

# “BOTTOM OF THE WELL” SEMI-CLASSICAL TRACE INVARIANTS

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ABSTRACT. Let  $\hat{H}$  be an  $\hbar$ -admissible pseudodifferential operator whose principal symbol,  $H$ , has a unique non-degenerate global minimum. We give a simple proof that the semi-classical asymptotics of the eigenvalues of  $\hat{H}$  corresponding to the “bottom of the well” determine the Birkhoff normal form of  $H$  at the minimum. We treat both the resonant and the non-resonant cases.

## 1. INTRODUCTION

Let  $X$  be an  $n$ -dimensional manifold and  $\hat{H}$  a self-adjoint zeroth order semi-classical  $\Psi$ DO acting on the space of half-densities,  $|\Omega|^{1/2}(X)$ . We will assume that the principal symbol,  $H(x, \xi)$ , of  $\hat{H}$  has a unique non-degenerate global minimum,  $H = C$ , at some point  $(x_0, \xi_0)$ , and that outside a small neighborhood of  $(x_0, \xi_0)$   $H$  is bounded from below by  $C + \delta$ , for some  $\delta > 0$ . We will also assume that at  $(x_0, \xi_0)$  the subprincipal symbol of  $\hat{H}$  vanishes. From these assumptions one can deduce that on an interval

$$C < E < C + \epsilon, \quad \epsilon < \delta,$$

the spectrum of  $\hat{H}$  is discrete and consists of eigenvalues:

$$(1.1) \quad E_i(\hbar), \quad 1 \leq i \leq N(\hbar),$$

where

$$(1.2) \quad N(\hbar) \sim (2\pi\hbar)^{-n} \text{Vol} \{ (x, \xi) ; H(x, \xi) \leq C + \epsilon \}.$$

In addition, we will make a non-degeneracy assumption on the Hessian of  $H(x, \xi)$  at  $(x_0, \xi_0)$ . Choose a Darboux coordinate system centered at  $(x_0, \xi_0)$  such that

$$(1.3) \quad H(x, \xi) = C + \sum_{i=1}^n \frac{u_i}{2} (x_i^2 + \xi_i^2) + \dots$$

In this paper we present a short proof of the following theorem:

**Theorem 1.1.** *Assume that the  $u_i$ 's are linearly independent over the rationals and that the subprincipal symbol of  $\hat{H}$  vanishes at  $(x_0, \xi_0)$ . Then the eigenvalues, (1.1), determine the Taylor series of  $H$  at  $(x_0, \xi_0)$  up to symplectomorphism or, in other words, determine the Birkhoff canonical form of  $H$  at  $(x_0, \xi_0)$ .*

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Our results are closely related to some recent results of [4] on the Schrödinger operator,  $\hat{H} = \hbar^2 \Delta + V$ , which show that the “bottom of the well” spectral asymptotics determines the Taylor series of  $V$  at  $x_0$ . They are also related to inverse spectral results of [2], [6], [3] and [5]. In these papers it is shown that if

$$(1.4) \quad \exp tv_H, \quad v_H = \sum \frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial \xi_i}$$

is the classical dynamical system on  $T^*M$  associated with  $H$  and  $T_\gamma$  is the period of a periodic trajectory  $\gamma$  of this system, the asymptotic behavior of the wave trace,  $\text{trace}\left(\exp \frac{-it\hat{H}}{\hbar}\right)$  at  $NT_\gamma$ ,  $N \in \mathbb{Z}$ , determines the Birkhoff canonical form of (1.4) in a formal neighborhood of  $\gamma$ . (In the first two papers we have cited, results of this type are proved for standard  $\Psi$ DOs, and [3] and [5] are versions of these results in the semiclassical setting.) It turns out that if the trajectory,  $\gamma$ , is replaced by a fixed point of the system (1.4) and, in particular, if this fixed point is a non-degenerate minimum of  $H(x, \xi)$ , the recovery of the Birkhoff canonical form from the spectral data, (1.1), can be greatly simplified. Our goal, in this short note, is to show why.

We also obtain nearly-optimal results in the resonant case, see §4.

## 2. THE SEMI-CLASSICAL BIRKHOFF CANONICAL FORM

We quickly review here the construction of the semi-classical Birkhoff canonical form of  $\hat{H}$ . We follow the exposition in [1] and refer to that paper for details.

