. excellent schemes assuming resolution. One idea (partial algebraization) comes from the proof of affine coh. dim. in loc. cit.

1. Weak local uniformization

1.1 Weak version

Let X be a quasi excellent scheme, $Z \subset X$ nowhere dense closed subset. Then there are

$$\begin{array}{lccc} \text{regular} & X_i & \xrightarrow{\text{gen. finite}} & X \\ & \cup & \square & \cup \\ \text{N.C.D.} & Z_i & \longrightarrow & Z \end{array}$$

covering family for the h-topology.

For a scheme S, the h-topology on the category of schemes locally of finite pres. over S is the topology generated by proper surjective maps and Zariski open coverings.

1.2 Suppose X is integral (qc, sep) $K = R(X)$

$$ZRS(X) = \{\text{valuation rings of } K \text{ dominating some } \mathcal{O}_{X,x}\} = \varprojlim_{X' \to X \text{ proper birational}} X'$$

$$ZRS_{\bar{K}}(X) = \varprojlim X'$$

X' integral, projective gen. finite over X

$\underbrace{R(X) \subset R(X')}_{\text{normal}} \subset \bar{K}$, Galois group acts on X'.

Weak version $\Leftrightarrow$ every $v \in ZRS_{\bar{k}}(X)$ dominates a good local model

$\Leftrightarrow \exists$ proj. model X' (equiv.) and an open cover U_i of X' s.t. $U_i \to X$ factorizes through a good local model.

1.3 For $v \in ZRS_{\bar{k}}(X)$ define the inertia and decomposition group. These are $\varprojlim$ of inertia and dec. groups for p.e. models. The latter are upper semi continuous for the Zariski resp. constructible topology.

Strong form of Theorem: $\forall\ V \in ZRS_{\bar{k}}$ l invertible on X S an l-sylow of $D =$ dec. gp of V, V^S dominates a good local model.

$\Leftrightarrow \exists$ p.e. X', $U_i \subset X'$ open, $H_i \subset Gal(X'/X)$ acts on U_i, s.t. U_i/H_i dominates a good local model and

$\forall x \in X' \ \exists i \ (x \in U_i$ and $H_i \supset \ell$-Sylow of $D(x)$).

$\Longleftrightarrow$ there is the following

$$\begin{tikzcd}
 & \Omega'_i \arrow[dl, "\text{finite flat}"', "(\deg, \ell) = 1"] \arrow[rr] & & Y_i \arrow[dd, "f_i"] \\
\Omega_i \arrow[dr, "\text{Nisnevich cover}"'] & & & \\
 & X' \arrow[rr, "\text{proper birational}"'] & & X
\end{tikzcd}$$

regular with $f_i^{-1}(Z)$ N.C.D.

Similarly when the condition is restricted to valuations center at $x \in X$.

Thm for $(X, x) \Longleftrightarrow$ Thm for (X^h, x)

$\Longleftrightarrow$ Thm for $(X^\wedge, x)$

X^h = Spec of henselization of local ring

$X^\wedge$ = " completion "

2. Approximation.

Let R be an excellent henselian local ring. Given an algebro geometric datum of finite presentation over $\hat{R}$ it comes from a datum over a f.g. subalgebra R_1 of $\hat{R}$. By Popescu's thm ($\hat{R} = \varinjlim$ smooth R algebras) $R_1 \longrightarrow \hat{R}$ can be approximated by $R_1 \longrightarrow R$. Want to preserve properties of schemes and morphisms.

2.1 Let $I \subset A$ be an ideal, M, M' A-modules. An (I, c) isomorphism $M \underset{(I,c)}{\simeq} M'$ is an isomorphism

$$\bigoplus_{n \in \mathbb{Z}} I^n M / I^{n+c} M \longrightarrow \bigoplus_{n \in \mathbb{Z}} I^n M' / I^{n+c} M'$$

over $\bigoplus_{n \in \mathbb{Z}} I^n$.

Similarly for algebras.

2.2 (Artin-Rees) A noetherian, $M' \xrightarrow{f} M \xrightarrow{g} M''$ (*) an exact sequence of f.g. A-modules, I an ideal. Then for $n \gg 0$ if $M_1' \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_1''$ is (I, n)-isomorphic to (*) & $g_1 f_1 = 0$ then it is exact and all kernels, cokernels and images are $(I, n-c)$-isomorphic to those of (*).

