Remarks on Boundary Layer Expansions

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Abstract

A systematic mathematical methodology for derivation of boundary layer expansions is presented. An explicit calculation of boundary layer sizes is given and proved to be coordinates system independent. It relies on asymptotic properties of symbols of operators. Several examples, including the quasigeostrophic model, are discussed.

1 Introduction and main result

Boundary layers appear in various physical contexts, such as fluid mechanics, thermodynamics, or ferromagnetic media. From the mathematical point of view, they are related to singular perturbation problems in bounded domains. The singular perturbation is due to the presence of small parameters in the dimensionless governing equations. For instance, in magnetohydrodynamics (MHD), the so-called Rossby number (ratio between the angular velocity of the fluid and the angular velocity of the Earth) can be as low as 10^{-7}. Similarly, the Prandtl

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number (ratio between hydrodynamic and magnetic diffusivity) is about 10^{-6} in the Earth's core. In the interior of the domain, these small parameters lead to some reduced dynamics of the equations. For instance, in highly rotating fluids, the velocity field does not vary along the rotation axis, see textbook [11]. This reduced dynamics is often incompatible with boundary conditions. This yields boundary layers, in which the solutions of the equations have strong gradients, in order to satisfy the boundary conditions (typically a Dirichlet condition at a rigid surface).

In a formal setting, boundary layer problems are connected to systems of the type

$$\mathcal{A}^\varepsilon(x, D_x)u^\varepsilon \; + \; \mathcal{Q}^\varepsilon\,(u^\varepsilon) \; = \; f^\varepsilon, \;\; x \in \Omega, \tag{1.1}$$

where $\varepsilon \in \mathbb{R}^d$ describes the small parameters of the system, $\Omega \subset \mathbb{R}^n$ is the domain, $u^\varepsilon(x)$ the unknown, $f^\varepsilon(x)$ the data, $\mathcal{A}^\varepsilon$ is a linear differential operator (most often of elliptic or parabolic type), and $\mathcal{Q}^\varepsilon$ the nonlinear part. One must of course supply these equations with appropriate boundary conditions.

The basic idea, which underlies the mathematical study of all boundary layers, is that the solution u^ε of (1.1) should satisfy an asymptotics of the form

$$\begin{aligned} & u^\varepsilon(x) \;\sim\; u(x) \;+\; u_1^{bl}\left(x, \frac{d(x,\partial\Omega)}{\alpha_1(\varepsilon)}\right) \;+\ldots+\; u_r^{bl}\left(x, \frac{d(x,\partial\Omega)}{\alpha_r(\varepsilon)}\right), \quad \varepsilon \to 0, \\ & u_i^{bl}\,(\theta_i) \;\to 0, \quad \theta_i \to +\infty, \quad \forall i. \end{aligned} \tag{1.2}$$

This means that u^ε should have a regular part, depending on x, but also a singular part, depending on stretched variables $\theta_i = d(x, \partial\Omega)/\alpha_i(\varepsilon)$, with $\alpha_i(\varepsilon) \to 0$ as $\varepsilon \to 0$. This singular part should be localized near the boundary and express the strong gradients of boundary layers. Broadly speaking, the aim of boundary layers studies is to answer the following questions:

i) What are the possible $\alpha_i(\varepsilon)$ (the boundary layer sizes)?

ii) What are the possible profiles u, u_i^{bl} ?

iii) Is the asymptotics (1.2) correct (in a sense to be determined) ?

The first two questions are related to the derivation of boundary layers, whereas the third one is connected to stability issues.

The aim of this note is to give some insight into the derivation problem. In some cases, the derivation is quite easy, and the stability analysis is the most difficult part (see for instance [10, 15] on viscous perturbations of hyperbolic systems). But in most situations of physical interest, it may involve a variety of length scales and equations. A typical example is the description of water in a highly rotating tank. If the water is at rest in the rotating frame attached to the tank, it is well modeled by Stokes equations with Coriolis term:

$$\begin{aligned} \mathbf{e} \times u \;+\; \nabla p \;-\; E\Delta u \;&=\; f, \\ \nabla \cdot u \;&=\; 0, \end{aligned} \tag{1.3}$$

where $\mathbf{e} = (0, 0, 1)$ is the rotation vector, and E is a small parameter called the Ekman number. For such system (with appropriate source term f and boundary conditions), the structure of the solutions near a flat horizontal boundary is understood: there is a boundary layer of size $E^{1/2}$, the Ekman layer (see [6]). In this case, one can even carry an analysis of the full Navier-Stokes equations with rotation: we refer to [12, 14, 4] among others. But

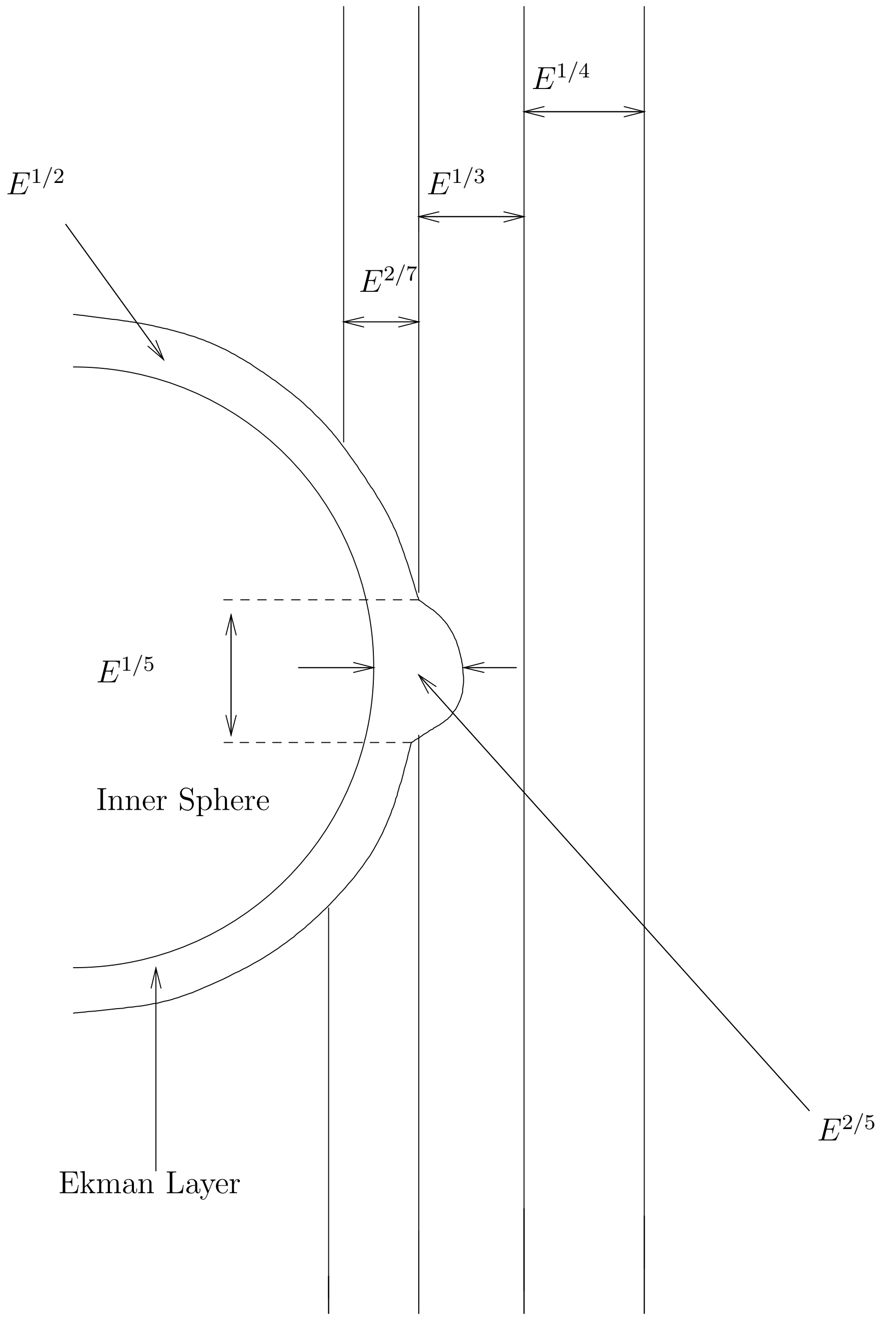

Figure 1: Boundary layers of rotating fluids, near a sphere and at the circumscribing cylinder (following Stewartson [19])

in more elaborate geometries, following the articles of Stewartson [18, 19], many other layers develop: for instance, between two concentric spheres with slightly different rotation speeds, boundary layers of size $E^{1/2}$, $E^{1/3}$, $E^{1/4}$, $E^{2/5}$, ... are expected near the inner sphere and the cylinder circumscribing it (see figure 1).

Note that equations (1.3) are linear. In many cases, the derivation of the boundary layers (at least of the boundary layer sizes) is not affected by the nonlinearity $\mathcal{Q}^\varepsilon$ of (1.1): it only matters in stability questions. In short, one can say that the nonlinear term does not create the boundary layer, but may destabilize it. Therefore, we restrict ourselves to linear equations

$$\mathcal{A}^\varepsilon(x, D_x)\, u^\varepsilon \;=\; f^\varepsilon. \tag{1.4}$$

Classically, the problem of the derivation is tackled through one of the following ways:

i) an explicit calculation, where the exact solution is computed and expanded. However, such technique is restrictive (an analytical computation is rarely tractable) and often tedious (see for instance [19], with Bessel functions).

ii) the so-called "method of matched asymptotics", see [5, 22, 23] for a general presentation. The idea of this method is to patch two asymptotic expansions: an "outer" regular one, far from the boundary, and an "inner" singular one, close to it. Such expansions must coincide in an overlap domain, which provides boundary conditions for both outer and inner profiles. The structure of the inner expansion (including the layer sizes) is sought by trial, through the principle of least degeneracy (*c.f.* [5] for a detailed explanation). This method has been applied with success to various physical systems, including singularities near edges. However, the determination of layer sizes leads to heavy computations, and must often be supplied with refined physical arguments (see [21] on rotating fluids).

A new method of derivation has been recently introduced in [8] (see also the proceedings [9]). It is based on the asymptotic analysis of the symbol $\mathcal{A}^\varepsilon(x, \xi)$ as ε goes to zero, for $x \in \partial\Omega$. Briefly, up to use local coordinates near $x \in \partial\Omega$, one can consider that equation (1.4) holds in a neighborhood of $x = (x', 0)$ in $\mathbb{R}^n_+$. Then, the leading idea of the derivation is to carry a Fourier-Laplace analysis: boundary layer sizes near x are deduced from modal solutions

$$u^\varepsilon(\tilde{x}) \;=\; \exp\left(i\xi^\varepsilon \cdot \tilde{x}\right)\, V^\varepsilon, \quad V^\varepsilon \neq 0, \tag{1.5}$$

of the equation with frozen coefficients

$$\mathcal{A}^\varepsilon(x, \partial_{\tilde{x}})\, u^\varepsilon \;=\; 0.$$

In other words, one must consider the characteristic manifold of $\mathcal{A}^\varepsilon$,

$$\sigma\left(a^\varepsilon\right) = \left\{ \left(x, \xi^\varepsilon\right),\, a^\varepsilon(x, \xi^\varepsilon) = 0 \right\}$$

where $a^\varepsilon(x, \xi) = \det \mathcal{A}^\varepsilon(x, \xi)$. Broadly speaking, if $\xi = (\zeta, \xi_n)$ is the dual variable of $x = (x', x_n) \in \mathbb{R}^n_+$, boundary layers correspond to (x, ξ^ε) in $\sigma\left(a^\varepsilon\right)$ that satisfy

$$\mathcal{I}m\, \xi_n^\varepsilon \;\to +\infty, \quad \varepsilon \to 0.$$

The size of the boundary layers is then given by $|\mathcal{I}m\, \xi_n^\varepsilon|^{-1}$. In most cases, boundary layer equations follow, by appropriate rescaling of the symbol. This method has been used in [8] at a fully formal level. It has been applied with efficiency to various geophysical systems, including rotating fluids or MHD. It has allowed to recover the main boundary layers of the physical literature, with very few and simple algebraic computations (see [8] for all details).

The aim of the present paper is to provide this formal method with some mathematical basis. We will limit ourselves to scalar equations. In this reduced (but still large!) setting, we will show how the microlocal analysis of (1.4) is linked to boundary layer expansions (1.2). We will present a set of conditions (assumptions (H1) to (H5)) that ensures the existence of such expansions, *with clearly identified boundary layer sizes and equations.* Hypothesis (H1) to (H5) involve symbols derived from a^ε. They will be shown to be *intrinsic*, although these symbols depend on the choice of local coordinates.

Let us specify the framework of the study. Let X be a smooth Riemannian manifold, $n = \dim X$. In all situations of physical interest, X will be of the type $\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, where the torus $\mathbb{T}^{n_1}$ models periodic boundary conditions in n_1 physical variables. The Riemannian metric will be the Euclidean metric induced by $\mathbb{R}^{2n_1} \times \mathbb{R}^{n_2}$. Let Ω a domain of X. We denote $Y = \partial\Omega$ the boundary of Ω, which is a $n-1$ dimensional submanifold of X. We assume that Y has a finite number of connected components, Y_1 to Y_N. We endow Y with the Riemannian structure induced by X. We assume that there exist a tubular neighborhood of Y in X, $T > 0$, and a smooth diffeomorphism

$$\mathcal{T} \approx Y \times (-T, T), \quad x \approx (y, t),$$

such that $t > 0$ in $\mathcal{T} \cap \Omega$, $t < 0$ in $\mathcal{T} \cap (X \setminus \bar{\Omega})$. For instance, such assumption is satisfied when $X = \mathbb{R}^n$ and Ω is a bounded open subset. *In the whole sequel, we identify $\mathcal{T}$ and $Y \times (-T, T)$.*

We consider the scalar problem,

$$a^\varepsilon(x, D_x)\, u^\varepsilon \;=\; f, \quad x \in \Omega, \tag{1.6}$$

$$b_l^\varepsilon(x, D_x)\, u^\varepsilon \;=\; g_l, \quad x \in Y, \quad l \in \{1, \ldots, L\}, \tag{1.7}$$

where $a^\varepsilon(x, D_x)$, respectively $b_l^\varepsilon(x, D_x)$, is a differential operator depending smoothly on $x \in \Omega \cup \mathcal{T}$, respectively $x \in Y$, and polynomial in ε. The source term f, respectively g_l, is smooth on $\Omega \cup \mathcal{T}$, resp. on Y. Let $m = \deg(a^\varepsilon) > 0$.

