SOME REMARKS ABOUT SEMICLASSICAL TRACE INVARIANTS AND QUANTUM NORMAL FORMS

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Abstract. In this paper we explore the connection between semi-classical and quantum Birkhoff canonical forms (BCF) for Schrödinger operators. In particular we give a "non-symbolic" operator theoretic derivation of the quantum Birkhoff canonical form and provide an explicit recipe for expressing the quantum BCF in terms of the semi-classical BCF.

1. Introduction

Let X be a compact manifold and $H:\ L^2(X) \to L^2(X)$ a self-adjoint first order elliptic pseudodifferential operator with leading symbol $H(x,\xi)$. From the wave trace

$$\sum_{E_k \in Spec(H)} e^{itE_k}, \tag{1.1}$$

one can read off many properties of the "classical dynamical system" associated with H, i.e. the flow generated by the vector field

$$\xi_H \ = \ \sum \frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}. \tag{1.2}$$

For instance it was observed in the *'70's'* by Colin de Verdière, Chazarain and Duistermaat-Guillemin that (1.1) determines the period spectrum of (1.2) and the linear Poincaré map about a non-degenerate periodic trajectory, γ, of (1.2) ([3], [2], [4]).

More recently it was shown by one of us [5] that (1.1) determines the entire Poincaré map about γ, i.e. determines, up to isomorphism, the classical dynamical system associated with H in a formal neighborhood of γ. The proof of this result involved a microlocal Birkhoff canonical form for H in a formal neighborhood of γ and an algorithm for computing the wave trace invariants associated with γ from the microlocal Birkhoff canonical form. Subsequently a more compact and elegant algorithm for computing these invariants from the Birkhoff canonical form was discovered by Zelditch [11] [12] making the computation of these local trace invariants extremely simple and explicit.

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In this paper we will discuss some semiclassical analogues of these results. In our set-up H can either be the Schrödinger operator on $\mathbb{R}^n$

$$-\hbar^2\Delta \ + \ V$$

with $V \to \infty$ as x tends to infinity, or more generally a self-adjoint semiclassical elliptic pseudodifferential operator

$$H(x, \hbar D_x)$$

whose symbol, $H(x,\xi)$, is proper (as a map from T^*X into $\mathbb{R}$). Let E be a regular value of H and γ a non-degenerate periodic trajectory of period T_γ lying on the energy surface $H = E$ [1].

Consider the Gutzwiller trace (see [6])

$$\sum \psi\left(\frac{E - E_i}{\hbar}\right) \tag{1.3}$$

where ψ is a C^∞ function whose Fourier transform is compactly supported with support in a small neighborhood of T_γ and is identically one in a still smaller neighborhood. As shown in [8], [9] (1.3) has an asymptotic expansion

$$e^{i\frac{S_\gamma}{\hbar}+\sigma_\gamma}\sum_{k=0}^{\infty} a_k\hbar^k \tag{1.4}$$

and we will show below how to compute the terms of this expansion to <u>all</u> orders in terms of a microlocal Birkhoff canonical form for H in a formal neighborhood of γ by means of a Zelditch-type algorithm [2].

If γ is non-degenerate so are all its iterates γ^r. Then, for each of these iterates one gets an expansion of (1.3) similar to (1.4)

$$e^{i\frac{S_{\gamma^r}}{\hbar}+\sigma_{\gamma^r}}\sum_{k=0}^{\infty} a_{k,r}\hbar^k \tag{1.5}$$

and for these expansions as well the coefficients $a_{k,r}$ can be computed from the microlocal Birkhoff canonical form theorem for H in a formal neighborhood of γ. Conversely one can show

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[1]For simplicity we will consider periodic trajectories of elliptic type in this paper however our results are true for non-degenerate periodic of all types, hyperbolic, mixed elliptic hyperbolic, focus-focus, etc. Unfortunately however the Zelditch algorithm depends upon the type of the trajectory and in dimension n there are roughly as many types of trajectories as there are Cartan subalgebras of $Sp(2n)$ (See for instance [1]) i.e. the number of types can be quite large

[2]For elliptic trajectories non-degeneracy means that the numbers

$$\theta_1, ..., \theta_n, 2\pi$$

are linearly independent over the rationals, $e^{i\theta_\kappa}, \kappa = 1, ..., n$ being the eigenvalues of the Poincaré map about γ. the results above are true to order $O(\hbar^r)$ providing

$$(\kappa_1\theta_1 + ... + \kappa_n\theta)n + l2\pi \neq 0$$

for all $\mid \kappa_1 \mid + ... + \mid \kappa_n \mid \leq r$, i.e. providing there are no resonances of order $\leq r$.

Theorem 1.1. *the constants $a_{k,r}, \kappa, r = 0, 1, ...$ determine the microlocal Birkhoff canonical form for H in a formal neighborhood of γ (and hence, a fortiori, determine the the classical Birkhoff canonical form).*

One of the main goal of this paper will be to give a proof of this result. Our proof, in sections 2, 3 and 6 is, with semiclassical modifications, more or less the same as the proof of the Guillemin-Zelditch results [5, 11, 12] alluded to above. An alternative proof based on Grushun reductions, flux norms and trace formulas for monodromy operators can be found in [7].

Another main goal of this paper is develop a purely quantum mechanical approach to the theory of Birkhoff canonical forms in which symbolic expansions get replaced by operators theoretic expansions and estimates involving Hermite functions. This can be seen as a "local" version of the Rayleigh-Schrödinger perturbation formalism where no "ϵ" parameter is involved. The virtue of this approach is that the $\hbar$ dependence of the normal form is an intrinsic part of the theory, and avoids any additional semiclassical computation. This approach is developed in section 4 and the connection of this to the symbolic approach of section 2-3 is described in section 5.

To conclude these prefatory remarks we would like to thank Cyrille Heriveuax for his perusal of the first draft of the our manuscript and we would also like to express our gratitude to the referee for his careful line-by-line reading of the manuscript and his many helpful suggestions.

2. The classical Birkhoff canonical form theorem

Let M be a $2n+2$ dimensional symplectic manifold, H a C^∞ function and

$$\xi_H = \sum \frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i} \tag{2.1}$$

the Hamiltonian vector field associated with H. Let E be a regular value of H and γ a non-degenerate elliptic periodic trajectory of ξ_H lying on the energy surface, $H = E$. Without loss of generality one can assume that the period of γ is 2π. In this section we will review the statement (and give a brief sketch of the proof) of the classical Birkhoff canonical form theorem for the pair (H, γ).

Let (x, ξ, t, τ) be the standard cotangent coordinates on $T^*(\mathbb{R}^n \times S^1)$ and let

$$\begin{aligned} p_i &= x_i^2 + \xi_i^2 \\ &\text{and} \\ q_i &= \arg(x_i + \sqrt{-1}\, y_i) \end{aligned} \tag{2.2}$$

Theorem 2.1. *There exists a symplectomorphism, φ, of a neighborhood of γ in M onto a neighborhood of $p = \tau = 0$ such that φ o $\gamma(t) = (0, 0, t, 0))$ and*

$$\varphi^* H = H_1(p, \tau) + H_2(x, \xi, t, \tau) \tag{2.3}$$

H_2 vanishing to infinite order at $p = \tau = 0$.