Given M with a resolution $F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \to M \to 0$ by f.g. free-modules, if we approximate the matrix entries of ∂ s.t. $\partial\partial = 0$ holds we get an approximation of M. Conversely, approx. of M $\Rightarrow$ approx. of truncated resolutions.

2.3. A noetherian excellent $B = A/J$, $B' = A/J'$ is (I, n) close to B. Then B reduced near $V(I)$ $\Rightarrow$ B' reduced near $V(I)$. Same for normal.

The assertion for "reduced" is shown by blowing up I. When $I = (f)$ use

Lemma. B reduced Japanese, $h \in B$ non zero divisor $\exists n \ (x^2 \in h^n B \Rightarrow x \in hB)$.

For the above type of approximation, $\forall x \in \mathrm{Spec}(A/I+J)$ $\dim B_x = \dim B'_x$, B_x regular $\Leftrightarrow B'_x$ regular.

3. Log regular schemes and quotients.

A locally noetherian fs log scheme X is log regular iff $\forall x \in X$ the completion (of the strict hensel.) is isomorphic to $k[[x_1, \ldots, x_n]][[M]]$ (M a sharp toric monoid) in the equal-characteristic case, to $I[[x_1, \ldots, x_n]][[M]]/(f)$ ($f \mapsto p \in I$, I cohen ring) in the mixed characteristic case.

A canonical desingularization procedure for excellent schemes of char. 0 gives a canonical desingularization for toric varieties over $\mathbb{Z}$ and for log regular schemes.

A log regular scheme is normal and the log structure is determined by the locus of triviality of the log structure

$j : U \hookrightarrow X$ (dense) $\quad M_X = \mathcal{O}_X \cap j_* \mathcal{O}_U^*$.

(étale top)

We say $(X, X-U)$ is log regular.

3.1 Let X be a separated scheme equipped with an action of a finite group G. We say the action is tame iff $\forall x \in X$ the order of the inertia group G_x is invertible in $k(x)$.

3.2 Lemma. Let G be a finite group acting on a noetherian ring A with $|G|^{-1} \in A$. Then A^G is noetherian and A is finite over A^G

3.3. Let G be a finite group acting generically freely on a sep. log regular scheme (X, Z). We say the action is very tame at $x \in X$ iff G_x is of order prime to the char. exp. $k(x)$, G_x acts trivially on $\bar{M}_x$ and G_x acts trivially on the

connected stratum containing x.
[Connected component of $\{x \in X \mid \mathrm{rk}\, \overline{M}_{\bar{x}} = i\}$]
Then G_x is abelian, $\exists$ structure thm for completion, condition is open.

If action is very tame $(X/G, Z/G)$ is log regular and action is free on $X - Z$.
$X - Z \longrightarrow (X-Z)/G$ is a G torsor tamely ramified at maximal points of Z/G.

Conversely, if (X', Z') is log regular and $V \longrightarrow X' - Z'$ is a G-torsor tamely ramified in cod. 1 then by FK purity it extends to a finite Kummer log étale map $X \longrightarrow X'$, X the normalization of X' in V and G acts very tamely on (X, Z).

3.4 Thm. Let G act generically freely and tamely on a $\underbrace{\text{log regular}}_{\text{sep, qc}}$ (X, Z).
Then there is a projective birational map $X' \xrightarrow[p]{} X$, s.t. if $Z' = p^{-1}(Z \cup \text{locus of non free action})$

Then (X', Z') is log regular, G acts very tamely on X'.

<u>sketch of proof</u>. Use canonical desing. of X.

WMA X regular, Z N.C.D.

To ensure G_x act trivially on $\bar{M}_x$ need to blow up k-uple intersections of components of Z $k \geq 2$. (Blow up N-uple intersections, then proper transform of $(N-1)$-uple intersections etc.)

(étale locally WMA simple NCD)

In a similar way blow up along X^H $\forall H \neq \{1\}$ and increase Z. This leads to a situation with abelian inertia groups and free action on $X - Z$. Étale locally can increase Z s.t. action is very tame. This depends on choosing eigenfunctions and is not unique. This gives a log regular structure on X/G and the canonical desingularization of X/G is shown independent of local choices by lifting to char. 0, $Y \xrightarrow{p} X/G$. Show $(Y, p^{-1}(Z/G))$ is log regular. Normalize Y in $X - Z$.

