Entropy and the localization of eigenfunctions.

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$$\|\psi_k\|_{L^2(X)} = 1$$

in the limit $\lambda_k \longrightarrow +\infty$.

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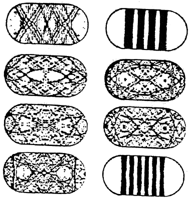

FIG.: A few eigenfunctions of the Bunimovich billiard (Heller, 89).

We study the weak limits of the probability measures on X,

$$|\psi_k(x)|^2 d\mathrm{Vol}(x)$$

$\lambda_k \longrightarrow +\infty.$

For convenience, we use semiclassical notations,

$$-\hbar_k^2 \Delta \psi_k = \psi_k,$$

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The operator $-\hbar^2 \frac{\Delta}{2}$ is the Schrödinger operator associated with the classical Hamiltonian

$$H(x, \xi) = \frac{\|\xi\|^2}{2}$$

on T^*X, which generates the geodesic flow $g^t : T^*X \longrightarrow T^*X$.
$\hbar \longrightarrow 0$ is the semiclassical limit.

$|\psi_k(x)|^2 d\mathrm{Vol}(x)$ is a probability measure on X.

To keep the information of the frequency vector of ψ_k, we use a **microlocal lift** of these measures to T^*X. To do so, we use a **quantization procedure**

$$a \in C_c^\infty(T^*X) \mapsto \mathrm{Op}_\hbar(a) \in \mathcal{L}(L^2(X)).$$

We study the distributions $\mu_k(a) = \langle \psi_k, \mathrm{Op}_{\hbar_k}(a)\psi_k \rangle$, $\mu_k \in \mathcal{D}'(T^*X)$, which project on X to the measure $|\psi_k(x)|^2 d\mathrm{Vol}(x)$.

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NB : the definition of μ_k depends on the choices made for $\mathrm{Op}_{\hbar_k}$, but not the collection of its weak limits when $k \longrightarrow +\infty$.

- $\| \operatorname{Op}_\hbar(a) \operatorname{Op}_\hbar(b) - \operatorname{Op}_\hbar(ab)\|_{\mathcal{L}} = O(\hbar)$;
- $\|e^{it\hbar\frac{\Delta}{2}} \operatorname{Op}_\hbar(a) e^{-it\hbar\frac{\Delta}{2}} - \operatorname{Op}_\hbar(a \circ g^t)\|_{\mathcal{L}} = O_t(\hbar)$; **(Egorov)**

- $\| \mathrm{Op}_\hbar(a) \, \mathrm{Op}_\hbar(b) - \mathrm{Op}_\hbar(ab) \|_{\mathcal{L}} = O(\hbar)$;
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Consequences

Let μ_k be microlocal lifts of $|\psi_k(x)|^2 d\mathrm{Vol}(x)$.
As $k \longrightarrow +\infty$,

- *one can extract from (μ_k) a converging subsequence in $\mathcal{D}'$;*
- *any limit point μ is a probability measure carried by the unit cotangent bundle,*
- *and μ is invariant under the geodesic flow g^t : $g^t_* \mu = \mu$.*

Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(X)$, with

$$-\hbar_k^2\Delta\psi_k = \psi_k, \qquad \hbar_k \geq \hbar_{k+1}.$$

Denote (μ_k) the microlocal lifts of the measures $|\psi_k(x)|^2 d\mathrm{Vol}(x)$.

Theorem (Snirelman, Zelditch, Colin de Verdière)

Assume that the action of the geodesic flow on the energy layer S^*X is **ergodic** *for the Liouville measure.* **Then, there exists a subset $\mathcal{S}\subset\mathbb{N}$ of density 1, such that**

$$\mu_k \underset{k\longrightarrow\infty, k\in\mathcal{S}}{\longrightarrow} \mathrm{Liouville}.$$

We now assume that X has negative sectional curvature.
$\longrightarrow$ *the geodesic flow has the Anosov property = is uniformly hyperbolic : non-zero Lyapunov exponents, strong mixing properties...*

Conjecture (Rudnick, Sarnak 94)

On a negatively curved manifold,

the whole sequence $\mu_k \underset{k\longrightarrow\infty}{\longrightarrow} \mathrm{Liouville}.$
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Conjecture (Rudnick, Sarnak 94)

