A finiteness theorem for non abelian H^1 of excellent schemes

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1. Statements

Theorem 1.0 Let $f: X \to Y$ be a morphism of finite type of noetherian schemes and F a constructible sheaf of sets on X_{et}.
Then $f_* F$ is constructible.

The proof is by reducing to the case f proper (SGA 4) and f an open immersion (easier when normalizations are finite)

Theorem 1.1 Let $\mathbb{L}$ be a finite set of primes, $f: X \to Y$ a morphism of finite type between quasi-excellent schemes on which every $p \in \mathbb{L}$ is invertible. Let F be a constructible $\mathbb{L}$-torsion sheaf of groups on X.
Then $R^1 f_* F$ is constructible.

Also stack version.
Proved here using ultraproducts.

Theorem 1.2. $X, Y, \mathbb{L}, f$ as in 1.1. Y qc. If F is a constructible $\mathbb{L}$-torsion abelian sheaf on X, then the sheaves $R^i f_* F$ are constructible and 0 for $i \gg 0$.

(Planned for Deligne's conference. Using alterations)

2. Reductions.

Th. 1.1 known for proper f, so enough to consider open immersions. Reduce to Y normal affine, $X = Y - Z$ (Z reduced) and F constant.

Enough to find $Y' \to Y$, normalization of Y in a finite extension of $R(Y)$ which kills $R^1 f_* F$.

This exists outside $\mathrm{Sing}(Y) \cup \mathrm{Sing}(Z)$ (Abh. Lemma). Reduce to

Th. 2.1. Let Y be a normal excellent scheme, Z a closed subscheme of cod ≥ 2, $j : Y - Z \to Y$, G a finite group. Then $R^1 j_* G$ is constructible.

(The order of G not necessarily inv. on Y)

This will be reduced to the following case:

Th. 2.2. Let A be a strictly henselian excellent normal local ring of dimension 2. For every finite group G

$$H^1(\mathrm{Spec}(A) - \{m\}, G) \quad \text{is finite.}$$

3. Lefschetz

Recall (SGA 2 XIII §2)

Th. 3.0. A cnlr $f \in m$, f nonzerodivisor, $X = \mathrm{Spec}(A)$ $X' = X - \{m\}$, $\mathrm{depth}\, \mathcal{O}_{X',x} \geq 2$ for closed points of X'. Then

$$\Gamma(X', \mathcal{O}) \xrightarrow{\sim} \Gamma(\hat{X}', \mathcal{O}) \quad \text{so}$$

$$\pi_0(X' \cap V(f)) \xrightarrow{\sim} \pi_0(X').$$

Cor. 1. A excellent normal nlr of dim ≥ 3 $f \in A$ nonzero nonunit, then the punctured spectrum of $\underset{= Y}{\mathrm{Spec}(A/fA)}$ is connected.

Y is connected in cod. 1, i.e. if $Z \subset Y$ is of cod ≥ 2 then $Y - Z$ is connected.

<u>Cor. 2.</u> Let X be normal excellent scheme

$D \subset X$ effective Cartier divisor

$Z \subset D$ closed, $\operatorname{cod}_X Z \geq 3$

$$\begin{tikzcd} D - Z \arrow[r, "j'"] \arrow[d] & D \arrow[d] \\ X - Z \arrow[r, "j"] & X \end{tikzcd}$$

If L is a locally constant constr sheaf on $X - Z$

$$(j_* L)|_D \xrightarrow{\sim} j'_*(L|_{D-Z}).$$

$(R^1 j_* G)|_D$ injects into $R^1 j'_* G$

for every finite group G.

4. Use of desingularization of 2 dimensional schemes.

<u>Lemma 4,1</u>. In the situation of Th. 2.1 if Z_i is an irreducible component of Z of codim 2 in Y then there is an open dense $U_i \subset Z_i$ s.t. the sheaf of pointed sets $F = R^1 j_* G$ on U_i has specialization maps $F_s \longrightarrow F_t$ (for $t \to s$ map of geom. pts of U_i) $\overset{\shortparallel}{\text{topos pts}}$ with trivial kernel.

