ON THE VANISHING OF SOME NON SEMISIMPLE ORBITAL INTEGRALS

GAËTAN CHENEVIER AND DAVID RENARD

Abstract. We prove the vanishing of the (possibly twisted) orbital integrals of certain functions on real Lie groups at non semisimple elliptic elements. This applies to Euler-Poincaré functions and makes some results of [CCl] unconditionnal.

Introduction

Let G be a connected reductive algebraic group over a number field F and θ an automorphism of finite order of G defined over F. View $G\theta$ as a connected component of the linear group $G \rtimes \langle\theta\rangle$, and let $f = \prod_v f_v$ be a smooth bi-K-finite function with compact support on $G(\mathbb{A}_F)\theta$. In his monumental work [A1], J. Arthur shows that under some "cuspidality" assumption on f_v at two places v, a simple form of his trace formula holds for f. Under an extra assumption on f_v at some place v, he even shows that the geometric side of his formula reduces to the sum of the orbital integrals of f (times some volume) over the $G(F)$-conjugacy classes of semisimple F-elliptic elements of $G(F)\theta$. This extra assumption is the following ([A1, Cor. 7.4]):

$\mathrm{H}(f_v)$: *For any element $\gamma \in G(F_v)\theta$ which is not semisimple and F_v-elliptic, the orbital integral $O_\gamma(f_v)$ of f_v at γ vanishes.*

Recall that for $L = F$ or $L = F_v$, a semisimple element $\gamma \in G(L)\theta$ is *L-elliptic* if the split component of the center of the connected centralizer of γ in G coincides with the split component of $Z(G)^\theta$, where $Z(G)^\theta$ is the subgroup of the center $Z(G)$ of G which is invariant under θ ([A1, §3, p. 508]).

This condition $\mathrm{H}(f_v)$ actually implies that f_v is cuspidal in the sense of Arthur (see §7, p. 538 *loc.cit.*). Historically, $\mathrm{H}(f_v)$ has been first imposed by assuming that the support of f_v lies in the regular elliptic subset of $G(F_v)\theta$, for instance for a finite place v, and in conjonction with a supercuspidality assumption at another place, in which case it is actually possible to give a much simpler proof of Arthur's results, as explained in [DKV, A.1]. This option has the disadvantage to kill the

Date: October 13, 2009.
Gaëtan Chenevier is supported by the C.N.R.S.

non-regular terms of the geometric side of the trace formula, which is a problem for some applications.

When v is a finite place, and in the non-twisted case $\theta = 1$, Kottwitz had shown that the Euler-Poincaré functions f_v of $G(F_v)$ satisfy $\mathrm{H}(f_v)$ ([Ko2]). These f_v are pseudo-coefficients of the Steinberg representation and a key property that Kottwitz proves, using the Bruhat-Tits building of $G(F_v)$ and results of Serre on Euler-Poincaré measures, is first that their orbital integrals $O_\gamma(f_v)$ vanish at every semisimple non-F_v-elliptic element $\gamma \in G(F_v)$, and second, that $O_\gamma(f_v)$ is constant on the set of semisimple elliptic elements γ in $G(F_v)$ (the Haar measure on their connected centralizers being the Euler-Poincaré measures). Using an argument of Rogawski relying on results of Harish-Chandra on Shalika germs, Kottwitz concludes the non-vanishing of $O_\gamma(f_v)$ for all non semisimple γ. These results were extended to the twisted case in [CCl, §3] (see also [BLS]).

In this paper, we are interested in the case where v is archimedean. Euler-Poincaré functions f_v still exist and have been constructed by Labesse [L] and Clozel-Labesse [CL]. The orbital integrals of these functions f_v vanish at all the semisimple non-elliptic elements and satisfy some stability properties at the semisimple elliptic ones. Our aim in this paper is to show that these properties imply the vanishing of all the non semisimple orbital integrals. Although this might well be known to some specialists, we have not been able to find a proof of this precise statement in the literature. After this paper was written, J.-L. Waldspurger told us that our main result is very close, in the non-twisted (but essential) case, to Arthur's theorem 5.1 in [A3]. Our assumptions are slightly different but the main ideas of the proof are the same. Nevertheless, we think that for past and future applications this paper might be a useful reference. Our main motivation for this question is a recent paper [CCl] of Clozel and the first author, in which they prove some results concerning the orthogonal/symplectic alternative for selfdual automorphic representations of $\mathbf{GL}_{2n}$ assuming $\mathrm{H}(f_v)$ for certain pseudo-coefficients f_v as above, in the special case $G = \mathbf{GL}_{2n}$ and $\theta(g) = {}^t g^{-1}$. Before discussing the statements that the present work make unconditional, let us explain our results in more detail.

