Transforms associated to square integrable group representations. I. General results

A. Grossmann, J. Morlet,[a] and T. Paul[b]
Centre de Physique Théorique,[c] *Section II, Centre National de la Recherche Scientifique, Luminy, Case 907, F-13288 Marseille Cedex 9, France*

(Received 20 December 1984; accepted for publication 12 April 1985)

Let G be a locally compact group, which need not be unimodular. Let $x \to U(x)$ $(x \in G)$ be an irreducible unitary representation of G in a Hilbert space $\mathscr{H}(U)$. Assume that U is square integrable, i.e., that there exists in $\mathscr{H}(U)$ at least one nonzero vector g such that $\int |(U(x)g,g)|^2\, dx < \infty$. We give here a reasonably self-contained analysis of the correspondence associating to every vector $f \in \mathscr{H}(U)$ the function $(U(x)g,f)$ on G, discussing its isometry, characterization of the range, inversion, and simplest interpolation properties. This correspondence underlies many properties of generalized coherent states.

I. INTRODUCTION

This paper is the first of a series concerned with applications of various families of "generalized coherent states" to quantum mechanics, wave propagation, and signal analysis.

Many properties of the classical (canonical) coherent states[1,2] are closely tied to the Weyl–Heisenberg group. In particular, the fundamental formula

$$\mathbb{1} = \int |z\rangle d^2z \langle z| \tag{1.1}$$

is a way of writing the orthogonality relations[3,4] for the irreducible representation of that group.

Aslaksen and Klauder[5] have considered the analogous states for the two-parameter group of shifts and dilations and found that the "fiducial vector" ("analyzing wavelet" in our terminology) cannot be arbitrary, in contrast to the Weyl–Heisenberg case.

The same two-parameter group appeared in Ref. 6 in the study of decomposition of signals into "wavelets of constant shape"; the restriction on the analyzing wavelet was there called an "admissibility condition."

Another (equivalent) representation of the same group, together with an appropriate choice of the analyzing wavelet, has given rise to a realization of quantum mechanics on a Hilbert space of function analytic on a half-plane.[7,8] Other groups were used to define coherent states: SU(2) for spin coherent states,[9] SU(1,1) in Ref. 10, and a general definition was proposed by Perelomov.[11]

In this paper we shall be concerned with "coherent states" associated with certain representations of arbitrary (in particular not necessarily unimodular) locally compact groups.

Let G be a locally compact group, U a continuous irreducible representation of G in a Hilbert space $\mathscr{H}(U)$, and g a vector in $\mathscr{H}(U)$.

We consider the family of vectors

$$|x\rangle = U(x)g \quad (x \in G) \tag{1.2}$$

in $\mathscr{H}(U)$. This family depends on the choice of g.

Since U is irreducible, the linear span of the vectors $|x\rangle$ is dense in $\mathscr{H}(U)$.

One can then ask the question whether there exists a (suitably normalized) invariant measure $d\mu(x)$ on G, such that

$$\int |x\rangle d\mu(x) \langle x| = \mathbb{1}, \tag{1.3}$$

where $|x\rangle\langle x|$ is defined by $|x\rangle\langle x|f\rangle = (U(x)g,f)U(x)g$, and $\mathbb{1}$ is the identity operator in $\mathscr{H}(U)$.

The answer, in general, is no; this can be seen by taking $G = \mathbb{R}$ (additive), and by considering the one-dimensional irreducible representation space $\mathbb{C}$, with $U(x)$ the operator of multiplication by e^{ix}.

However, if the representation U is "square integrable" in a sense that will be defined below, then there exists a dense set of vectors in $\mathscr{H}(U)$, which give rise to (1.3). If g is such a vector (called "admissible") then the correspondence $f \to \psi$, with

$$\psi(x) = \langle x|f\rangle = (U(x)g,f) \quad (x \in G, \quad f \in \mathscr{H}(U)), \tag{1.4}$$

can be shown to be a multiple of an isometry between $\mathscr{H}(U)$ and $L^2(G,d\mu(x))$ and so (1.3) holds. The range of this transform is a closed subspace of $L^2(G,d\mu(x))$, and can be characterized by a reproducing kernel. If the group is unimodular, the set of admissible vectors is the whole space $\mathscr{H}(U)$, but this is not the case if the group is not unimodular (e.g., the affine group).

The purpose of this first paper is to give general results about transformations defined by (1.4). All the results derived here can be found in the mathematical literature and are part of the study of orthogonality relations for generalized square integrable representations. See, in particular, Refs. 12 and 13. We write them here in a form that is covenient for the applications we have in mind, using tools familiar to mathematical physicists (e.g., we give another proof of orthogonality relations with the help of quadratic forms).

The second paper of the series will be devoted to the particular case of the "$ax + b$" group, which has given rise to applications in applied mathematics.[6]

Further papers will be concerned with discrete versions of (1.2) and (1.3), with analyticity properties, special cases, and applications.

[a] Permanent address: ELF Aquitaine Company, ORIC Lab. 370 Bis Av. Napoleon Bonaparte, Rueil-Malmaison, France.
[b] Allocataire D.G.R.S.T.
[c] Laboratoire propre.

 0022-2488/85/102473-07$02.50

II. SQUARE INTEGRABLE REPRESENTATIONS; ADMISSIBLE VECTORS

In this section we present basic notions that will be used in the construction of transforms defined in Sec. IV.