In the region $Y \times [0, T')$, $T' > 0$ small enough, we can write

$$a^\varepsilon(x, D_x) \;=\; a^\varepsilon(x, D_y, D_t) \;=\; \sum_{j=0}^{m} a_j^\varepsilon(x, D_y)\, D_t^j$$

where a_j^ε is a differential operator on Y, of order $m - j$, with coefficients smooth in x, polynomial in ε. We assume that the leading coefficient reads

$$a_m^\varepsilon(x) \;=\; \varepsilon^M\, a_m(x), \quad M \in \mathbb{N}, \quad a_m(x) \neq 0, \; \forall x.$$

Before we state our main result, we still need to precise what we mean by boundary layer expansion. Let φ be a cut-off near the boundary. Precisely, $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$, $\varphi = 1$ on $Y \times [0, T/4)$, and $\varphi = 0$ on $\overline{\Omega} \setminus (Y \times [0, T/2))$.

Definition 1. *A family of functions $u^\epsilon(x)$ is* of boundary layer type *if it reads*

$$u^\varepsilon(x) \;=\; u_r^\varepsilon(x) \;+\; \varphi(x) v_{bl}^\varepsilon(x) \tag{1.8}$$

where the regular part u_r^ε and the singular part v_{bl}^ε have the following asymptotic expansions, for some positive δ and γ_j^i:

$$u_r^\varepsilon(x) \;\sim\; \sum_{k=0}^{\infty} \varepsilon^{\delta k}\, u^k(x), \quad u^k \in \mathcal{C}^\infty\left(\overline{\Omega}\right), \tag{1.9}$$

uniformly on every compact subset of $\overline{\Omega}$, *and and*

$$v_{bl}^{\varepsilon}(y,t) \sim \sum_{k=0}^{\infty} \varepsilon^{\delta k} \sum_{\substack{1\le i\le N \\ 1\le j\le r_i}} \mathbf{1}_{Y^i}(y)\, v_j^{i,k}\left(y, \frac{t}{\varepsilon^{\gamma_j^i}}\right), \tag{1.10}$$

uniformly on every compact subset of $Y \times [0,T)$, *where* $v_j^{i,k} = v_j^{i,k}(y,\theta)$ *are not all zero and belong to* $\mathcal{X}^i = \mathcal{S}\left(\mathbb{R}_t^+; \mathcal{C}^\infty(Y^i)\right)$.

We recall that $\mathcal{S}$ stands for the Schwartz space of fast decreasing functions. Spaces χ^i express the localization of boundary layer profiles. Note that in our definition, the singular part is non zero (at least one $v_j^{i,k}$ is non zero). Our main result resumes to

Theorem 1. *Suppose that assumptions (H1) to (H5), c.f. next sections, hold. Then, the system*

$$\begin{cases} a^\varepsilon(x,D_x)\, u^\varepsilon \; = \; f, & x \in \Omega, \\ b_l^\varepsilon(x,D_x)\, u^\varepsilon \; = \; g_l, & x \in Y, \quad l \in \{1,\dots,L\}, \end{cases} \tag{1.11}$$

has a solution of boundary layer type, up to remainder terms that are $O(\varepsilon^\infty)$ *uniformly on every compact subset.*

We remind that "$O(\varepsilon^\infty)$" means $O(\varepsilon^m)$ for any power m (here for the supremum norm on every compact subset). Theorem 1 is the rigorous translation of the formal method described in article [8]. In the following, we will present assumptions (H1) to (H5), which allow to build solutions of boundary layer type. Broadly, assumptions (H1) to (H3) (see section 2) relate to the determination of exponents γ_j^i. Assumption (H4) (section 3) is linked to the derivation of boundary layer profiles $v_j^{i,k}$. Finally, (H5) (section 4) is connected to the construction of the regular part u_r^ε. The proof of theorem 1 follows. Application to the quasigeostrophic equation is given in section 5.

2 The singular perturbation

2.1 Local characteristic manifolds and assumptions (H1)-(H2)

The aim of the paper is to describe the structure of the solutions u^ε of (1.6) near the boundary. Let $y \in Y$, and $\left(\mathcal{O}, \chi = (x', x_n)\right)$ a local chart in X around y, with

$$x_n(y) = 0, \quad \frac{\partial x_n}{\partial t}(y) \neq 0.$$

The local coordinates (x', x_n) lie in $\mathbb{R}^{n-1} \times \mathbb{R}$. We denote by $a_\chi^\varepsilon = a_\chi^\varepsilon(x,\xi)$ the symbol of the operator a^ε in this local chart: precisely,

$$a_\chi^\varepsilon(x,\xi) \;:=\; e^{-i\chi(x)\cdot\xi}\left[a^\varepsilon(x, D_{\tilde{x}}) e^{i\chi(\tilde{x})\cdot\xi}\right]_{|\,\tilde{x}=x}.$$

Our idea is to deduce the singular structure of u^ε from the local symbols a_χ^ε, precisely from their characteristic manifolds. We are interested in modal solutions that are singular with respect to ε in the direction normal to the boundary. This means we wish to consider wavevectors $\xi^\varepsilon = (\zeta, \xi_n^\varepsilon)$ such that

$$|\xi_n^\varepsilon| \to +\infty, \quad \varepsilon \to 0.$$

We have the following

Proposition 1. *For all* $\zeta \in \mathbb{R}^{n-1}$,

i) *the* m *roots of* $a_\chi^\varepsilon(y, \zeta, \cdot)$ *can be written as* m *functions* $\xi_1^\zeta(\varepsilon)$, $\ldots$, $\xi_m^\zeta(\varepsilon)$ *for* $\varepsilon > 0$ *small enough.*

ii) *There exists* $p = p(\zeta) \in \mathbb{N}^*$ *such that, for* $i = 1, \ldots, m$, $\xi_i^\zeta(\varepsilon^p)$ *has an extension meromorphic in* ε.

This proposition follows from standard results of complex analysis. For fixed ζ, $a_\chi^\varepsilon(y, \zeta, \cdot)$ is a polynomial in ξ_n with coefficients holomorphic in ε, and we refer to Kato [13] for a detailed study.

Corollary 1. *For all* $\zeta \in \mathbb{R}^{n-1}$, $1 \leq i \leq m$, *one of the two following possibilities occurs:*

1. $\xi_i^\zeta(\varepsilon)$ *has a limit as* $\varepsilon > 0$ *goes to zero, that we denote* $\xi_{i,0}^\zeta$.
2. *there exists a unique* $p_i = p_i(\zeta) \in \mathbb{Q}_*^+$ *such that* $\varepsilon^{p_i}\, \xi_i^\zeta(\varepsilon)$ *has a non-zero limit as* $\varepsilon > 0$ *goes to zero, that we denote* $\eta_{i,0}^\zeta$.

Note that if the leading coefficient of $a_\chi^\varepsilon(y, \zeta, \cdot)$ does not vanish when $\varepsilon \to 0$, the second possibility does not occur. Roots with singular behaviour exist only if the highest order term goes to zero with ε, which is typical of a singular perturbation.

We denote by $s = s(\zeta)$ the number of roots ξ_i^ζ satisfying 2. Several indices i may correspond to the same value of p_i. Let γ_1, $\ldots$, γ_r, $r \leq s$, be the distinct values of p_i. For $1 \leq j \leq r$, we call m_j the number of indexes i such that $p_i = \gamma_j$. We make the following assumption:

(H1) There exists $R = R(y) > 0$**, such that for all** $|\zeta| > R$**, the values of** r**,** (γ_1, m_1) **up to** (γ_r, m_r) **are independent of** ζ**.**

This also implies that $s = \sum m_j$ is independent on ζ. Note that, using notation γ_r, we implicitly assume $r \geq 1$. When $r = 0$, the statements in the sequel become empty, so that we do not pay attention to this case.

Assumption (H1) allows to state

Proposition 2. *Let* $|\overline{\zeta}| > R$. *There exists* $\mathcal{W} = \mathcal{W}\left(\overline{\zeta}\right)$ *a neighborhood of* $\overline{\zeta}$, *and constants* $\varepsilon_0 = \varepsilon_0\left(\overline{\zeta}\right) > 0$, $\delta = \delta\left(\overline{\zeta}\right) > 0$, *such that: for all* ζ *in* $\mathcal{W}$, *for all* $0 < \varepsilon \leq \varepsilon_0$, *the roots of* $a_\chi^\varepsilon(y, \zeta, \cdot)$ *can be divided into* $r+1$ *disjoint sets* $\mathcal{Z}_j^\varepsilon(\zeta)$, $0 \leq j \leq r$ *satisfying:*

i) *card* $\mathcal{Z}_0^\varepsilon(\zeta) = m - s$, *and for all* ξ_n *in* $\mathcal{Z}_0^\varepsilon(\zeta)$,

$$0 \;\leq\; |\xi_n| \;\leq\; \delta^{-1}.$$

ii) *For all* $1 \leq j \leq r$, *card* $\mathcal{Z}_j^\varepsilon(\zeta) = m_j$, *and for all* ξ_n *in* $\mathcal{Z}_j^\varepsilon(\zeta)$,

$$\frac{\delta}{\varepsilon^{\gamma_j}} \;\leq\; |\xi_n| \;\leq\; \frac{\delta^{-1}}{\varepsilon^{\gamma_j}}.$$

Remark 1. *Suppose proposition 2 holds. Let* $\zeta \in \mathcal{W}$. *By propositions 1 and 1, for* ε *small enough, the roots of* $a_\chi^\varepsilon(y, \zeta, \cdot)$ *can be written* $\xi_i^\zeta(\varepsilon)$, $1 \leq i \leq m$. *Up to reindex, the roots satisfy:*

$$\xi_i^\zeta \;\xrightarrow[\varepsilon \to 0]{}\; \xi_{i,0}^\zeta, \quad s+1 \leq i \leq m,$$
$$\varepsilon^{\gamma_j}\, \xi_i^\zeta \;\xrightarrow[\varepsilon \to 0]{}\; \eta_{i,0}^\zeta \neq 0, \quad 1 \leq j \leq r, \quad p_i = \gamma_j.$$

Therefore, there exists ε_0^ζ such that , for all $0 < \varepsilon < \varepsilon_0^\zeta$,

$$\mathcal{Z}_0^\varepsilon(\zeta) \;=\; \left\{\xi_i^\zeta(\varepsilon), \quad s+1 \leq i \leq m\right\},$$

$$\mathcal{Z}_j^\varepsilon(\zeta) \;=\; \left\{\xi_i^\zeta(\varepsilon), \quad p_i = \gamma_j\right\}$$

for $1 \leq j \leq r$. We emphasize that in proposition 2, the sets $\mathcal{Z}_j^\varepsilon(\zeta)$ are defined for $0 < \varepsilon \leq \varepsilon_0$, where ε_0 depends only on $\overline{\zeta}$. This is not the case for the $\xi_i^\zeta(\varepsilon)$, defined only up to ε_0^ζ, which depends a priori on ζ. Moreover, up to consider a smaller ε_0, sets $\mathcal{Z}_j^\varepsilon(\zeta)$ satisfying i) and ii) will be automatically disjoint.

Proof: Let $|\zeta| > R$. With the notations of remark 1, for all $0 < \varepsilon \leq \varepsilon_0^\zeta$, we can factorize

$$a_\chi^\varepsilon(y,\zeta,\xi_n) \;=\; a_0^{\varepsilon,\zeta}(\xi_n) \prod_{j=1}^r a_j^{\varepsilon,\zeta}\left(\varepsilon^{\gamma_j}\xi_n\right), \tag{2.12}$$

where

$$\begin{aligned} a_0^{\varepsilon,\zeta}(\xi_n) \;&=\; \varepsilon^M \, \varepsilon^{-\sum \gamma_j m_j} \, C_\chi \prod_{i=s+1}^m \left(\xi_n \;-\; \xi_i^\zeta(\varepsilon)\right), \\ a_j^{\varepsilon,\zeta}(\eta) \;&=\; \prod_{p_i=\gamma_j} \left(\eta \;-\; \varepsilon^{\gamma_j}\xi_i^\zeta(\varepsilon)\right), \quad \text{for } 1 \leq j \leq r, \end{aligned}$$

and $C_\chi \;:=\; \left(\dfrac{\partial x_n}{\partial t}(y)\right)^m \; a_m(y)$. We then define

$$a_{r,\chi}^\varepsilon(\zeta,\eta) \;:=\; \varepsilon^{-\beta_r} \, a_\chi^\varepsilon\left(\zeta, \frac{\eta}{\varepsilon^{\gamma_r}}\right),$$

with $\beta_r = M - m\gamma_r$. It is a polynomial in η, whose coefficients are functions of the type

$$\sum_{k=-k_0}^{k_0} b_k(\zeta)\, \varepsilon^{\alpha k},$$

for some fixed $k_0 \in \mathbb{N}$, $\alpha > 0$, and polynomials symbols b_k.

Using (2.12), it is easily seen that: for all ζ, locally uniformly in ξ_n,

$$a_{r,\chi}^\varepsilon(\zeta,\eta) \;\xrightarrow[\varepsilon\to 0]{}\; C_\chi\, \eta^{m-m_r} \prod_{p_i=\gamma_r} \left(\eta \;-\; \eta_{i,0}^\zeta\right).$$

It is *a priori* pointwise convergence in ζ. However, the pointwise convergence of the coefficients, of the type

$$\sum_{k=-k_0}^{k_0} b_k(\zeta)\, \varepsilon^{\alpha k} \;\xrightarrow[\varepsilon\to 0]{}\; b(\zeta)$$

holds if and only if $b_k = 0$ for $k < 0$, and $b_0 = b$. This implies that the coefficients of

$$a_{r,\chi}^0(\zeta,\eta) \;:=\; C_\chi\, \eta^{m-m_r} \prod_{p_i=\gamma_r} \left(\eta \;-\; \eta_{i,0}^\zeta\right)$$

are polynomials, and that the convergence is in fact locally uniform in ζ.

Let $|\overline{\zeta}| > R$. Let Γ_r a curve enclosing the $\eta_{i,0}^{\overline{\zeta}}$ and no other root of $a_{r,\chi}^0(\overline{\zeta},\cdot)$. As this polynomial has smooth coefficients, Rouché's theorem yields a neighborhood $\mathcal{W}$ of $\overline{\zeta}$ such that, for all ζ in $\mathcal{W}$, Γ_r encloses the $\eta_{i,0}^{\zeta}$ (and no other root of $a_{r,\chi}^0(\zeta,\cdot)$). Now, the convergence of $a_{r,\chi}^\varepsilon$ to $a_{r,\chi}^0$ is locally uniform in ζ. Still by Rouché's theorem, and up to take smaller $\mathcal{W}$, we get: for all ζ in $\mathcal{W}$, Γ_r encloses exactly m_r roots of $a_{r,\chi}^\varepsilon(\zeta,\cdot)$.