On a negatively curved manifold, (or, more generally, for Anosov Hamiltonian systems ?)
the whole sequence $\mu_k \underset{k\longrightarrow\infty}{\longrightarrow} \text{Liouville.}$
("equidistribution of eigenfunctions")

Theorem (Arithmetic quantum unique ergodicity, Lindenstrauss 02)

Let X be a congruence (compact) arithmetic surface (curvature -1). Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(X)$, formed of eigenfunctions of the laplacian
Then the whole sequence $\mu_k \underset{k\longrightarrow\infty}{\longrightarrow} \text{Liouville}.$

Theorem (Arithmetic quantum unique ergodicity, Lindenstrauss 02)

Let X be a congruence (compact) arithmetic surface (curvature -1). Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(X)$, formed of eigenfunctions of the laplacian **and of the Hecke operators**. *Then the whole sequence $\mu_k \underset{k\longrightarrow\infty}{\longrightarrow} \mathrm{Liouville}$.*

Multiplicity of eigenvalues is a difficult open problem.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

acts linearly on the torus $\mathbb{T}^2$; this diffeomorphism has the Anosov property. This system can be quantized thanks to the **Weyl** formalism : $\hat{A}_\hbar : \mathcal{H}_\hbar \longrightarrow \mathcal{H}_\hbar$.

Counter-example (Faure, Nonnenmacher, De Bièvre 03)

For any $\beta \in [0, 1/2]$, for any periodic orbit γ of A, there exists a sequence (ψ_k) of eigenfunctions such that, for any $a \in C^\infty(\mathbb{T}^2)$,

$$\mu_k(a) = \left\langle \psi_k, \mathrm{Op}^W(a)\psi_k \right\rangle \underset{k \longrightarrow +\infty}{\longrightarrow} (1-\beta)\int a\, dpdq + \beta \int a\, d\gamma.$$

We denoted $d\gamma$ the A-invariant probability measure carried by the periodic orbit γ.

Theorem (Faure, Nonnenmacher 04)

β cannot be greater than $\frac{1}{2}$ in these counter-examples.

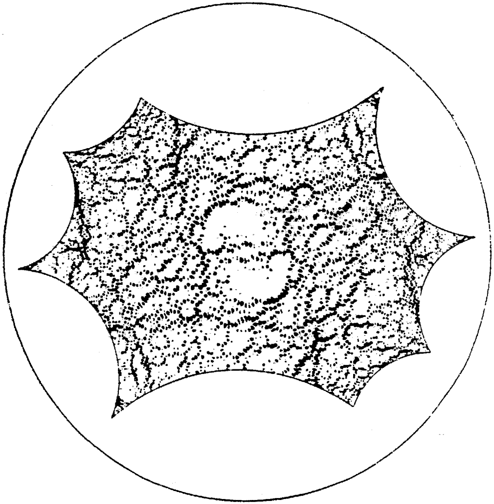

FIG.: An eigenfunction on a negatively curved surface.

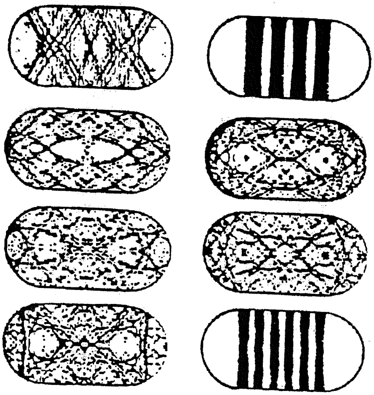

FIG.: A few eigenfunctions for the Bunimovich billiard (Heller, 89).

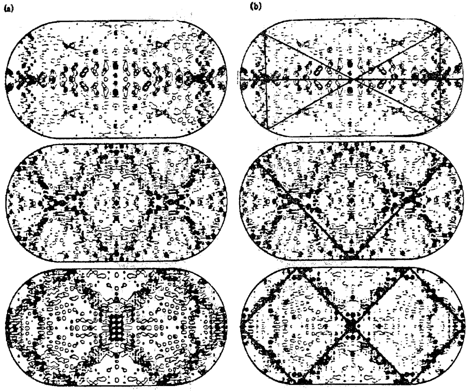

FIG.: Scars for the Bunimovich billiard (Heller, 89).