A. Notations

G will denote a locally compact group, with identity e, and $x, y,\ldots$ elements of G. It is well known that there exists in such a group a left-invariant and a right-invariant (Haar) measure.[3,4] The left-invariant measure, with a fixed normalization, will be written dx. So

$$d(yx) = dx \quad (y \in G). \tag{2.1}$$

The right-invariant measure will be denoted by $d_R x$:

$$d_R(xy) = d_R(x) \quad (y \in G). \tag{2.2}$$

One has

$$d_R x = \Delta^{-1}(x)dx, \quad d(xy) = \Delta(y)dx, \tag{2.3}$$

where $\Delta(x)$ (the modular function) is a positive-valued character

$$\Delta(e) = 1, \quad \Delta(x) > 0, \quad \Delta(xy) = \Delta(x)\Delta(y). \tag{2.4}$$

If $\Delta(x) \equiv 1$, the group G is said to be unimodular. Notice a collision of terminologies: the group $\mathrm{GL}(n,\mathbb{R})$ of nonsingular $n \times n$ matrices is unimodular, even though it contains nonunimodular matrices.

The inhomogeneous group $\mathrm{IGL}(n,\mathbb{R})$, a semidirect product of translations in $\mathbb{R}^n$ and of $\mathrm{GL}(n;\mathbb{R})$, is not unimodular.

When one deals with semidirect products, it is convenient to use the right-invariant measure $d_R x$, since it is the product of the right-invariant measures of the factors (see Ref. 14, p. 210).

We have

$$d(x^{-1}) = \Delta^{-1}(x)dx = d_R x. \tag{2.5}$$

If $x \rightarrow \Phi(x)$ is any function on the group, define

$$\check{\Phi}: \check{\Phi}(x) = \Phi(x^{-1}). \tag{2.6}$$

Notice that $\check{\check{\Phi}} = \Phi$.

Then

(Φ is left integrable)$\Leftrightarrow$($\check{\Phi}$ is right integrable)

and the corresponding integrals are equal, since

$$\int \check{\Phi}(x) d_R x = \int \Phi(x^{-1}) d(x^{-1})$$

$$= \int \Phi(x')dx'.$$

We shall need the following statement: Let $\Phi(x)$ be a complex-valued function on the group G, such that

$$\bar{\Phi}(x) = \Phi(x^{-1}) \quad (x \in G) \tag{2.7}$$

[here $\bar{\Phi}(x)$ is the complex conjugate of $\Phi(x)$]. If, for some p, with $1 \leqslant p \leqslant +\infty$, we have $\Phi \in L^p(G,dx)$ and if Φ satisfies (2.7), then also $\check{\Phi} \in L^p(G,dx)$.

Here $\Phi \in L^p(G,dx)$ means

$$\int |\Phi(x)|^p \, dx < \infty,$$

the integral being taken over G.

B. Left and right regular representation

Left regular: If $\Phi \in L^2(G,dx)$ and $a \in G$, we define $\lambda(a)\Phi \in L^2(G,dx)$ by

$$(\lambda(a)\Phi)(x) = \Phi(a^{-1}x) \quad (x \in G). \tag{2.8}$$

Right regular: If $\Psi \in L^2(G,d_R x)$ and $a \in G$, we define $\rho(a)\Psi \in L^2(G,d_R x)$ by

$$(\rho(a)\Psi)(x) = \Psi(xa) \quad (x \in G). \tag{2.9}$$

The two representations λ and ρ act unitarily—in different spaces in general, namely $L^2(G,dx)$ and $L^2(G,d_R x)$.

C. Definition of square integrable representations and of admissible vectors

Let $x \rightarrow U(x)$ be a strongly continuous unitary representation of the locally compact group G in a complex Hilbert space $\mathscr{H}(U)$.

A vector $g \in \mathscr{H}(U)$ is said to be *admissible* if

$$\int |(U(x)g,g)|^2 \, dx < +\infty. \tag{2.10}$$

In (2.10) the left-invariant measure dx can be replaced by the right-invariant measure $d_R x$, since

$$(U(x)g,g) = (g, U(x^{-1})g);$$

so

$$\int |(U(x)g,g)|^2 \, dx = \int |(g, U(x^{-1})g)|^2 \, dx$$

$$= \int |(U(x^{-1})g,g)|^2 \, dx$$

$$= \int |(U(x)g,g)|^2 \, d_R x.$$

Definition (2.1): U will be called square integrable if (i) U is irreducible and, (ii) there exists in $\mathscr{H}(U)$ at least one nonzero admissible vector.

Remarks: (a) Any representation unitarily equivalent to a square-integrable representation is also square-integrable.

(b) If G is compact, any irreducible representation U of G is square integrable. We shall see below examples of square-integrable representations of groups that are not compact.

(c) As an example of an irreducible representation that is *not* square-integrable, consider the one-dimensional representation $x \rightarrow e^{iax}$ of $\mathbb{R}$.

(d) If G is unimodular and if U is a square-integrable representation of G, then every vector in $\mathscr{H}(U)$ is admissible. (See, e.g., Ref. 3.) We shall see that the situation is different if G is not unimodular.