We break the proof of this up into the following five steps.

Step 1 For ϵ small there exists a periodic trajectory, γ_ϵ, on the energy surface,$H = E+\epsilon$, which depends smoothly on ϵ and is equal to γ for $\epsilon = 0$. The union of these trajectories is a 2 dimensional symplectic submanifold , Σ, of M which is invariant under the flow of ξ_H. Using the Weinstein tubular neighborhood theorem one can map a neighborhood of γ symplectically onto a neighborhood of $p = \tau = 0$ in $T^*(\mathbb{R}^n \times S^1)$ such that Σ gets mapped onto $p = 0$ and $\varphi\ o\ \gamma(t) = (0, 0, t, 0)$. Thus we can henceforth assume that $M = T^*(\mathbb{R}^n \times S^1)$ and Σ is the set, $p = 0$.

Step 2 We can assume without loss of generality that the restriction of H to Σ is a function of τ alone, i.e. $H = E + h(\tau)$ on Σ. With this normalization,

$$H = E + h(\tau) + \sum \theta_i(\tau)p_i + O(p^2) \tag{2.4}$$

where $h(\tau) = \tau + O(\tau^2)$ and

$$\theta_i = \theta_i(0),\ \ i = 1, ..., n \tag{2.5}$$

are the rotation angles associated with γ. Since γ is non- degenerate, $\theta_1, ..., \theta_n, 2\pi$ are linearly independent over the rationals.

Step 3 Theorem 2.1 can be deduced from the following result (which will also be the main ingredient in our proof of the "microlocal" Birkhoff canonical theorem in the next section).

Theorem 2.2. *Given a neighborhood, $\mathcal{U}$, of $p = \tau = 0$ and $G = G(x, \xi, t, \tau) \in C^\infty(\mathcal{U})$, there exist functions F, G_1, $R \in C^\infty(\mathcal{U})$ with the properties*

i. $G_1 = G_1(p, \tau)$

ii. $\{H, F\} = G + G_1 + R$

iii. R vanishes to infinite order on $p = \tau = 0$.

Moreover, if G vanishes to order κ on $p = \tau = 0$, on can choose F to have this property as well.

Proof of the assertion: Theorem 2.2 $\Rightarrow$ Theorem 2.1:
By induction one can assume that H is of the form, $H = H_0(p, \tau) + G(x, \xi, t, \tau)$, where G vanishes to order κ on $p = \tau = 0$. We will show that H can be conjugated to a Hamiltonian of the same form with G vanishing to order $\kappa + 1$ on $p = \tau = 0$. By Theorem 2.2 there exists an F, G and R such that F vanishes to order κ and R to order ∞ on $p = \tau = 0$, $G_1 = G_1(p, \tau)$ and

$$\{H, F\} = G + G_1 + R.$$

Thus

$$\begin{aligned} (\exp\xi_F)^* H &= H + \{F, H\} + \frac{1}{2!}\{F, \{F, H\}\} + ... \\ &= H_0(p, \tau) - G_1(p, \tau) + ... \end{aligned}$$

the "dots" indicating terms which vanish to order $\kappa + 1$ on $p = \tau = 0$.

Step 4 Theorem 2.2 follows (by induction on κ) from the following slightly weaker result:

Lemma 2.3. *Given a neighborhood,* $\mathcal{U}$*, of* $p=\tau=0$ *and a function,* $G\in C^\infty(\mathcal{U})$ *, which vanishes to order* κ *on* $p=\tau=0$*, there exists functions* $F,\ G_1,\ R\ \in C^\infty(\mathcal{U})$ *such that*
i. $G_1=G_1(p,\tau)$
ii. $\{H,F\}=G+G_1+R$
iii. F *vanishes to order* κ *and* R *to order* $\kappa+1$ *on* $p=\tau=0$*.*

Step 5 Proof of Lemma 2.3: In proving Lemma 2.3 we can replace H by the Hamiltonian

$$H_0=E+\tau+\sum\theta_i p_i$$

since $H(p,q,t,\tau)-H_0(p,q,t,\tau)$ vanishes to second order in τ,p. Consider now the identity

$$\{H_0,F\}=G+G_1(p,\tau)+O(p^\infty).$$

Introducing the complex coordinates, $z=x+\sqrt{-1}\xi$, and $\overline{z}=x-\sqrt{-1}\xi$, this can written as

$$\sqrt{-1}\sum_{i=1}^n\theta_i\left(z_i\frac{\partial}{\partial z_i}-\overline{z_i}\frac{\partial}{\partial\overline{z_i}}\right)F+\frac{\partial F}{\partial t}=G+G_1+O(p^\infty).$$

Expanding $F,\ G$ and G_1 in Fourier-Taylor series about $z=\overline{z}=0$:

$$\begin{aligned}F&=\sum_{\mu\neq\nu}a_{\mu,\nu,m}(\tau)z^\mu\overline{z}^\nu e^{2\pi imt}\\ G&=\sum b_{\mu,\nu,m}(\tau)z^\mu\overline{z}^\nu e^{2\pi imt}\\ G_1&=\sum_\mu c_\mu(\tau)z^\mu\overline{z}^\mu\end{aligned}$$

one can rewrite this as the system of equations

$$\sqrt{-1}\left(\sum_{i=1}^n\theta_i\left(\mu_i-\nu_i\right)+2\pi m\right)a_{\mu,\nu,m}(\tau)=b_{\mu,\nu,m}(\tau)\tag{2.6}$$

for $\mu\neq\nu$ or $\mu=\nu$ and $m\neq 0$, and

$$-c_\mu(\tau)=b_{\mu,\mu,0}(\tau)\tag{2.7}$$

for $\mu=\nu$ and $m=0$. By assumption the numbers, $\theta_1,...,\theta_n,2\pi$, are linearly independent over the rationals, so this system has a unique solution. Moreover , for μ and ν fixed

$$\sum b_{\mu,\nu,m}(\tau)e^{2\pi imt}$$

is the (μ,ν) Taylor coefficient of $G(z,\overline{z},t,\tau)$ about $z=\overline{z}=0$; so, with μ and ν fixed and $j>>0$

$$\mid b_{\mu,\nu,m}(\tau)\mid\leq C_{\mu,\nu,j}m^{-j}$$

for all m. Hence, by (2.6)

$$\mid a_{\mu,\nu,m}(\tau)\mid\leq C'_{\mu,\nu,j}m^{-j-1}$$

for all m. Thus

$$a_{\mu,\nu}(t,\tau) = \sum a_{\mu,\nu,m}(\tau) e^{2\pi imt}$$

is a C^∞ function of t and τ. Now let $F(z,\overline{z},t,\tau)$ and $G_1(p,\tau)$ be C^∞ functions with Taylor expansion:

$$\sum_{\mu\neq\nu} a_{\mu,\nu}(t,\tau) z^\mu \overline{z^\nu}$$

and

$$\sum_{\mu} c_\mu(\tau) z^\mu \overline{z^\mu}$$

about $z = \overline{z} = 0$. Note, by the way that, if G vanishes to order κ on $p = \tau = 0$, so does F and G; so we have proved Theorem 2.2 (and, a fortiori Lemma 2.3) with H replaced by H_0.