Performing a preliminary microlocalization and conjugation by an  $\hbar$ -FIO, we can assume that  $\hat{H}$  is an operator on  $\mathbb{R}^n$ , and that the global minimum  $(x_0, \xi_0)$  is the origin  $(0, 0) \in T^*\mathbb{R}^n$ . Let us denote by

$$[\hat{H}] = \sum_{\alpha, \beta, k} x^\alpha \xi^\beta \hbar^k$$

the Taylor series of the full Weyl symbol of  $\hat{H}$ , where the monomial  $x^\alpha \xi^\beta \hbar^j$  has degree  $|\alpha| + |\beta| + 2j$ . In fact we can assume that

$$[\hat{H}] = \sum_i \frac{u_i}{2} (x_i^2 + \xi_i^2) + \dots$$

where the dots indicate terms of degree three and higher. Notice that

$$[\hat{H}]|_{\hbar=0} = \text{the Taylor series of the principal symbol of } \hat{H}.$$

Let  $\hat{H}_2$  denotes the Weyl quantization of

$$H_2 := \sum_i \frac{u_i}{2} (x_i^2 + \xi_i^2),$$

and set

$$\hat{H} = \hat{H}_2 + \hat{L}.$$

To construct the quantum Birkhoff canonical form of  $\hat{H}$ , one conjugates  $\hat{H}$  by suitable Fourier integral operators in order to successively make higher-order terms

in  $L$  commute with  $\hat{H}_2$ . The resulting series is the quantum Birkhoff canonical form,  $H_{\text{can}}$  of  $H$ .

The non-resonance condition implies that we can write  $H_{\text{can}}$  in the form:

$$(2.1) \quad \widehat{H}_{\text{can}} = \hat{H}_2 + F(P_1, \dots, P_n, \hbar), \quad P_i = \hbar^2 D_i^2 + x_i^2$$

with  $F$  an  $\hbar$ -admissible symbol whose Taylor series is of the form

$$(2.2) \quad [F] = \sum_{|r| \geq 1} c_r(\hbar) p^r,$$

where  $p_i = \xi_i^2 + x_i^2$ ,  $r = (r_1, \dots, r_n)$ ,

$$(2.3) \quad c_r(\hbar) = \hbar^{|r|-1} (c_{r,0} + \dots)$$

and  $c_r(0) = 0$  for  $|r| = 1$  (so that all the monomials in  $[F] - H_2$  have degree  $\geq 3$ ).

Theorem 1.1 is a direct consequence of the following:

**Theorem 2.1.** *Under the assumptions of Theorem 1.1, the eigenvalues, (1.1), determine the semi-classical Birkhoff canonical form of  $\hat{H}$ .*

### 3. THE PROOF OF THEOREM 2.1

The first step in our argument is more or less identical with that of [4], [3] and [5]. Assume without loss of generality that  $C = 0$ , and let  $\rho \in C_0^\infty(\mathbb{R})$  be equal to one on the interval  $[-1/2, 1/2]$  and zero outside the interval  $[-1, 1]$ .

**Proposition 3.1.** *There exists  $\tau > 0$  such that for all sufficiently small  $\epsilon$  and for all  $t \in (0, \tau)$ , the  $\rho$ -truncated wave trace*

$$(3.1) \quad \text{trace } \rho\left(\frac{\hat{H}}{\epsilon}\right) \exp \frac{-it\hat{H}}{\hbar}$$

*is equal modulo  $O(\hbar^\infty)$  to the  $\rho$ -truncated wave trace for the Birkhoff canonical form,*

$$(3.2) \quad \text{Tr}(t, \hbar) := \text{trace } \rho\left(\frac{\widehat{H}_{\text{can}}}{\epsilon}\right) \exp\left(\frac{-it\widehat{H}_{\text{can}}}{\hbar}\right).$$

*Moreover, the truncated wave trace admits an asymptotic expansion as  $\hbar \rightarrow 0$  of the form*

$$\text{Tr}(t, \hbar) \sim a_0(t) + a_1(t)\hbar + \dots$$

*The coefficients  $a_j(t)$  are real-analytic functions of  $t$  and are independent of  $\rho$ , provided  $\rho$  is chosen as above.*