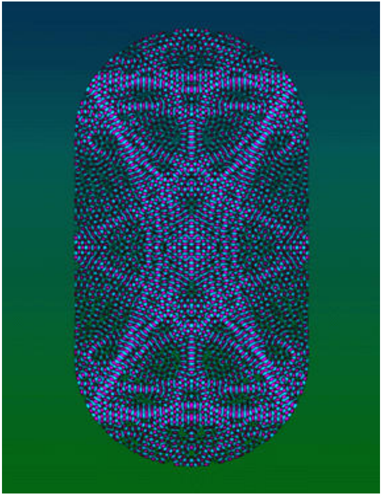

FIG.: Scar.

http ://www.ericjhellergallery.com/

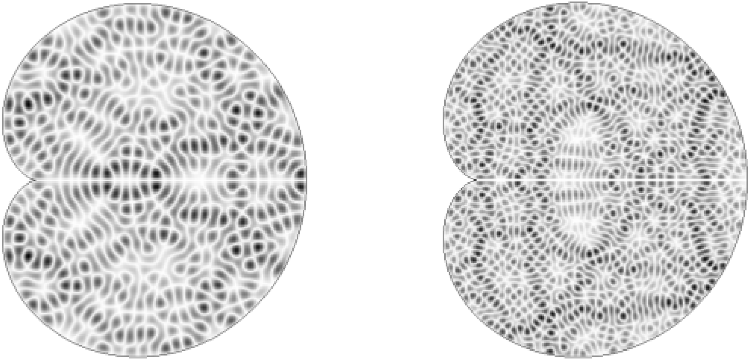

FIG.: Two typical eigenstates for the cardioid billiard (Arnd Bäcker).

http ://www.physik.tu-dresden.de/ baecker/

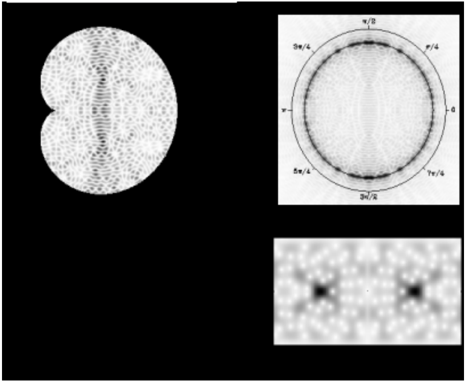

FIG.: Scar for the cardioid billiard (Arnd Bäcker).

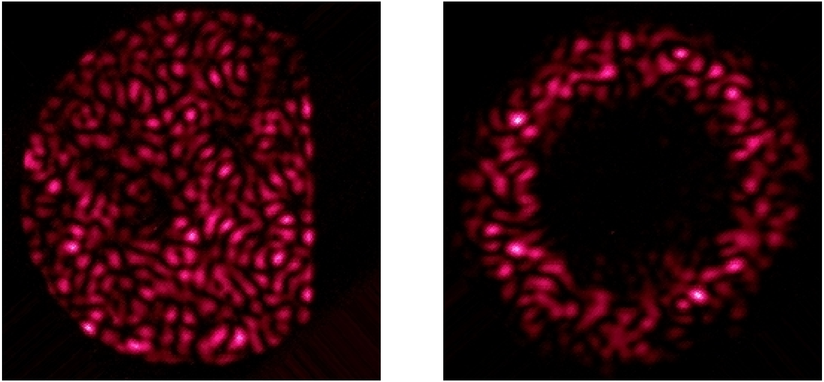

FIG.: *D*-shaped optical fiber, typical state.

Valérie Doya, Olivier Legrand, Claire Michel, Fabrice Mortessagne, Laboratoire de Physique de la Matière Condensée, Université de Nice. Christian Miniatura, Laboratoire Ondes et Désordre, Sophia-Antipolis.