3. The semiclassical version of the Birkhoff canonical form theorem

Let X be an $(n+1)$-dimensional manifold and $H : C_0^\infty(X) \to C^\infty(X)$ a semi-classical elliptic pseudo-differential operator with leading symbol, $H(x,\xi)$, and let γ be a periodic trajectory of the bicharacteristic vector field (2.1). As in Section 1 we will assume that γ is elliptic and non-degenerate, with rotation numbers (2.4). Let P_i and D_t be the differential operators on $\mathbb{R}^n \times S^1$ associated with the symbols (2.2) and τ i.e.

$$P_i = -\hbar^2 \partial_{x_i}^2 + x_i^2$$

and

$$D_t = -i\hbar\partial_t$$

We will prove below the following semiclassical version of Theorem 2.1

Theorem 3.1. *There exists a semiclassical Fourier integral operator* $A_\varphi : C_0^\infty(X) \to C^\infty\left(\mathbb{R}^n \times S^1\right)$ *implementing the symplectomorphism (2.3) such that microlocally on a neighborhood,* $\mathcal{U}$*, of* $p = \tau = 0$

$$A_\varphi^* = A_\varphi^{-1} \tag{3.1}$$

and

$$A_\varphi H A_\varphi^{-1} = H'\left(P_1, ..., P_n, D_t, \hbar\right) + H'' \tag{3.2}$$

the symbol of H'' *vanishing to infinite order on* $p = \tau = 0$.

Proof. Let B_φ be any Fourier integral operator implementing φ and having the property (3.1). Then, by Theorem 2.1, the <u>leading</u> symbol of $B_\varphi H B_\varphi^{-1}$ is of the form

$$H_0'(p,\tau) + H_0''(p,q,t,\tau) \tag{3.3}$$

$H_0''(p,q,t,\tau)$ being a function which vanishes to infinite order on $p=\tau=0$. Thus the symbol,H_0, of $B_\varphi H B_\varphi^{-1}$ is of the form

$$H_0'(p,\tau)+H_0''(p,q,t,\tau)+\hbar H_1(p,q,t,\tau)+O(\hbar^2). \tag{3.4}$$

By Theorem 2.2 there exists a function, $F(p,q,t,\tau)$, with the property

$$\{H_0,F\}=H_1(p,q,t,\tau)+H_1'(p,\tau)+H_1''(p,q,t,\tau) \tag{3.5}$$

where H_1'' vanishes to infinite order on $p=\tau=0$.

Let Q be a self-adjoint pseudo-differential operator with leading symbol F and consider the unitary pseudo-differential operator

$$\mathcal{U}_s \;=\; e^{isQ}.$$

Let

$$\begin{aligned} H_s &= (\mathcal{U}_s B_\varphi)\, H\, (\mathcal{U}_s B_\varphi)^{-1} \\ &= \mathcal{U}_s \left(B_\varphi H B_\varphi^{-1}\right) \mathcal{U}_{-s} \end{aligned}$$

Then

$$\frac{\partial}{\partial s} H_s = i[Q,H_s] \tag{3.6}$$

so $\frac{\partial}{\partial s}H_s$ is of order -1, and hence the leading symbol of H_s is independent of s. In particular the leading symbol of $\frac{\partial}{\partial s}H_s$ is equal, by (3.6) to the leading symbol of $i[Q,H_s]$ which, by (3.5), is:

$$-\hbar\left(H_1(p,q,t,\tau)+H_1'(p,\tau)+H_1''(p,q,t,\tau)\right).$$

Thus by (3.4) and (3.5) the symbol of

$$(\mathcal{U}_1 B_\varphi)\, H\, (\mathcal{U}_1 B_\varphi)^{-1} = B_\varphi H B_\varphi^{-1} + \int_0^1 \frac{\partial}{\partial s} H_s ds$$

is of the form

$$H_0'(p,\tau)+\hbar H_1'(p,\tau)+\left(H_0''+\hbar H_1''\right)+O(\hbar^2) \tag{3.7}$$

the term in parenthesis being a term which vanishes to infinite order on $p=\tau=0$.

By repeating the argument one can successively replace the terms of order $\hbar^2,\dots,\hbar^r$ etc in (3.7) by expressions of the form

$$\hbar^r\left(H_r'(p,\tau)+H_r''(p,q,t,\tau)\right)$$

with H_r'' vanishing to infinite order on $p=\tau=0$. □

4. A direct construction of the quantum Birkhoff form

In this section we present a "quantum" construction of the quantum Birkhoff normal form which is in a sense algebraically equivalent to the classical one of Section 2. To do this we will need to define for operators the equivalent of "a Taylor expansion which vanishes at a given order".

We will first start in the $L^2(\mathbb{R}^n\times S^1)$ setting, and show at the end of the section the link with Theorem 3.1.

Definition 4.1. Let us consider on $L^2(\mathbb{R}^n\times S^1,dxdt)$ the following operators:

- $a_i^-=\frac{1}{\sqrt{2}}(x_i+\hbar\partial_{x_i})$
- $a_i^+=\frac{1}{\sqrt{2}}(x_i-\hbar\partial_{x_i})$
- $D_t=-i\hbar\frac{\partial}{\partial t}$

We will say that an operator A on $L^2(\mathbb{R}^n\times S^1)$ is an "ordered polynomial of order greater than $p\in\mathbb{N}$" (OPOG(p)) if there exists $P\in\mathbb{N}$ such that:

$$A=\sum_{i=p}^{P}\sum_{j=0}^{[\frac{i}{2}]}\alpha_{ij}(t,\hbar)D_t^j\prod_{l=1}^{i-2j}b_l \tag{4.1}$$

with, $\forall l, b_l\in\{a_1^-,a_1^+,\dots,a_n^-,a_n^+\}$ and $\alpha_{ij}\in C^\infty(S_1\times[0,1[)$.

In (4.1) $\prod\limits_{l=1}^{i-2j}b_l$ is meant to be the ordered product $b_1\dots b_{i-2j}$.

The meaning of this definition is clarified by the following:

Lemma 4.2. *Let H_μ denote the basis of $L^2(\mathbb{R}^n\times S^1)$ defined by $H_\mu(x,t)=\hbar^{-n/4}h_{\mu_1}(x_1/\sqrt{\hbar})\dots h_{\mu_n}(x_n/\sqrt{\hbar})e^{i\mu_{n+1}t}$ where the h_j are the (normalized) Hermite functions. Let us define moreover $|\mu\hbar|:=\sqrt{\mu^2\hbar^2}$.*

Let A be an OPOG(p). Then:

$\forall\mathcal{M}<+\infty$ (microlocal "cut-off"), $\exists\ C=C(A,\mathcal{M})$ such that,

$$||AH_\mu||_{L^2}\leq C|\mu\hbar|^{\frac{p}{2}},\quad \forall\mu\in\mathbb{N}^{n+1}s.t\ |\mu\hbar|\leq\mathcal{M}.$$

Proof. The proof follows immediately from the two well known facts (expressed here in one dimension):

$$a^\pm H_\mu=\sqrt{(\mu\pm 1)\hbar}H_{\mu\pm1}$$

and

$$D_te^{imt}=m\hbar e^{imt}.$$

□

For the rest of this section we will need the following collection of results.