*Proof.* The operator  $\rho(\epsilon^{-1}\hat{H})e^{it\hbar^{-1}\hat{H}}$  is an  $\hbar$ -Fourier integral operator, associated to the graph of the Hamilton flow of  $H$  restricted to the portion of phase space  $\Sigma_\epsilon = \{H < \epsilon\}$ . If  $\epsilon$  is sufficiently small, for some  $\tau > 0$  there are no periodic trajectories in  $\Sigma_\epsilon$  with period in  $(0, \tau)$ , other than the fixed point  $(x_0, \xi_0)$ . Therefore, writing the truncated trace as an oscillatory integral and applying the method of stationary

phase, for each  $t$  the phase has a unique critical point, corresponding to the absolute minimum  $(x_0, \xi_0)$  which is a fixed point of the classical flow. This proves the existence of the asymptotic expansion of (3.1). Since the cutoff operator  $\rho\left(\frac{\hat{H}}{\epsilon}\right)$  is microlocally equal to the identity in a neighborhood of  $(x_0, \xi_0)$ , the asymptotic expansion of the cutoff trace is independent of  $\rho$ , provided  $\rho \in C_0^\infty$  is identically equal to one near zero.

For each  $\hbar$ , (3.1) is an analytic function of  $t$  (it's a finite weighted sum of exponentials), and the expansion is locally uniform in  $t$ . Therefore, the  $a_j(t)$  are analytic in  $t$  as well.

The operator  $\hat{H}$  is unitarily equivalent to an operator of the form  $\widehat{H}_{\text{can}} + \hat{R}$ , where  $\hat{R}$  is an operator whose full symbol vanishes at  $(x_0, \xi_0)$  to infinite order. Since the coefficients of the truncated trace of  $\widehat{H}_{\text{can}} + \hat{R}$  only depend on the derivatives of the full symbol of this operator evaluated at  $(x_0, \xi_0)$ , the asymptotic expansions of (3.1) and (3.2) are the same.  $\square$

*Remark 3.2.* The truncated trace of the Birkhoff canonical form equals

$$(3.3) \quad \text{Tr}(t, \hbar) = \sum_{k \in (\mathbb{Z}_+)^n} \rho\left(\frac{H_{\text{can}}(\hbar(k+1/2), \hbar)}{\epsilon}\right) e^{it\hbar^{-1}H_{\text{can}}(\hbar(k+1/2), \hbar)}.$$

However, since  $\rho$  is identically equal to one in a neighborhood of zero, as a power series in  $\hbar$

$$(3.4) \quad \text{Tr}(t, \hbar) \sim \sum_{k \in (\mathbb{Z}_+)^n} e^{it\hbar^{-1}H_{\text{can}}(\hbar(k+1/2), \hbar)}.$$

Notice that, although the asymptotic expansion of the truncated trace (3.1) equals the right-hand side of (3.4) only for  $t \in (0, \tau)$ , by analyticity our spectral data determine the right-hand side of (3.4) for all  $t \in \mathbb{R}$ .

We now rewrite (3.4) in a more amenable form using a variant of the ‘‘Zelditch trick’’ (see [6]).

**Proposition 3.3.** *For any choice of  $\rho$  as above and for  $\epsilon$  small, as  $\hbar \rightarrow 0$*

$$(3.5) \quad \text{Tr}(t, \hbar) \sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \left( \sum_{|r| \geq 1} \hbar^{|r|-1} c_r(\hbar) \left(\frac{1}{t} D_\theta\right)^r \right)^m \frac{e^{it\frac{1}{2}\sum_j \theta_j}}{\prod_j (1 - e^{it\theta_j})} \Big|_{\theta=u},$$

where

$$D_\theta = -i \left( \frac{\partial}{\partial \theta_1}, \dots, \frac{\partial}{\partial \theta_n} \right)$$

and the right-hand side of (3.5) is understood as a power series in  $\hbar$ .