Back to a_χ^ε, this yields a $\delta = \delta(\overline{\zeta}) > 0$, such that $a_\chi^\varepsilon(y,\zeta,\cdot)$ has m_r roots satisfying

$$\frac{\delta}{\varepsilon^{\gamma_r}} \;\le\; |\xi_n| \;\le\; \frac{\delta^{-1}}{\varepsilon^{\gamma_r}}.$$

This provides us with the set $\mathcal{Z}_r^\varepsilon(\zeta)$.

The construction of the other sets is made inductively, using the polynomials

$$a_{j,\chi}^\varepsilon(\zeta,\eta) \;:=\; \varepsilon^{-\beta_j}\, a_\chi^\varepsilon\left(y,\zeta,\frac{\eta}{\varepsilon^{\gamma_j}}\right),$$

with $\beta_j = M - m\gamma_j + \sum_{k\ge j+1}(\gamma_j-\gamma_k)$. We only indicate how to build $\mathcal{Z}_{r-1}^\varepsilon(\zeta)$. The general induction argument is left to the reader.

The coefficients of the polynomial $a_{r-1,\chi}^\varepsilon(\zeta,\cdot)$ are of the same type as $a_{r,\chi}^\varepsilon(\zeta,\cdot)$. Using again (2.12), we have this time, locally uniformly in ζ,η,

$$\begin{aligned} a_{r-1,\chi}^\varepsilon(\zeta,\eta) &\xrightarrow[\varepsilon\to 0]{} a_{r-1,\chi}^0(\zeta,\eta),\\ a_{r-1,\chi}^0(\zeta,\eta) &:= C_\chi\, \eta^{m-m_r-m_{r-1}} \prod_{p_i=\gamma_{r-1}} \left(\eta - \eta_{i,0}^\zeta\right)\left(\prod_{p_i=\gamma_r} -\eta_{i,0}^\zeta\right), \end{aligned}$$

and $a_{r-1,\chi}^0$ has smooth symbols in ζ as coefficients. Note that

$$\prod_{p_i=\gamma_r} -\eta_{i,0}^\zeta \;\ne\; 0$$

by definition of the $\eta_{i,0}^\zeta$. Reasoning as above with Rouché's theorem, up to reduce $\mathcal{W}$, ε_0 and δ, we find a curve Γ_{r-1} (independent on ζ in $\mathcal{W}$, $\varepsilon \le \varepsilon_0$) which encloses exactly m_{r-1} roots of $a_\chi^\varepsilon(y,\zeta,\cdot)$, all satisfying

$$\frac{\delta}{\varepsilon^{\gamma_{r-1}}} \;\le\; |\xi_n| \;\le\; \frac{\delta^{-1}}{\varepsilon^{\gamma_{r-1}}}, \quad \zeta\in\mathcal{W}.$$

They define the set $\mathcal{Z}_{r-1}^\varepsilon(\zeta)$. □

Let $\overline{\zeta}$, $\mathcal{W}$ and ε_0 as in proposition 2. For all ζ in $\mathcal{W}$, for all $0<\varepsilon\le\varepsilon_0$, we can factorize the symbol as

$$a_\chi^\varepsilon(y,\zeta,\xi_n) \;=\; a_0^\varepsilon(\zeta,\xi_n) \prod_{j=1}^{r} a_j^\varepsilon\left(\zeta, \varepsilon^{\gamma_j}\xi_n\right), \tag{2.13}$$

where

$$a_0^\varepsilon(\zeta,\xi_n) \;:=\; C_\chi^\varepsilon \prod_{z\in\mathcal{Z}_0^\varepsilon(\zeta)} \left(\xi_n - z\right),$$

$$a_j^\varepsilon(\zeta,\eta) \;:=\; \prod_{z\in\mathcal{Z}_j^\varepsilon(\zeta)} \left(\eta - \varepsilon^{\gamma_j} z\right),$$

for $1 \leq j \leq r$. Note that for all ζ in $\mathcal{W}$, and for $0 < \varepsilon \leq \varepsilon_0^{\zeta}$, we have

$$a_j^{\varepsilon}(\zeta, \cdot) \;=\; a_j^{\varepsilon,\zeta}$$

where $a_j^{\varepsilon,\zeta}$ was introduced in (2.12). We can extend a_j^{ε} to $\varepsilon = 0$ by

$$a_0^0(\zeta, \cdot) \;:=\; \prod_{i=s+1}^{m} \left(\cdot \,-\, \xi_{i,0}^{\zeta}\right), \quad a_j^0(\zeta, \cdot) \;:=\; \prod_{p_i=\gamma_j} \left(\cdot \,-\, \eta_{i,0}^{\zeta}\right),$$

for $1 \leq j \leq r$. We emphasize that all a_j^0 are globally defined in ζ, *i.e.* on $\{|\zeta| > R\}$. On the contrary, the a_j^{ε} are only locally defined in (ε, ζ), on an open subset $[0, \varepsilon_0(\overline{\zeta})) \times \mathcal{W}(\overline{\zeta})$. The regularity of these various polynomials is given in

Proposition 3. *For all $0 \leq j \leq r$:*

i) *The coefficients of a_j^0 are symbols in ζ, (restricted to $|\zeta| > R$).*

ii) *Let $|\overline{\zeta}| > R$. The coefficients of a_j^{ε} are smooth functions of (ε, ζ) in a neighborhood of $(\varepsilon = 0, \overline{\zeta})$.*

Proof: We use the notations introduced in the proof of proposition 2. Again, we deal only with $j = r, r-1$, and leave the general induction argument to the reader.

i) In the course of the previous proof, we have shown that the coefficients of

$$a_{r,\chi}^0(\zeta, \eta) \;=\; C_\chi\, \eta^{m-m_r} \prod_{p_i=\gamma_r} \left(\eta \,-\, \eta_{i,0}^{\zeta}\right) \;=\; C_\chi\, \eta^{m-m_r}\, a_r^0(\zeta, \eta)$$

are polynomial in ζ, $|\zeta| > R$, so that the coefficients of a_r^0 share the same property. Also, the coefficients of

$$\begin{aligned} a_{r-1,\chi}^0(\zeta, \eta) \;&=\; C_\chi\, \eta^{m-m_r-m_{r-1}} \prod_{p_i=\gamma_{r-1}} \left(\eta \,-\, \eta_{i,0}^{\zeta}\right) \left(\prod_{p_i=\gamma_r} -\eta_{i,0}^{\zeta}\right). \\ &=\; C_\chi\, \eta^{m-m_r-m_{r-1}}\, a_{r-1}^0(\zeta, \eta) \left(\prod_{p_i=\gamma_r} -\eta_{i,0}^{\zeta}\right). \end{aligned}$$

are polynomial in ζ. As

$$C_\chi \left(\prod_{p_i=\gamma_r} -\eta_{i,0}^{\zeta}\right)$$

is the coefficient of order $m - m_r$ of $a_{r,\chi}^0$, it is a smooth symbol, which does not cancel by definition of the $\eta_{i,0}^{\zeta}$. We deduce that the coefficients of $a_{r-1}^0(\zeta, \cdot)$ are smooth symbols in ζ as well.

ii) In the course of previous proof, we have shown the existence of a curve Γ_r (independent on $\zeta \in \mathcal{W}$, $\varepsilon \leq \varepsilon_0$), enclosing the set $\varepsilon^{\gamma_r}\, \mathcal{Z}_r^{\varepsilon}(\zeta)$, and none of the other roots of $a_{r,\chi}^{\varepsilon}(\zeta, \cdot)$. By Cauchy's formula, we deduce for all $k \in \mathbb{N}$,

$$\sum_{\eta \,\in\, \varepsilon^{\gamma_r - 1}\, \mathcal{Z}_r^{\varepsilon}} \eta^k \;=\; \frac{1}{2i\pi} \int_{\Gamma_r} \tilde{\eta}^k\, \frac{\partial_{\tilde{\eta}}\, a_{r,\chi}^{\varepsilon}(\cdot, \tilde{\eta})}{a_{r,\chi}^{\varepsilon}(\cdot, \tilde{\eta})}\, d\tilde{\eta}.$$

The right-hand side, so the left-hand side, defines a smooth function of (ε, ζ) in $[0, \varepsilon_0) \times \mathcal{W}$. Now, it is well known that the coefficients of a polynomial are themselves polynomial in such symmetric functions of the roots. We deduce that the coefficients of $a_r^\varepsilon(\zeta,\cdot)$ are equally smooth. We can proceed similarly for a_{r-1}^ε, through the formula

$$\sum_{\eta \in \varepsilon^{\gamma_{r-1}} \mathcal{Z}_{r-1}^\varepsilon} \eta^k \;=\; \frac{1}{2i\pi} \int_{\Gamma_{r-1}} \tilde{\eta}^k \, \frac{\partial_{\tilde{\eta}} \, a_{r-1,\chi}^\varepsilon(\cdot,\tilde{\eta})}{a_{r-1,\chi}^\varepsilon(\cdot,\tilde{\eta})} \, d\tilde{\eta},$$

and the smoothness of a_{r-1}^ε follows as well. □

On the basis of (H1), we hope for an asymptotics of type (1.2) with $\alpha_j = \varepsilon^{\gamma_j}$ for some indices j. Nevertheless, for such an asymptotics to be true, it is reasonable that r, (γ_1, m_1), …, (γ_r, m_r) depend neither on the local chart χ, nor on $y \in Y$. With above notations, we have : for all $|\zeta| > R$, for all $1 \le j \le r$,

$$a_j^0(\zeta, 0) \;=\; (-1)^{m_j} \prod_{p_i = \gamma_j} \eta_{i,0}^\zeta \neq 0. \tag{2.14}$$

By proposition 3, $a_j^0(\zeta, 0)$ is a symbol in ζ (restricted to $|\zeta| > R$). By equation 2.14, it does not cancel. In general, this property of no cancellation is not preserved by a change of variable. Therefore, the assumption (H1), as well as r,(γ_1, m_1), …, (γ_r, m_r) depend on the local charts. To overcome this problem, we need to make a stronger assumption:

(H2) For all $1 \le j \le r$, $a_j^0(\cdot, 0)$ is elliptic.

Ellipticity is here understood in the classical sense: for all j, the principal symbol of $a_j^0(\cdot, 0)$ satisfies

$$\sigma\left(a_j^0(\cdot,0)\right)(\zeta) \neq 0, \quad \forall \zeta \in \mathbb{R}^{n-1} \setminus \{0\}.$$

Note that, up to consider a larger R, (H2) implies (2.14). Note also that (H2) is preserved by a change of the tangential variable x'.

2.2 Asymptotic invariance and assumption (H3)

We can now state the following invariance result:

Theorem 2. (Invariance through diffeomorphism)
Assume (H1)-(H2). Let $(\mathcal{O}_\Psi, \Psi)$ a local chart in X around y such that

$$\Psi_n(y) = 0, \quad \frac{\partial \Psi_n}{\partial t}(y) \neq 0.$$

Let (H1)$_\psi$-(H2)$_\psi$ be the same as (H1)-(H2), with Ψ in place of χ. Then, (H1)$_\psi$-(H2)$_\psi$ holds, with the same r, (γ_1, m_1), …, (γ_r, m_r).

Remark 2. *In short, theorem 2 states that the conjunction (H1)-(H2) is intrinsic, as well as the main features of the singularities (exponents γ_j and multiplicities r, m_j). We stress that (H1)-(H2) involves the whole symbol $a_\chi^\varepsilon(y,\cdot)$ and not only its principal symbol. Therefore, it is not obvious that it should be preserved by another choice of coordinates.*

Proof: Let us first set a notation: for any smooth $\phi = (\phi', \phi_n) : U \to V$, $0 \in U$, for any symbol $P(\zeta, \eta)$ defined on $\mathbb{R}^{n-1} \times \mathbb{R}$, and for any $\gamma \in \mathbb{R}$, we introduce

$$\phi_*^\gamma P(\zeta,\eta) \;:=\; e^{i\phi(0)\cdot\left(\zeta, \varepsilon^{-\gamma}\eta\right)} \left[P(D_{x'}, \varepsilon^\gamma D_{x_n}) \, e^{-i\phi(x)\cdot\left(\zeta,\varepsilon^{-\gamma}\eta\right)} \right]_{|\,x=0}$$

We also denote

$$\phi_* P(\zeta,\eta) \;:=\; \phi_*^0 P(\zeta,\eta).$$

Let $(\mathcal{O},\chi)$ and $(\mathcal{O}_\Psi,\Psi)$ the local charts of the theorem. It is clear that (H1) and (H2) (and r, $(\gamma_1,m_1),\ldots,(\gamma_r,m_r)$) are not affected by a translation of each coordinate. Therefore, with no loss of generality, we can assume that $\chi(y)=\Psi(y)=0$. Let $\phi=\Psi\circ\chi^{-1}$. It is a smooth diffeomorphism between two neighborhoods U,V of 0. Let now $P^\varepsilon(\zeta,\eta)$ a polynomial in ε,ζ,η. The proof of theorem 2 will follow from the study of $\phi_*^\gamma P^\varepsilon$. Precisely, we will show:

Proposition 4. *Let $\gamma>0$. Then, uniformly in each compact subset of $\mathbb{R}^{n-1}\times\mathbb{R}$,*

$$\lim_{\varepsilon\to 0}\phi_*^\gamma\, P^\varepsilon(\zeta,\eta)\;=\;\overline{\phi}_*\,P^0(\zeta,\eta), \tag{2.15}$$

where $\overline{\phi}(x',x_n)\;=\;\big(\phi'(x',0),\;\partial_{x_n}\phi_n(x',0)\,x_n\big)$.

Remark 3. *If we denote*

$$\sum_j P_j^0(\zeta)\,\eta^j$$

the expansion of P^0 with respect to η, then

$$\overline{\phi}_*\,P^0(\zeta,\eta)\;=\;\sum_j Q_j^0(\zeta)\,\eta^j,$$

with

$$Q_j^0(\zeta)\;=\;\left[P_j^0(D_{x'})\left(e^{-i\phi'(x',0)\cdot\zeta}\,\frac{\partial\phi_n}{\partial x_n}(x',0)^j\right)\right]_{|\,x'=0}. \tag{2.16}$$

The symbols Q_j^0 are polynomial in ζ. More precisely, classical symbolic calculus (see [2]) provides the following (finite) expansion:

$$Q_j^0(\zeta)\;\sim\;\sum_{\alpha\geq 0}\frac{1}{\alpha!}\,\partial_{x'}^\alpha\left[e^{ir'(x')}\,\frac{\partial\phi_n}{\partial x_n}(x',0)^j\right]_{|x'=0}\;D_\zeta^\alpha P_j^0\left(d_{x'}\phi'(0)\zeta\right), \tag{2.17}$$

where

$$r'\left(x'\right)\;:=\;\phi'\left(x',0\right)\;-\;d_{x'}\phi'(0)\;\left(x'\right).$$

The terms of rank α in this expansion are polynomial in ζ, of degree less than deg $\left(P_j^0\right)-|\alpha|/2$. (see again [2] for details).

