A mini-course on spectral measures

Abstract

Lecture 1. Let X be a compact Riemannian manifold and let f_n, $n = 1, 2, \ldots$, be the eigenfunctions of the Laplace operator, normalized so that $\int |f_n|^2\, dx = 1$. If X is negatively curved, then, by a theorem of Shnirelman–Zelditch–Colin de Verdiere, the measures,

$$\mu_i = |f_{n_i}|^2\, dx$$

tend weakly to dx as i tends to infinity for "most" sequences of integers, $n_1, n_2, \ldots$. For Riemannian manifolds in general no such result seems to be true, however in some instances, because of the presence of symmetries or for other geometric reasons, the eigenvalues of Δ tend to group into clusters

$$\lambda_{i,n}\,, \qquad 1 \le i \le N(n)$$

and the averages of the corresponding measures

$$(*) \qquad \frac{1}{N(n)} \sum \mu_{n_i}$$

have, in some of these cases, interesting asymptotic properties. One example of this is when X is a *Zoll* manifold. A Zoll manifold has, by definition, the property that the eigenvalues of $\sqrt{\Delta}$ lie on bands

$$an + b - O\left(\frac{1}{n}\right) < \sqrt{\lambda_{i,n}} < an + b + O\left(\frac{1}{n}\right)$$

and in this lecture I will describe some beautiful results of Colin de Verdiere from the mid-eighties on the asymptotics of the measures $(*)$ associated with these bands.

Lecture 2. One of Colin's results is that for Zoll manifolds the function, $N(n)$, is a polynomial. Boutet de Monvel and I were able to obtain an explicit formula for this polynomial by noting some similarities between Zoll manifolds and complex projective varieties. We also discovered that Colin's results have analogues for projective varieties: If X is a non-singular projective variety and $\mathbb{L} \to X$ its canonical line bundle one can associate a spectral measure to $\mathbb{L}^n$ by setting

$$\mu_n(f) = \operatorname{trace} \pi_n M_f$$

where, for $f \in C^0(X)$,

$$M_f : \Gamma_{\text{hol}}(\mathbb{L}^n) \to L^2(\mathbb{L}^n)$$

is multiplication by f and

$$\pi_n : L^2(\mathbb{L}^n) \to \Gamma_{\text{hol}}(\mathbb{L}^n)$$

is orthogonal projection and one gets an asymptotic expansion of μ_n in inverse powers of n as n tends to infinity. In general this asymptotics isn't as explicitly computable as in Colin's case except in a few rare instances.

One such instance is if X is a toric variety and Δ its defining polytope. I will show that in this case the asymptotics of μ_n is described by an Euler–Maclaurin formula for Δ.

Lecture 3. Another setting in which one encounters a natural clustering of eigenvalues is in semi-classical analysis. If P_h is a self-adjoint semi-classical differential operator on $\mathbb{R}^n$ with symbol, p, and $p^{-1}([a,b])$ is compact for some interval, $[a,b]$, then P_h has a finite number of eigenvalues

$$\lambda_i(h)\,, \quad i = 1, \dots, N(h)$$

on this interval with

$$N(h) \sim (2\pi h)^{-n} \operatorname{vol}\,(p^{-1}[a,b])\,.$$

In lecture 3 I will show that these are analogues of the results I discussed in lectures 1 and 2 for the spectral measures associated with these eigenvalues and discuss some inverse spectral theorems that one can extract from these results.