III. ORTHOGONALITY RELATIONS

A. Historical comments

Orthogonality relations were derived by Schur (beginning of this century) for finite groups, Weyl (in the 1920's) for compact groups, and Bargmann and Godement for (square-integrable representations of) unimodular groups in the 1950's; nonunimodular groups have been investigated more recently.[12,13]

Orthogonality relations in the unimodular case are expressed by the following equality [3,4]:

$$\int_G \overline{(U(x)g_1,f_1)}(U(x)g_2,f_2)dx = \lambda\,(g_2,g_1)(f_1,f_2)$$

for every g_1, g_2, f_1, f_2 in $\mathscr{H}(U)$, where λ depends only on the square-integrable representation U.

B. Statement of orthogonality relations

We have the following[12,13] theorem.

Theorem 3.1: Let U be a square integrable representation of G, acting on the Hilbert space $\mathscr{H}(U)$. Then there exists in $\mathscr{H}(U)$ a unique self-adjoint positive operator C such that the following hold.

(i) The set of admissible vectors coincides with the domain of C.

(ii) Let g_1 and g_2 be any two admissible vectors. Let f_1 and f_2 be any two vectors in $\mathscr{H}(U)$. Then

$$\int_G \overline{(U(x)g_1,f_1)}(U(x)g_2,f_2)dx = (Cg_2,Cg_1)(f_1,f_2). \tag{3.1}$$

(iii) If the group G is unimodular, then C is a multiple of the identity.

The proof in the general case uses an extension of the Schur lemma and is given in the Appendix.

C. A special case

If $g_1 = g_2 = f_1 = f_2$, then (3.1) gives that for any admissible vector g, one has

$$(Cg,Cg) = \frac{1}{\|g\|^2}\int |(U(x)g,g)|^2\,dx, \tag{3.2}$$

where $\|\cdot\|$ denotes the norm in $\mathscr{H}(U)$.

If $g_1 = g_2 = g$, one has

$$\int \overline{(U(x)g,f_1)}(U(x)g,f_2)dx = \frac{\int|(U(x)g,g)|^2\,dx}{\|g\|^2}(f_1,f_2). \tag{3.3}$$

IV. L_g TRANSFORM AND R_g TRANSFORM

In this section, we define the transforms described in the Introduction and prove their isometry.

A. Definitions

Let U be a square-integrable representation of G acting on $\mathscr{H}(U)$, and let g be a nonzero admissible vector [see (2.10)].

Associate to g the positive number

$$c_g = \frac{1}{\|g\|^2}\int |(U(x)g,g)|^2\,dx, \tag{4.1}$$

which, by Sec. II A, is also

$$c_g = \frac{1}{\|g\|^2}\int |(U(x)g,g)|^2\,d_R x.$$

Notice that, by the results of Sec. II A one has

$$c_{U(x_0)g} = \Delta(x_0)^{-1}c_g, \tag{4.2}$$

where $\Delta(x_0)$ is the modular function.

For any $f\in\mathscr{H}(U)$ consider the complex-valued functions $L_g f$ and $R_g f$ on G, defined by

$$(L_g f)(x) = (1/\sqrt{c_g})(U(x)g,f) \quad (x\in G), \tag{4.3}$$

$$(R_g f)(x) = (1/\sqrt{c_g})(g,U(x)f) \quad (x\in G). \tag{4.4}$$

One has

$$(R_g f)(x) = (L_g f)(x^{-1}). \tag{4.5}$$

The function $(L_g f)(x)$ will be called the L_g *transform* of f. It depends on the representation U and on the choice of the admissible vector g. For reasons which will become clear, we shall sometimes call g the *analyzing wavelet*.

Similarly, $(R_g f)(x)$ will be called the R_g transform of f.

Remark: By (4.2) and (4.3) we have

$$(L_{U(x_0)g}f)(x) = \Delta(x_0)^{1/2}(L_g f)(xx_0). \tag{4.6}$$

By the same argument we have also

$$(R_{U(x_0)g}f)(x) = \Delta(x_0)^{1/2}(R_g f)(x_0^{-1}x). \tag{4.7}$$

B. Continuity and boundedness

By the continuity of $U(x)$ and continuity of scalar product, the functions $(L_g f)(x)$ and $(R_g f)(x)$ are continuous on G.

By the Schwarz inequality, $L_g f$ and $R_g f$ are bounded on G:

$$|(L_g f)(x)| \leqslant (1/\sqrt{c_g})\|f\|\,\|g\|, \tag{4.8}$$

$$|(R_g f)(x)| \leqslant (1/\sqrt{c_g})\|f\|\,\|g\|, \tag{4.9}$$

for every $x\in G$.

C. Intertwining (covariance)

By the definition of L_g, we have, for every $f\in\mathscr{H}(U)$, $a\in G$, $x\in G$,

$$(L_g f)(a^{-1}x) = (U(a^{-1}x)g,f) = (U(x)g,U(a)f),$$

which can be written as

$$\lambda(a)L_g = L_g U(a). \tag{4.10}$$

Similarly we have

$$(R_g f)(xa) = (g,U(x)U(a)f),$$

giving

$$\rho(a)R_g = R_g U(a). \tag{4.11}$$

D. Isometry of L_g and of R_g

We have the following proposition.