Proof. W.M.A. $Z = Z_i$ and that there is $p : Y' \longrightarrow Y$ proper, birational, isomorphism outside Z, Y' regular, $p^{-1}(Z)_{red}$ is a normal crossings divisor with irr components dominating Z and $p^{-1}(Z) \longrightarrow Z$ is (univ) locally 0-acyclic.
(connected Milnor fibres)

5. Proof of Th. 2.1 assuming Th. 2.2.

WMA $Y = \mathrm{Spec}(A)$, A excellent normal domain and that the result is known

for (~~proper subschemes of Y and for~~)

normalizations of $Y' \subsetneq Y$, Y' irr, in

finite extensions of $R(Y')$.

Using section 4 there is

$\mathrm{Spec}(A') \xrightarrow[\text{finite}]{} \mathrm{Spec}(A)$ which kills

$R^1 j_* G$ outside a cod ≥ 3 locus,

so reduce to Z of cod ≥ 3 and

use section 3.

6. Discriminants.

Lemma 6.1. Let $A = (A_{ij})$ be an $n \times n$ matrix of $m \times m$ matrices over a commutative ring k, with commuting A_{ij}. Let A be the resulting block $nm \times nm$ matrix.

Then $\det A = \det \det A$.

[Hint: replace A_{11} by $A_{11} + t$, invertible over $k' \supset k[t]$.]

Lemma 6.2. Let $A \subset B \subset C$ be comm. rings with C a free B-module with basis f_j $(1 \le j \le n)$ and B a free A-module with basis e_i $(1 \le i \le m)$. (so $(e_i f_j)$ is an A basis of C). We use these bases to define discriminants, e.g. $\mathrm{disc}_{B/A} = \det \mathrm{Tr}_{B/A}(e_i e_j)$.

Then $\mathrm{disc}_{C/A} = (\mathrm{disc}_{B/A})^n \, \mathrm{Norm}_{B/A}(\mathrm{disc}_{C/B})$.

7. Rigidity.

Th. 7.1. Let (A, I) be a henselian pair with I finitely generated, $U \subset \mathrm{Spec}(A)$ a quasi-compact open $U \supset \mathrm{Spec}(A) - V(I)$, $\hat{A}$ the I-adic completion of A, $\hat{U} \subset \mathrm{Spec}(\hat{A})$ the inverse image of U. Let F be a sheaf of sets (resp. an ind-finite sheaf of groups) on U. Then $H^0(U, F) \xrightarrow{\sim} H^0(\hat{U}, F)$ (resp. $H^1(U, F) \xrightarrow{\sim} H^1(\hat{U}, F)$).

version for stacks.

Can reduce to case F constant, which holds by Elkik's approximation when A is noetherian and also when I is principal and $U = \mathrm{Spec}(A) - V(I)$. In general blow up a f.g. ideal defining the complement of U and use proper base change for stacks (Girand + extension to non noeth. case) and affine base change to reduce to the principal ideal case.

<u>Th. 7.2.</u> (close to SGA4 XV)

$$\begin{tikzcd} p: X' \arrow[r] & X \\ U' \arrow[u, "j'"] & U \arrow[u, "j"'] \end{tikzcd}$$

p smooth, X normal excellent, $U' = p^{-1}(U)$
U open containing all points of cod ≤ 1.
Then $p^* R^1 j_* G \xrightarrow{\sim} R^1 j'_* G$ (G finite).

Lemma 7.3. Let Z be a nowhere dense closed subscheme of a noetherian scheme X. Then the following conditions are equivalent

(1) Let $p : X' \to X_{red}$ be the normalization. Then $p^{-1}(Z)$ is of codim ≥ 2 in X'.