Assume from now on that G is a connected reductive algebraic group over $\mathbb{R}$, and that θ is an automorphism of finite order of G defined over $\mathbb{R}$. Let $f : G(\mathbb{R})\theta \longrightarrow \mathbb{C}$ be a smooth function with compact support and consider the two following properties :

(i) $O_\gamma(f) = 0$ *if* $\gamma \in G(\mathbb{R})\theta$ *is semisimple, strongly regular, but non-elliptic.*

(ii) *if γ and γ' are stably conjugate strongly regular semisimple elements of $G(\mathbb{R})\theta$, then $O_\gamma(f) = O_{\gamma'}(f)$, the measure on the connected centralizers of γ and γ' being compatible (see§1.3).*

Theorem A: *If $f \in \mathcal{C}_c^\infty(G(\mathbb{R})\theta)$ satisfies* (i) *and* (ii)*, then*

$$O_\gamma(f) = 0$$

for any $\gamma \in G(\mathbb{R})\theta$ which is not semisimple and elliptic.

The proof of this statement follows two steps. The first one is to use a version of Harish-Chandra's descent to the connected centralizer of the semisimple part of γ. This reduces the problem to $\theta = 1$ and to a statement on the real Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$. We rely on results of one of us thesis [R]. We are then led to prove the following infinitesimal variant of Theorem A.

Let us consider now the adjoint action of $G(\mathbb{R})$ on $\mathfrak{g}$. Let f be a smooth complex function on $\mathfrak{g}$ with compact support and denote by $O_X(f)$ the integral of f over the $G(\mathbb{R})$-orbit of $X \in \mathfrak{g}$, equipped with some $G(\mathbb{R})$-invariant measure. (We actually use Duflo-Vergne normalization to simultaneously fix such an invariant measure for the semisimple orbits. Moreover, we consider in the proof another normalization of $O_X(f)$ that we denote by $J_\mathfrak{g}(f)(X)$.) Recall that a regular semisimple $X \in \mathfrak{g}$ is elliptic if the Cartan subalgebra of $\mathfrak{g}$ containing X only has imaginary roots in $\mathfrak{g}_\mathbb{C}$. If $X, Y \in \mathfrak{g}$ are two regular semisimple elements, they are said to be stably conjugate if $\mathrm{Ad}(g)(X) = Y$ for some $g \in G(\mathbb{C})$.

Theorem B : *Let $f : \mathfrak{g} \longrightarrow \mathbb{C}$ be a smooth function with compact support. Assume that there exists a neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$ such that:*

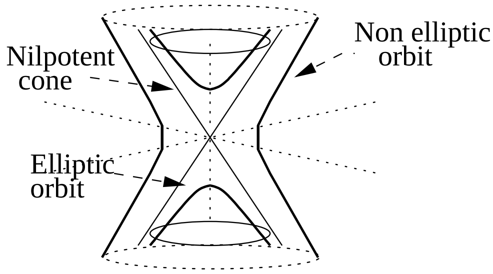
(i) $O_X(f) = 0$ *for all regular, semisimple, and non elliptic $X \in \mathcal{V}$.*

(ii) $O_X(f) = O_Y(f)$ *for all regular, semisimple, elliptic, and stably conjugate elements $X, Y \in \mathcal{V}$.*

Then $O_X(f) = 0$ if $X \in \mathfrak{g}$ is nilpotent, unless perhaps if $X = 0$ and $\mathfrak{g}$ has an elliptic Cartan subalgebra.

Let us give a proof of this result in the simple case $G = \mathrm{SL}_2$. In this case, $\mathfrak{g}$ is three-dimensional and the nilpotent elements $\mathcal{N}$ form a quadratic cone over which $G(\mathbb{R})$ has 3 orbit : 0, $\mathcal{O}$ and $-\mathcal{O}$. The complement of $\mathcal{N}$, consisting of semisimple elements, has three connected components : two inside the cone consisting of the elliptic orbits, each one being a sheet of an hyperboloid with two sheets, and one outside the cone consisting of non elliptic orbits, which are hyperboloids with one sheet.

Choose $G(\mathbb{R})$-invariant measures on $\mathcal{O}$ and $-\mathcal{O}$ which are symmetric with respect to zero. The idea is to view the nilpotent orbits as



limits of these hyperboloid sheets, either from inside or from outside the cone, which gives two relations. First, approaching $\mathcal{O} \cup -\mathcal{O}$ with hyperboloids with one sheet, over which $O_*(f)$ vanishes by (i), we obtain that $O_X(f) + O_{-X}(f) = 0$ for any nonzero nilpotent X. Second, approaching now $\pm\mathcal{O}$ using elliptic orbits, condition (ii) ensures that $O_X(f) = O_{-X}(f)$. These two relations lead to $O_X(f) = O_{-X}(f) = 0$, what we had to prove.

Note that if we had chosen $G = \mathrm{PGL}_2$, then $\mathcal{N}\backslash\{0\}$ would be a single orbit, hence the first step would have been enough to conclude. This is compatible with the fact that (ii) is automatically satisfied in $G(\mathbb{R})$ (stable conjugacy and conjugacy coincide). For a general group G, the strategy that we use is a little different as the geometry of the nilpotent cone is more complicated. However, part of this geometry is encoded in the so-called jump relations of orbital integrals, due to Harish-Chandra, which actually essentially reflect the situation in this $\mathfrak{sl}_2$-case. Another ingredient will be a description by Bouaziz of the invariant distributions on $\mathfrak{g}$ with support in the nilpotent cone, as well as a more refined version for measures on nilpotent orbits due to Hotta-Kashiwara [HoKa] (we use [Ko3] as a reference for these results).