Proposition 4.1: (i) The correspondence $f\to L_g f$ is isometric from $\mathscr{H}(U)$ into $L^2(G,dx)$; that is, for every $f_1\in\mathscr{H}(U)$, $f_2\in\mathscr{H}(U)$, we have

$$\int \overline{(L_g f_1)(x)}(L_g f_2)(x)dx = (f_1,f_2). \tag{4.12}$$

(ii) The correspondence $f\to R_g f$ is isometric from $\mathscr{H}(U)$ into $L^2(G,d_R x)$: for every $f_1\in\mathscr{H}(U)$, $f_2\in\mathscr{H}(U)$, we have

$$\int \overline{(R_g f_1)(x)}\,(R_g f_2)(x)d_R x = (f_1,f_2). \tag{4.13}$$

Proof: This proposition is a corollary of Theorem 3.1; by (3.3), we have

$$(f_1,f_2)=\frac{\|g\|^2}{\int|(U(x')g,g)|^2\,dx'}\int\overline{(U(x)g,f_1)}\,(U(x)g,f_2)dx$$
$$=\frac{1}{c_g}\int\overline{(U(x)g,f_1)}\,(U(x)g,f_2)dx$$
$$=\int\overline{(L_gf_1)(x)}\,(L_gf_2)(x)dx.$$

The change of variable $x\to x^{-1}$, $d(x^{-1})=d_R(x)$ gives

$$(f_1,f_2)=\frac{1}{c_g}\int\overline{(U(x^{-1})g,f_1)}\,(U(x^{-1})g,f_2)d(x^{-1})$$
$$=\frac{1}{c_g}\int\overline{(g,U(x)f_1)}(g,U(x)f_2)d_R(x)$$
$$=\int\overline{(R_gf_1)(x)}\,(R_gf_2)(x)d_R(x),$$

and the proposition is proved.

Remark: Since L_g is isometric from $\mathscr{H}(U)$ into $L^2(G,dx)$, L_g is unitary from $\mathscr{H}(U)$ to $L_g\mathscr{H}(U)\subset L^2(G,dx)$. By (4.10) we see that $L_g\mathscr{H}(U)$ is invariant under the left regular representation. So L_g is an intertwining operator between U and the restriction of the left regular representation of G.

This construction allows us to consider the representation U as a subrepresentation of the left regular representation of G. The following section will give a characterization of the range of L_g.

The same remark is also valid for R_g.

V. CHARACTERIZATION OF THE RANGES OF R_g and L_g

Let g be an admissible vector for U. Consider, on G, the complex-valued function $p_g(x)$, defined by

$$p_g(x)=(1/c_g)(U(x)g,g)=(1/\sqrt{c_g})(L_gg)(x)$$
$$=(1/\sqrt{c_g})(R_gg)(x^{-1}). \tag{5.1}$$

The function p_g satisfies $\bar p_g(x)=p_g(x^{-1})$ and belongs, by Sec. II A, to $L^2(G;dx)\cap L^2(G,d_Rx)$.

Proposition 5.1: (i) Let Φ belong to $L^2(G,dx)$. Then Φ belongs to $L_g\mathscr{H}(U)\subset L^2(G,dx)$ if and only if the equation

$$\Phi(x)=\int p_g(y^{-1}x)\Phi(y)dy \tag{5.2}$$

holds for every $x\in G$.

(ii) Let Ψ belong to $L^2(G,d_Rx)$. Then Ψ belongs to $R_g\mathscr{H}(U)\subset L^2(G,d_Rx)$ if and only if the equation

$$\Psi(x)=\int p_g(yx^{-1})\Psi(y)d_R(y) \tag{5.3}$$

holds for every $x\in G$.

Proof: Notice first that the integrals (5.2) and (5.3) converge for every x, since the integrand is the product of two square-integrable functions.

(i) Suppose Φ belongs to $L_g\mathscr{H}(U)$. This means that $\Phi(y)=(L_gf)(y)$ for some $f\in\mathscr{H}(U)$. By the definition of p_g [(5.1)] and of L_g [(4.3)], we have

$$\int p_g(y^{-1}x)\Phi(y)dy=\frac{1}{\sqrt{c_g}}\int(L_gg)(y^{-1}x)\Phi(y)dy$$
$$=\frac{1}{\sqrt{c_g}}\int\overline{L_gg(x^{-1}y)}\Phi(y)dy$$
$$=\frac{1}{\sqrt{c_g}}\int\overline{(\lambda(x)L_g)(y)}\Phi(y)dy.$$

From (4.10) we obtain

$$\int p_g(y^{-1}x)\Phi(y)dy=\frac{1}{\sqrt{c_g}}\int\overline{L_g(U(x)g)(y)}\Phi(y)dy$$
$$=(1/\sqrt{c_g})(U(x)g,f)$$
$$=(L_gf)(x)=\Phi(x)$$

by isometry and definition of L_g.

An analogous proof holds for R_g.

We must now prove the converse part of the proposition. To do this we must have an explicit expression of the inverse of L_g.

Lemma 5.2: For every $\Phi\in L^2(G,dx)$ the expression

$$\varphi=\frac{1}{\sqrt{c_g}}\int_G\Phi(x)U(x)g\,dx \tag{5.4}$$

defines a vector in $\mathscr{H}(U)$.

Proof: The integral (5.4) is weakly convergent. Indeed, for any $\psi\in\mathscr{H}(U)$, the function $x\to(\psi,U(x)g)$ is in $L^2(G,dx)$ [it is up to a constant $\overline{(L_g\Psi)(x)}$]. Since $\Phi(x)\in L^2(G,dx)$, the integral

$$\frac{1}{\sqrt{c_g}}\int(\psi,U(x)g)\Phi(x)dx=\int\overline{(L_g\psi)(x)}\Phi(x)dx$$

exists. Furthermore, by the Schwarz inequality in $L^2(G,dx)$ and isometry of L_g we have

$$\left|\frac{1}{\sqrt{c_g}}\int(\Psi,U(x)g)\Phi(x)dx\right|\leqslant\|\Phi\|_{L^2(G,dx)}\|\psi\|_{\mathscr{H}(U)}.$$

Then, by the Riesz theorem, the integral $(1/\sqrt{c_g})\times\int_G\Phi(x)U(x)g\,dx$ defines a vector in $\mathscr{H}(U)$.