Remark 4. *In the case $\gamma=0$, the limit* (2.15) *is easily replaced by*

$$\lim_{\varepsilon\to 0}\phi_* P^\varepsilon(\zeta,\eta)\;=\;\phi_* P^0(\zeta,\eta). \tag{2.18}$$

We postpone temporarily the proof of the proposition, and we show how to deduce theorem 2 from it. We must prove that $(\mathrm{H1})_\Psi$-$(\mathrm{H2})_\Psi$ holds, that is we can replace the local symbol a_χ^ε by the other local symbol a_Ψ^ε. With a slight abuse of notation, we identify x and $\chi(x)$ so that we work with $x\in U$, U a neighborhood of $y=0$ in $\mathbb{R}^n$. Hence,

$$a_\Psi^\varepsilon(0,\cdot)\;=\;\phi_*\,a_\chi^\varepsilon(0,\cdot). \tag{2.19}$$

Proof of $(\mathrm{H1})_\Psi$: We consider

$$a_{r,\Psi}^\varepsilon(\zeta,\eta):=\varepsilon^{-\beta_r}a_\Psi^\varepsilon(0,\zeta,\frac{\eta}{\varepsilon^{\gamma_r}}),$$

and, as in previous proofs,

$$a_{r,\chi}^\varepsilon(\zeta,\eta):=\varepsilon^{-\beta_r}\,a_\chi^\varepsilon(0,\zeta,\frac{\eta}{\varepsilon^{\gamma_r}}).$$

Equation (2.19) becomes

$$a^\varepsilon_{r,\Psi} \;=\; \phi_*^{\gamma_r}\, a^\varepsilon_{r,\chi}.$$

We have seen in the previous proofs that $a^\varepsilon_{r,\chi}$ is polynomial in $\varepsilon^\alpha, \zeta, \eta$ for some appropriate $\alpha > 0$. Moreover,

$$a^0_{r,\chi}(\zeta,\eta) \;=\; C_\chi\, \eta^{m-m_r} a^0_r(\zeta,\eta), \quad |\zeta| > R. \tag{2.20}$$

If we expand

$$a^0_{r,\chi}(\zeta,\eta) \;=\; \sum_j a^0_{r,j}(\zeta)\, \eta^j,$$

we get from proposition 4 that, for $|\zeta|$ large enough,

$$\lim_{\varepsilon\to 0}\; a^\varepsilon_{r,\Psi}(\zeta,\eta) \;=\; \sum_j b^0_{r,j}(\zeta)\, \eta^j,$$

with

$$b^0_{r,j}(\zeta) = \left[a^0_{r,j}(D_{x'}) \left(e^{-i\phi_0'(x')\cdot\zeta}\, \frac{\partial \phi_n}{\partial x_n}(x',0)^j \right) \right]_{|\, x'=0}$$

Using (2.20), we deduce that the term of lowest degree (with respect to η) in $\lim_{\varepsilon\to 0} a^\varepsilon_{r,\Psi}(\zeta,\eta)$ is $b^0_{r,m-m_r}(\zeta)\,\eta^{m-m_r}$. Moreover, by (2.17), the principal symbol of $b^0_{r,m-m_r}$ is

$$\begin{aligned}\sigma\left(b^0_{r,m-m_r}\right)(\zeta,\eta) \;&=\; \sigma\left(a^0_{r,m-m_r}\right)(d\phi_0'(0)\,\zeta)\\ &= C_\chi\, \sigma\left(a^0_r(\cdot,0)\right)(d\phi_0'(0)\,\zeta).\end{aligned}$$

Note that $d_{x'}\phi'(0)$ is a diffeomorphism in $\mathbb{R}^{n-1}$. By assumption (H2), $a^0_r(\cdot,0)$ is elliptic, hence $b^0_{r,m-m_r}$ is still elliptic. As a consequence, it does not vanish for $|\zeta|$ large enough. Therefore, at fixed ζ, the polynomial $\lim_{\varepsilon\to 0} a^\varepsilon_{r,\Psi}(\zeta,\cdot)$ has m_r roots away from zero. By Rouché's theorem, it is still the case for $\varepsilon \le \varepsilon_0$ small enough. Back to the original variables, we find m_r roots of $a^\varepsilon_\Psi(0,\zeta,\cdot)$ satisfying

$$\frac{\delta}{\varepsilon^{\gamma_r}} \;\le\; |\xi_n| \le \frac{\delta^{-1}}{\varepsilon^{\gamma_r}}.$$

With similar arguments, we can find m_{r-1} roots of $a^\varepsilon_\Psi(0,\zeta,\cdot)$ which scale like $\varepsilon^{-\gamma_{r-1}}$. We consider this time

$$a^\varepsilon_{r-1,\Psi} \;:=\; \varepsilon^{-\beta_{r-1}}\, a^\varepsilon_\Psi(0,\zeta,\frac{\eta}{\varepsilon^{\gamma_{r-1}}}),$$

as well as $a^\varepsilon_{r-1,\chi}$. Again, $a^\varepsilon_{r-1,\chi}$ is a polynomial in $(\varepsilon^\alpha,\zeta,\eta)$ and

$$a^0_{r-1,\chi}(\zeta,\eta) \;=\; C_\chi\, \eta^{m-m_r-m_{r-1}}\, a^0_{r-1}(\zeta,\eta)\, a^0_r(\zeta,\eta).$$

We then apply the same reasoning as above, replacing $a^0_r(\cdot,0)$ by the product $a^0_{r-1}(\cdot,0)a^0_r(\cdot,0)$, which is still elliptic.

Proceeding recursively, we find, for all $1 \le j \le r$, m_j roots of $a^\varepsilon_\Psi(0,\zeta,\cdot)$ such that

$$\frac{\delta}{\varepsilon^{\gamma_j}} \;\le\; |\xi_n| \le \frac{\delta^{-1}}{\varepsilon^{\gamma_j}},$$

for $\varepsilon \le \varepsilon_0$ small enough, $\delta = \delta(\varepsilon_0)$ small enough. Finally, by (2.18),

$$\lim_{\varepsilon\to 0}\, a^\varepsilon_\Psi(0,\zeta,\xi_n) \;=\; \phi_*\left(a^0_0(\zeta,\xi_n)\, \prod_{j=1}^r a^0_j(\zeta,0)\right).$$

At fixed ζ, the right-hand side defines a polynomial in ξ_n, of degree $m-s$. By Rouché's theorem, this yields $m-s$ roots of $a^\varepsilon_\Psi(0,\zeta,\cdot)$ with

$$0 \le |\xi_n| \le \delta^{-1}.$$

As the degree of $a^\varepsilon_\Psi(0,\zeta,\cdot)$ is $m = m - s + \sum m_j$, we obtain in this way all the roots. Assumption $(\mathrm{H1})_\Psi$ follows with the same r, $\gamma_1,\ldots,\gamma_r$.

Proof of $(\mathrm{H2})_\Psi$: We have just established $(\mathrm{H1})_\Psi$. Just as for a^ε_χ, we can factorize

$$a^\varepsilon_\Psi(y,\zeta,\xi_n) = \tilde{a}^\varepsilon_0(\zeta,\xi_n)\prod_{j=1}^{r}\tilde{a}^\varepsilon_j(\zeta,\varepsilon^{\gamma_j}\xi_n) \tag{2.21}$$

(see factorization (2.13)), and we have to show that: for all $1\le j\le r$, $\tilde{a}^0_j(\zeta,0)$ is elliptic in a neighborhood of y. But from above identities, we have got easily:

$$C_\Psi\,\sigma\left(\tilde{a}^0_r(\cdot,0)\right)(\zeta) = C_\chi\,\sigma\left(a^0_r(\cdot,0)\right)(d\phi_0'(0)\zeta),$$

and

$$C_\Psi\ \sigma\left(\tilde{a}^0_{r-1}(\cdot,0)\right)(\zeta)\ \sigma\left(\tilde{a}^0_r(\cdot,0)\right)(\zeta) = \\ C_\chi\ \sigma\left(a^0_{r-1}(\cdot,0)\right)(d_{x'}\phi'(0)\,\zeta)\ \sigma\left(a^0_r(\cdot,0)\right)(d_{x'}\phi'(0)\,\zeta),$$

and so on. As ellipticity is preserved by inversion and product, we deduce $(\mathrm{H2})_\Psi$ from a simple recursion. □

Proof of proposition 4: The proof relies on the representation of $\phi^\gamma_* P^\varepsilon$ as an oscillatory integral. The asymptotic analysis of such integral will be performed through "stationary phase type" theorems. This approach is very classical in symbolic calculus, to establish the properties of conjugation, product, or transformation under a diffeomorphism (see, among many, textbooks [3, 2]). However, in the standard setting, one is mainly concerned with the principal symbol, so that the small parameter in the asymptotics is the wavelength $1/|\xi|$. In our framework, the natural idea is to work with the parameter ε instead of the wavelength.

Let $u\in\mathcal{C}^\infty_c(U)$, $\zeta\in\mathbb{R}^{n-1}$, $\eta\neq 0$. We wish to compute

$$I^\varepsilon = \left[P^\varepsilon(D_{x'},\varepsilon^\gamma D_{x_n})\left(e^{-i\phi(x)\cdot(\zeta,\varepsilon^{-\gamma}\eta)}u(x)\right)\right]_{|\,x=0}$$

$I^\varepsilon = I^\varepsilon(\zeta,\eta)$ is defined by the oscillatory integral:

$$I^\varepsilon := \frac{1}{(2\pi)^n}\int e^{-ix\cdot(\tilde\zeta,\tilde\eta)}\,P^\varepsilon\left(\tilde\zeta,\varepsilon^\gamma\tilde\eta\right)\,u(x)\,e^{i\phi(x)\cdot(\zeta,\varepsilon^{-\gamma}\eta)}\,dx\,d(\tilde\zeta,\tilde\eta).$$

Following a scheme of proof used in [2], we make the change of variables:

$$\tilde\eta := \tilde\eta - \frac{\partial_{x_n}\phi_n(x',0)}{\varepsilon^\gamma}\eta.$$

By the Fubini theorem for oscillatory integrals, we can write

$$I^\varepsilon = \frac{1}{(2\pi)^n}\int e^{-ix'\cdot\tilde\zeta}\,J^\varepsilon(x',\tilde\zeta)\,dx'\,d\tilde\zeta,$$

with

$$J^\varepsilon(x',\tilde\zeta) := \int e^{-ix_n\,\tilde\eta}\, e^{i\,\varepsilon^{-\gamma}\, r_\eta(x)}\, u(x)\, e^{i\phi'(x)\cdot\zeta}$$

$$P^\varepsilon\left(\tilde\zeta, \varepsilon^\gamma\tilde\eta + \partial_{x_n}\phi_n(x',0)\eta\right)\, dx_n d\tilde\eta,$$

where

$$r_\eta(x) \;:=\; \eta\,\left(\phi_n(x) - \partial_{x_n}\phi_n(x',0)\right).$$

We set

$$\Phi^\varepsilon(x',\tilde\eta) := \int e^{-ix_n\,\tilde\eta}\, e^{i\frac{r_\eta(x)}{\varepsilon^\gamma}}\, u(x)\, e^{i\phi'(x)\cdot\zeta}\, dx_n,$$

so that

$$J^\varepsilon(x',\tilde\zeta) \;=\; \int P^\varepsilon\left(\tilde\zeta,\, \varepsilon^\gamma\tilde\eta \,+\, \partial_{x_n}\phi_n(x',0)\,\eta\right)\, \Phi^\varepsilon(x',\tilde\eta)\, d\tilde\eta.$$

For fixed η, x', we introduce the phase

$$\varepsilon^{-\gamma}\, f(x_n) \;:=\; \varepsilon^{-\gamma}\,(r_\eta(x) \;-\; \varepsilon^\gamma\tilde\eta\, x_n)\,,$$

with derivative

$$f'(x_n) \;=\; \eta\left(\partial_{x_n}\phi_n(x) - \partial_{x_n}\phi_n(x',0)\right) - \varepsilon^\gamma\tilde\eta.$$

- If $|\,\eta\,\partial_{x_n}\phi_n(x',0) \,+\, \varepsilon^\gamma\tilde\eta\,| < 1/C$, C such that $|\,\eta\,\partial_{x_n}\phi(x)\,| \geq 2/C$ for x in a neighborhood of 0, then $|f'(x_n)| \;\geq\; 1/C$.
- If $|\,\eta\,\partial_{x_n}\phi_n(x',0) \,+\, \varepsilon^\gamma\tilde\eta\,| \;\geq\; C$, C such that $|\,\eta\partial_{x_n}\phi(x)\,| \;\leq\; C/2$ for x in a neighborhood of 0, then $|\,f'(x_n)\,| \;\geq\; C/2$.

We deduce from the non-stationary phase theorem (see [2]) that for large enough C, outside

$$\frac{1}{C} \;\leq\; \left|\eta\partial_{x_n}\phi_n(x',0) + \varepsilon^\gamma\tilde\eta\right| \;\leq\; C,$$

for all k,

$$\left|\Phi^\varepsilon(x',\tilde\eta)\right| \;\leq\; C_k\,\left(1 + |\tilde\eta| + \frac{\eta}{\varepsilon^\gamma}\right)^{-k}. \tag{2.22}$$

Still following [2], we introduce a truncation function α such that $\alpha \in \mathcal{C}^\infty_c(\mathbb{R})$, $\alpha(\xi_n) = 1$ for $C^{-1} \leq |\xi_n| \leq C$, $\alpha(\xi_n) = 0$ for $|\xi_n| \leq (2C)^{-1}$. We divide

$$J^\varepsilon(x',\tilde\zeta) = J_1^\varepsilon(x',\tilde\zeta) + J_2^\varepsilon(x',\tilde\zeta)$$

with

$$J_2^\varepsilon \;=\; \int \alpha\left(\partial_{x_n}\phi_n(x',0) + \frac{\varepsilon^\gamma}{\eta}\tilde\eta\right) P^\varepsilon\left(x,\, \tilde\zeta,\, \varepsilon^\gamma\tilde\eta \,+\, \partial_{x_n}\phi_n(x',x_n)\,\eta\right)\, \Phi^\varepsilon(x',\tilde\eta)\, d\tilde\eta.$$

Note that J_1^ε and J_2^ε are polynomials in $\tilde\zeta$ (as the only dependence on $\tilde\zeta$ is in P^ε). From the estimate (2.22), we deduce

$$\left|\partial^\beta_{(x',\tilde\zeta)}\, J_1^\varepsilon(x',\tilde\zeta)\right| \;\leq\; C_{\beta,k}\,|\tilde\zeta|^{N-\beta}\,\varepsilon^k, \quad N\in\mathbb{N}, \quad \forall\beta, k.$$

Hence,

$$I_1^\varepsilon \;:=\; \frac{1}{(2\pi)^n}\int e^{-ix'\cdot\tilde\zeta}\, J_1^\varepsilon(x',\tilde\zeta)\, dx'\, d\tilde\zeta$$

satisfies $I_1^\varepsilon = O(\varepsilon^k)$ for all k. Moreover, a rapid look at the proof shows that all estimates hold uniformly in every compact subset of $\mathbb{R}^{n-1}\times\mathbb{R}_*$.