Let us recall now the two statements of [CCl, §4.18] that our work makes unconditional (see Thm. 4.20 and Thm. 4.22 of *loc. cit.*).

Theorem C : *Let F be a totally real number field and π an automorphic cuspidal representation of $\mathbf{GL}_{2n}(\mathbb{A}_F)$. Assume that π is selfdual, essentially square-integrable at one finite place w at least, and cohomological at all archimedean places. Then :*

(i) *For all places v of F, π_v is symplectic.*

(ii) *If V_ℓ is an ℓ-adic Galois representation of $\mathrm{Gal}(\overline{F}/F)$ associated to $\pi|.|^{(2n-1)/2}$, with ℓ prime to w, then there exists a non degenerate, Galois-equivariant, symplectic pairing $V_\ell \otimes V_\ell \longrightarrow \overline{\mathbb{Q}}_\ell(2n-1)$.*

By π_v *is symplectic* we mean that the Langlands' parameter[1] of π_v

$$L(\pi_v) : WD_{F_v} \longrightarrow \mathbf{GL}_{2n}(\mathbb{C})$$

[1]For v non archimedean, we use the $\mathbf{SU}(2)$-form of the Weil-Deligne group $WD_{F_v} = W_{F_v} \times \mathbf{SU}(2)$.

defined by Langlands for archimedean v, and by Harris-Taylor in general, may be conjugate to fall into the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$. Of course, we may define similarly an orthogonal representation (and there are plenty of them), and the interesting fact is that an essentially discrete π_w cannot be both orthogonal and a local component component of a automorphic π as in the statement. Actually, assertion (i) for archimedean v is automatic by definition, and its truth at all v is suggested by the hypothetical existence of the global Langlands' group: see the discussion in [CCl, §4.18]. An important feature of the automorphic representations π of the statement is their relation with algebraic number theory. Indeed, as conjectured by Langlands, Clozel-Kottwitz and Harris-Taylor attached to such a π a system of ℓ-adic representations V_ℓ satisfying some strong compatible conditions with the local Langlands correspondence (see [HT],[T],[TY]). Assertion (ii) of the theorem above implies a previously missing property of these representations.

Actually, (i) is an simple consequence of (ii) and of the irreducibility of V_ℓ, and it is easy to show (ii) when π is furthermore a Steinberg representation at an auxiliary finite place (see [CCl, §4.18]). The theorem above is then a consequence of the following other one. There w is a finite place of a totally real number field F, θ is the automorphism $g \mapsto w^t g^{-1} w^{-1}$ of $\mathbf{GL}_{2n}$ where

$$w = \begin{pmatrix} & & & & 1 \\ & & & \iddots & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix} \in \mathbf{GL}_{2n}(\mathbb{Q}).$$

Note that the element $\theta \in \mathbf{GL}_{2n}(\mathbb{Q})\theta$ is semisimple, $\mathbb{Q}$-elliptic and with centralizer $\mathbf{Sp}_{2n}$.

Theorem D : *Let π_w be an essentially discrete, selfdual, irreducible representation of $\mathbf{GL}_{2n}(F_w)$, and let $f_w \in \mathcal{C}_c^\infty(\mathbf{GL}_{2n}(F_w)\theta)$ be a twisted pseudo-coefficient of π_w. The following conditions are equivalent:*

(i) *π_w is a local component of a cuspidal, selfdual, automorphic representation π of $\mathbf{GL}_{2n}(\mathbb{A}_F)$, which is cohomological at all archimedean places.*

(ii) *π_w is a local component of a cuspidal, selfdual, automorphic representation π of $\mathbf{GL}_{2n}(\mathbb{A}_F)$, which is cohomological at all archimedean places, and which is moreover a Steinberg representation at another finite place (that we may choose).*

(iii) *$O_\theta(f_w) \neq 0$.*

If these conditions are satisfied, then π_w is symplectic.

This theorem was proved *loc. cit.* under some assumption called (H) ([CCl, §4.17]) which was used[2] to apply Arthur's simple form of the trace formula for the reasons discussed in the first paragraph of this introduction: this assumption (H) is property H(f) for f a twisted pseudo-coefficient of a θ-discrete cohomological representation of $\mathbf{GL}_{2n}(\mathbb{R})$. These pseudo-coefficients are studied in detail in [CCl, §2.7]. In particular, it is shown there they satisfy the assumptions of Theorem A, hence its conclusion (H).

We have to mention now that Theorem C is actually as special case of a more general recent result of Joël Bellaïche and the first author [BeC]. Their proof is however more demanding than the one obtained here, and we think that the method of [CCl], completed here, is still of interest. In any case, it is natural to expect that Theorem A may have other applications in the future.

The first author would like to thank Laurent Clozel for sharing his knowledge of the trace formula. We are grateful to J.-L. Waldspurger for drawing our attention on Arthur's paper [A3].