*Proof.* Recalling that  $[\hat{H}_2, \hat{F}] = 0$ ,

$$\text{Tr}(t, \hbar) \sim \sum_{k \in (\mathbb{Z}_+)^n} e^{itu \cdot (k+1/2)} \langle k | e^{it\hbar^{-1}\hat{F}} | k \rangle,$$

where  $\{|k\rangle\}$  is an orthonormal basis of eigenvectors of the canonical  $n$ -torus representation on  $L^2(\mathbb{R}^n)$ , and  $u \cdot (k + 1/2) = \sum_{j=1}^n u_j(k_j + 1/2)$ . For each  $k$ , the Taylor expansion,  $[F]$ , gives us an asymptotic expansion

$$(3.6) \quad \langle k | e^{it\hat{h}^{-1}\hat{F}} | k \rangle = \sum_{m=0}^{\infty} \frac{(it)^m}{\hbar^m m!} F(\hbar(k+1/2), \hbar)^m \sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \left( \sum_r \hbar^{|r|-1} c_r(\hbar) (k+1/2)^r \right)^m.$$

Let us introduce the variables  $\theta = (\theta_1, \dots, \theta_n)$  and write:

$$(k+1/2)^r e^{itu \cdot (k+1/2)} = \left( \frac{1}{t} D_\theta \right)^r e^{it\theta \cdot (k+1/2)} \Big|_{\theta=u}.$$

Then

$$\text{Tr}(t, \hbar) \sim \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \sum_{k \in \mathbb{Z}_+} \rho \left( \frac{F(\hbar(k+1/2), \hbar)}{\epsilon} \right) \left( \sum_r \hbar^{|r|-1} c_r(\hbar) \left( \frac{1}{t} D_\theta \right)^r \right)^m e^{it\theta \cdot (k+1/2)} \Big|_{\theta=u}.$$

Finally, for each  $m$  (summing a geometric series)

$$\begin{aligned} & \sum_{k \in \mathbb{Z}_+} \left( \sum_r \hbar^{|r|-1} c_r(\hbar) \left( \frac{1}{t} D_\theta \right)^r \right)^m e^{it\theta \cdot (k+1/2)} \Big|_{\theta=u} = \\ & = \left( \sum_r \hbar^{|r|-1} c_r(\hbar) \left( \frac{1}{t} D_\theta \right)^r \right)^m \frac{e^{it\frac{1}{2} \sum_j \theta_j}}{\prod_j (1 - e^{it\theta_j})} \Big|_{\theta=u}, \end{aligned}$$

and the result follows.  $\square$

We will show that the  $m = 0$  term in the series on the right-hand side of (3.5) suffices to determine the  $c_r(\hbar)$ . More precisely:

**Theorem 3.4.** *From the coefficients of  $\hbar^s$ ,  $s \leq \ell$ , in the series in  $\hbar$*

$$(3.7) \quad V(t, \hbar) = \sum_{|r| \geq 1} \hbar^{|r|-1} c_r(\hbar) \left( \frac{1}{t} D_\theta \right)^r \frac{e^{it\frac{1}{2} \sum_j \theta_j}}{\prod_j (1 - e^{it\theta_j})} \Big|_{\theta=u}$$

*one can determine the coefficients of  $\hbar^s$ ,  $s \leq \ell$ , in  $c_r(\hbar)$  for all  $r$ .*

*Proof.* Let  $\rho$  be a cutoff function as before, and  $\hat{\varphi} = \rho$ . Integrating (3.7) against  $\epsilon^n \varphi(\epsilon t)$  and essentially reversing the previous calculation, we find:

$$(3.8) \quad \begin{aligned} \tilde{V}(\epsilon, \hbar) &= \sum_k \left( \sum_r \hbar^{|r|-1} c_r(\hbar) (k+1/2)^r \right) \rho \left( \epsilon^{-1} u \cdot (k+1/2) \right) = \\ &= \sum_k \hbar^{-1} F(\hbar(k+1/2)) \rho \left( \epsilon^{-1} u \cdot (k+1/2) \right). \end{aligned}$$

Letting

$$c_r(\hbar) = \sum_{i=0}^{\infty} c_{r,i} \hbar^{|r|-1+i},$$

we can rearrange (3.8) in increasing powers of  $\hbar$  (using the variable  $\ell = 2|r| - 2 + i$  for the exponent of  $\hbar$ ):

$$(3.9) \quad \tilde{V}(\epsilon, \hbar) = \sum_{\ell=0}^{\infty} \hbar^{\ell} \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{|r|=j+1} c_{r, \ell-2j} \left( \sum_k (k+1/2)^r \rho \left( \epsilon^{-1} u \cdot (k+1/2) \right) \right).$$