It remains to estimate the "stationary terms" J_2^ε and I_2^ε. For this, we will use a lemma whose statement and proof can be found in [2]:

Lemma 1. *([2])*
Let $r = r(z,\theta) \in \mathcal{C}^\infty(\mathbb{R}\times\mathcal{O})$, $\mathcal{O}$ open subset of $\mathbb{R}^p$, such that $\partial_z r(0,\theta) = 0$. Let $b^\lambda = b^\lambda(z,\eta,\theta) \in \mathcal{C}^\infty(R\times\mathbb{R}\times\mathcal{O})$, depending on a parameter $\lambda > 0$ so that, for some M,

$$\forall\,\beta,\beta', \quad \left|\partial_z^\beta\partial_\eta^{\beta'} b^\lambda(z,\eta,\theta,\lambda)\right| \le C_{\beta,\beta'}\,\lambda^{M-\beta'}. \tag{2.23}$$

Assume moreover that b^λ has compact support in z, uniformly with respect to η, θ, λ. Then, the integral

$$J(\lambda) \;=\; (2\pi)^{-n}\int e^{-i\,z\,\eta} e^{i\lambda r(z,\theta)}\, b^\lambda(z,\eta,\theta)\,dz\,d\eta$$

has the following asymptotic expansion for $\lambda\to+\infty$

$$J(\lambda) \;\sim\; \sum_{\alpha\ge 0}\frac{1}{\alpha!}\partial_z^\alpha\left(e^{i\lambda r(z,\theta)} D_\eta^\alpha b^\lambda(z,0,\theta)\right)|_{z=0},$$

where terms of rank α are bounded by $\lambda^{M-\alpha/2}$. This expansion holds locally uniformly with respect to θ.

The function $J_2^\varepsilon(x,\tilde\zeta)$ is polynomial in $\tilde\zeta$, with coefficients that read

$$\int e^{-ix_n\,\tilde\eta}\; e^{i\varepsilon^{-\gamma} r_\eta(x)\,\eta}\; b_j^\varepsilon(x_n,\tilde\eta,\zeta,\eta,x')\,dx_n.$$

Such integrals can be easily put in the framework of lemma 1, with $\lambda := \varepsilon^{-\gamma}$, $z := x_n$, $\eta := \tilde\eta$, and $\theta := (\zeta,\eta,x')$. Thus, $r(z,\theta) := r_\eta(x)$ satisfies $\partial_z r(0,\theta) = 0$. Besides, the function

$$b^\lambda(z,\eta,\theta) := b_j^\varepsilon(x_n,\tilde\eta,\zeta,\eta,x')$$

satisfies estimate (2.23) with $M = 0$. Its support in z is contained in the support of u, so uniform with respect to the other variables. Hence, we can apply lemma 1. The first term of the expansion provides

$$J_2^\varepsilon(x,\tilde\zeta) \underset{\varepsilon\to 0}{\longrightarrow} J_2^0(x,\tilde\zeta) \;:=\; e^{i\,\phi'(x',0)}\,P^0\left(\tilde\zeta,\,\partial_{x_n}\phi_n(x',0)\,\eta\right)\,u(x',0)$$

and consequently

$$I_2^\varepsilon \;\underset{\varepsilon\to 0}{\longrightarrow}\; \frac{1}{(2\pi)^n}\int e^{-ix'\cdot\tilde\zeta}\,J_2^0(x,\tilde\zeta)\,dx'\,d\tilde\zeta$$

with convergence in every compact subset of $\mathbb{R}^{n-1}\times\mathbb{R}_*$. As we handle polynomials in η, the uniform convergence extends to compact subsets of $\mathbb{R}^{n-1}\times\mathbb{R}$. By taking u equal to 1 in a neighborhood of 0 in U, we obtain the result. $\square$

We end this section on asymptotic invariance with a definition:

Definition 2. *Let $y \in Y$. We say that a^ε is uniformly singular at y if (H1)-(H2) holds at y. The corresponding γ_j's, $1\le j\le r$ are called singular exponents (of a^ε) at y, and the m_j's are their multiplicity.*

These definitions make sense because of theorem 2. We recall that the singular exponents are positive rational numbers (see corollary 1).

Definition 3. *Let $Y' \subset Y$. We say that a^ε is uniformly singular on Y', if it is uniformly singular at all $y \subset Y'$, with constant singular exponents and multiplicities.*

In the whole sequel, we will make the following assumption:

(H3) The operator a^ε is uniformly singular on each connected component of the boundary Y.

3 Boundary layer sizes and equations

3.1 Definitions and hypothesis (H4)

Let Y' a connected component of Y. By assumption (H3), a^ε is uniformly singular on Y'. We denote by γ_1 to γ_r the singular exponents, and m_1 to m_r their respective multiplicity. In a region $Y' \times (0, T')$, equation (1.6) reads

$$a^\varepsilon(x, D_y, D_t)u^\varepsilon \;=\; 0.$$

Let $y \in Y'$. Let $(\mathcal{O}', \chi')$ a local chart in Y' around y. The application

$$\chi : \mathcal{O}' \times (-T', T') \mapsto \mathbb{R}^n, \quad (y', t) \mapsto (\chi'(y'), t)$$

defines a local chart in X around y. Hence, we can define a local operator $a^\varepsilon_\chi(x, D_{x'}, D_t)$, with symbol $a^\varepsilon_\chi(x, \zeta, \xi_n)$.

On one hand, we have a finite expansion of the type

$$a^\varepsilon\left(y, D_y, \frac{D_\theta}{\varepsilon^{\gamma_j}}\right) \;=\; \sum_{k=-k_0}^{k_0} \sum_{k'} \underset{\chi'}{A_{k,k'}}(y, D_y)\, D_\theta^{k'}\, \varepsilon^{\alpha k} \tag{3.24}$$

for some appropriate $\alpha > 0$ and smooth differential operators $A_{k,k'}$ on Y'. Hence,

$$a^\varepsilon_\chi\left(y, \zeta, \frac{\eta}{\varepsilon^{\gamma_j}}\right) \;=\; \sum_{k=-k_0}^{k_0} \sum_{k'} A_{k,k'}(y, \zeta)\, \eta^{k'}\, \varepsilon^{\alpha k}. \tag{3.25}$$

On the other hand, with notations introduced in (2.13), we have for $|\zeta| > R(y)$,

$$\lim_{\varepsilon \to 0} \varepsilon^{\beta_j} a^\varepsilon_\chi\left(y, \zeta, \frac{\eta}{\varepsilon^{\gamma_j}}\right) \;=\; C_\chi\, \eta^{m - \sum_{j' \geq j} m_{j'}}\; a^0_j(\zeta, \eta) \prod_{j' > j} a^0_{j'}(\zeta, 0) \tag{3.26}$$

From (3.25) and (3.26) we deduce

$$\lim_{\varepsilon \to 0} \varepsilon^{\beta_j} a^\varepsilon\left(y, D_y, \frac{D_\theta}{\varepsilon^{\gamma_j}}\right) \;=\; D_\theta^{m - \sum_{j' \geq j} m'_j} a_{\gamma_j}(y, D_y, D_\theta), \tag{3.27}$$

where a_{γ_j} is a smooth differential operator. It satisfies, for $|\zeta| > R(y)$,

$$a_{\gamma_j, \chi}(y, \zeta, \eta) \;=\; a^0_j(\zeta, \eta) \prod_{j' > j} a^0_{j'}(\zeta, 0)$$

For $1 \leq j \leq r$, we say that the operator a_{γ_j} is a singular operator (associated to the singular exponent γ_j).

Special attention will be paid to localized solutions of equation

$$a_\gamma \, u \; = \; f, \tag{3.28}$$

where a_γ is a singular operator, associated to the singular exponent γ. Indeed, the functions $v_j^{i,k}$ in expansion (1.10) will satisfy equations of type (3.28). Therefore, we introduce the Fréchet space

$$\mathcal{X} \; := \; \mathcal{S}\left(\mathbb{R}_t^+; \mathcal{C}^\infty(Y')\right).$$

We also denote

$$\mathcal{X}_0 \; := \; \{u \in \mathcal{X}, \quad a_\gamma u = 0\}.$$

We set the following definition

Definition 4. *We say that γ is a boundary layer exponent (and ε^γ a boundary layer size) on Y', if there exist subspaces $\mathcal{X}_0' \neq \{0\}$, $\mathcal{X}'$ with:*

i) $\mathcal{X}_0' \subset \mathcal{X}_0, \quad \mathcal{X}_0' \subset \mathcal{X}' \subset \mathcal{X}$.

ii) $D_\theta \mathcal{X}' = \mathcal{X}', \quad D_y \mathcal{X}' \subset \mathcal{X}', \quad \theta \mathcal{X}' \subset \mathcal{X}', \quad f\,\mathcal{X}' \subset \mathcal{X}'$ *for all smooth functions* $f \in \mathcal{C}^\infty(Y')$.

iii) *For all $f \in \mathcal{X}'$, (3.28) has a solution in $\mathcal{X}'$.*

Remark 5. *Note that identity $D_\theta \mathcal{X}' = \mathcal{X}'$ implies that $\mathcal{X}'$ is preserved by both D_θ and its inverse D_θ^{-1} (which is obviously defined on $\mathcal{X}$). Note also that $\mathcal{X}_0'$ and $\mathcal{X}'$ are not uniquely determined. However, the subspaces $+\,\mathcal{X}_0'$ and $+\,\mathcal{X}'$ where the algebraic sum is taken over all subspaces $\mathcal{X}_0'$ and $\mathcal{X}'$ satisfying i)-iii), are well-defined, and are the largest subspaces satisfying i)-iii)*

Remark 6. *The definition of a boundary layer exponent is at first sight dependent on the diffeomorphism*

$$\chi : \mathcal{T}' \approx Y' \times (-T', T'), \quad x \approx (y, t),$$

where $\mathcal{T}'$ is a tubular neighborhood of Y'. However, if

$$\psi : \tilde{\mathcal{T}}' \approx Y \times (-\tilde{T}', \tilde{T}'), \quad x \approx (\tilde{y}, \tilde{t})$$

is another such diffeomorphism, we can link the singular operators $a_{\gamma,\chi}$ and $a_{\gamma,\psi}$ following computations of theorem 2. Indeed, we have

$$a_{\gamma,\psi}(y, D_y, D_t) \; = \; \overline{\phi}_* \, a_{\gamma,\chi}(y, D_{\tilde{y}}, D_{\tilde{t}}),$$

where $\phi = (\phi', \phi_n) = \psi \circ \chi^{-1} \in Y' \times (-T_{min}, T_{min})$, where $T_{min} \; := \; \min(T', \tilde{T}')$. , and

$$\overline{\phi} : (y', t') \; \mapsto \; \left(\phi'(y', 0), \, \frac{\partial \phi_n}{\partial t}(y', 0)\, t'\right).$$

Thus, solutions of (3.28) with $a_\gamma = a_{\gamma,\chi}$ and $a_\gamma = a_{\gamma,\psi}$ are deduced from each other by the change of variable $\overline{\phi}$. As soon as

$$\frac{\partial \phi_n}{\partial t} \; \geq \; C_0 > 0,$$

this change of variables preserves the properties of a boundary layer exponent.

We end this section with an assumption that will ensure the existence of a singular (boundary layer part) in (1.8):

(H4) Among the singular exponents given by (H3), there exists at least one boundary layer exponent.

3.2 Solvability

The verification of (H4) relies on solutions of (3.28) in $\mathcal{X}$. In some particular cases, one can give some effective criteria that ensure the solvability of this equation.

3.2.1 Order zero operators

For many physical systems, the coefficients of the singular operators are just smooth functions

$$a_\gamma(y, D_y, D_\theta) \; = \; a_\gamma(y, D_\theta).$$

Hence, (3.28) resumes to a collection of linear ODE's with constant coefficients, indexed by y in Y'. Thus, to determine if γ is a boundary layer exponent is relatively easy. For instance,

Proposition 5. *Assume that there exists a closed curve*

$$\Gamma \subset \{\mathcal{I}m\,(\eta) > 0\}$$

enclosing the set

$$\{\eta, \quad \exists y, a_\gamma(y,\eta) = 0\} \,\cap\, \{\mathcal{I}m\,(\eta) > 0\}$$

of all roots with positive imaginary part. If this set is not empty, then γ is a boundary layer exponent, otherwise it is not.

Proof: By Rouché's theorem and smoothness of a_γ, the number $m^+(y)$ of roots of $a_\gamma(y,\cdot)$ lying inside Γ is locally constant, As Y' is connected, it is constant. It is then well-known (see [1]) that the homogeneous space $\mathcal{X}_0$ is spanned by functions of the type

$$f(y)\,\omega_j(y,\theta),\; f \in \mathcal{C}^\infty(Y'),\; \omega_j(y,\theta) \; = \; \int_\Gamma e^{i\,\eta\,\theta}\, a_\gamma^{-1}(y,\eta)\,\eta^j\,d\eta, \quad j \in \mathbb{N}.$$

Moreover, the dimension of $\text{Vect}\;\{\omega_j(y,\theta),\; j \in \mathbb{N}\}$ is m^+. The result follows, using $\mathcal{X}'_0 = \mathcal{X}_0$, and

$$\mathcal{X}' \;=\; \text{Vect}\;\left\{ f(y)\;\int_\Gamma e^{i\,\eta\,\theta}\,a_\gamma^{-l}(y,\eta)\,\eta^j\,d\eta\,, \quad f \in \mathcal{C}^\infty(Y'), \quad l \in \mathbb{N}, j \in \mathbb{Z} \right\}$$

□

As shown in proposition 5, the boundary layer is connected to roots with positive imaginary part. This echoes the formal method used in [8], where roots ξ_n^ε with $\mathcal{I}m\;\xi_n^\varepsilon \to +\infty$ were considered.