Contents

[2]More precisely, this was used in the proof of (i) $\Rightarrow$ (iii), possibly after some well-chosen real quadratic base change of π if $F = \mathbb{Q}$.

1. Notation and preliminaries

1.1. **Reductive groups and their Lie algebras.** Let G be a group, X a set on which G acts, and Y a subset of X. Set:

$$Z_G(Y) = \{g \in G \,|\, \forall x \in Y,\, g \cdot x = x\},$$
$$N_G(Y) = \{g \in G \,|\, \forall x \in Y,\, g \cdot x \in Y\}.$$

When $N_G(Y)$ is a group, we denote by $W(G,Y)$ the quotient group $N_G(Y)/Z_G(Y)$. We will also use the notation G^Y for $Z_G(Y)$.

If G is an algebraic group defined over $\mathbb{R}$, we denote by $G(\mathbb{R})$ and $G(\mathbb{C})$ respectively the groups of real and complex points of G and by $\mathfrak{g}$ the Lie algebra of $G(\mathbb{R})$. We denote by G^0 the identity component of G and by $Z(G)$ the center of G.

For any real Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{g}_{\mathbb{C}}$ its complexification.

1.2. **Regular elements.** Let G be a connected algebraic reductive group defined over $\mathbb{R}$, let θ be an automorphism of finite order of G defined over $\mathbb{R}$, and consider the (non necessarily connected) algebraic group $G^+ = G \rtimes \langle\theta\rangle$. We will denote by $G\theta$ the connected component of G^+ containing θ and $G(\mathbb{R})\theta$ its real points. The group G acts by conjugacy on $G\theta$. In particular, if $\gamma \in G\theta$, G^γ denotes the centralizer in G of γ.

Since G^+ is linear, there is a well-defined notion of semisimple and unipotent element in G^+, and any element $\gamma \in G^+(\mathbb{R})$ can be written uniquely as $\gamma = su = us$ with $s \in G^+(\mathbb{R})$ semisimple, and $u \in G(\mathbb{R})$ unipotent.

Definition 1.1. An element $X \in \mathfrak{g}$ is regular if $\mathfrak{g}^X$ is a Cartan subalgebra of $\mathfrak{g}$. An element $\gamma \in G(\mathbb{R})\theta$ is *regular* (resp. *strongly regular*) if $(G^\gamma)^0$ is a torus (resp. if it is semisimple and if G^γ is abelian).

Proposition 1.2. *(i) If an element $X \in \mathfrak{g}$ is regular, then it belongs to a unique Cartan subalgebra of $\mathfrak{g}$.*

(ii) If $\gamma \in G(\mathbb{R})\theta$ is regular, then γ is semisimple and $\mathfrak{a} := \mathfrak{g}^\gamma$ is an abelian subalgebra of $\mathfrak{g}$ whose elements are semisimple. Moreover, $\mathfrak{g}^\gamma$ contains regular elements of $\mathfrak{g}$, thus $\mathfrak{h} := \mathfrak{g}^{\mathfrak{a}}$ is a Cartan subalgebra of $\mathfrak{g}$. In this setting, we denote by T_γ the centralizer in G of $\mathfrak{h}$. This is a maximal torus in G.

(iii) A strongly regular element is regular and an element $\gamma \in G(\mathbb{R})\theta$ is strongly regular if and only if $G^\gamma \subset T_\gamma$.

(iv) Let $\gamma \in G(\mathbb{R})\theta$ be semisimple. There exists a $G(\mathbb{R})^\gamma$-invariant neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}^\gamma$ such that if $X \in \mathcal{V}$ is regular in $\mathfrak{g}^\gamma$, then it is regular in $\mathfrak{g}$ and $\gamma \exp X$ is strongly regular. Furthermore $\mathfrak{a} = \mathfrak{g}^{\gamma \exp X}$ is a Cartan subalgebra of $\mathfrak{g}^\gamma$.

Proof. The assertion (i) is well-known. For (ii), see for instance [R], Propositions 2.1 and 2.4. For (iii), it is obvious that if $G^\gamma \subset T_\gamma$, then γ is semisimple, hence strongly regular since T_γ is abelian. Assume now that γ is strongly regular. In particular, γ is semisimple, and $(G^\gamma)^0$ is both reductive and abelian, hence is a torus in G. This shows that γ is regular. We use the notation of (ii). If $g \in G^\gamma$, then $g \in Z_G(\mathfrak{a})$, and since $\mathfrak{a}$ contains regular elements in $\mathfrak{g}$, we get $g \in Z_G(\mathfrak{h})$. This shows that $G^\gamma \subset T_\gamma$.

For (iv), choose a $G(\mathbb{R})^\gamma$-invariant neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}^\gamma$ such that

— for all $X \in \mathcal{V}$, $G(\mathbb{R})^{\gamma \exp X} \subset G(\mathbb{R})^\gamma$.

— exp is injective on $\mathcal{V}$.