Now arrange the numbers

$$u_k = u \cdot (k+1/2), \quad k \in (\mathbb{Z}_+)^n$$

in strictly increasing order (which is possible because there are no repetitions among them):

$$(3.10) \quad 0 < \nu_1 = u_k|_{k=0} < \nu_2 < \dots$$

Let us write:  $\nu_s = u_{k^{(s)}}$ . Now vary  $\epsilon$  in (3.9), starting with a very small value. Gradually increasing  $\epsilon$ , we can arrange that the coefficient of  $\hbar^{\ell}$  in (3.9) is

$$\sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{|r|=j+1} c_{r, \ell-2j} \sum_{s=1}^m (k^{(s)} + 1/2)^r \rho \left( \epsilon^{-1} \nu_s \right)$$

for any given  $m$ . Therefore, by an inductive argument on  $m$  we can recover the values of the polynomial

$$\mathfrak{p}_{\ell}(x) = \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{|r|=j+1} c_{r, \ell-2j} (x+1/2)^r$$

at all  $k \in (\mathbb{Z}_+)^n$ . But these values determine the polynomial, and therefore its coefficients.  $\square$

Now we show that the asymptotic expansion of the trace,  $\text{Tr}(t, \hbar)$ , determines  $V$ :

**Theorem 3.5.** *From the coefficients of  $\hbar^s$ ,  $s \leq \ell$ , in the expansion (3.5) one can determine the coefficients of  $\hbar^s$ ,  $s \leq \ell$  in the series  $V(t, \hbar)$ .*

*Proof.* We proceed by induction on  $\ell$ .

The coefficient of  $\hbar$  in  $V$  coincides with the coefficient of  $\hbar$  in (3.5), since all terms in the sum (3.5) except the first are of order  $O(\hbar^m)$ ,  $m > 1$ .

By theorem 3.4 the coefficient of  $\hbar$  in  $V$  enables us to determine the coefficient of  $\hbar$  in  $c_r(\hbar)$ , and hence the coefficient of  $\hbar^2$  in the second summand of (3.5). But the coefficient of  $\hbar^2$  in the first summand coincides with the coefficient of  $\hbar^2$  in  $V$ , so the coefficients of  $\hbar$  and  $\hbar^2$  in (3.5) determine the coefficient of  $\hbar^2$  in  $V$ . It is clear that this procedure can be continued indefinitely.  $\square$

Theorem 2.1 is an immediate consequence of theorems 3.4 and 3.5.

4. THE RESONANT CASE

We now consider the case when the frequencies  $u_1, \dots, u_n$  are not linearly independent over  $\mathbb{Q}$ . Following [1], let us introduce the number

$$d = \min\{|\alpha|, \alpha \in \mathbb{Z}^n \setminus \{0\} \mid \alpha \cdot u = 0\}$$

which is a measure of the rational relations among the frequencies (here  $|\alpha| = \sum_{j=1}^n |\alpha_j|$ ). We will make use below of the following observation:

**Lemma 4.1.** *Among the eigenvalues of  $\hat{H}_2$  of the form:*

$$(4.1) \quad \lambda_k = k \cdot (u + 1/2) \quad \text{with} \quad |k| < d/2$$

*there are no repetitions (i. e. the mapping  $k \mapsto \lambda_k$  is 1-1 in the range  $|k| < d/2$ ).*

*Proof.* If  $k \cdot (u + 1/2) = k' \cdot (u + 1/2)$ , then  $(k - k') \cdot u = 0$  and therefore  $|k - k'| \geq d$ . The conclusion now follows from the triangle inequality.  $\square$

Continuing to assume that the subprincipal symbol vanishes at the absolute minimum, the semi-classical Birkhoff canonical form in the resonant case has the following structure (see [1]):

$$H_{can} = H_2 + F + K,$$

where

- (1)  $F = F(p_1, \dots, p_n, \hbar)$  where  $F$  is a polynomial in all variables of degree at most  $\lfloor \frac{d-1}{2} \rfloor$ .
- (2)  $[K]$  is a power series with monomials  $\hbar^j x^\alpha \xi^\beta$  where  $|\alpha| + |\beta| + 2j > d$  and  $[\hat{H}_2, \hat{K}] = 0$ .