Proposition 4.3. *Let A be a (Weyl)pseudodifferential operator on $L^2(\mathbb{R}^n\times S^1)$ with symbol of type $S_{1,0}$. Then, $\forall L\in\mathbb{N}$ and $\forall\mathcal{M}<+\infty$, there exists an OPOG(1) A_L and a constant $C=C(A,L,\mathcal{M})$ such that,*

$$||(A-A_L)H_\mu||_{L^2} \leq C|\mu\hbar|^{\frac{L+1}{2}}), \quad \forall \mu \in \mathbb{N}^{n+1} s.t.\ |\mu\hbar| \leq \mathcal{M}.$$

Moreover, if the principal symbol of A is of the form:

$$a_0(x_1,\xi_1,\dots,x_n,\xi_n,t,\tau) = \sum \theta_i(x_i^2+\xi_i^2) + \tau + h.o.t.,$$

(or is any function whose symbol vanishes to first order at $x=\xi=\tau=0$) then A_L is an OPOG(2).

Proof. Let us take the Lth order Taylor expansion of the (total) symbol of A in the variables $x,\xi,\tau,\hbar$ near the origin. Noticing that a pseudodifferential operator with polynomial symbol in $x,\xi,\tau,\hbar$ is an OPOG, we just have to estimate the action, on H_μ, of a pseudo-differential operator whose symbol vanishes at the origin to order L in the variable x,ξ,τ. The result is easily obtained for the τ part, as the "t" part of H_μ is an exponential. For the μ part we will prove this result in one dimension, the extension to n dimensions being straightforward.

Let us define a coherent state at (q,p) to be a function of the form $\psi^a_{qp}(x) := \hbar^{-1/4} a\left(\frac{x-q}{\sqrt{\hbar}}\right) e^{i\frac{px}{\hbar}}$, for a in the Schwartz class and $||a||_{L^2}=1$. Let us also set $\varphi_{qp} = \psi^a_{qp}$ for $a(\eta) = \pi^{-1/2}e^{-\eta^2/2}$. It is well known, and easy to check using the generating function of the Hermite polynomials, that:

$$H_\mu = \hbar^{-\frac{1}{4}} \int_{S^1} e^{-i\frac{t}{2}} \varphi_{q(t)p(t)} dt,$$

where $q(t)+ip(t) = e^{it}(q+ip)$, $q^2+p^2 = (\mu+\frac{1}{2})\hbar$. Therefore, for any operator A,

$$||AH_\mu|| = O(\sup_{p^2+q^2=(\mu+\frac{1}{2})\hbar} \hbar^{-\frac{1}{4}}||A\varphi_{qp}||). \tag{4.2}$$

Lemma 4.4. *let H a pseudodifferential operator whose (total) Weyl symbol vanishes at the origin to order M. Then, if $\frac{\hbar}{q^2+p^2} = O(1)$:*

$$||H\psi^a_{qp}|| = O\left((p^2+q^2)^{\frac{M}{2}}\right).$$

Before proving the Lemma we observe that the proof of the Proposition follows easily from the Lemma using (4.2).

Proof. An easy computation shows that, if h is the (pseudodifferential) symbol of H, then $H\psi^a_{qp} = \psi^b_{qp}$ with

$$b(\eta) = \int_{\mathbb{R}} h(q+\sqrt{\hbar}\eta, p+\sqrt{\hbar}\nu) e^{i\eta\nu}\hat{a}(\nu)d\nu, \tag{4.3}$$

where $\hat{a}$ is the ($\hbar$ independent) Fourier transform of a.

Developing (4.3) we get that $H\psi^a_{qp} = \sum_{k=M}^{k=K} \hbar^{\frac{k}{2}} D_k h(q,p)\psi^{b_k}_{qp} + O(\hbar^{\frac{k+1}{2}})$, where $b_k \in \mathcal{S}$ and D_k is an homogeneous differential operator of order k. It is easy to conclude, thanks to the hypothesis $\frac{\hbar}{q^2+p^2} = O(1)$, that

$$\hbar^{\frac{k}{2}}(q^2+p^2)^{\frac{M-k}{2}} = O((q^2+p^2)^{\frac{M}{2}}),\ \hbar^{\frac{M+1}{2}} = O((q^2+p^2)^{\frac{M}{2}}).$$

□

□

This Proposition is crucial for the rest of this Section, as it allows us to reduce all computations to the polynomial setting. For example A may have a symbol bounded at infinity (class $S(1)$, an assumption which we will need for the application below of Egorov's Theorem in the proof of Theorem 4.9), but, with respect to the algebraic equations we will have to solve, one can consider it as a "OPOG" (see Theorem 4.9 below).

Lemma 4.5. *Let A be a OPOG(1) on $L^2(\mathbb{R}^n \times S^1)$. Let us suppose that A is a symmetric operator. For $P \in \mathbb{N}$ (large), let*

$$A_P := A + (|D_\theta|^2 + |x|^2 + |D_x|^2)^P \tag{4.4}$$

Then A_P is an elliptic selfadjoint pseudo-differential operator. Therefore $e^{is\frac{A_P}{\hbar}}$ is a family of unitary Fourier integral operators.

Proof. it is enough to observe that A_P is, defined on the domain of $|D_\theta|^2 + |x|^2 + |D_x|^2$, a selfadjoint pseudodifferential operator with symbol of type $S_{1,0}$. □

Lemma 4.6. *Let H_0 the operator*

$$H_0 = \sum_1^n \theta_i a_i^- a_i^+ + D_t$$

then, if W is an OPOG(r), so is $\frac{[H_0,W]}{i\hbar}$.

Proof. $\frac{[H_0,W]}{i\hbar} = \frac{d}{ds} e^{isH_0/\hbar} W e^{isH_0/\hbar}|_{s=0}$ which, since H_0 is quadratic, is the same polynomial as W modulo the substitution $a_i^- \to e^{is} a_i^-$, $a_i^+ \to e^{-is} a_i$ and shifting of the coefficients in t by s. Therefore the result is immediate. □

More generally:

Lemma 4.7. *For any H and W of type OPOG(m) and OPOG(r) respectively, $\frac{[H,W]}{i\hbar}$ is an OPOG($m+r-2$).*

The proof is immediate noting that $[a_i^-, a_j^+] = \hbar\delta_{ij}$ and that, for any C^∞ function $a(t)$, $[D_t, a] = i\hbar a'$.