The existence of such a neighborhood is proved in [R], Section 6. Let $g \in G(\mathbb{R})^{\gamma \exp X}$. Then $g \in G(\mathbb{R})^\gamma$, so $\exp X = g(\exp X)g^{-1} = \exp g \cdot X$, and since exp is injective on $\mathcal{V}$, $g \cdot X = X$. Now, as X is regular in $\mathfrak{g}$, we have $g \in T_{\gamma \exp X}$, so by (iii) $\gamma \exp X$ is strongly regular. The remaining statements follow immediately. □

1.3. **Normalization of measures.** We recall Duflo-Vergne normalization of Haar measures on reductive Lie groups, defined as follows: let A be a reductive group (complex or real), and pick an A-invariant symmetric, non-degenerate bilinear form κ on $\mathfrak{a}$. Then $\mathfrak{a}$ will be endowed with the Lebesgue measure dX such that the volume of a parallelotope supported by a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{a}$ is equal to $|\det(\kappa(X_i, X_j))|^{\frac{1}{2}}$ and A will be endowed with the Haar measure tangent to dX. If M is a closed subgroup of A such that κ is non-degenerate on $\mathfrak{m}$, such as centralizers of semisimple elements, we endow M with the Haar measure determined by κ as above. If $M' \subset M$ are two closed subgroups of A such that κ is non-degenerate on their respective Lie algebras, we endow M/M' with the M-invariant measure, which is the quotient of the Haar measures on M and M' defined as above. We will denote it by $d\dot{m}$.

2. Results on the Lie algebra

2.1. **Invariant distributions and orbital integrals.** Let G be an algebraic connected reductive group defined over $\mathbb{R}$. Let us denote by

— $\mathcal{C}_c^\infty(\mathfrak{g})$ the space of smooth compactly supported functions on $\mathfrak{g}$.

— $\mathcal{D}(\mathfrak{g})$ the space of distributions on $\mathfrak{g}$.

— $\mathcal{D}(\mathfrak{g})^{G(\mathbb{R})}$ the space of invariant distributions on $\mathfrak{g}$, with respect to the adjoint action of $G(\mathbb{R})$ on $\mathfrak{g}$.

We now recall the definition of orbital integrals on $\mathfrak{g}$, and their characterization ([Bou]). For any subset Ω of $\mathfrak{g}$, we denote by Ω_{reg} the set of semisimple regular elements in Ω.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $H = Z_G(\mathfrak{h})$. We fix a non degenerate $G(\mathbb{R})$-invariant symmetric bilinear form κ on $\mathfrak{g}$, which determines, according to §1.3, Haar measures dg, dh and $d\dot{g}$ respectively on $G(\mathbb{R})$, $H(\mathbb{R})$ and $G(\mathbb{R})/H(\mathbb{R})$.

If X is regular in $\mathfrak{h}$, then $G(\mathbb{R})^X = H(\mathbb{R})$, the measure μ_X on $\mathfrak{g}$ is defined by

$$O_X(f) = \int_{G(\mathbb{R})/H(\mathbb{R})} f(g \cdot X)\, d\dot{g}.$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{g})$.

For all $f \in \mathcal{C}_c^\infty(\mathcal{V})$, one defines a function on $\mathfrak{g}_{reg}$, called the orbital integral of f, by

$$X \mapsto J_{\mathfrak{g}}(f)(X) = |\det(\mathrm{ad}(X)_{\mathfrak{g}/\mathfrak{h}})|^{\frac{1}{2}} O_X(f).$$

It is easy to see that $J_{\mathfrak{g}}(f) \in \mathcal{C}^\infty(\mathfrak{g}_{reg})^{G(\mathbb{R})}$. Furthermore, Harish-Chandra has given some properties of orbital integrals. These properties are listed in [Bou] as I_1, I_2, I_3, I_4. Bouaziz has considered the subspace $\mathcal{I}(\mathfrak{g})$ of $\mathcal{C}^\infty(\mathfrak{g}_{reg})^{G(\mathbb{R})}$ consisting of functions satisfying I_1, I_2, I_3, I_4, endowed with the natural induced topology (an inductive limit of Fréchet spaces) and showed the following

Theorem 2.1. *The map $J_{\mathfrak{g}} : \mathcal{C}_c^\infty(\mathfrak{g}) \to \mathcal{I}(\mathfrak{g})$ is a continuous surjective linear map. Its transpose*

$${}^t J_{\mathfrak{g}} : \mathcal{I}(\mathfrak{g})' \to \mathcal{D}(\mathfrak{g})$$

realizes a topological isomorphism between the dual $\mathcal{I}(\mathfrak{g})'$ of $\mathcal{I}(\mathfrak{g})$ and the space of invariant distributions $\mathcal{D}(\mathfrak{g})^{G(\mathbb{R})}$.

We will need to describe the most subtle of the properties of orbital integrals, namely the jump relations I_3, but also I_2. For this, we need to introduce more material from [Bou].