3.2.2 Operators with constant coefficients

Suppose that $Y' = \mathbb{R}^{n-1}$ (or $\mathbb{T}^{n-1}$) and a_γ has constant coefficients. One can use the Fourier transform over $\mathbb{R}^{n-1}$ (or T^{n-1}), which turns (3.28) into

$$a_\gamma(\zeta, D_\theta)\hat{u} \; = \; \hat{f}.$$

This is still a collection of linear ODE's, with constant coefficients, indexed by $\zeta \in \mathbb{R}^{n-1}$ (or $\mathbb{T}^{n-1}$).

3.2.3 General case and link with spectral problems

In general the singular operator $a_\gamma(y, D_y, D_\theta)$ will be a "true" differential operator. But in many cases it will give rise to a (spectral) decomposition which will allow us to solve the problem. In this paragraph we will describe briefly the methodology without focusing on technical details.

Let us remind that a solution of a problem of type

$$a_\gamma(y, D_y, D_\theta)\, u(y,\theta) \;=\; f(y,\theta) \tag{3.29}$$

that decreases as $\theta \to +\infty$ can be handled by Fourier-Laplace transform in θ. Denoting $\tilde{u}(y,\xi) \;:=\; \int_0^\infty e^{-\xi\theta} u(y,\theta) d\theta$, (3.29) is equivalent to an equation of the form :

$$a_\gamma(y, D_y, \xi)\, \tilde{u}(y,\xi) \;=\; \tilde{f}(y,\xi) + F(y,\xi) := G(y,\xi) \tag{3.30}$$

where F involves boundary terms issued from integrations by parts in the Laplace transform. Let us suppose that $A_\xi := a_\gamma(y, D_y, \xi)$ admits a (sometimes spectral) decomposition weakly on L^2 of the form:

$$A_\xi \;=\; \int g(\lambda,\xi) |\varphi_{\xi,\lambda} >< \varphi_{\xi,\lambda}|\, d\lambda \tag{3.31}$$

where $|\varphi_{\xi,\lambda} >< \varphi_{\xi,\lambda}|$ denotes the orthogonal projection on $\varphi_{\xi,\lambda}$, and the (spectral) parameter λ is either complex or real. Let us suppose moreover that the function $g(\lambda,\xi)$ and the family of vectors $\varphi_{\xi,\lambda}$ are analytic in ξ (let us mention here that a naive definition of analytic vectors is enough for the examples we have in mind: for a precise definition see [17]). We can solve (3.30) by using the decomposition (3.31) . Indeed denoting

$$\hat{G}(\lambda,\xi) \;:= < \varphi_{\xi,\lambda} | G(\lambda, .) > := \int \overline{\varphi_{\xi,\lambda}} G(\lambda,\xi) d\xi$$

we get:

$$u(y,\theta) \;=\; \frac{1}{2\pi i} \int d\lambda \int_{x-i\infty}^{x+i\infty} d\xi\, \frac{1}{g(\lambda,\xi)}\, \hat{G}(\lambda,\xi)\, e^{\xi\theta}\, \varphi_{\xi,\lambda}(y). \tag{3.32}$$

Therefore by the Cauchy theorem it is enough to know the zeros of g. Let us show how these ideas apply in the case of a non-differential operator, a pure differential one and in general when the boundary is a flat manifold.

Let $a_\gamma(y, D_y, D_\theta) = b(y, D_\theta)$. In this case $\varphi_{\xi,\lambda}(y) = \delta(y-\lambda)$ and $g(\lambda,\xi) = b(\lambda,\xi)$. Therefore the l.h.s of (3.32) can be computed easily: let $\xi_i(y)$ a family of non-degenerate zeros of $b(y,\xi)$, that is $b(y,\xi_i(y)) = 0$, $\partial_\xi b(y\xi_i(y)) \neq 0$. then:

$$\begin{aligned}
u(y,\theta) &= \frac{1}{2\pi i} \int d\lambda \int_{x-i\infty}^{x+i\infty} d\xi \frac{1}{g(\lambda,\xi)} \hat{G}(\lambda,\xi) e^{\xi\theta} \varphi_{\xi,\lambda}(y) \\
&= \frac{1}{2\pi i} \int d\lambda \int_{x-i\infty}^{x+i\infty} d\xi\, \frac{1}{b(\lambda,\xi)}\, \hat{G}(\lambda,\xi)\, e^{\xi\theta}\, \delta(y-\lambda) \\
&= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} d\xi\, \frac{1}{b(y,\xi)}\, \hat{G}(y,\xi)\, e^{\xi\theta} \\
&= \sum_i \frac{1}{\partial_\xi b(y,\xi)|_{\xi=\xi_i(y)}}\, \hat{G}(y,\xi_i(y))\, e^{\xi_i(y)\theta}
\end{aligned}$$

One sees immediately on this example that the boundary layer will exist as soon as one of the $\xi_i(y)$ satisfies $\mathcal{R}e(\xi_i(y)) \geq c > 0$, and there is no crossing between the different branches.

Let us suppose now that $a_\gamma(y, D_y, D_\theta) = c(D_y, D_\theta)$. In this case $\varphi_{\xi,\lambda}(y) = e^{i\lambda y}$ and $g(\lambda,\xi) = c(\lambda,\xi)$. Let as before $\xi_i(\lambda)$ a family of non-degenerate zeros of $b(\lambda,\xi)$, that is $b(\lambda,\xi_i(\lambda)) = 0$, $\partial_\xi b(\lambda,\xi_i(\lambda)) \neq 0$. We get:

$$
\begin{aligned}
u(y,\theta) &= \frac{1}{2\pi i}\int d\lambda \int_{x-i\infty}^{x+i\infty} d\xi\, \frac{1}{g(\lambda,\xi)}\,\hat G(\lambda,\xi)\, e^{\xi\theta}\varphi_{\xi,\lambda}(y)\\
&= \frac{1}{2\pi i}\int d\lambda \int_{x-i\infty}^{x+i\infty} d\xi\, \frac{1}{c(\lambda,\xi)}\,\hat G(\lambda,\xi)\, e^{\xi\theta}\, e^{i\lambda y}\\
&= \sum_i \frac{1}{\partial_\xi b(\lambda,\xi)|_{\xi=\xi_i(\lambda)}}\,\hat G(\lambda,\xi_i(\lambda))\, e^{\xi_i(\lambda)\theta}\, e^{i\lambda y}.
\end{aligned}
$$

Therefore the boundary layer will exist as soon as there exists a family $\xi_i(\lambda)$ with

$$\mathcal{R}e(\xi_i(\lambda)) \geq c > 0,\ \forall\lambda.$$

In the case where the boundary is $\mathbb{R}^n$ (or $\mathbb{T}^n$) and that the operator $a_\gamma(y, D_y, D_\theta)$ is a differential operator in y with polynomial coefficients a decomposition similar to the preceding one can be handled with the so-called coherent states. Let us briefly describe the construction on $\mathbb{R}$, the generalization to $\mathbb{R}^n$ (and $\mathbb{T}^n$ by periodization) is straightforward. Let $(p,q)\in\mathbb{R}^2$ and let $\varphi_{p,q}$ the family of functions defined in [7]

$$\varphi_{p,q}(y) := \pi^{-1/4}e^{-(y-q)^2/2}e^{ipy}.$$

Let us now consider the operator $a_\gamma(y, D_y, \xi) = a(y, D_y, \xi)$. Being with polynomial coefficients it can be surely rewritten as

$$a(y, D_y, \xi) = \sum_{k,j} \alpha_{k,j}(\xi)(y+\partial_y)^k(y-\partial_y)^j \tag{3.33}$$

Let us define $a_{AW}(q,p,\xi) := \sum_{k,j}\alpha_{k,j}(\xi)(q+ip)^k(y-ip)^j$

Lemma 2.

$$a(y, D_y, \xi) = \int_{\mathbb{R}^2} a_{AW}(q,p,\xi)|\varphi_{p,q}><\varphi_{p,q}|dpdq$$

where the integral is understood in the weak sense.

Proof: an easy computation shows that we have the following decomposition of identity

$$\sum_{\mathbb{R}^2} |\varphi_{p,q}><\varphi_{p,q}|dpdq = 1$$

Moreover it is easy to check that

$$(y+\partial_y)\varphi_{p,q} = (q+ip)\varphi_{p,q}$$

and therefore

$$<\varphi_{p,q}|(y-\partial_y) = (q-ip)<\varphi_{p,q}|.$$

Inserting in (3.33) the decomposition of the identity we get:

$$
\begin{aligned}
(y, D_y, \xi) &= \sum_{k,j}\alpha_{k,j}(\xi)(y+\partial_y)^k(y-\partial_y)^j\\
&= \sum_{k,j}\alpha_{k,j}(\xi)(y+\partial_y)^k\int_{\mathbb{R}^2}|\varphi_{p,q}><\varphi_{p,q}|dpdq(y-\partial_y)^j\\
&= \int_{\mathbb{R}^2}\sum_{k,j}\alpha_{k,j}(q+ip)^k|\varphi_{p,q}><\varphi_{p,q}|(q-ip)^j dpdq\\
&= \int_{\mathbb{R}^2} a_{AW}(q,p,\xi)|\varphi_{p,q}><\varphi_{p,q}|dpdq.
\end{aligned}
$$

It is easy to check that the same argument as for the preceding case holds, and that the boundary layer exists as soon as it exists a family of non-degenerate zeros, $a_{AW}(q, p, \xi(q,p)) = 0$ with real part strictly positive (uniformly in (q,p)). A detailed description of this method will be given elsewhere.

3.2.4 Almost solvability

Contrary to what previous cases suggest, the solvability of (3.28) can not be deduced in general from a symbolic analysis. Indeed, to go from a local analysis (with a local symbol near $y \in Y'$) to a global one (on all Y') requires a "patching process", involving a covering of the boundary and a partition of unity. Such process introduces regularizing operators, responsible for a loss of information at low frequencies. One can clarify this idea in the light of classical results by Trèves [20, Chapitre 3]. These results were obtained to describe the regularity properties of elliptic operators. However, they give some insight into our boundary layer problem, illustrating the smallness of the gap between symbolic properties and hypothesis (H4). Let us consider the "tangential principal symbol"

$$A_\gamma(y, D_y, D_\theta) \;:=\; \sum_{k=0}^{m} \sigma(a_k)(y, D_y)\, D_\theta^k,$$

where $a_\gamma := \sum_{k=0}^m a_k(y, D_y)\, D_\theta^k$. We suppose that

i) Y' is compact

ii) There exists $q > 0$, such that

$$\text{degree } \sigma(a_k) \;=\; \text{degree } \sigma(a_m) \,+\, q(m-k), \quad \forall k.$$

iii) There are two integers, m^+ and m^- such that $m^+ + m^- = m$, $m^+ \geq 1$, and for all (y, ζ) in $T^*Y' - \{0\}$, the polynomial $A_\gamma(y, \zeta, \eta)$ with respect to η has exactly m^+ roots with positive imaginary part, m^- roots with negative imaginary part.

Under these assumptions, we state

Theorem 3. *There exists a family R_k, $k = 0, \dots, m$ of regularizing operators over Y' such that*

$$\tilde{a}_\gamma(y, D_y, D_\theta) \;:=\; a_\gamma(y, D_y, D_\theta) \,+\, \sum_{k=0}^{m} R_k(y, D_y)\, D_\theta^k$$

satisfies: $\{u \in \mathcal{X},\ \tilde{a}_\gamma u = 0\} \neq \{0\}$, and for all $f \in \mathcal{X}$, equation $\tilde{a}_\gamma u = f$ has a solution in $\mathcal{X}$.

Broadly speaking, this theorem says that a singular operator whose "symbol has good properties" (assumptions i) and ii)) is "close" to a boundary layer operator, *i.e.* up to additional regularizing operators. However, as these regularizing operators do not vanish, one can not conclude to the solvability of (3.28) in $\mathcal{X}$.

Proof: We start noticing that

$$a_\gamma(y, D_y, D_\theta) \;=\; a_m(y, D_y)\, D_\theta^m \,+\, \sum_{k=0}^{m-1} a_k(y, D_y)\, D_\theta^k$$

with a_m elliptic. Up to compose by a parametrix of a_m, which would yield an additional $R(y, D_y) D_\theta^m$ term with R regularizing, we can assume that $a_m = 1$.

This simplification, together with ii) and iii) enter the framework studied in [20]. Trèves considered ii) with $q = 1$, but its arguments extend straightforwardly to $q \in \mathbb{N}^*$. In particular, we have the "almost factorization" (see [20, pages 157-162]):

$$a_\gamma(y, D_y, D_\theta) \;=\; a_\gamma^-(y, D_y, D_\theta)\, a_\gamma^+(y, D_y, D_\theta) \;+\; R(y, D_y, D_\theta),$$

where $a_\gamma^\pm := \sum_{k=0}^{m^\pm} a_k^\pm(y, D_y)\, D_\theta^k, \quad R := \sum_{k=0}^{m^+} R_k(y, D_y)\, D_\theta^k.$

The $a_k^\pm$, resp. R_k, are pseudo-differential operators of degree $q(m^\pm - k)$ resp. $-\infty$. Moreover, for all (y, ζ), the roots of

$$A_\gamma^\pm(y, \zeta, \eta) \;:=\; \sum_{k=0}^{m^\pm} \sigma(a_k^\pm)(y, \zeta)\, \eta^k$$

are the $m^\pm$ roots of $A_\gamma(y, \zeta, \cdot)$ with $\gtrless$ imaginary part. Thus, from above decomposition, we can rewrite equation (3.28) as a triangular system:

$$\begin{aligned} a_\gamma^+ v \;+\; Ru \;&=\; f, \\ a_\gamma^- u \;&=\; v \end{aligned} \tag{3.34}$$

Still following [20], we denote $-\Delta_y$ the Laplace-Beltrami operator on Y' and set $\Lambda = (1 - \Delta_y)^{1/2}$. Through the change of variables

$$\begin{aligned} D_\theta v^j \;&=\; \Lambda^q v^{j+1}, \quad j = 1, \ldots, m^+ - 1, \\ D_\theta u^j \;&=\; \Lambda^q u^{j+1}, \quad j = 1, \ldots, m^- - 1, \end{aligned} \tag{3.35}$$

we can write (3.34) as a first order system

$$\begin{aligned} D_\theta V \;-\; \mathcal{A}_\gamma^+ V \;&=\; \mathcal{R} U \;+\; F, \\ D_\theta U \;-\; \mathcal{A}_\gamma^- U \;&=\; \mathcal{J} V \end{aligned} \tag{3.36}$$

with $V = (v^j),\ U = (u^j)$,

$$\mathcal{R}U = (0, \ldots, \Lambda^{1-m^+} R u^1), \quad F = (0, \ldots, \Lambda^{1-m^+} f), \quad \mathcal{J}V = (0, \ldots, \Lambda^{1-m^-} v^1).$$