Lemma 5.3: If Φ satisfies (5.2) then

$$L_g\left(\frac{1}{\sqrt{c_g}}\int\Phi(x)U(x)g\,dx\right)(y)=\Phi(y). \tag{5.5}$$

Proof: The computation of the left-hand side of (5.5) gives

$$\frac{1}{\sqrt{c_g}\sqrt{c_g}}\int\Phi(x)(U(y)g,U(x)g)dx$$
$$=\frac{1}{c_g}\int\Phi(x)(U(x^{-1}y)g,g)dx$$
$$=\int\Phi(x)p_g(x^{-1}y)dx=\Phi(y)\ \text{by hypothesis.}$$

Lemma 5.3 implies that if Φ satisfies (5.2), then Φ belongs to $L_g\mathscr{H}(U)$.

The same proof holds for R_g.

During the proof of the proposition we have proved the following.

Proposition 5.4: The inverses of L_g and R_g, on their respective domains $L_g\mathscr{H}(U)$ and $R_g\mathscr{H}(U)$ are given by

$$L_g^{-1}\Phi = \frac{1}{\sqrt{c_g}}\int \Phi(x)U(x)g\,dx, \tag{5.6}$$

$$R_g^{-1}\psi = \frac{1}{\sqrt{c_g}}\int \psi(x)U(x^{-1})g\,d_R x, \tag{5.7}$$

where the integrals are taken in the weak sense.

Remarks: (1) The formulas (5.6) and (5.7) express the fact that, on the range of an isometric operator, the adjoint coincides with the inverse.

(2) The condition (5.2) can be rephrased by saying that the space $L_g\mathscr{H}(U)\subset L^2(G,dx)$ is a Hilbert space with reproducing kernel[15,16] (functional Hilbert space in the terminology of Ref. 16). The evaluation functional in $L_g\mathscr{H}(U)$ is

$$\Phi \rightarrow \Phi(x) = (e_x,\Phi)_{L^2(G,dx)}, \tag{5.8}$$

with

$$e_x(y) = \bar{p}_g(y^{-1}x) = p_g(x^{-1}y). \tag{5.9}$$

Similarly, (5.3) says that $R_g\mathscr{H}(U)\subset L^2(G,d_R x)$ is a Hilbert space with reproducing kernel in which the evaluation functional is

$$\Psi \rightarrow \Psi(x) = (h_x,\Psi)_{L^2(G,d_R x)}, \tag{5.10}$$

with

$$h_x(y) = \bar{p}_g(yx^{-1}) = p_g(xy^{-1}). \tag{5.11}$$

(3) In terms of the dyadic notation used in the Introduction [(1.2)], the isometry property (4.12) can be written as

$$\int \frac{dx}{c_g}|x\rangle\langle x| = 1, \tag{5.12}$$

where the integral is taken in the weak sense.

Indeed, from (5.12) we get

$$\int \frac{\langle f_1|x\rangle}{\sqrt{c_g}}\frac{\langle x|f_2\rangle}{\sqrt{c_g}}\,dx = (f_1,f_2),$$

which is (4.12).

The reproducing property (5.2) can be also deduced from (5.12), since (5.12) implies

$$\int \frac{\langle y|x\rangle}{c_g}\frac{\langle x|f\rangle}{\sqrt{c_g}}\,dx = \frac{\langle y|f\rangle}{\sqrt{c_g}},$$

which is (5.2).

The same remark holds for R_g.

VI. COVARIANT INTERPOLATION

A. Interpolation for the left transform

Proposition: Let $x_1,\ldots,x_n$ be n points in G. For $1\leqslant i,j\leqslant n$ consider the number

$$M_{ij} = (U(x_j)g,U(x_i)g) = (U(x_i^{-1}x_j)g,g) = c_g p_g(x_i^{-1}x_j). \tag{6.1}$$

Let M be the $n\times n$ positive definite Hermitian matrix with entries M_{ij}. Assume that

$$\det M \neq 0. \tag{6.2}$$

For $1\leqslant j\leqslant n$, define a function $\Phi_j(x)$ on G, by

$$\Phi_j(x) = (U(x_j^{-1}x)g,g) = c_g p_g(x_j^{-1}x). \tag{6.3}$$

Let $\zeta_1,\ldots,\zeta_n$ be any n complex numbers.

Define on G the function $\Phi(x) = \Phi(x_1,\ldots,x_n;\zeta_1,\ldots,\zeta_n;x)$ as

$$\Phi(x) = -\frac{1}{\det M}\times\det\begin{vmatrix} 0 & \Phi_1(x) & \cdots & \Phi_n(x)\\ \zeta_1 & M_{11} & \cdots & M_{1n}\\ \vdots & & & \vdots\\ \zeta_n & M_{n1} & \cdots & M_{nn}\end{vmatrix}.$$

Then we have the following.

(i) $\Phi(x)$ belongs to the range of L_g,

$\Phi\in L_g\mathscr{H}(U)$.