3.5. If in 3.4 (X, Z) is log smooth over a base S with a trivial G action then (X', Z') and its qt by G are log smooth over S.

3.6. Let (X, Z) be log regular $X' \xrightarrow{f} X$ a nodal curve smooth over $X - Z$, $D \subset X'$ divisor in smooth locus étale over X. Then $(X', D \cup f^{-1}(Z))$ is log regular

4. Absolute cohomological purity.

Recall (Azumino) That for a regular immersion $Z \subset X$ of cod. c have a global fundamental class in $H^{2c}_Z(X_{et}, \Lambda(c))$ $\Lambda = \mathbb{Z}/n$, n invertible on X.

All schemes below are assumed to have an ample line bundle.

$f: X \to Y$ relative complete intersection $\Leftrightarrow$ factorizable $X \underset{\text{reg. imm.}}{\hookrightarrow} M \xrightarrow[\text{lisse}]{} Y$ $\quad$ $\mathrm{cod}(f) = \mathrm{cod}_M X - \dim(M/Y)$.

Get a gysin map $\Lambda_X \to f^! \Lambda_Y (c) [2c]$
$c = \mathrm{cod}(f)$

satisfying transitivity.

$\mathrm{Tr} : Rf_! \Lambda_X \to \Lambda_Y (c) [2c]$.

For f flat coincides with SGA 4 XVIII 2.9.

4.1. Suppose

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

rel c.i., $\mathrm{cod}(f) = 0$

Gysin for f gives $f^* F \to f^! F \quad \forall F \in D^+$.

Get $f_! f^* F \to F$. For f proper

$F \to f_* f^* F \to F$. This is shown to be multiplication by $\deg(f) = \chi_{\eta}(Rf_* \mathcal{O}_X)$.

Let $K_X = p^! \Lambda_S (c) [2c] \quad c = \mathrm{cod}(p)$, $K_Y = \ldots$

$$f^*(\Lambda_Y \to K_Y) \to (\Lambda_X \to K_X) \to f^!(\Lambda_Y \to K_Y).$$

commutes.

For f finite get $(\Lambda_Y \to K_Y) \rightleftarrows f_*(\Lambda_X \to K_X)$

$\circlearrowleft$ = multip by $\deg(f)$ on 2 terms.

Hence if f is finite of constant generic degree prime to n and $\Lambda_X \xrightarrow{\sim} K_X$ then $\Lambda_Y \xrightarrow{\sim} K_Y$.

The problem of ACP in the mixed characteristic case is reduced to the case of schemes of finite type over a trait S (complete) and for such X, X is punctually pure iff $\Lambda_X \xrightarrow{\sim} K_X$.

By de Jong $\exists$ Galois alteration

$$\begin{tikzcd} G \circlearrowright X' \arrow[r] \arrow[d] & X \arrow[d] \\ G \circlearrowright T \arrow[r] & S \end{tikzcd}$$

$R(X')$ normal $/R(X)$, $Gal = G$.

s.t. X' has a G-invariant log regular structure, log lisse over T with canonical log structure. (By Vidal we can even take $X' \to T$ projective regular with semistable reduction.)

$n = \ell^i$, $S_\ell \subset G$ ℓ-sylow.

Can modify X' s.t. S_ℓ action is very tame and $Y = X'/S_\ell$ regular, and log lisse over T/S_ℓ.

Then X'/S_ℓ is étale locally of the form

$\mathrm{Spec}\ \mathcal{O}_{T/S_\ell}[x_1, \ldots, x_N]/(\prod x_i^{e_i} - \pi)$ ← unif. of T/S_ℓ

Such schemes are p.p. by [Rap. Zink], [Ill]

NB. punctual purity for such schemes is equivalent to $R^* j_* \Lambda = \Lambda^* R^1 j_* \Lambda$, j inclusion of general fiber. This is reduced to the case all $e_i = 1$.

Since $Y \longrightarrow X$ is of generic degree prime to ℓ, Y p.p $\Rightarrow$ X p.p.

5. Applying Gpp.

Th. 5. 1. Let R be a complete normal n.l.r. (of dim. ≥ 2), I_i ($i \in$ finite set) ideals of R.