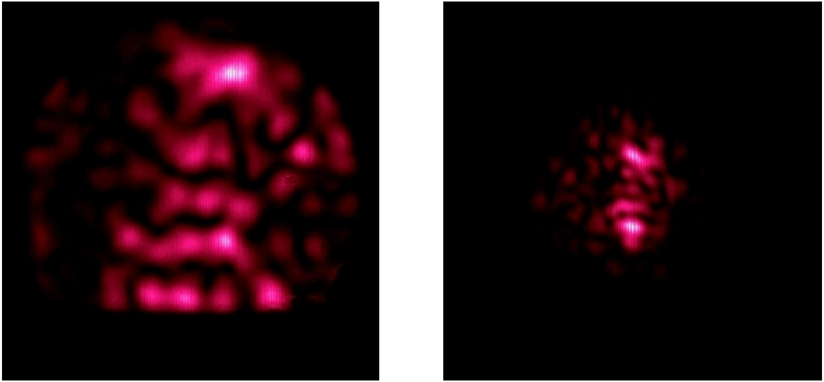

FIG.: *D*-shaped optical fiber, scar.

We now look at the **Kolmogorov-Sinai entropy** of the limit measures,

$$h_{KS} : \mathcal{M}_g^1 \longrightarrow \mathbb{R}^+.$$

- if μ is carried by a periodic orbit, then $h_{KS}(\mu) = 0$.
- $h_{KS}(\mu) \leq \int \sum_{k=1}^{d-1} \lambda_k^+ d\mu$, with equality only for the Liouville measure **(Ruelle-Pesin inequality)**.
- In curvature $\equiv -1$, $h_{KS}(\mu) \leq d - 1$, with equality only for the Liouville measure.
- h_{KS} is affine.

$$-\hbar_k^2 \Delta \psi_k = \psi_k, \qquad \hbar_k \longrightarrow 0.$$

Denote (μ_k) the microlocal lifts of $|\psi_k(x)|^2 d\mathrm{Vol}(x)$.

Theorem (A- 05)

Assume that X has negative curvature (or assume only that the geodesic flow is Anosov). Let μ be a limit point of the sequence (μ_k). Then

$$h_{KS}(\mu) > 0.$$

Corollary

μ is not entirely carried by a union of closed geodesics.

Corollary

(of the full quantitative thm)
In constant curvature -1, we have $\dim \operatorname{supp} \mu \geq d$.

Theorem (A-Nonnenmacher 06, A-N-Koch)

In constant curvature −1,

$$h_{KS}(\mu) \geq \frac{d-1}{2}.$$

In variable negative curvature,

$$h_{KS}(\mu) \geq \int \sum_{k=1}^{d-1} \lambda_k^+ d\mu - \frac{1}{2}\lambda_{\max}.$$

We expect the following :

$$h_{KS}(\mu) \geq \frac{1}{2} \int \sum_{k=1}^{d-1} \lambda_k^+ d\mu.$$

Definition of entropy. $S^*X = \sqcup_{k=1}^{K} P_k$ a measurable partition of S^*X.

$$\mu(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \ldots g^{-(n-1)} P_{\alpha_{n-1}})$$

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$$\mu(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \ldots g^{-(n-1)} P_{\alpha_{n-1}})$$

$$h_n(\mu, P) = -\sum_{\alpha_0,\ldots,\alpha_{n-1}} \mu(P_{\alpha_0} \ldots g^{-(n-1)} P_{\alpha_{n-1}}) \log \mu(P_{\alpha_0} \ldots g^{-(n-1)} P_{\alpha_{n-1}})$$

$$h_{KS}(\mu, P) := \lim \frac{h_n(\mu, P)}{n} = \inf \frac{h_n(\mu, P)}{n}$$

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Remark

- If, for any sequence $\alpha_0, \dots, \alpha_{n-1}$, $\mu(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \dots g^{-(n-1)} P_{\alpha_{n-1}}) \leq C e^{-\beta n}$, then $h_{KS}(\mu, P) \geq \beta$.
- Upper semicontinuity.

$X = \sqcup_{k=1}^{K} P_k$ as in definition of entropy.

$$\hat{P}_k = \mathrm{Op}(\mathbb{1}_{P_k}^{sm}) = \times \mathbb{1}_{P_k}^{sm}, \qquad \sum_k \hat{P}_k^2 = I$$

$$\hat{P}_k(t) = e^{it\hbar\frac{\Delta}{2}} \hat{P}_k e^{-it\hbar\frac{\Delta}{2}}$$

Proposition (The main inequality)

(Here in constant curvature $\equiv -1$).

$$\| \qquad \qquad \|_{\mathcal{L}(L^2(X))}$$

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for any sequence

$\alpha_0, \ldots, \alpha_{n-1}$.