We can now state the main result of this section:

Theorem 4.8. *Let H be a (Weyl) pseudo-differential operator on $L^2(\mathbb{R}^n \times S^1)$ whose principal symbol is of the form:*

$$H_0(x,\xi;t,\tau) = \sum_1^n \theta_i(x_i^2+\xi_i^2) + \tau + H_2,$$

where H_2 vanishes to third order at $x=\xi=\tau=0$ and $\theta_1,...,\theta_n, 2\pi$ are linearly independent over the rationals. Let us define, as before, $P_i = -\hbar^2\frac{\partial^2}{\partial x_i^2} + x_i^2$ and $D_t = -i\hbar\frac{\partial}{\partial t}$.

Then, $\forall \mathcal{M} < +\infty$, there exists a family of unitary operators $(U_L)_{L=3\dots}$ and constants $(C_L)_{L=3\dots}$, and a C^∞ function $h(p_i,\dots,p_n,\tau,\hbar)$ such that:

$$|| \left(U_L H U_L^{-1} - h(P_1,\dots,P_n,D_t,\hbar)\right) H_\mu||_{L^2(\mathbb{R}^n\times S^1)} \leq C_L|\mu\hbar|^{\frac{L+1}{2}}\ \forall \mu\in\mathbb{N}^{n+1} s.t\ |\mu\hbar| \leq \mathcal{M}.$$

Proof. The proof of Theorem 4.8 will be a consequence of the following:

Theorem 4.9. *Let H be as before, and let G be an OPOG(3). Then there exists a function $G_1(p_1,\dots,p_n,\tau,\hbar)$, an OPOG F and an operator R such that:*

i. $\frac{[H,F]}{i\hbar} = G + G_1 + R$

ii. R satisfies: $||RH_\mu|| = O(|\mu\hbar|^{\frac{L+1}{2}})$, $\forall\mu\in\mathbb{N}^{n+1}$, $|\mu\hbar| = O(1)$ *and* $\forall L\in\mathbb{N}$

iii if G is an OPOG(κ) so is F

iv. if G is a symmetric operator, so is F and G_1 is real.

Let us first prove that Theorem 4.9 implies Theorem 4.8:

by induction, as in the "classical" case and thanks to Proposition 4.3, one can assume that H is of the form $H = H_0 + G$, where G is an OPOG(κ). Let us consider the operators $e^{i\frac{F_P}{\hbar}}He^{-i\frac{F_P}{\hbar}}$ and $H(s) := e^{is\frac{F_P}{\hbar}}He^{-is\frac{F_P}{\hbar}}$, where F satisfies Theorem 4.9 and F_P is defined by (4.4) for P large enough.

Since we are in an iterative perturbative setting, it is easy to check by taking P large enough that we can omit the subscript P in $H(s)$ and let $e^{\pm i\frac{F_P}{\hbar}}$ stand for $e^{\pm i\frac{F}{\hbar}}$ in the rest of the computation. We have:

$$\begin{aligned} e^{i\frac{F}{\hbar}}He^{-i\frac{F}{\hbar}} &= H + \frac{[F,H]}{i\hbar} + \frac{[F,\frac{[F,H]}{i\hbar}]}{i\hbar} + \frac{[F,[F,[F,\int_0^1\int_0^t\int_0^s H(u)dudsdt]/i\hbar]/i\hbar]}{i\hbar} \\ &= H_0 + G + \frac{[F,H]}{i\hbar} + \frac{[F,\frac{[F,H]}{i\hbar}]}{i\hbar} + \tilde{R} \qquad (4.5) \\ &= H_0 - G_1 + R + \frac{[F,\frac{[F,H]}{i\hbar}]}{i\hbar} + \tilde{R}. \end{aligned}$$

Since we are interested in letting all the operators acting on the H_μ for $|\mu\hbar| = O(1)$ we can microlocalize near $x=\xi=\tau=0$ and replace F and H by their microlocalized versions $\tilde{F}$ and $\tilde{H}$. $e^{i\frac{\tilde{F}}{\hbar}}$ is a Fourier integral operator and, by

Egorov's Theorem, $\tilde{H}(s)$ is a family of pseudodifferential operators, and so is $\int_0^1\int_0^t\int_0^s\tilde{H}(u)dudsdt$. By Proposition 4.3, Lemma 4.7 and Lemma 4.2 we have, since G is an OPOG(κ),

$$||\tilde{R}H_\mu|| = O(|\mu\hbar|^{\kappa+1}).$$

By the same argument, $\frac{[F,\frac{[F,H]}{i\hbar}]}{i\hbar}$ satisfies the same estimate. Developing $\tilde{R}$ by the Lagrange formula (4.5) to arbitrary order, we get, thanks to Lemma 4.7, $\tilde{R}=\tilde{G}+R$ where $\tilde{G}$ is an OPOG($\kappa+1$) and

$$||RH_\mu|| = O(|\mu\hbar|^{\frac{L+1}{2}}).$$

Therefore, letting $G'=\frac{[F,\frac{[F,H]}{i\hbar}]}{i\hbar}+\tilde{G}$, we have:

$$e^{i\frac{F_P}{\hbar}}He^{-i\frac{F_P}{\hbar}} = H_0+G_1+G'+R,$$

with G' an OPOG($\kappa+1$). By induction Theorem 4.8 follows.

Proof of Theorem 4.9:

let us first prove the following

Lemma 4.10. *Let H_0 be as before and let G be an OPOG(r). Then there exists a OPOG(r) F and $G_1=G_1(p_1,\dots,p_n,D_t,\hbar)$, such that:*

$$\frac{[H_0,F]}{i\hbar} = G+G_1. \tag{4.6}$$

Proof. By Lemma 4.6, if F is an OPOG, it must be an OPOG(r), since the left hand side of (4.6) is an OPOG(r). Let us take the matrix elements of (4.6) relating the H_μs. We get:

$$-i\Theta.(\mu-\nu) < \mu|F|\nu > = < \mu|G+G_1|\nu > + < \mu|R|\nu >,$$

where $\Theta.(\mu-\nu) := \sum_1^n\theta_i\mu_i+\mu_{n+1}$ and $< \mu|.|\nu > = (H_\mu,.H_\nu)$. We get immediately that $G_1(\mu\hbar,\hbar) = -<\mu|G|\mu>$. Moreover, let us define F by:

$$< \mu|F|\nu > := \frac{< \mu|G+G_1|\nu >}{-i\Theta.(\mu-\nu)},$$

which exists by the non-resonance condition. To show that F is an OPOG one just has to decompose $G=\sum G_l$ in monomial OPOGs $G^l=\alpha(t)D_t^jb_1\dots b_m,\ b_i\in\{a_1,a_1^+,\dots,a_n,a_n^+\}$.