Then $\exists$ $R' \supset R$ finite extension s.t. R' is normal and is the completion of a ring (R'') ess. of finite type over a regular complete n.l.r. of dimension $\dim(R) - 1$, and $I_i R'$ come from R''.

In the equal characteristic case can take $R' = R$ and holds for $\dim R = 1$.

Th. 5. 2. Let $T \longrightarrow R$ be an extension of complete DVR's. When the residue char. is $p > 0$ assume that the residue field k_T is perfect and that the maximal perfect subfield of k_R is algebraic over k_T. Then $\exists$ finite extn $T \subset T'$ s.t. $(T' \hat{\otimes}_T R)^\nu_{red}$ has reduced special fibre over T'.

(Trivial in char. 0)

5.3. Consider the situation of Th. 5.1 in the case of mixed characteristic $(0, p)$. Let k_0 be the maximal perfect subfield of k_R.

$W = W(k_0) \longrightarrow R$.

$\exists\ W \to W'$ finite, $R' = (R \hat{\otimes} W')^\nu$ reduced fibre over max. ideal of W'. For every connected component of R' choose a coefficient field k of the special fibre $\bar{R}'$ over which $\bar{R}'$ is analytically separable: $\exists\ k[[t_1, \ldots, t_{d-1}]] \longrightarrow \bar{R}'$ gen. étale, finite

~~étale at generic point of special fibre.~~

~~Changing coordinates, WMA étale outside $V(p)$~~

~~for $W'[[t_1, \ldots, t_{d-2}]][t_{d-1}]$ monic~~

Extend k' to a Cohen ring I mapping to R'. $I[[t_1, \quad , t_{d-1}]] \longrightarrow R'$ finite, changing coordinates WMA it is étale at pt $(\pi, t_1, \quad , t_{d-2})$, hence étale outside $V(f)$ for $f \in I[[t_1, \quad , t_{d-2}]][t_{d-1}]$ monic. Using Elkik's approximation descend to $I[[t_1, \ldots, t_{d-2}]]\{t_{d-1}\}$ (henselian power series).

5.4. Prove weak form ~~of~~ (1.1) by induction on the dimension.

6. First proof of Th. 0.1.

This proof gives that each $R^i f_* F$ is constructible. For schemes of finite Krull dimension have bounds (cd$_\ell$ open in Spec(R), R str. local $\leq 2\dim(R) - 1$), [Hu].

6.1. Let $X_. \xrightarrow[\varepsilon]{} X$ be an h-hypercover.
Then $F \xrightarrow{\sim} R\varepsilon_* \varepsilon^* F \quad \forall\ F \in D^+(X, \mathbb{Z}/n)$.

If
$$\begin{tikzcd} X_. \arrow[r] \arrow[d, "f_."'] & X \arrow[d, "f"] \\ Y_. \arrow[r, "\varepsilon"] & Y \end{tikzcd}$$
cartesian, rows h-hypercov.

$$Rf_* F = R\varepsilon_* (Rf_{.*} F).$$

Analyzing the proof of this get also that this holds for pull-backs by an arbitrary $T \to Y$.

Reduce 0.1 to open immersions f, can choose hypercover s.t. $\forall\ i \leq N$ Y_i is regular and $X_i \subset Y_i$ is for every connected component empty or complement of N.C.D.

Use generic constructibility (SGA $4\frac{1}{2}$).

7. Second proof of Th. 0.1.

(P_c) For open immersion $U \overset{j}{\hookrightarrow} X$, $F \in D_c^b(U)$,

$\exists$ $T \subset X$ of Cod. $> c$ s.t. $Rj_* F|_{X-T} \in D_c^b$.

prove this inductively.

$c=0$ trivial

$c=1$ reduce to X normal and constant coefficients. Use ACP.

$\forall c$ (P_c) $\Rightarrow$ Theorem easy

Assume (P_{c-1}). Let T_α be an irr. comp. of cod. c of T. We allow to restrict to neighborhoods of the generic points of T.

Consider the diagram of page 3. Let $T' \subset X'$ be the locus coming from (P_{c-1}) for X'. We may assume $T' = \coprod T'_{\alpha\beta}$ $T'_{\alpha\beta}$ irreducible finite over T_α. $\forall \beta$ $\exists i$ $\Omega_i \longrightarrow X'$ is an isomorphism over $T'_{\alpha\beta}$.