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$$\|\hat{P}_{\alpha_{n-1}}(n-1)\ldots\hat{P}_{\alpha_1}(1)\hat{P}_{\alpha_0} \qquad \|_{\mathcal{L}(L^2(X))}$$

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Let $\chi_\hbar \in C_c^\infty(T^*X)$ be a smooth cut-off function, supported in a $\hbar^{1-\delta}$-neighbourhood of S^*X.

Proposition (The main inequality)

(Here in constant curvature $\equiv -1$).

$$\|\hat{P}_{\alpha_{n-1}}(n-1)\dots\hat{P}_{\alpha_1}(1)\hat{P}_{\alpha_0}\,\mathrm{Op}_\hbar(\chi_\hbar)\|_{\mathcal{L}(L^2(X))}$$

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Proposition (The main inequality)

(Here in constant curvature $\equiv -1$).

$$\|\hat{P}_{\alpha_{n-1}}(n-1)\ldots\hat{P}_{\alpha_1}(1)\hat{P}_{\alpha_0}\,\mathrm{Op}_\hbar(\chi_\hbar)\|_{\mathcal{L}(L^2(X))} \leq C\hbar^{-(d-1+\delta)/2} e^{-\left(\frac{d-1}{2}\right)n}(1+\mathrm{diam}\,P)^n$$

for $n \leq \mathcal{K}|\log\hbar|$ ($\mathcal{K}$ arbitrary) and $\hbar \leq \hbar_{\mathcal{K}}$, for any sequence $\alpha_0, \ldots, \alpha_{n-1}$.

Define

$$h_n^+(\psi_\hbar, \hat{P}) := -\sum_{\alpha_0,\dots,\alpha_{n-1}} \|\hat{P}_{\alpha_{n-1}}(n-1)..\hat{P}_{\alpha_0}\psi_\hbar\|^2 \log \|\hat{P}_{\alpha_{n-1}}(n-1)..\hat{P}_{\alpha_0}\psi_\hbar\|^2$$

and

$$h_n^-(\psi_\hbar, \hat{P}) := -\sum_{\alpha_0,\dots,\alpha_{n-1}} \|\hat{P}_{\alpha_0}..\hat{P}_{\alpha_{n-1}}(n-1)\psi_\hbar\|^2 \log \|\hat{P}_{\alpha_0}..\hat{P}_{\alpha_{n-1}}(n-1)\psi_\hbar\|^2.$$

Using Maassen–Uffink's **entropic uncertainty principle**, we prove

$$\frac{1}{2n}[h_n^+(\psi_\hbar, \hat{P}) + h_n^-(\psi_\hbar, \hat{P})] \geq (d-1) - \frac{(d-1+\delta)}{2}\frac{|\log\hbar|}{n}$$

for $n \leq \mathcal{K}|\log\hbar|$ ($\mathcal{K}$ arbitrary) and $\hbar \leq \hbar_{\mathcal{K}}$.

Maassen–Uffink's entropic uncertainty principle.

Theorem

Let $\mathcal{H}$ be a Hilbert space.
Suppose we have two partitions of unity $(\pi_k)_{k=1}^N$ and $(\tau_j)_{j=1}^M$, that is, two families of operators on $\mathcal{H}$ such that

$$\sum_{k=1}^{N} \pi_k \pi_k^* = Id, \qquad \sum_{j=1}^{M} \tau_j \tau_j^* = Id.$$

For any normalized $\psi \in \mathcal{H}$, define
$h_\pi(\psi) = -\sum_{k=1}^N \|\pi_k^* \psi\|^2 \log \|\pi_k^* \psi\|^2$ *and*
$h_\tau(\psi) = -\sum_{j=1}^M \|\tau_j^* \psi\|^2 \log \|\tau_j^* \psi\|^2$
Let U be an isometry of $\mathcal{H}$.
Define $c_{\tau,\pi}(U) \stackrel{\mathrm{def}}{=} \sup_{j,k} \|\tau_j^* \, U \, \pi_k\|_{\mathcal{L}(\mathcal{H})}$.
Then, for any normalized $\psi \in \mathcal{H}$, we have

$$h_\tau(U\psi) + h_\pi(\psi) \geq -2 \log c_{\tau,\pi}(U) \,.$$