In this section we prove the following:

**Theorem 4.2.** *If  $d$  is even the eigenvalues (1.1) determine the entire semi-classical canonical form,  $F(x, \hbar)$ . If  $d$  is odd, those eigenvalues determine the semi-classical canonical form except for the monomials of maximal degree,  $\lfloor \frac{d-1}{2} \rfloor$ .*

Except for a few additional complications, the method of proof is the same as in the non-resonant case. We begin by checking that the asymptotic expansion of the truncated trace can be treated by the same methods as before, up to a sufficiently high order in  $\hbar$ :

**Proposition 4.3.** *In the resonant case, the expansion (3.5) is valid modulo  $O(\hbar^{\lfloor \frac{d}{2} \rfloor})$ .*

*Proof.* Once again we write the trace as a sum of diagonal matrix elements over a normalized basis  $\{|k\rangle\}$  of eigenfunctions of the standard representation of the  $n$ -torus, splitting off the  $H_2$  part (which is possible since  $\hat{F} + \hat{K}$  commutes with  $H_2$ ):

$$\text{Tr}(t, \hbar) \sim \sum_{k \in (\mathbb{Z}_+)^n} e^{itu \cdot (k+1/2)} \langle k | e^{it\hbar^{-1}(\hat{F} + \hat{K})} | k \rangle.$$

We next expand the exponential in its Taylor series. We want to show that every term involving  $\hat{K}$  is  $O(\hbar^{\lfloor \frac{d}{2} \rfloor})$ .

A term involving

$$\langle k | (\hat{F} + \hat{K})^m | k \rangle$$

is a sum of terms of the form

$$(4.2) \quad \langle k | \hat{F}_1 \hat{K}_1 \cdots \hat{F}_s \hat{K}_s | k \rangle$$

where the  $F_j$  are powers of  $F$  and the  $K_j$  are powers of  $K$ . Therefore, the  $K_j$  are sums of monomials  $\hbar^j x^\alpha \xi^\beta$  where  $2j + |\alpha| + |\beta| > d$ , just as is  $K$ . Let us express those monomials in terms of raising and lowering operators,

$$A_{\alpha\beta} = z^\alpha \bar{z}^\beta, \quad z = x + i\xi.$$

Then (4.2) is a linear combination of terms of the form

$$(4.3) \quad \hbar^{\sum_{i=1}^s j_i} \langle k | \hat{F}_1 \widehat{A_{\alpha^1 \beta^1}} \cdots \widehat{A_{\alpha^s \beta^s}} | k \rangle$$

where, for each  $i$ ,

$$2j_i + |\alpha^i| + |\beta^i| > d.$$

Now recall that (i) the  $\hat{F}_j$  are diagonal in the basis  $\{|k\rangle\}$  and (ii) the  $\widehat{A_{\alpha\beta}}$  act on the basis vectors by:

$$\widehat{A_{\alpha\beta}} |k\rangle = \hbar^{|\beta|} c_{\alpha\beta} |k + \alpha - \beta\rangle$$

where  $c_{\alpha\beta}$  is a constant whose value we won't need. Therefore, a diagonal matrix element of the sort (4.3) is zero unless

$$\sum_{i=1}^s \alpha^i - \beta^i = 0,$$

in which case (4.3) is  $O(\hbar^{j + \sum_i |\beta^i|})$  where  $j = \sum_i j_i$ . However,  $|\alpha^i| + |\beta^i| > d - 2j_i$  for each  $i$  and

$$\sum_{i=1}^s \alpha^i - \beta^i = 0 \quad \Rightarrow \quad \left| \sum_{i=1}^s \alpha^i \right| = \left| \sum_{i=1}^s \beta^i \right|.$$

Therefore,  $\sum_i |\beta^i| \geq [sd/2] - j$  and so (4.3) is  $O(\hbar^{\lfloor sd/2 \rfloor})$ . It follows that all diagonal matrix elements to which  $\hat{K}$  contributes are at least  $O(\hbar^{\lfloor \frac{d}{2} \rfloor})$ .  $\square$