(2) for every $y \in Z$, the irreducible components of $\mathrm{Spec}(\mathcal{O}^h_{X,y})$ are of dim ≥ 2

(3) same for $\hat{\mathcal{O}}_{X,y}$.

If this holds we say that Z is C2 in X.

Prop. 7.4. Let $A \longrightarrow B$ be a local homomorphism of nlr, formally smooth for m-adic topologies. Then There is a direct system of local essentially smooth A-algebras B_i with

$$\hat{B} \simeq (\varinjlim B_i)^{\wedge}$$

$$B_i \longrightarrow B_j \text{ flat } \quad m_i B_j = m_j .$$

<u>Th. 7.5.</u> Let Z be C2 in X

$p : X' \longrightarrow X$ a flat morphism of noetherian schemes, the fibres of p above points of Z are geometrically regular.

Conclusion as in 7.2.

8. Separable projection.

Lemma 8.1. Let R be a complete noetherian local ~~ring~~ domain. Then R has a regular subring R_0, R finite over R_0 with $\mathrm{Frac}(R)$ separable over $\mathrm{Frac}(R_0)$.

Only problem in equal characteristic $p > 0$.
Let k be the residue field of R and $\{b_i\}_{i \in I}$ a p-basis of k. There is a bijection

$$\{\text{coefficient fields of } R\} \longleftrightarrow \{\text{liftings of } b_i\}.$$

Fix a coefficient field.
By the proof of Nagata's Jacobian criterion there is a finite subset $J \subset I$ s.t. if $k' = k^p(b_i,\ i \notin J)$ then

$$\Omega = \Omega^1_{R/\overline{k' R^p}}$$ has generic rank

$\dim(R) + \mathrm{card}\,(J)$.
Then change the liftings of b_i $i \in J$ s.t. their differentials are linearly independent in Ω. With this coefficient field

one can take $k' = k$.

Let $t_1, \ldots, t_d$ be a system of parameters for R and let $f_1, \ldots, f_d \in R$ be s.t. df_i form a basis of $\Omega \otimes_R \mathrm{frac}(R)$.

WMA $f_i \in m$.

Let $t_i' = t_i^p (1 + f_i)$.

Can take

$$R_0 = k[[t_1', \ldots, t_d']].$$

9. Ultraproducts

Let I be a set. There is a bijection

$$\{\text{ultrafilters on } I\} \longleftrightarrow \mathrm{Spec}\left((\mathbb{Z}/2\mathbb{Z})^I\right)$$

$$F \longmapsto p = \{\chi_A \mid A \notin F\}.$$

If R_i $(i \in I)$ are rings then the ultraproduct $\prod_{/F} (R_i)$ is $\prod_i (R_i)/\sim$ where

$$(r_i) \sim (s_i) \Leftrightarrow \{i \mid r_i = s_i\} \in F.$$

In the commutative case there is a map of topological spaces

$$f : \mathrm{Spec}\,(\prod_i R_i) \longrightarrow \mathrm{Spec}\,((\mathbb{Z}/2)^I)$$

defined by "$f^* \chi_A = \chi_A$".

The fibres of f (with the restriction of the structure sheaf) are Spec of the ultraproduct.

R_i all fields, domains, local rings,
$\Rightarrow$ same for $\prod_{/F}(R_i) = R_\infty$

(*) If R_i are local rings whose maximal ideals are generated by n elements then R_∞ has the same property, so its completion is a $\underbrace{\ }_{\text{complete}}$ noetherian local ring.

- If furthermore F is not ω complete R_∞ maps onto $\hat{R}_\infty$
- If $_{(*),}$ F is ω complete and R_i are noetherian

then R_∞ is noetherian.

<u>Lemma</u>. If $R_i \to S_i$ $(i \in I)$ are finite maps of noetherian local rings, $\exists n$ s.t. $\forall i$ the maximal ideal of R_i and S_i have n generators as an R_i-module, F is an ultrafilter on I, $R_\infty \to S_\infty$ the corresponding map on ultraproducts then $R_\infty \to S_\infty$ ~~thenthe~~ is finite and $\ker(R_\infty \to \hat{R}_\infty)$ generates $\ker(S_\infty \to \hat{S}_\infty)$.