2.2. **Cayley transforms.** Let $\mathfrak{g}$ be a real reductive Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. We denote by σ the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Let $\mathfrak{b} \subset \mathfrak{g}$ be a Cartan subalgebra. We denote by $R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ the root system of $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Let $\alpha \in R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ be an imaginary root, that is a root such that

$$\sigma(\alpha) = -\alpha.$$

Choose a root vector X_α for α and fix a root vector $X_{-\alpha}$ of $-\alpha$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, where H_α is another notation for the coroot $\check{\alpha} \in \mathfrak{b}_{\mathbb{C}}$. Then

$$\mathfrak{s}_{\mathbb{C}} = \mathbb{C} \cdot X_\alpha + \mathbb{C} \cdot X_{-\alpha} + \mathbb{C} \cdot H_\alpha$$

is a simple complex Lie algebra invariant under σ, $\sigma(H_\alpha) = -H_\alpha = H_{-\alpha}$ and $\sigma(X_\alpha) = cX_{-\alpha}$ for some $c \in \mathbb{R}^*$. If $c < 0$, we can renormalize to get $\sigma(X_\alpha) = -X_{-\alpha}$ or if $c > 0$, to get $\sigma(X_\alpha) = X_{-\alpha}$. In the former case, $\mathfrak{s} = \mathfrak{s}_{\mathbb{C}}^{\sigma} \simeq \mathfrak{su}(2)$ (and we say that α is a *compact root*). In the latter case, $\mathfrak{s} \simeq \mathfrak{sl}(2,\mathbb{R})$ and α is a *non-compact root.*

Suppose that α is non-compact. We define a standard Cayley transform with respect to α as an element of the adjoint group of $\mathfrak{s}_{\mathbb{C}}$ of the form $c_\alpha = \exp(-i\pi(X_\alpha + X_{-\alpha})/4)$, where X_α, $X_{-\alpha}$ are normalized as explained above. They are unique up to a scalar factor of absolute value 1, and all the standard Cayley transforms for α are conjugate in the adjoint group of $\mathfrak{s}_{\mathbb{C}}$. We have :

$$\begin{aligned} \mathfrak{b}_{\mathbb{C}} &= \ker \alpha \oplus \mathbb{C} \cdot H_\alpha \\ \mathfrak{b} &= \ker \alpha_{|\mathfrak{b}} \oplus i\mathbb{R} \cdot H_\alpha. \end{aligned}$$

Let $\mathfrak{a}_{\mathbb{C}} := c_\alpha \cdot \mathfrak{b}_{\mathbb{C}} = \ker \alpha \oplus \mathbb{C} \cdot (X_\alpha - X_{-\alpha})$. This is a Cartan subalgebra defined over $\mathbb{R}$ and its σ-invariant subspace is

$$\mathfrak{a} = \ker \alpha_{|\mathfrak{b}} \oplus i\mathbb{R} \cdot (X_\alpha - X_{-\alpha}).$$

The root $\beta := c_\alpha \cdot \alpha$ of $R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ is real and $c_\alpha \cdot H_\alpha = H_\beta = i(X_\alpha - X_{-\alpha})$. Furthermore :

$$\sigma(c_\alpha) = \exp(i\pi(X_\alpha + X_{-\alpha})/4) = c_\alpha^{-1}.$$

It is easy to check that $\sigma(c_\alpha)^{-1} c_\alpha = c_\alpha^2$ realizes the Weyl reflection s_α with respect to the root α.

2.3. **Jump data.** We say that $X \in \mathfrak{g}$ is *semi-regular* when the derived algebra of $\mathfrak{g}^X$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$. Suppose it is $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathfrak{b}$ be a fundamental Cartan subalgebra of $\mathfrak{g}^X$, and $\pm\alpha$ the roots of $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^X$. They are non-compact imaginary. Let c_α be a Cayley transform with respect to α as in the previous paragraph and let us also denote by $\mathfrak{a}$ the maximally split Cartan subalgebra of $\mathfrak{g}^X$ obtained from the Cayley transform (ie. $\mathfrak{a}_{\mathbb{C}} = c_\alpha \cdot \mathfrak{b}_{\mathbb{C}}$). We refer to these notations by saying that $(X, \mathfrak{b}, \mathfrak{a}, c_\alpha)$ is a jump datum for $\mathfrak{g}$.

2.4. **Jump relations.** Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We will denote by $\mathfrak{h}_{I-reg}$ the set of $X \in \mathfrak{h}$ such that the root system of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^X$ has no imaginary root. Equivalently, $\mathfrak{h}_{I-reg} \subset \mathfrak{h}$ is the complement of the union of the kernels of imaginary roots.