Then, for each ν there is only one μ for which $< \mu|G+G_1|\nu > \neq 0$ and the difference $\mu-\nu$ depends obviously only on G^l, not on ν. Let us call this difference ρ_{G_l}. Then F is given by the sum:

$$F=\sum\frac{1}{-i\Theta.\rho_{G_l}}G_l.$$

□

It is easy to check that one can pass from Lemma 4.10 to Theorem 4.9 by induction, writing $[H,F+F']=[H,F]+[H_0,F']+[H-H_0,F]+[H-H_0,F']$. $\square$

We will show finally that Theorems 4.8 and 3.1 are equivalent. Once again we can start by considering an Hamiltonian on $L^2(\mathbb{R}^n\times S^1)$ since any Fourier integral operator B_φ, as defined in the beginning of the proof of Theorem 3.1, intertwines the original Hamiltonian $H: C_0^\infty(X)\to C^\infty(X)$ of Section 3 with a pseudodifferential operator on $L^2(\mathbb{R}^n\times S^1)$ satisfying the hypothesis of Theorem 4.8.

Let us remark first of all that if $U_L=e^{i\frac{W_3}{\hbar}}e^{i\frac{W_4}{\hbar}}\dots e^{i\frac{W_L}{\hbar}}$, all $e^{i\frac{W_l}{\hbar}}$ being Fourier integral operators, then so is U_L. Secondly we have

Proposition 4.11. *Let A be a pseudodifferential operator of total Weyl symbol $a(x,\xi,t,\tau,\hbar)$. Then*

a vanishes to infinite order at $p=\tau=0$ if and only if $||AH_\mu||_{L^2(\mathbb{R}^n\times S^1)}=O(|\mu\hbar|^\infty)$.

Proof. the “if” part is exactly Proposition 4.3. For the “only if” part let us observe that, if the total symbol didn't vanish to infinite order, then it would contain terms of the form $\alpha_{kmnr}(t)\hbar^k(x+i\xi)^m(x-i\xi)^n\tau^r$. Let us prove this can't happen in dimension 1, the extension to dimension n being straightforward.

Each term of the form $(x+iD_x)^m(x-iD_x)^n=a^m(a^+)^n$ gives rise to an operator $A_{m,n}$ such that:

$$\begin{aligned} A_{m,n}H_\mu &= \hbar^{\frac{|m+n|}{2}}\sqrt{(\mu+1)\dots(\mu+n)(\mu+n-1)\dots(\mu+n-m)}H_{\mu+m-n} \\ &\sim |\mu\hbar|^{\frac{m+n}{2}}H_{\mu+m-n}. \end{aligned}$$

Therefore $\sum c_{mn}A_{m,n}H_\mu=\sum c_{m,m-l}H_{\mu+l}\sim\sum c_{m,m-l}|\mu\hbar|^{\frac{2m-l}{2}}H_{\mu+l}$. In particular:

$$||\sum c_{mn}A_{m,n}H_\mu||^2\sim\sum|\mu\hbar|^{2m-l}$$

so $||\sum c_{mn}A_{mn}H_\mu||=O(|\mu\hbar|^\infty)$ implies $C_{mn}=0$. It is easy to check that the same argument is also valid for any ordered product of a's and a^+'s. $\square$

In the next section we will show how the functions H' of Theorem 4.8 and h of Theorem 3.1 are related.

5. Link between the two quantum constructions

Consider a symbol (on $\mathbb{R}^{2n}$) of the form

$$h(p_1,...,p_n)$$

with $p_i=\frac{\xi_i^2+x_i^2}{2}$. The are several ways of quantizing h: one of them consists in associating to h, by the spectral theorem, the operator

$$h(P_1,...,P_n)=h(P)$$

where $P_i=\frac{-\hbar^2\partial_{x_i}^2+x_i^2}{2}$. Another one is the Weyl quantization procedure.

In this section we want to compute the Weyl symbol h^{we} of $h(P_1, ..., P_n)$ and apply the result to the situation of the preceding sections. By the metaplectic invariance of the Weyl quantization and the fact that $h(P_1, ..., P_n)$ commutes with all the P_i's we know that h^{we} has the form

$$h^{we}(p_1, ..., p_n) = h^{we}(p),$$

that is, is a function of the classical harmonic oscillators $p_i := \xi_i^2 + x_i^2$.

To see how this h^{we} is related to the h above we note that H is diagonal on the Hermite basis h_j. Therefore

$$h((j+\frac{1}{2})\hbar)) =< h_j, Hh_j >= \int h^{we}\left((\frac{x+y}{2})^2+\xi^2\right) e^{i\frac{x\xi}{\hbar}} h_j(x)h_j(\xi)\frac{dxd\xi}{\hbar^{n/2}}.$$

We now claim

Proposition 5.1. *let h be either in the Schwartz class, or a polynomial function. Let $\hat{h}(s) = \frac{1}{(2\pi)^n}\int h(p)e^{-is.p}dp$ be the Fourier transform of h. Then*

$$h^{we}(p) = \int \hat{h}(s)e^{\frac{2i\tan(s\hbar/2).p}{\hbar}}\Phi(s)ds \tag{5.1}$$

where $\tan(s\hbar/2).p$ stands for $\sum_i \tan(s_i\hbar/2)p_i$ and $\Phi(s) = \prod_{i=1}^n (1-2i\tan(s_i\hbar/2))$, and where (5.1) has to be interpreted in the sense of distribution, that is, for each φ in the Schwartz's class of $\mathbb{R}$,

$$\int h^{we}(p)\varphi(p)dp = \int \hat{h}(s)e^{\frac{2i\tan(s\hbar/2).p}{\hbar}}\Phi(s)ds\varphi(p)dp = \int \hat{h}(s)\Phi(s)\hat{\varphi}\left(\frac{2i\tan(s\hbar/2)}{\hbar}\right)ds.$$

Finally, as $\hbar \to 0$,

$$h^{we} \sim h + \sum_{l=1}^{\infty} c_l\hbar^{2l} \tag{5.2}$$

Proof. Let $h(P) = \int \hat{h}(s)e^{is.P}ds$, where $e^{is.P}$ is a zeroth order semiclassical pseudo-differential operator whose Weyl symbol will be computed from its Wick symbol (see 5.5 below for the definition). Let us first remark that since $e^{is.P} = \Pi_{i=1}^n e^{is_i.P_i}$ it is enough to prove the Theorem in the one-dimensional case.

Let $\varphi_{x\xi}$ be a coherent state at (x, ξ), that is

$$\varphi_{x\xi}(y) = (\pi\hbar)^{-\frac{1}{4}}e^{i\frac{\xi y}{\hbar}}e^{-\frac{(y-x)^2}{2\hbar}}$$

Let $z = \frac{\xi+ix}{\sqrt{2}}$, $z' = \frac{\xi'+ix'}{\sqrt{2}}$ and $z(t) = \frac{\xi(t)+ix(t)}{\sqrt{2}}$. A straightforward computation gives

$$\left(\varphi_{x\xi}, \varphi_{x'\xi'}\right) = e^{\frac{2z\overline{z'}-|z|^2-|z'|^2}{2\hbar}}. \tag{5.3}$$

Moreover decomposing $\varphi_{x\xi}$ on the Hermite basis leads to

$$e^{isP}\varphi_{x\xi} = e^{i\frac{s}{2}\hbar}\varphi_{x(\hbar s)\xi(\hbar s)} \tag{5.4}$$

where $P = \frac{-\hbar^2\partial_x^2+x^2}{2}$ and $z(t) = e^{it}z$.