The proof uses that in all cases $R_{\infty,sep} = \operatorname{Im}(R_\infty \to \hat{R}_\infty)$ is noetherian.

We will use this only for complete local rings, in which case it is easily seen that R_∞ maps onto $\hat{R}_\infty$,

10. Proof of Th. 2.2.

We may assume A is complete. View A as a finite generically étale extension of a 2 dimensional regular complete local ring R. A/R finite free rk m.

If the assertion is false there is $n > 0$ and connected pairwise non isomorphic finite étale maps $E_i \longrightarrow \operatorname{Spec}(A) - \{m_A\}$ of degree n, $((i \in \mathbb{N}))$. Let B_i be the normalization of A in E_i. B_i is complete normal local ring and E_i is the punctured spectrum of B_i. B_i is finite free rk nm as an R-module.

$$\operatorname{disc}_R B_i = (\operatorname{disc}_R A)^n \quad \text{up to unit.}$$

Let F be an ultrafilter on $\mathbb{N}$. Consider the ultraproducts

$$R_\infty \longrightarrow A_\infty \longrightarrow B_\infty$$

$R_\infty \to A_\infty$ free rk m
$R_\infty \to B_\infty$ free rk nm
B_∞ a finitely presented A_∞-module
information on $\mathrm{disc}_R B$ retained

Same for $\hat{R_\infty} \longrightarrow \hat{A_\infty} \longrightarrow \hat{B_\infty}$.

Note $A \longrightarrow \hat{A_\infty}$ is flat : mod m_A^ℓ this reduces to the fact that over a coherent ring infinite products of flat modules are flat.
The residue field of $\hat{A_\infty}$ is a regular (in particular separable) extension of the residue field of A. By "localization de la lissité formelle" the fibre rings of $A \to \hat{A_\infty}$ are geometrically regular. Hence $\hat{A_\infty}$ is normal.
$\hat{B_\infty}$ is torsion free over $\hat{R_\infty}$, hence over $\hat{A_\infty}$.
Hence $\hat{B_\infty} / \hat{A_\infty}$ is finite flat over the punctured spectrum, necessarily of rank n.
By the discriminant information we get that

$\hat{B}_\infty / \hat{A}_\infty$ is finite étale on the punctured spectrum. Hence $\forall\, \xi \in \mathrm{Spec}(\hat{A_\infty}) - V(m_A)$, the criterion of flatness by fibres gives that $(B_\infty)_\xi$ is flat over $(A_\infty)_\xi$, hence finite étale. But every point of $\mathrm{Spec}(A_\infty) - V(m_A)$ is a generization of a point of $\mathrm{Spec}(\hat{A_\infty}) - V(m_A)$.

[If $I \subset \mathrm{Rad}(A)$ is a f.g. ideal in a ring A and $\hat{A}$ the I-adic completion of A then all closed points of $\mathrm{Spec}(A) - V(I)$ are in the image of $\mathrm{Spec}(\hat{A}) \longrightarrow \mathrm{Spec}(A)$.]

This proves

<u>Theorem.</u> $p : \mathrm{Spec}(\prod B_i) \longrightarrow \mathrm{Spec}(\prod_i A)$ is finite étale of deg n on $V(m_A)^c$.

Let F be a non principal ultrafilter on $\mathbb{N}$. By the rigidity facts the restriction of p to $\mathrm{Spec}(A_\infty) - V(m_A)$ comes from a finite etale cover of $\mathrm{Spec}(A) - V(m_A)$. By passage to the limit $\exists\, T \in F$ s.t. same holds for $p|_{\mathrm{Spec}(\prod_{i \in T} A) - V(m_A)}$. Contradiction.