(ii) $\Phi(x)$ satisfies

$$\Phi(x_i) = \zeta_i \quad (i = 1,\ldots,n),$$

i.e., is a solution of the interpolation problem.

(iii) $\Phi(x)$ is of minimal norm, in the following sense: If Φ^{other} is any other function on G satisfying (i) and (ii), then

$$\|\Phi^{\text{other}}\| > \|\Phi\|,$$

the norm being taken in $L^2(G;dx)$.

(iv) The interpolation procedure is invariant under left multiplication in G, in the following sense: Let a be any element of G. Then

$$M(ax_1,\ldots,ax_n) = M(x_1,\ldots,x_n) \tag{6.4}$$

and

$$\Phi(ax_1,\ldots,ax_n;\zeta_1,\ldots,\zeta_n;ax) = \Phi(x_1,\ldots,x_n;\zeta_1,\ldots,\zeta_n;x), \tag{6.5}$$

so that the left-displaced interpolation problem

$$\Phi_1(ax_i) = \zeta_i \tag{6.6}$$

is solved by the function

$$\Phi_1(x) = \Phi(a^{-1}x). \tag{6.7}$$

B. Interpolation for the right transform

Proposition: Let $x_1,\ldots,x_n$ be n points in G and $\zeta_1,\ldots,\zeta_n$ n complex numbers. Consider the $n\times n$ Hermitian matrix $N = N(x_1,\ldots,x_n)$,

$$N_{ij} = c_g p_g(x_i x_j^{-1}) = (U(x_j^{-1})g,U(x_i^{-1})g).$$

Assume that N_{ij} is invertible, and define on G the function $\Psi(x) = \Psi(x_1,\ldots,x_n;\zeta_1,\ldots,\zeta_n;x)$ as

$$\psi(x) = -\frac{1}{\det N}\times\det\begin{vmatrix} 0 & \Psi_1(x) & \cdots & \Psi_n(x)\\ \zeta_1 & & & \\ \vdots & & N & \\ \zeta_n & & & \end{vmatrix}, \tag{6.8}$$

where $\Psi_j(x) = (U(xx_j^{-1})g,g) \quad (j = 1,\ldots,n)$.

Then, (i) Ψ belongs to $R_g\mathscr{H}(U)\subset L^2(G,d_R x)$; (ii) Ψ takes the prescribed values $\zeta_1,\ldots,\zeta_n$ at the prescribed points $x_1,\ldots,x_n$,

$$\Psi(x_j) = \zeta_j \quad (j = 1,\ldots,n); \tag{6.9}$$

(iii) Ψ is of minimal norm, subject to (i) and (ii); and (iv) for any $a\in G$, the function

$$\Psi_1(x) = \Psi(xa^{-1}) \tag{6.10}$$

takes the values $\zeta_1,...,\zeta_n$ at the points $x_1 a,...,x_n a$,

$$\Psi_1(x_j a) = \zeta_j \quad (j = 1,...,n). \tag{6.11}$$

An alternative way of writing the interpolating functions $\Phi(x)$ and $\Psi(x)$ is

$$\Phi(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \zeta_i (M^{-1})_{ij} p_g(x_j^{-1}x), \tag{6.12}$$

$$\Psi(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \zeta_i (N^{-1})_{ij} p_g(xx_j^{-1}), \tag{6.13}$$

where M^{-1} is the matrix inverse of M, and N^{-1} is the matrix inverse of N.

The interpolations are typical of Hilbert spaces with reproducing kernels and the proof can be adapted from Meschkowsky.[15]

APPENDIX: PROOF OF THEOREM 3.1

In the proof of Theorem 3.1, we shall use the following extension of Schur's lemma.

Proposition A.1: Suppose that (i) G is a group; (ii) U is a unitary irreducible representation of G in a Hilbert space $\mathscr{H}$; (iii) π is a unitary (not necessarily irreducible) representation of G in a Hilbert space $\mathscr{H}'$; (iv) T is a closed operator from $\mathscr{H}$ to $\mathscr{H}'$ with domain $\mathscr{D} \subset \mathscr{H}$ dense in $\mathscr{H}$ and stable under U; and (v) $TU(x) = \pi(x)T$ on $\mathscr{D}$ for every x in G; then T is a multiple of an isometry, and $\mathscr{D} = \mathscr{H}$.

Proof: Let us denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ the scalar products in $\mathscr{H}$ and $\mathscr{H}'$, and by $\|\cdot\|$ and $\|\cdot\|'$ the associated norms.

Consider on $\mathscr{D}$ the scalar product

$$(g,f)_T = (g,f) + (Tg,Tf)'$$

and the associated norm

$$\|g\|_T^2 = \|g\|^2 + \|Tg\|'^2.$$

Then $\mathscr{D}$, equipped with the scalar product $(\cdot,\cdot)_T$, is a Hilbert space which we call $\mathscr{D}_T$.

Since

$$\frac{(\|Tg\|')^2}{\|g\|_T^2} = \frac{(\|Tg\|')^2}{(\|Tg\|')^2 + \|g\|^2} \leqslant 1,$$

T is bounded from $\mathscr{D}_T$ to $\mathscr{H}'$.