The Wick symbol of e^{isP} is defined as

$$\sigma^{wi}(e^{isP})(x,\xi) := \left(\varphi_{x\xi}, e^{isP}\varphi_{x\xi}\right) \tag{5.5}$$

which, by (5.3) and (5.4), is equal to

$$e^{-\frac{1-e^{-i\hbar s}}{\hbar}\left(\frac{x^2+\xi^2}{2}\right)+i\frac{s}{2}\hbar}$$

Moreover, using the Weyl quantization formula, it is immediate to see that the Weyl and Wick symbols are related by

$$\sigma^{wi} = e^{-\frac{\hbar\Delta}{4}}\sigma^{we}$$

where $\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2}\right)$.

It is a standard fact the the Wick symbol determines the operator: indeed the function $e^{\frac{-2z\overline{z'}+|z|^2+|z'|^2}{2\hbar}}(\varphi_{x\xi}, e^{isP}\varphi_{x'\xi'})$ obviously determines e^{isP}. Moreover it is easily seen to be analytic in z and $\overline{z'}$. Therefore it is determined by its values on the diagonal $z=\overline{z'}$ i.e., precisely, the Wick symbol of e^{isP}. A straightforward calculation shows that, for $\frac{s}{2\hbar} \neq \frac{(2k+1)\pi}{2}$, $k\in\mathbb{Z}$,

$$(1-2i\tan(s\hbar/2))e^{-\frac{\hbar\Delta}{4}}e^{\frac{2i\tan(s\hbar/2)\frac{x^2+\xi^2}{2}}{\hbar}} = e^{-\frac{1-e^{-i\hbar s}}{\hbar}\left(\frac{x^2+\xi^2}{2}\right)+i\frac{s}{2}\hbar} \tag{5.6}$$

This shows that, for $\frac{s}{2\hbar} \neq \frac{(2k+1)\pi}{2}$, $k\in\mathbb{Z}$, we have

$$\sigma^{we}(e^{isP})(p) = (1-2i\tan(s\hbar/2))e^{\frac{2i\tan(s\hbar/2)\frac{x^2+\xi^2}{2}}{\hbar}}.$$

Let us now take φ in the Schwartz's class of $\mathbb{R}$, and B_φ be the operator of (total) Weyl symbol $\varphi(\frac{x^2+\xi^2}{2})$. Let:

$$f(s) := 2\pi\int \sigma^{we}(e^{isP})(p)\varphi(p)pdp = \mathrm{Trace}[e^{isP}B_\varphi].$$

Lemma 5.2.

$$f\in C^\infty(\mathbb{R}).$$

Proof. by metaplectic invariance we know that B_φ is diagonal on the Hermite basis. Therefore, $\forall k\in\mathbb{N}$,

$$(-i)^k\frac{d^k}{ds^k}f(s) := \mathrm{Trace}[e^{isP}P^kB_\varphi] = \sum < h_j, B_\varphi h_j > ((j+\frac{1}{2})\hbar)^k e^{is(j+\frac{1}{2})\hbar}.$$

Since h_j is microlocalized on the circle of radius $(j+\frac{1}{2})\hbar$ and φ is in the Schwartz class, the sum is absolutely convergent for each k. □

Therefore $f(s) = 2\pi\int(1-2i\tan(s\hbar/2))e^{\frac{2i\tan(s\hbar/2)\frac{x^2+\xi^2}{2}}{\hbar}}\varphi(p)pdp$ and (5.6) is valid in the sense of distribution (in the variable p) for all $s\in\mathbb{R}$. This expression gives (5.1) immediately for h in the Schwartz class. When h is a polynomial function it is straightforward to check that, since $\hat{h}$ is a sum of derivatives of the Dirac mass and e^{isP} is a Weyl operator whose symbol is C^∞ with respect of s, the formula also holds in this case. The asymptotic expansion (5.2) is obtained by expanding $e^{\frac{2itg(s\hbar/2)\frac{x^2+\xi^2}{2}}{\hbar}}$ near $e^{is\frac{x^2+\xi^2}{2}}$. □

Formula (5.1) shows clearly that h^{we} depends only on the $\frac{2\pi}{\hbar}$periodization of $\hat{h}(s)e^{i\frac{s\hbar}{2}}$, therefore

Corollary 5.3. *h^{we} depends only of the values $h\left((k+\frac{1}{2})\hbar\right)$, $k \in \mathbb{N}$.*

We mention one application of formula (5.1). Let us suppose first that we have computed the quantum normal form at order K, that is

$$h_K(p) = \sum_{|k|=k_1+\dots k_n \le K} c_k p^k := \sum_{|k|=k_1+\dots k_n \le K} c_k p_1^{k_1} \dots p_n^{k_n}$$

and let us define h_K^{we} as the Weyl symbol of $h_K(P)$.

Corollary 5.4.

$$\begin{aligned} h_K^{we}(p) &= \sum_{|k|=k_1+\dots k_n \le K} c_k \frac{\partial^K}{\partial_s^k} \left(\Phi(s) e^{\frac{2i\tan(s\hbar/2)p}{\hbar}}\right)|_{s=0} \\ &:= \sum_{|k|=k_1+\dots k_n \le K} c_k \frac{\partial^K}{\partial_{s_1}^{k_1} \dots \partial^{k_n} s_n} \left(\Phi(s) e^{2i\frac{\tan(s_1\hbar/2)p_1+\dots+\tan(s_n\hbar/2)p_n}{\hbar}}\right)|_{s=0}. \end{aligned}$$

Let us come back now to the comparison between the two constructions of Sections 2 and 3.

Clearly the "θ" part doesn't play any role, as the Weyl quantization of any function $f(\tau)$ is exactly $f(D_\theta)$. therefore we have the following

Theorem 5.5. *The functions H' of Theorem 3.1 and h of Theorem 4.8 are related by the formula*

$$H'\left(P_1, ..., P_n, D_t, \hbar\right) = \int \hat{h}(s, D_t, \hbar) e^{\frac{2i\tan(s\hbar/2).p}{\hbar}} \Phi(s) ds$$

where $\hat{h}$ is the Fourier transform of h with respect to the variables p_i. In particular

$$H' - h = O(\hbar^2).$$

Proof. the proof follows immediately from Proposition 5.1, and the unicity of the (quantum) Birkhoff normal form. □

6. The computation of the semiclassical Birkhoff canonical form from the asymptotics of the trace formula

In this section we will abandon the quantum approach to Birkhoff canonical forms developed in sections 4-5 and revert to the symbolic approach of sections 2-3. Using this approach we will prove that the wave trace data coming from the Gutzwiller formula determine the Quantum Birkhoff canonical form constructed in section 3. Our goal will be by "mimicking" (with semiclassical modifications) the proof of this result by Zelditch in [11, 12] and in particular avoid the method of "Grushin reduction" used in [7] to equate the trace formula of [6, 8, 9] with the trace formula for a monodromy operator.