Given $\quad Z \subset Y \xrightarrow{p} X$

$$p^{-1}(U) \xrightarrow[j_Y]{} Y$$

Denote

$$R\varphi_*\left((Rj_{Y*}F)|_Z\right) = \varphi(Z, Y)$$

$$\begin{tikzcd}
T''_\beta \subset \Omega'_\beta \arrow[d] \arrow[rr] & & Y_\beta \supset T'''_\beta \arrow[dd] \\
T'_\beta \subset \Omega_\beta \arrow[dr] & & \\
& X' \supset T'_\beta \arrow[r] & X \supset T
\end{tikzcd}$$

Using inductive assumption

$$\varphi(T, X) \longrightarrow \bigoplus_\beta \varphi(T'_\beta, X')$$

has Cone in D^b_c.

$$\varphi(T'_\beta, X') \xrightarrow{\sim} \varphi(T'_\beta, \Omega_\beta) \longrightarrow \varphi(T''_\beta, \Omega'_\beta)$$

has left inverse

$\varphi(T, X)$.

$\varphi(T, X) \longrightarrow \varphi(T''_\beta, \Omega'_\beta)$ factorizes through an object in D^b_c since on Y_β have good open immersion (N.C.D)

8. Affine morphisms.

$f : X \longrightarrow Y$ affine, f.t., assume Y universally catenarian and $\exists$ dimension function δ on Y ($\exists$ etale locally on a quasi excellent scheme), i.e $\delta(y') = \delta(y)+1$ if y is an immediate specialization of y'.

Want to prove $\delta(R^i f_* F) \leq \delta(F) - i$

(cf. SGA4 XIX).

Reduce to

<u>Th. 8.1</u> $Y = \mathrm{Spec}(R)$ strictly local excellent $\dim = d$, $f \in R$, then $H^i(\underbrace{\mathrm{Spec}\, R[f^{-1}]}_{U}, \mathbb{Z}/n) = 0$ $\forall\, i > d$.

Let $Y_\cdot \longrightarrow Y$ be a truncated h-hypercover as in first proof of finiteness.

$U_\cdot \longrightarrow U$ $\quad U_\cdot \subset \overline{U}_\cdot$ have log structure

$H^i(U) = H^i($ closed fibre of $\overline{U}_\cdot$ with Kummer étale topology $)$

By formally smooth base change w.m.a. R complete, quotient of $I[[X_1, \ldots, X_m]]$.

Approximate to a quotient of $I\{X_1, \ldots, X_m\}$.

9. Top ~~[illegible]~~ cohomology of the punctured spectrum

9.1 Transition maps.

Let $\bar{y} \to \bar{x}$ be an immediate specialization of geometric points of a quasi excellent scheme X.

So $\bar{y} \longrightarrow X(\bar{x})$ is centered at y corresponding to a 1-dimensional $C \subset X(\bar{x})$. C^{ν} is a trait.

$F \in D^{+}(X, \mathbb{Z}/n)$.

$$H^{i}_{\bar{y}}(F) \longrightarrow H^{1}(\mathrm{Gal}(\bar{y}/y), H^{i}_{\bar{y}} F)(1) \longrightarrow H^{i+1}_{y} F(1) \qquad (\text{sign!})$$

$$\longrightarrow H^{i+1}(X(\bar{x}) - \{\bar{x}\}, F(1)) \longrightarrow H^{i+2}_{\bar{y}}(F)(1)$$

divided by the degree of the (inseparable) residue field extension for $C^{\nu} \longrightarrow C$.

9.2. Thm. For every strictly local normal excellent X $d = \dim(X)$, n invertible on X have an isomorphism

$$H^{2d}_{\xi}(X, \mathbb{Z}/n(d)) \longrightarrow \mathbb{Z}/n$$

compatible with transition maps

compatible with ~~[illegible]~~ trace for finite $X' \longrightarrow X$ (up to residue extension).

For $\dim(X) = 2$ let $X' \xrightarrow[p]{} X$ be a desingularization $p^{-1}(\xi)_{\mathrm{red}}$ (simple) NCD $\bigcup_i D_i$

$j : U = X - \{\xi\} \hookrightarrow X'$.

$$d_2 : H^0(X'_\xi, R^2 j_* \mathbb{Z}/n) \longrightarrow H^2(X'_\xi, R^1 j_* \mathbb{Z}/n)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \bigoplus \Lambda_{D_i}(-1)$$

can compute d_2,

$\operatorname{Coker}(d_2) = H^3(U)$.