Moreover $U(x)$ is unitary in $\mathscr{D}_T$

$$\begin{aligned}\|U(x)g\|_T^2 &= \|U(x)g\|^2 + (\|TU(x)g\|')^2 \\ &= \|g\|^2 + (\|\pi(x)Tg\|')^2 \\ &= \|g\|^2 + (\|Tg\|')^2 = \|g\|_T^2,\end{aligned}$$

for every x in G and g in $\mathscr{D}$, and $U(x)|_{\mathscr{D}}$ is surjective, since for every $g \in \mathscr{D}$, $g = U(x)U(x^{-1})g$, and $\mathscr{D}$ is stable under $U(x^{-1})$.

By the usual Schur's lemma[17] we have that T is a multiple of an isometry from $\mathscr{D}_T$ to $\mathscr{H}'$, that is

$$(\|Tg\|')^2 = \lambda \|g\|_T^2,$$

for every g in $\mathscr{D}$, which gives

$$(\|Tg\|')^2 = \lambda \|g\|^2 + \lambda (\|Tg\|')^2.$$

From this equality we see that $\lambda \neq 1$ and that

$$(\|Tg\|')^2 = [\lambda/(1-\lambda)]\|g\|^2,$$

so T is a multiple of an isometry from $\mathscr{D}$ to $\mathscr{H}'$ and consequently extends to a multiple of an isometry from $\mathscr{H}$ to $\mathscr{H}'$. Since T was assumed closed, one has $\mathscr{D} = \mathscr{H}$, and the proposition is proved.

If we take $\mathscr{H}' = \mathscr{H}$ and $\pi = U$, the same argument together with the classical lemma of Schur shows that T is a multiple of the identity.

In order to prove Theorem 3.1, we first compute the integral (3.1).

For g admissible, consider the following operator T_g from $\mathscr{H}(U)$ to $L^2(G,dx)$: the domain $\mathscr{D}$ of T_g is the set of vectors f in $\mathscr{H}(U)$ such that

$$\int |(U(x)g,f)|^2 dx < +\infty;$$

for f in $\mathscr{D}$, $T_g f$ is defined by

$$(T_g f)(x) = (U(x)g,f).$$

For f in $\mathscr{D}$ and y in G we have

$$\begin{aligned}\int |(U(x)g,U(y)f)|^2\,dx &= \int |(U(y^{-1}x)g,f)|^2\,dx \\ &= \int |(U(x)g,f)|^2\,dx < +\infty,\end{aligned}$$

by the left invariance of dx. So $U(y)f$ belongs to $\mathscr{D}$ for every y in G, i.e., $\mathscr{D}$ is stable under U and

$$T_g U(y) = L(y) T_g \quad \text{on } \mathscr{D}.$$

We see that $\mathscr{D}$ contains the linear span of the set of vectors $U(x)g$, $x \in G$. This linear span is dense in $\mathscr{H}(U)$ by irreducibility of U, so $\mathscr{D}$ is dense in $\mathscr{H}(U)$.

We prove now that T_g is closed: Take a sequence $\{f_n\}$, with $f_n \in \mathscr{D}$ for every n, converging to f in $\mathscr{H}(U)$, and such that $T_g f_n$ converges in $L^2(G,dx)$ to $\varphi \in L^2(G,dx)$. Then $T_g f_n$ converges to φ weakly in $L^2(G,dx)$ and the sequence of $L^2(G,dx)$ norms $\|T_g f_n\|$ is bounded.

By the continuity of the scalar product in $\mathscr{H}(U)$, the sequence of numbers $(U(x)g,f_n)$ converges to $(U(x)g,f)$ for every $x \in G$. Then we have (see Ref. 18, p. 207) that

$$(U(x)g,f) = \varphi(x)$$

so

$$\int |(U(x)g,f)|^2\,dx = \int |\varphi(x)|^2 dx < +\infty,$$

which implies that f belongs to $\mathscr{D}$ and $T_g f = \varphi$. So T_g is closed.

By the extended Schur's lemma (A.1), T_g is a multiple of an isometry: so $\mathscr{D} = \mathscr{H}(U)$ and T_g is bounded.

Now take g_1 and g_2 admissible; then T_{g1} and T_{g2} are bounded and $T_{g_1}^* T_{g_2}$ is a bounded operator in $\mathscr{H}(U)$. Since

$$L(x) T_{g_i} = T_{g_i} U(x) \quad (i = 1,2),$$

for every x in G, we have

$$U(x) T_{g_1}^* T_{g_2} = T_{g_1}^* T_{g_2} U(x),$$

for every x in G.

So, by Schur's lemma, $T_{g_1}^* T_{g_2}$ is a multiple of the identity:

$$T_{g_1}^* T_{g_2} = C_{g_1 g_2} \mathbf{1}.$$

This means that, for every f_1, f_2 in $\mathscr{H}$,

$$\int \overline{(U(x)g_1, f_1)}(U(x)g_2, f_2)dx = (T_{g_1} f_1, T_{g_2} f_2)_{L^2(G,dx)}$$
$$= (f_1, T^*_{g_1} T_{g_2} f_2)_{\mathscr{H}(U)}$$
$$= C_{g_1 g_2}(f_1, f_2). \qquad \text{(A1)}$$

Let us now consider the number $C_{g_1 g_2}$ defined by

$$C_{g_1 g_2} = \frac{\int \overline{(U(x)g_1 f)}(U(x)g_2 f)dx}{\|f\|^2}. \qquad \text{(A2)}$$

Here, $C_{g_1 g_2}$ is, by (A1), independent of $f \neq 0$.

Let us denote by $\mathscr{A}$ the set of admissible vectors.