We denote by $S(\mathfrak{h}_{\mathbb{C}})$ the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$, and we identify it with the algebra of differential operators with constant coefficients on $\mathfrak{h}$. We denote by $\partial(u)$ the differential operator corresponding to $u \in S(\mathfrak{h}_{\mathbb{C}})$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $Y \in \mathfrak{h}$ and ϕ a function on $\mathfrak{h}_{reg}$. Let β be an imaginary root of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, and $H_\beta \in i\mathfrak{h}$ its coroot. Then, when the limits in the following formula exist we set:

$$[\phi]_\beta^+(Y) = \lim_{t\to 0^+} \phi(Y + tiH_\beta) + \lim_{t\to 0^-} \phi(Y + tiH_\beta).$$

Let ψ be a function in $\mathcal{C}^\infty(\mathcal{V}_{reg})^{G(\mathbb{R})}$ and denote by $\psi_{|\mathfrak{h}}$ its restriction to $\mathfrak{h} \cap \mathcal{U}_{reg}$ for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Property I_2 of orbital integrals is the following :

I_2 : *for any Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$, $\psi_{|\mathfrak{h}}$ *has a smooth extension to* $\mathfrak{h}_{I-reg}$, *and for all semi-regular elements* $X \in \mathfrak{h}$ *such that the roots* $\pm\alpha$ *of* $\mathfrak{h}_{\mathbb{C}}$ *in* $\mathfrak{g}_{\mathbb{C}}^X$ *are compact imaginary, for all* $u \in S(\mathfrak{h}_{\mathbb{C}})$,

$$[\partial(u) \cdot \psi_{|\mathfrak{h}}]_\alpha^+(X) = 0.$$

The jump relation property I_3 is :

I_3 : *for all jump data* $(X, \mathfrak{b}, \mathfrak{a}, c_\alpha)$ *and for all* $u \in S(\mathfrak{b}_{\mathbb{C}})$,

$$[\partial(u) \cdot \psi_{|\mathfrak{b}}]_\alpha^+(X) = d(X)\, \partial(c_\alpha \cdot u) \cdot \psi_{|\mathfrak{a}}(X),$$

where $d(X)$ *is equal to 2 if the reflection* s_α *is realized in* $G(\mathbb{R})$ *and 1 otherwise.*

2.5. **Invariant distributions in the nilpotent cone.** Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$ and let us denote by $\mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$ the space of invariant distributions on $\mathfrak{g}$ with support in $\mathcal{N}$. This space can be explicitly described in terms of limits of orbital integrals as follows ([Bou], Section 6).

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a connected component Γ of $\mathfrak{h}_{I-reg}$. For all $f \in \mathcal{C}_c^\infty(\mathfrak{g})$ and all $u \in S(\mathfrak{h}_{\mathbb{C}})$, set

$$\Theta_{u,\Gamma}(f) = \lim_{X \to 0, X \in \Gamma} \partial(u) \cdot J_{\mathfrak{g}}(f)(X).$$

Then $\Theta_{u,\Gamma}$ is an invariant (tempered) distribution on $\mathfrak{g}$ with support in $\mathcal{N}$.

Now, we fix a system of representatives $\mathfrak{h}_i$ of conjugacy classes (under $G(\mathbb{R})$) of Cartan subalgebras, and for each of them, we fix a connected component Γ of $\mathfrak{h}_{i,I-reg}$. Let us denote by

$$\varepsilon_i^I : W(G(\mathbb{R}), \mathfrak{h}_i) \longrightarrow \{\pm 1\}$$

the imaginary signature of the real Weyl group of $\mathfrak{h}_i$, defined as follows. Consider the root system $R_I(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_i)_{\mathbb{C}})$ of imaginary roots (a subsystem of $R(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_i)_{\mathbb{C}})$, and fix a choice of positive roots $R_I^+(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_i)_{\mathbb{C}})$. The action of $W(G(\mathbb{R}), \mathfrak{h}_i)$ on $R(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_i)_{\mathbb{C}})$ preserves $R_I(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_i)_{\mathbb{C}})$, and if $w \in W(G(\mathbb{R}), \mathfrak{h}_i)$, $\varepsilon_i^I(w)$ is $(-1)^{l_I(w)}$ where $l_I(w)$ is the number of positive imaginary roots α such that $w \cdot \alpha$ is negative.

Let $S((\mathfrak{h}_i)_{\mathbb{C}})^{\varepsilon_i^I}$ be the subspace of elements $u \in S((\mathfrak{h}_i)_{\mathbb{C}})$ satisfying $w \cdot u = \varepsilon_i^I(w) u$ for all $w \in W(G(\mathbb{R}), \mathfrak{h}_i)$. Then

Theorem 2.2. *The map*

$$\Phi : \bigoplus_i S((\mathfrak{h}_i)_{\mathbb{C}})^{\varepsilon_i^I} \to \mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$$
$$(u_i)_i \mapsto \sum_i \Theta_{u_i,\Gamma_i}$$

is an isomorphism of vector spaces.