**Lemma 4.4.**  *$F(\hbar(x + 1/2), \hbar)$  is a polynomial in  $\hbar$  of degree at most  $\lfloor \frac{d-1}{2} \rfloor$ , and if we write*

$$F(\hbar(x + 1/2), \hbar) = \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \hbar^j F_j(x)$$

*the power series expansion of  $\text{Tr}(t, \hbar)$  determines the values  $F_j(k)$  for all  $k \in (\mathbb{Z}_+)^n$  such that  $|k| < d/2$ , for all  $j \leq \lfloor \frac{d-1}{2} \rfloor$  if  $d$  is even and for all  $j < \lfloor \frac{d-1}{2} \rfloor$  if  $d$  is odd.*

*Proof.* The first statement follows from the general form of  $F$ .

By theorems 3.5 and 3.4, for any  $\ell$  the first  $\ell$  terms of the expansion of  $\text{Tr}(t, \hbar)$  determine the first  $\ell$  terms of (3.8), provided we replace  $F$  by  $F + K$ . But, by the previous proposition, the expansion of  $\text{Tr}(t, \hbar) \bmod O(\hbar^{\lfloor \frac{d}{2} \rfloor})$  is insensitive to what  $K$  is. Therefore, (3.8) remains valid  $\bmod O(\hbar^{\lfloor \frac{d}{2} \rfloor})$ , where  $F$  now stands for the part of the canonical form we are determining from the spectrum.

If  $d$  is even

$$\lfloor \frac{d-1}{2} \rfloor < \lfloor \frac{d}{2} \rfloor,$$

and so it follows that the expansion of  $\text{Tr}$  determines the sums

$$\sum_k \hbar^{-1} F(\hbar(k+1/2)) \rho(\epsilon^{-1} u \cdot (k+1/2)).$$

Now we proceed as before, letting  $\epsilon$  grow starting at a very small value. Since the eigenvalues (4.1) are all different, we can determine the polynomial in  $\hbar$ ,  $F(\hbar(x+1/2))$ , evaluated at each  $k$  with  $|k| < d/2$ . If  $d$  is odd we must discard the term  $F_j$  with  $j = \lfloor \frac{d-1}{2} \rfloor$ .  $\square$

Since  $F_j$  is a polynomial of degree at most  $\lfloor \frac{d-1}{2} \rfloor$ , the proof of theorem 4.2 is completed by the following result:

**Lemma 4.5.** *Let  $f(x_1, \dots, x_n)$  be a polynomial of degree  $N$ . Then  $f$  is completely determined by its values at the points*

$$(k_1 + 1/2, \dots, k_n + 1/2),$$

for all  $k$  such that  $|k| \leq N$  and  $k_j \geq 0$ .

*Proof.* The proof is by induction on the number of variables. The case  $n = 1$  is trivial. Assume the result is true for polynomials of  $n - 1$  variables, and let

$$f = f_N(x_2, \dots, x_n) + f_{N-1}(x_2, \dots, x_n)x_1 + \dots + f_0 x_1^N.$$

Note that degree  $f_i = i$ .

Evaluating  $f$  at  $(k+1/2, 1/2, \dots, 1/2)$ ,  $0 \leq k \leq N$  determines  $f_i(1/2, 1/2, \dots, 1/2)$ ,  $i = 0, \dots, N$ , and in particular determines  $f_0$ .

Evaluating  $f - f_0 x_1^N$  at  $(k+1/2, k_2+1/2, \dots, k_n+1/2)$ ,  $0 \leq k \leq N-1$ ,  $k_2 + \dots + k_n \leq 1$  determines  $f_i(k_2+1/2, \dots, k_n+1/2)$  at all  $k_2 + \dots + k_n \leq 1$  and in particular determines  $f_1$ .

Evaluating  $f - f_1 x_1^{N-1} - f_0 x_1^N$  at  $(k+1/2, k_2+1/2, \dots, k_n+1/2)$ ,  $0 \leq k \leq N-2$ ,  $k_2 + \dots + k_n \leq 2$  determines  $f_i(k_2+1/2, \dots, k_n+1/2)$  at all  $k_2 + \dots + k_n \leq 2$  and in particular determines  $f_2$ . Etc.  $\square$

When  $d$  is odd our methods recover the classical Birkhoff normal form of  $H$  except for its monomials of top degree,  $\frac{d-1}{2}$ .

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