The correspondence q: $\mathscr{A} \times \mathscr{A} \to \mathbb{C}$ defined by $q(g_1, g_2) = C_{g_1 g_2}$ for g_1, g_2 in $\mathscr{A}$ is by (A2) a positive, symmetric quadratic form with form domain $\mathscr{A}$. Moreover q is closed: indeed consider on $\mathscr{A}$ the norm $\|\cdot\|_q$ defined by

$$\|g\|_q^2 = \|g\|^2 + q(g,g).$$

Take a $\|\cdot\|_q$-Cauchy sequence of vectors g_n in $\mathscr{A}$. This implies that (i) $\{g_n\}$ is a Cauchy sequence with respect to the $\mathscr{H}(U)$ norm, [so $\{g_n\}$ converges to g in $\mathscr{H}(U)$], and (ii) that

$$\lim_{n,m \to \infty} q(g_n - g_m, g_n - g_m) = 0,$$

which implies that the sequence of functions φ_n of $L^2(G,dx)$ defined by $\varphi_n(x) = (U(x)g_n f)$ is a Cauchy sequence in $L^2(G,dx)$ and so converges strongly to $\varphi \in L^2(G,dx)$. Consequently φ_n converges weakly to φ and the sequence of norms $\|\varphi_n\|$ is bounded.

Moreover the sequence $(U(x)g_n f)$ converges for each x in G to $(U(x)g,f)$. Then (see Ref. 18, p. 207), $(U(x)g, f) = \varphi(x)$ which means that, for every f in $\mathscr{H}$, $\int |(U(x)g, f)|^2\, dx < +\infty$. In particular $\int |(U(x)g,g)|^2\, dx < +\infty$ and so g is admissible. Furthermore

$$\lim_{n \to \infty} \|g_n - g\|_q^2 = \lim_{n \to \infty} \|g_n - g\|^2 + \lim_{n \to \infty} q(g_n - g, g_n - g)$$
$$= 0 + \lim_{n \to \infty} \frac{\|\varphi_n - \varphi\|^2_{L^2(G,dx)}}{\|f\|^2} = 0,$$

which shows that g_n converges in $\|\cdot\|_q$ norm to $g \in \mathscr{A}$.

So $\mathscr{A}$ is complete and then,[19] q is closed. Being as q is a densely defined closed symmetric positive form, by the second representation theorem,[20] there exists a unique positive operator C with domain $\mathscr{A}$ such that

$$q(g_1, g_2) = C_{g_1 g_2} = (Cg_1, Cg_2).$$

This proves parts (i) and (ii) of Theorem 3.1.

Suppose now G is unimodular; then we can see that

$$q(U(y)g_1, U(y)g_2) = \frac{1}{\|f\|^2} \int \overline{(U(xy)g_1, f)}(U(xy)g_2, f)dx$$
$$= \frac{1}{\|f\|^2} \int \overline{(U(x)g_1, f)}(U(x)g_2, f)d(xy^{-1})$$
$$= q(g_1, g_2).$$

This implies

$$(U(y)^{-1}CU(y)g_1, U(y)^{-1}CU(y)g_2) = (Cg_1, Cg_2),$$

and then $U(y)^{-1}CU(y) = C$ on $\mathscr{D}$.

By the remark at the end of the proof of Proposition A.1, it follows that C is a multiple of the identity since C is closed, $\mathscr{D}$ dense in $\mathscr{H}(U)$ and stable under U.

This proves point (iii) of Theorem 3.1.

In particular, if G is unimodular, then $\mathscr{A} = \mathscr{H}(U)$; that is, if one vector is admissible, then all vectors are admissible.

[1] J. R. Klauder and E. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).
[2] J. R. Klauder and B. S. Skagerstam, *Coherent States. Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1984).
[3] G. Warner, *Harmonic Analysis on Semi-simple Lie Groups. I and II* (Springer, Berlin, 1972).
[4] S. A. Gaal, *Linear Analysis and Representation Theory* (Springer, Berlin, 1973).
[5] E. W. Aslaksen, and J. R. Klauder, J. Math. Phys. **10**, 2267 (1969).
[6] A. Grossmann and J. Morlet, SIAM J. Math. Anal. **15**, 723 (1984).
[7] A. Grossmann and J. Morlet, "Decomposition of functions into wavelets of constant shape and related transforms," to appear in *Mathematics + Physics, Lectures on Recent Results* (World Scientific, Singapore).
[8] T. Paul, J. Math. Phys. **25**, 3252 (1984).
[9] T. Paul, "Affine coherent states for the radial Schrödinger equation," in preparation.
[10] A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1972).
[11] A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).
[12] M. Duflo and C. C. Moore, J. Funct. Anal. **21**, 209 (1976).
[13] A. L. Carey, Bull. Austral. Math. Soc. **15**, 1 (1976).
[14] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. I.* (Springer, Berlin, 1963).
[15] H. Meschkowsky, *Hilbertsche Räume mit Kernfunktion* (Springer, Berlin, 1962).
[16] R. Young, *An Introduction to Non-harmonic Fourier Series* (Academic, New York, 1980).
[17] A. A. Kirillov, *Elements of the Theory of Representations* (Springer, Berlin, 1976).
[18] E. Hewitt and K. Stromberg, *Real and Abstract Analysis* (Springer, Berlin, 1969).
[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis* (Academic, New York, 1980).
[20] T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1976).