The $\mathcal{A}_\gamma^\pm$ are matricial pseudo-differential operators of order q over Y'. Moreover, they satisfy

$$\det\big(\eta - \sigma(\mathcal{A}_\gamma^\pm)(y, \zeta)\big) \;=\; A_\gamma^\pm(y, \zeta, \eta).$$

Hence, we can apply [20, theorem 1.1, page 134] on parametrices for parabolic type operators: one can find a function $\mathcal{U}^\pm(\theta)$, with values in matricial pseudo-differential operators of order zero over Y', such that

$$D_\theta \mathcal{U}^\pm \;\mp \mathcal{A}_\gamma^\pm \mathcal{U}^\pm \;\sim\; 0 \text{ in } Y' \times \{\theta > 0\}, \quad \mathcal{U}^\pm|_{\theta=0} = I^\pm$$

where $I^\pm$ is the identity of $\mathbb{C}^{m^\pm}$. In each local chart $\big(\mathcal{O}', \chi'\big)$ of Y', the local symbol $(y, \zeta) \mapsto \mathcal{U}_{\chi'}^\pm(y, \theta, \zeta)$ of $\mathcal{U}^\pm(\theta)$ satisfies

i) $\mathcal{U}_{\chi'}^\pm$ is $\mathcal{C}^\infty$ on $\mathcal{O}' \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$.

ii) To every compact subset $\mathcal{K}$ of $\mathcal{O}'$, to every pair of $(n-1)$-tuples $\alpha, \beta \in \mathbb{Z}_+^{n-1}$, and to every pair of integers $r, N \geq 0$, there is a constant $C > 0$ such that for all y in $\mathcal{K}$, $\theta > 0$, $\zeta \in \mathbb{R}^{n-1}$,

$$\|\partial_y^\alpha\, \partial_\zeta^\beta\, \partial_\theta^r\, \mathcal{U}_{\chi'}^\pm(y, \theta, \zeta)\| \;\leq\; C\, \theta^{-N}\, (1 + |\zeta|)^{rq - |\beta| - Nq} \tag{3.37}$$

We quote that in theorem 1.1 of Trèves, estimate (3.37) holds only locally in θ, because the symbol $\mathcal{A} = \mathcal{A}(\theta)$ under consideration has only local control in θ. In our case, $\mathcal{A}_\gamma^\pm$ are independent on θ, so that (3.37) extends to θ in $\mathbb{R}_+$.

Suppose now that $f \in \mathcal{X}$ so that $F \in \mathcal{X}^{m^+}$. Let $V^0 \in (\mathcal{C}^\infty(Y'))^{m_+}$. Set

$$\begin{aligned} V \;&=\; \mathcal{U}^+(\theta)\, V^0 \;-\; \int_0^\theta \mathcal{U}^+(\theta - s)\, F(s)\, ds, \\ U \;&=\; -\int_\theta^{+\infty} \mathcal{U}^-(s-\theta)\, \mathcal{J} V(s)\, ds. \end{aligned} \tag{3.38}$$

By estimates (3.37), it is clear that $(U, V) \in \mathcal{X}^m$, and as $\mathcal{U}^\pm$ are parametrices,

$$\begin{aligned} D_\theta V \;-\; \mathcal{A}_\gamma^+ V \;&=\; \mathcal{R} U \;+\; \mathcal{R}^+ V \;+\; F, \\ D_\theta U \;-\; \mathcal{A}_\gamma^- U \;&=\; \mathcal{J}\, V \;+\; \mathcal{R}^- U \end{aligned} \tag{3.39}$$

for regularizing operators $\mathcal{R}^\pm$. Back to original variables, we get the result. □

4 Boundary layer expansions

4.1 W.K.B. Ansatz

The notions and hypothesis introduced above are linked to boundary layer expansions for solutions of (1.6)-(1.7). Let Y^1 to Y^N the connected components of $Y = \partial\Omega$. To each $i = 1, \dots, N$, we can associate $r_i \in \mathbb{N}$ boundary layer exponents $\gamma_1^i < \ldots < \gamma_{r_i}^i$. Remind that these exponents are positive rational numbers. Hence, replacing for simplicity ε by ε^δ with some appropriate $\delta > 0$, we can assume that $\delta = 1$, $\gamma_j^i \in \mathbb{N}^*$, $\forall i, j$.

As usual in WKB approach, we inject approximations (1.8)-(1.10) into (1.6). The resulting equations are expanded and ordered according to powers of ε, and coefficients of the different powers of ε are set equal to zero. It leads to a family of equations on the u^k and $v_j^{i,k}$. To lighten notations, we set $u^k = 0$, $v_k^{i,j} = 0$ for $k < 0$.

i) *Interior terms.* Denoting $a^\varepsilon(x, D_x) := \sum_{l=0}^h \varepsilon^l a_l(x, D_x)$, we have

$$\sum_{l=0}^h a_l(x, D_x)\, u^{k-l} \;=\; \delta_{0k}\, f, \quad \forall k \in \mathbb{N}. \tag{4.40}$$

where δ_{0k} is the Kronecker symbol. This reads in short

$$\mathcal{A}(x, D_x) U = F \tag{4.41}$$

where $U := (u^k) \in \mathcal{C}^\infty(\overline{\Omega})^{\mathbb{N}}$, $F := (f, 0, \ldots)$, and

$$\mathcal{A} : \mathcal{C}^\infty(\overline{\Omega})^{\mathbb{N}} \mapsto \mathcal{C}^\infty(\overline{\Omega})^{\mathbb{N}}, \quad (\mathcal{A}U)^k := \sum a_l u^{k-l}.$$

ii) *Boundary layer terms.* We deduce from (3.27)

$$D_\theta^{n_j^i}\, a_{\gamma_j^i}(y, D_y, D_\theta)\, v_j^{i,k} \;+\; L(y, \theta, D_y, D_\theta)\left(v_j^{i,0}, \ldots, v_j^{i,k-1}\right) = 0 \tag{4.42}$$

for $y \in Y^i$ and $\theta \in \mathbb{R}_+$. L is the linear differential operator coming from lower order terms, and n_j^i is an integer. L has smooth coefficients, polynomial in θ. As D_θ is invertible from $\mathcal{X}^i$ to $\mathcal{X}^i$, we rewrite such equations as:

$$a_{\gamma_j^i}(y, D_y, D_\theta)\, v_j^{i,k} \;+\; M(y, \theta, D_y, D_\theta)\left(v_j^{i,0}, \ldots, v_j^{i,k-1}\right) = 0. \tag{4.43}$$

In particular,

$$a_{\gamma_j^i}(y, D_y, D_\theta)\, v_j^{i,0} \;=\; 0 \tag{4.44}$$

Moreover, as γ_j^i is a boundary layer exponent, equation (4.44) has a non-trivial solution in $\left(\mathcal{X}^i\right)$. Recursively, equation (4.43) (hence equation (4.42)) has a solution in $\left(\mathcal{X}^i\right)$. Again, equations (4.42) can be written

$$\mathcal{A}_{bl}(y, \theta, D_y, D_\theta) V \;=\; 0, \quad V = \left(v_j^{1,k}, \ldots, v_j^{N,k}\right)_{j,k} \in \left(\prod_{i=1}^{N} \mathcal{X}^i\right)^{\mathbb{N}}.$$

Remark 7. *No other scalings in boundary layer expansion* (1.10) *could lead to "reasonable" localized solutions. This means that, in view of factorization* (2.13)*, any boundary layer term of the type* $v(y, t/\varepsilon^\gamma)$ *where* γ *is not a singular exponent yields equations of the type*

$$e(y, D_y)\, D_\theta^\alpha v \;=\; f$$

for some elliptic operator $e(y, D_y)$ *and* $\alpha \in \mathbb{N}$*. Except for some spurious source terms in the case where* $e(y, D_y)$ *is non-invertible, such equation has no nontrivial localized solution.*

4.2 Assumption (H5) and proof of Theorem 1

At this point, the boundary layer and regular parts have been described independently from one another. They are of course linked through the boundary conditions (1.7). It can be written

$$\sum_{k=0}^{m'} B_{kl}^{\varepsilon}(y, D_y)\, D_\theta^k u|_{t=0} \;=\; g_l, \quad y \in Y, \tag{4.45}$$

where $B_{kl}^{\varepsilon}(y, D_y)$ are smooth differential operators, polynomial in ε. Let us introduce

$$\partial\mathcal{V} \;=\; \left\{ \left(D_\theta^k V|_{t=0}\right), \quad k = 0, \ldots, m', \quad \mathcal{A}_{bl} V = 0 \right\} \;\subset\; \left(\prod_{i=1}^{N} \mathcal{C}^\infty(Y^i)^{\mathbb{N}}\right)^{m'+1}$$

Again, we can inject the Ansatz (1.8)-(1.10) into (1.7). This can be written as

$$\sum_{k=0}^{m} \mathcal{B}_{kl}(y, D_y) D_\theta^k U|_{t=0} \;+\; \mathcal{C}_l(y, D_y) W \;=\; G_l,$$

for some differential operators $\mathcal{B}_{kl} : \mathcal{C}^\infty(Y)^{\mathbb{N}} \mapsto \mathcal{C}^\infty(Y)^{\mathbb{N}}$, $W \in \partial\mathcal{V}$, and

$$\mathcal{C}_l : \partial\mathcal{V} \mapsto \left(\prod_{i=1}^{N} \mathcal{C}^\infty(Y^i)^{\mathbb{N}}\right)^{m+1}.$$

Introducing any projector Π_l on the range of $\mathcal{C}_l$, we can rewrite previous equation as

$$\sum_{k=0}^{m} \Pi_l\, \mathcal{B}_{kl}(y, D_y) D_\theta^k U|_{t=0} \;=\; \Pi_l G_l \tag{4.46}$$

The existence of the regular part resumes to:

(H5) One can find $U \in \mathcal{C}^\infty(\overline{\Omega})^{\mathbb{N}}$ such that

$$\begin{cases} \mathcal{A}(x, D_x)U = F, \\ \displaystyle\sum_{k=0}^{m} \Pi_l \, \mathcal{B}_{kl}(y, D_y) D_\theta^k U|_{t=0} \; = \; \Pi_l G_l \end{cases} \tag{4.47}$$

As assumptions (H1) to (H5) are fulfilled, one can find u^ε of type (1.8)-(1.10) satisfying

$$\begin{cases} a^\varepsilon(x, D_x)\, u^\varepsilon \; \sim \; f, \quad x \in \Omega, \\ b_l^\varepsilon(x, D_x)\, u \; \sim \; g_l, \quad x \in Y, \quad l \in \{1, \dots, L\}, \end{cases} \tag{4.48}$$

uniformly on the compact subsets. Theorem 1 follows.

One can see system (4.47) as the limit system of (1.6)-(1.7). It means that the small-scale effects due to boundary layers are contained in this large-scale system. A strong difficulty is that it is *infinite-dimensional*: correction terms u^i at any order may interact. Hence, in most situations, one does not treat directly system (4.47). The classical attempt is to turn it into *an infinite collection of finite-dimensional systems*, that are of the same type and solved recursively. This will be illustrated on the quasigeostrophic equation in the next section.

5 Application to the quasigeostrophic model

The stationary linearized quasigeostrophic equation reads, for x in a two-dimensional domain Ω

$$\beta \, \partial_{x_1} \psi \; + \; \nabla^\perp \psi \cdot \nabla \eta_B \; + \; r \Delta \psi \; - \; \frac{1}{\mathrm{Re}} \Delta^2 \psi \; = \; \beta \, \mathrm{curl} \; \tau \tag{5.49}$$

where $\psi = \psi(x) \in \mathbb{R}$, $x = (x_1, x_2)$ in cartesian coordinates, β, r and Re are positive parameters, and $\eta_B = \eta_B(x)$, $\tau = \tau(x)$ are smooth functions. It is a celebrated basic model for oceanic circulation: ψ is a stream function, associated to horizontal velocity field $u = \nabla^\perp \psi$, η_B is a bottom topography term. The expression $\beta \, \partial_{x_1} \psi$ comes from the variation of the Coriolis force with latitude, $r \Delta \psi$ and $-1/\mathrm{Re} \, \Delta^2 \psi$ are dissipative terms due resp. to friction and viscosity. Finally, $\beta \, \mathrm{curl} \; \tau$ is a vorticity term created by the wind. We refer to [16] for all necessary details. Equation (5.49) is fulfilled with Dirichlet boundary conditions

$$\psi = \partial_n \psi = 0, \quad x \in \partial\Omega. \tag{5.50}$$

We will apply previous analysis to system (5.49), (5.50), for several domains Ω and ranges of parameters β, r and Re.

5.1 Munk layers

We first investigate the case where

$$\beta \to +\infty, \quad r, \mathrm{Re}, \tau \text{ given} \tag{5.51}$$

which corresponds to strong forcing by the wind. We consider a domain of the type

$$\Omega \; := \; \{\chi_w(x_2) \le x_1 \le \chi_e(x_2)\} \, ,$$

where χ_w and χ_e are smooth, with all derivatives bounded. They describe the western and eastern coasts. We assume that $x_2 \in \mathbb{R}$ or $x_2 \in \mathbb{T}$. The case of a closed basin will be evoked in last subsection.

This boundary layer problem enters the framework given in section 2, with $\varepsilon = \beta^{-1}$. We start with the derivation of singular exponents for both connected components of $Y := \partial\Omega$,

$$Y_w := \{(\chi_w(x_2), x_2), \quad x_2 \in \mathbb{R} \text{ or } \mathbb{T}\}, \; Y_e := \{(\chi_e(x_2), x_2), \quad x_2 \in \mathbb{R} \text{ or } \mathbb{T}\}.$$

Western boundary layers. For any $y \in Y_w$, global coordinates $x'_1 = x_2$, $x'_2 = x_1 - \chi_w(x_2)$ define a local chart near y. In these coordinates, the corresponding symbol satisfies as $|\xi_1| + |\xi_2| \to +\infty$,

$$\begin{aligned} a_w(x', \xi_1, \xi_2) \; := \; & i\,\varepsilon^{-1}\,\xi_2 \, + \, O(|\xi_1| \, + \, |\xi_2|) \, - \, r\left(\xi_1^2 \, + \, (1 + \chi_w'^2)\,\xi_2^2 \right. \\ & \left. + \, O(|\xi_1| \, + \, |\xi_2|)\right) \, - \, \frac{1}{\mathrm{Re}}\left(\xi_1^4 \, + \, (1+\chi_w'^2)^2\xi_2^2 \, + \, O((|\xi_1| \, + \, |\xi_2|)^3)\right), \end{aligned}$$

with $\chi'_w := \chi'_w(x'_1)$. For all $\xi_1 \in \mathbb{R}$, the roots of $a_w(x', \xi_1, \cdot)$ with $|\xi_2| \to +\infty$ satisfy

$$i\,\varepsilon^{-1}\,\xi_2 \; \sim \; \frac{1}{\mathrm{Re}}\,(1+\chi_w'^2)^2\xi_2^4, \quad i.e. \;\; \xi_2 \; \sim \; \left(\frac{i\mathrm{Re}}{(1+\chi_w'^2)^2}\right)^{1/3} \varepsilon^{-1/3}.$$

Eastern boundary layers. The derivation is similar, with $x'_1 = x_2$, $x'_2 = \chi_e(x_2) - x_1$. Roots of $a_e(x', \xi_1, \cdot)$ with singular behaviour satisfy

$$\xi_1 \; \sim \; \left(\frac{-i\mathrm{Re}}{(1+\chi_e'^2)^2}\right)^{1/3} \varepsilon^{-1/3}.$$

Hence, assumptions (H1) to (H3) are fulfilled, with the singular exponent $\gamma = 1/3$.