Warning: the A_φ in display 6.3 below is not the family of U_L's figuring in Theorem 4.8 but is the "symbolic" A_φ figuring in Theorem 3.1. In particular the estimates in theorems 4.8 and 4.9 will not play any role in this proof.

Let X and H be as in the introduction. Let γ be a periodic trajectory of the vector field (2.1) of period 2π.

For $l \in \mathbb{Z}$ let ψ_l be a Schwartz function on the real line whose Fourier transform $\hat{\psi}_l$ is supported in a neighborhood of $2\pi l$ containing no other period of (2.1). The semiclassical trace formula gives an asymptotic expansion for Trace $\psi_l\left(\frac{H-E}{\hbar}\right)$ of the form:

$$\text{Trace } \psi_l\left(\frac{H-E}{\hbar}\right) \sim \sum_{m=0}^{\infty} d_l^m \hbar^m \tag{6.1}$$

where the d_l's are distributions acting on $\hat{\psi}_l$ with support concentrated at $\{2\pi l\}$.

We will show that the knowledge of the d_ls determine the quantum semiclassical Birkhoff form of Section 2, and therefore the classical one.

Let us first rewrite the l.h.s of (6.1) as

$$\text{Trace}\left(\int \hat{\varphi\psi}(t) e^{it\frac{H-E}{\hbar}} dt\right) \tag{6.2}$$

Since $\hat{\psi}$ is supported near a single period of (2.1) we know from the general theory of Fourier integral operators that one can microlocalize (6.1) near γ.

Therefore we can conjugate (6.2) by the semiclassical Fourier integral operator A_φ of Theorem 3.1. This leads to the computation of

$$(6.3) \qquad \text{Trace}\left(A_\varphi \int \hat{\psi}(t) e^{it\frac{H-E}{\hbar}} dt A_\varphi^{-1}\right)$$

$$= \text{Tr}\left(\int \hat{\psi}(t)\rho(P_1,...,P_n,D_t) e^{it\frac{H'(P_1,...,P_n,D_t,\hbar)+H''-E}{\hbar}} dt\right)$$

where $\rho \in C_0^\infty(\mathbb{R}^{n+1})$ with $\rho = 1$ in a neighborhood of $p = \tau = 0$ and Tr stands for the Trace in $L^2(\mathbb{R}^n \times S^1)$.

Let us note that, as is standard in the proof on trace formulas, by the independance condition of the Poincaré angles (see footnote (2)), γ is isolated on its energy shell $\{H = E\}$. By standard stationary phase technics this is enough to show that the contribution of H'' in (6.3) is of order $O(\hbar^\infty)$.

Let us write $H'(P_1,...,P_n,D_t,\hbar)$ as

$$E + D_t + \sum \theta_i P_i + \sum_{r\in\mathbb{N}^n, s\in\mathbb{Z}} c_{r,s}(\hbar) P^r D_t^s. \tag{6.4}$$

We will first prove

Proposition 6.1. *let $g^l_{r,s}(t,\theta)$ be the function defined by*

$$g^l_{r,s}(t,\theta)=\left(-i\frac{\partial}{t\partial\theta}\right)^r\left(-i\frac{\partial}{\partial t}\right)^s\left[\frac{e^{it\frac{\theta_1+\dots+\theta_n}{2}}}{\Pi_i(1-e^{it\theta_i})}t\hat{\psi}(t)\right] \tag{6.5}$$

Let us fix $l\in\mathbb{Z}$. Then the knowledge of all the d_l^ms for $m<M$ in (6.1) determines the following quantities

$$\sum_{|r|+s=m}c_{r,s}(\hbar)g^l_{r,s}(2\pi l,\theta) \tag{6.6}$$

for all $m<M$.

Proof. the r.h.s. of (6.3) can be computed thanks to (6.4) using

$$\begin{aligned}\text{spectrum } P_i &= \{(\mu_i+\frac{1}{2})\hbar,\ \mu_i\in\mathbb{N}\}\\ \text{spectrum } D_t &= \{\nu\hbar,\ n\in\mathbb{Z}\}\end{aligned}$$

Thus the r.h.s of (6.3) can be written as
(6.7)

$$\int\hat{\psi}_l(t)\sum_{\mu,\nu}\rho\left((\mu+\frac{1}{2})\hbar,\nu\hbar\right)e^{it\left[\nu+\theta.(\mu+\frac{1}{2})\right]}\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\left(\sum_{r,s}c_{r,s}(\hbar)\left(\mu+\frac{1}{2}\right)^r\nu^s\hbar^{|\mu|+s-1}\right)^k dt$$

since the support of $\hat{\psi}_l$ contains only one period, and therefore the trace can be microlocalized infinitely close to the periodic trajectory, making the role of H'' inessential.

Using the following remark of S. Zelditch:

$$(\mu+\frac{1}{2})^r\nu^s=\left(-i\frac{\partial}{t\partial\theta}\right)^r\left(-i\frac{\partial}{\partial t}\right)^s e^{it\left[\nu+\theta.(\mu+\frac{1}{2})\right]}$$

we get, mod($\hbar^\infty$),
(6.8)

$$\int\hat{\psi}_l(t)\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\left(\sum_{r,s}\hbar^{|\mu|+s-1}c_{r,s}(\hbar)\left(-i\frac{\partial}{t\partial\theta}\right)^r\left(-i\frac{\partial}{\partial t}\right)^s\right)^k\sum_{\mu,\nu}e^{it\left[\nu+\theta.(\mu+\frac{1}{2})\right]}dt.$$

Since $\sum_{\nu\in\mathbb{Z}}e^{it\nu}=2\pi\sum_l\delta(t-2\pi l)$, and $\sum_{\mu\in\mathbb{N}^n}e^{it\theta.(\mu+\frac{1}{2})}=\frac{e^{it\frac{\theta_1+\dots+\theta_n}{2}}}{\Pi_i\left(1-e^{it\theta_i}\right)}$, together with the fact that $\hat{\psi}$ is supported near $2\pi l$, we get that (6.8) is equal to

$$2\pi\left[\sum_{k=0}^{\infty}\frac{(i)^k}{k!}\left(\sum_{r,s}\hbar^{|\mu|+s-1}c_{r,s}(\hbar)\left(-i\frac{\partial}{t\partial\theta}\right)^r\left(-i\frac{\partial}{\partial t}\right)^s\right)^k\left(t^k\hat{\psi}_l(t)\frac{e^{it\frac{\theta_1+\dots+\theta_n}{2}}}{\Pi_i\left(1-e^{it\theta_i}\right)}\right)\right]_{t=2\pi l}.$$

Rearranging terms in increasing powers of $\hbar$ shows that the quantities (6.6) can be computed recursively. ∎

The fact that one can compute the $c_{r,s}(\hbar)$ from the quantities (6.6) is an easy consequence of the rational independence of the θ_is and the Kronecker theorem, and is exactly the same as in [5].

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