For $\dim(X) > 2$ let X' be the normalization of $bl_q(X)$, q an m-primary ideal.

$$E_2^{pq} = H^p(X'_\xi, R^q j_* \Lambda)$$

concentrated in $0 \le p \le 2d-2$, $0 \le q \le d$

$p \le 2(d-q)$ using §8.

Can avoid using this by showing that the limit of E_2^{pq} over all X' vanishes outside this range.

$$H^{2d-4}(X'_\xi, R^2 j_* \Lambda) \to H^{2d-2}(X'_\xi, R^1 j_* \Lambda) \to H^{2d-1}(U, \Lambda) \to 0.$$

Using the information from the 2 dimensional case show the image of d_2 identifies the contributions of the irreducible components of $X'_{\bar{s}}$.

Check compatibility with transition map for curves $C \subset X$ whose proper transform is regular and meets $(X'_{\bar{s}})_{red} = E$ transversally at a regular point.

For X regular every C becomes transversal on some X'.

10. Dualizing complexes.

10.1 Given any specialization $\bar{y} \longrightarrow \bar{x}$ define $H^i_{\bar{y}}(F) \longrightarrow H^{i+2c}_{\bar{x}}(F(c))$ by decomposing to immediate specializations.

It is independent of the choice. (scheme q. exc.)

Assume X universally catenarian with a dimension function δ.

Definition 10.2. A candidate dualizing (c.d.) complex on X is $K \in D^+(X, \Lambda)$ equipped with $R\Gamma_{\bar{x}}(K) \simeq \Lambda(\delta(x))[2\,\delta(x)]$ compatible with transition maps.

10.3. If $Y \xrightarrow{f} X$ is of finite type and K is c.d. on X then $f^! K$ is c.d. on Y w.r.t. $\delta(y) = \delta(f(y)) + \text{tr.deg.}(y/f(y))$.

~~[illegible]~~

10.4. If $Y \xrightarrow{f} X$ is flat ~~[illegible]~~ with geometrically regular fibres and K is c.d. on X, then $f^* K$ is c.d. on Y w.r.t.

$$\delta(y) = \delta(f(y)) - \dim\left(\mathcal{O}_{f^{-1}(f(y)),\, y}\right).$$

<u>Th. 10.5.</u> A c.d. complex exists and is unique up to a unique isomorphism.

$\Lambda_X \xrightarrow{\sim} \tau_{\leq 0} R\underline{Hom}(K, K)$.

Remark. K is (-2δ) - perverse for X normal irreducible with generic point $j : \eta \to X$, $\delta(\eta) = 0$, $K = \tau_{\leq \varphi} Rj_* \Lambda$

$\varphi(x) = \max(0, 2\dim(\mathcal{O}_{X,x}) - 2)$.

In the proof of 10.5 we may assume it is known for schemes finite over proper closed subschemes of X.

$$\begin{array}{ccc} Y' & \overset{i'}{\subset} & X' = X^{\nu}_{red} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \overset{i}{\subset} & X \end{array}$$

$X - Y$ ~~open~~ = normal locus of X_{red}.

Have $K_{Y'} = p'^! K_Y = i'^! K_{X'}$.

$$p'_* K_{Y'} \longrightarrow K_Y \oplus p_* K_{X'}$$

show Cone of this is c.d. for X.

10.6 $K \in D^b_{ctf}(X)$, compatible with change of Λ, $D_K : D^b_c \to D^b_c$. For a constructible sheaf F show $F \xrightarrow{\sim} \tau_{\leq 0} D_K D_K F$ by reducing to the case of constant sheaves.

10.7 show $\mathcal{H}^r(\mathrm{Cone}(F \to D_K D_K F)) = 0$

by induction on $(\dim X, r)$.

The proof of [Th. Finitude] extends to

(biduality for excellent schemes of dim $\leq d$)

$\Rightarrow$ (biduality for schemes of finite type over excellent schemes of dim $\leq d$).

Enough to embedd F in a sheaf for which biduality is known. Use 5.1.