To identify singular operators and boundary layer sizes is easy using the global coordinates. Indeed,

$$\begin{aligned} a_w(x'_1, 0, D_{x'_1}, \frac{D_\theta}{\varepsilon^{1/3}}) \; &\sim \; \varepsilon^{-4/3}\left(D_\theta - \frac{1}{\mathrm{Re}}\left(1+\chi_w'^2\right)^2 \, D_\theta^4\right), \\ a_e(x'_1, 0, D_{x'_1}, \frac{D_\theta}{\varepsilon^{1/3}}) \; &\sim \; \varepsilon^{-4/3}\left(-D_\theta - \frac{1}{\mathrm{Re}}\left(1+\chi_e'^2\right)^2 \, D_\theta^4\right) \end{aligned}$$

Thus, the singular operators are

$$a_{w,\gamma} := D_\theta - \frac{1}{\mathrm{Re}}\left(1+\chi_w'^2\right)^2 \, D_\theta^4, \quad a_{e,\gamma} := -D_\theta - \frac{1}{\mathrm{Re}}\left(1+\chi_e'^2\right)^2 \, D_\theta^4.$$

They have order zero coefficients. Denoting $\alpha_{w,e} := \mathrm{Re}^{1/3}/(1+\chi_{w,e}'^2)^2$, the roots of their symbols are $\alpha_w e^{i\pi/6}$, $\alpha_w e^{5i\pi/6}$, $\alpha_w e^{3i\pi/2}$, resp. $\alpha_e e^{-i\pi/6}$, $\alpha_e e^{i\pi/2}$, $\alpha_e e^{7i\pi/6}$. Proposition 5 applies (two roots, resp. one root with positive imaginary part). Thus, $\varepsilon^{1/3}$ is a boundary layer size on $Y_{w,e}$.

It remains to show the existence of solutions similar to (1.8), with

$$\begin{aligned} u^\varepsilon_{int}(x) \; &\sim \psi^0(x) \, + \, \varepsilon^{1/3}\,\psi^1(x) \, + \ldots \\ v^\varepsilon_{bl}(x) \; &\sim \psi^0_w\left(\frac{x_1 - \chi_w(x_2)}{\varepsilon^{1/3}}, x_2\right) \, + \, \psi^0_e\left(\frac{\chi_e(x_2) - x_1}{\varepsilon^{1/3}}, x_2\right) \\ & \quad + \, \varepsilon^{1/3}\psi^1_w\left(\frac{x_1 - \chi_w(x_2)}{\varepsilon^{1/3}}, x_2\right) \, + \, \ldots \end{aligned}$$

Equation (5.49) yields

$$\partial_{x_1}\psi^i \; = \; f^i \, + \, \delta_{0i}\,\mathrm{curl}\,\tau, \tag{5.52}$$

where f^i involves u^k, $k \le i-1$. Equation (5.50) yields

$$\begin{aligned}
&\psi^i_w|_{\theta=0} \,+\, \psi^i|_{Y_w} = 0, \quad \psi^i_e|_{\theta=0} \,+\, \psi^i|_{Y_e} = 0,\\
&D_\theta \psi^i_w|_{\theta=0} = g^i_w, \quad D_\theta \psi^i_e|_{\theta=0} = g^i_e,
\end{aligned}$$

where $g^i_{w,e}$ depends on u^k and $u^k_{w,e}$, $k \le i-1$. The western boundary layer operator has two characteristic roots with positive imaginary part. One can find localized solutions ψ_w of $a_{w,\gamma}\psi_w = f_w$ for arbitrary values of $\psi_w|_{\theta=0}$, $D_\theta\psi_w|_{\theta=0}$. This is not the case for the eastern boundary layer, for which only one condition can be given. The last boundary condition prescribes $D_\theta \psi^0_e|_{\theta=0} = 0$. Recursively, we easily deduce that the approximate solutions exist if

$$\partial_x \psi = f, \quad \psi|_{Y_e} = g, \quad f \in \mathcal{C}^\infty(\overline{\Omega}), \quad g \in \mathcal{C}^\infty(Y_e) \tag{5.53}$$

has a solution. As $\psi(x_1,x_2) = \int_{\chi_e(x_2)}^{x_1} f \,+\, g(x_1)$ is the unique solution of (5.53), this proves the existence of boundary layer expansions. These boundary layers are called *Munk layers*.

5.2 Stommel and friction layers

We now investigate the case where

$$\beta \to +\infty, \quad \frac{\beta^{2/3}}{r} \to 0, \quad \mathrm{Re}, \tau \text{ given}, \tag{5.54}$$

which emphasizes the role of friction. In terms of $\varepsilon := \beta^{-1}$ and $\varepsilon_s := r/\beta$, the last condition reads $\varepsilon^{1/3}/\varepsilon_s \to 0$. To fit exactly the context under consideration, we assume that $\varepsilon_s = \varepsilon^\delta$ for some rational number $\delta < 1/3$. The domain Ω remains the same as in previous subsection.

Remind that the western symbol reads

$$\begin{aligned}
a_w(x',\xi_1,\xi_2) \;:=\; & i\,\varepsilon^{-1}\,\xi_2 \,+\, O(|\xi_1| \,+\, |\xi_2|) \,-\, \varepsilon\,\varepsilon^{-1}\Big(\xi_1^2 \,+\, (1+\chi_w'^2)\,\xi_2^2 \\
&+\, O(|\xi_1| \,+\, |\xi_2|)\Big) \,-\, \frac{1}{\mathrm{Re}}\Big(\xi_1^4 \,+\, (1+\chi_w'^2)^2\xi_2^4 \,+\, O((|\xi_1| \,+\, |\xi_2|)^3)\Big).
\end{aligned}$$

For all $\xi_1 \in \mathbb{R}$, the singular roots of $a_w(x',\xi_1,\cdot)$ satisfy either

$$\begin{aligned}
&i\,\varepsilon^{-1}\,\xi_2 \,\sim\, \varepsilon_s\,\varepsilon^{-1}\,\left(1+\chi_w'^2\right)\xi_2^2, \;\; \textit{i.e.} \;\; \xi_2 \,\sim\, \frac{i}{1+\chi_w'^2}\,\varepsilon_s^{-1}, \qquad \text{or}\\
&\frac{\varepsilon_s}{\varepsilon}\left(1+\chi_w'^2\right)\xi_2^2 \,\sim\, \frac{-1}{\mathrm{Re}}\left(1+\chi_w'^2\right)^2\xi_2^4, \;\; \textit{i.e.} \;\; \xi_2 \,\sim\, \frac{(-\mathrm{Re})^{1/2}}{1+\chi_w'^2}\,\left(\frac{\varepsilon_s}{\varepsilon}\right)^{1/2}.
\end{aligned}$$

Similarly, for the eastern symbol,

$$\xi_2 \,\sim\, \frac{-i}{1+\chi_w'^2}\,\varepsilon_s^{-1}, \qquad \text{or } \xi_2 \,\sim\, \frac{(-\mathrm{Re})^{1/2}}{1+\chi_e'^2}\,\left(\frac{\varepsilon_s}{\varepsilon}\right)^{1/2}.$$

There are two singular exponents, uniformly on Y: $\gamma_1 = \delta$ (*i.e.* $\varepsilon^{\gamma_1} = \varepsilon_s$) and $\gamma_2 = \frac{1-\delta}{2}$ (*i.e.* $\varepsilon^{\gamma_2} = \sqrt{\varepsilon/\varepsilon_s}$).

The singular operators read

$$\begin{aligned}
a_{w,\gamma_1} \;&=\; i\,D_\theta \,-\, rD_\theta^2, \quad a_{e,\gamma_1} \;=\; -i\,D_\theta \,-\, rD_\theta^2,\\
a_{w,\gamma_2} \;&=\; -rId \,-\, \frac{1}{\mathrm{Re}}D_\theta^2, \quad a_{e,\gamma_2} \;=\; -rId \,-\, \frac{1}{\mathrm{Re}}D_\theta^2.
\end{aligned}$$

Thanks to proposition 5, we deduce that ε^{γ_2} is a boundary layer size on $Y_{w,e}$, and ε^{γ_1} is only a boundary layer size on Y_e (one root with positive imaginary part for $a^{\gamma_2}_{w,e}$ and $a^{\gamma_1}_w$, no such root for $a^{\gamma_1}_e$).

We look for approximate solutions with

$$\begin{aligned} u^\varepsilon_{int} \: &= \: \psi^0 \: + \: \varepsilon^{\gamma_2-\gamma_1}\,\psi^1 \: + \: \varepsilon^{2(\gamma_2-\gamma_1)}\,\psi^2 \: + \: \dots, \\ v^\varepsilon_{bl} \: &= \: \psi^0_{w,1}\left(\frac{x_1-\chi_w(x_2)}{\varepsilon^{\gamma_1}}, x_2\right) \: + \: \psi^0_{w,2}\left(\frac{x_1-\chi_w(x_2)}{\varepsilon^{\gamma_2}}, x_2\right) \\ &+ \: \psi^0_{e,2}\left(\frac{\chi_e(x_2)-x_1}{\varepsilon^{\gamma_2}}, x_2\right) \: + \: \varepsilon^{\gamma_2-\gamma_1}\psi^1_{w,1}\left(\frac{x_1-\chi_w(x_2)}{\varepsilon^{\gamma_1}}, x_2\right) \: + \: \dots \end{aligned}$$

Equation (5.49) yields (5.52). Equation (5.50) yields

$$\begin{aligned} &\psi^i_{w,1}|_{\theta=0} \: + \: \psi^i_{w,2}|_{\theta=0} \: + \: \psi^i|_{Y_w} = 0, \quad \psi^i_{e,2}|_{\theta=0} \: + \: \psi^i|_{Y_e} = 0, \\ &D_\theta\psi^i_{w,2}|_{\theta=0} = g^i_w, \quad D_\theta\psi^i_{e,2}|_{\theta=0} = g^i_e, \end{aligned}$$

where $g^i_{w,e}$ involve u^k and $u^k_{w,e,1,2}$ for $k \le i-1$. All boundary layer operators have one root with positive imaginary part, thus only one condition can be given at the boundary. The last line prescribes $D_\theta\psi^0_{w,e,2} = 0$. Reasoning recursively, the existence of approximate solutions relies on the solvability of (5.53). The boundary layers with size ε_s and $\varepsilon_f := \sqrt{\varepsilon_s/\varepsilon}$ are known as the Stommel and friction layers.

5.3 Geostrophic degeneracy

We consider again the asymptotics (5.51). In subsection 5.1, we have considered a domain with western and eastern boundaries, but no northern or southern boundaries. Let us examine here the case of a closed basin. We assume simply that

$$\Omega \: = \: \left\{x_1^2 + x_2^2 \: \le 1\right\}, \quad Y = \partial\Omega = \mathbb{S}^1.$$

Let us look for singular exponents near $y_0 := (0,1)$. We chose local coordinates $x'_1 = x_1$, $x'_2 = \sqrt{1-x'^2_1} - x_2$. The local symbol reads, as $|\xi_1|+|\xi_2| \to \infty$

$$\begin{aligned} a^\varepsilon_\chi(x', \xi_1, \xi_2) \: &= i\,\varepsilon^{-1}\left(\xi_1 - \frac{2x'_1}{\sqrt{1-x'^2_1}}\,\xi_2\right) \\ &+ \: i\begin{pmatrix} \xi_1 - \frac{2x'_1}{\sqrt{1-x'^2_1}}\xi_2 \\ -\xi_2 \end{pmatrix}^\perp \cdot \begin{pmatrix} \partial_{x'_1}\eta_B - \frac{2x'_1}{\sqrt{1-x'^2_1}}\partial_{x'_2}\eta_B \\ -\partial_{x'_2}\eta_B \end{pmatrix} \\ &- \: r\left(\xi_1^2 + \left(\frac{4x'^2_1}{1-x'^2_1}+1\right)\,\xi_2^2 \: + \: O(|\xi_1|+|\xi_2|)\right) \\ &- \: \frac{1}{\mathrm{Re}}\left(\xi_1^4 \: + \: \left(\frac{4x'^2_1}{1-x'^2_1}+1\right)^2\,\xi_2^4\right) \: + \: O((|\xi_1|+|\xi_2|)^3) \end{aligned}$$

Let $\xi_1 \in \mathbb{R}^*$. For $y \neq y_0$, *i.e.* $x'_1 \neq 0$, the roots ξ_2 with $|\xi_2| \to +\infty$ satisfy

$$2i\,\varepsilon^{-1}\,\frac{1}{\mathrm{Re}}\,\frac{x'_1}{\sqrt{1-x'^2_1}}\,\xi_2 \: \sim \: \left(\frac{4x'^2_1}{1-x'^2_1}+1\right)^2\,\xi_2^4,$$

which yields $\gamma = 1/3$ in agreement with subsection 5.1. For $y = y_0$, *i.e.* $x'_1 = 0$, this relation degenerates. The roots ξ_2 going to infinity satisfy

$$i\,\varepsilon^{-1}\,\xi_1 \: \sim \: \frac{1}{\mathrm{Re}}\,\xi_2^4, \; i.e.\,\xi_2 \: \sim \: (i\,\xi_1\mathrm{Re})^{1/4}\,\varepsilon^{-1/4}$$

so that $\gamma = 1/4$. In particular, there are no singular exponents uniformly on a neighborhood of $y_0 = (0, 1)$ and (H3) fails. This phenomenon is known as geostrophic degeneracy. It takes place in various systems of fluid mechanics, including rotating fluids (as can be seen on figure 1, near the equator of the inner sphere, following [19]). To understand the structure of the solutions near such "turning points" is an open question, both from physical and mathematical viewpoints.

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