The fact unipotent orbital integrals are in the image of Φ is due to Harish-Chandra (see also [A3]), appendix). Let us also recall Harish-Chandra's limit formula for the Dirac distribution at 0: suppose $\mathfrak{b}$ is fundamental Cartan subalgebra of $\mathfrak{g}$, so that all its roots are imaginary or complex, and let $\varpi = \prod_{\alpha\in R^+} H_\alpha$, where R^+ is any positive root system in $R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$. Then

$$\lim_{X\to 0, X\in\mathfrak{b}_{reg}} (\partial(\varpi)\cdot J_{\mathfrak{g}}(f))(X) = c\,\delta_0(f)$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{g})$, c being a non zero constant and δ_0 the Dirac distribution at 0. Denote by ε the signature character of the complex Weyl group $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ and by $S(\mathfrak{b}_{\mathbb{C}})_\varepsilon$ the subspace of $u \in S(\mathfrak{b}_{\mathbb{C}})$ be such that $w(u) = \varepsilon(w)u$ for all $w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$. Since $\mathfrak{b}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$, it is well-known that the restriction of the character ε to $W(G(\mathbb{R}), \mathfrak{b})$ coincides with ε^I. Thus,

$$S(\mathfrak{b}_{\mathbb{C}})_\varepsilon \subset S(\mathfrak{b}_{\mathbb{C}})^{\varepsilon^I}.$$

Lemma 2.3. *Let $u \in S(\mathfrak{b}_{\mathbb{C}})_\varepsilon$. Then $\Theta_{u,\Gamma}$ is a distribution with support in $\{0\}$ and any invariant distribution with support in $\{0\}$ is obtained in this way.*

Indeed, $w \cdot \varpi = \varepsilon(w)\varpi$ and the ε-isotypic component $S(\mathfrak{b}_{\mathbb{C}})_\varepsilon$ for the representation of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ in $S(\mathfrak{b}_{\mathbb{C}})$ is $\varpi\, S(\mathfrak{b}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})}$. Let $u \in S(\mathfrak{b}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})}$, and let D_u be the element of $S(\mathfrak{g}_{\mathbb{C}})$ corresponding to u by Harish-Chandra's natural isomorphism $S(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{g}_{\mathbb{C}}} \xrightarrow{\sim} S(\mathfrak{b}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})}$. We consider D_u as a differential operator on $\mathfrak{g}$. Then, from Harish-Chandra's limit formula and the property

$$(\partial(u)\cdot J_{\mathfrak{g}})(f) = J_{\mathfrak{g}}(D_u \cdot f), \qquad (f \in \mathcal{C}_c^\infty(\mathfrak{g})),$$

(see [Var], Prop. II.10.4) we get

$$\lim_{X\to 0, X\in\mathfrak{b}_{reg}} (\partial(\varpi u)\cdot J_{\mathfrak{g}}(f))(X) = c\,\delta_0(D_u\cdot f).$$

For the last assertion, it is well-known that any distribution on $\mathfrak{g}$ with support in $\{0\}$ is a derivative of the Dirac distribution. For such a distribution to be invariant, the constant coefficient differential operator has to be invariant under the action of $G(\mathbb{R})$, *i.e.* given by an element in $S(\mathfrak{g}_{\mathbb{C}})^{G(\mathbb{R})} = S(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{g}_{\mathbb{C}}}$.

If $\mathcal{D}(\mathfrak{g})_{\{0\}}^{G(\mathbb{R})}$ denotes the space of invariant distributions on $\mathfrak{g}$ with support in $\{0\}$, the lemma says that Φ realizes an isomorphism

$$S(\mathfrak{b}_{\mathbb{C}})_{\varepsilon} \xrightarrow{\sim} \mathcal{D}(\mathfrak{g})_{\{0\}}^{G(\mathbb{R})}.$$

We shall need to define a class of distributions $\Theta \in \mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$ which are somehow "orthogonal" to $\mathcal{D}(\mathfrak{g})_{\{0\}}^{G(\mathbb{R})}$. Note that for any $u \in S(\mathfrak{b}_{\mathbb{C}})$, we may write $u = \tilde{u} + u - \tilde{u}$ where

$$\tilde{u} = \frac{1}{|W|} \sum_{w \in W} \varepsilon(w) w(u) \qquad (W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}))$$

is the W-equivariant projection of u to the ε-isotypic component of $S(\mathfrak{b}_{\mathbb{C}})$.

Let $\mathfrak{b} = \mathfrak{h}_{i_0}$ be the representant of the fundamental Cartan subalgebra of $\mathfrak{g}$ among the $\mathfrak{h}_i$, and set $\varepsilon^I = \varepsilon^I_{i_0}$. By Theorem 2.2, each $\Theta \in \mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$ may be uniquely written as

$$\Theta = \Theta_{b,\Gamma} + \Theta'$$

where $b = b(\Theta) \in S(\mathfrak{b}_{\mathbb{C}})^{\varepsilon^I}$ and $\Theta' \in \Phi(\sum_{i \neq i_0} S(\mathfrak{h}_i)^{\varepsilon_i^I})$.

Definition 2.4. Define $\mathcal{D}^+(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})} \subset \mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$ as the subspace of distributions Θ whose component $b(\Theta)$ as above has a trivial ε-projection under $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$, *i.e.* such that $\widetilde{b(\Theta)} = 0$. We have a direct sum

$$\mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})} = \mathcal{D}(\mathfrak{g})_{\{0\}}^{G(\mathbb{R})} \oplus \mathcal{D}^+(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}.$$

Let $\mathcal{O} \subset \mathcal{N}$ be a $G(\mathbb{R})$-orbit. As is well-known, the centralizer in $G(\mathbb{R})$ of any $X \in \mathcal{O}$ is unimodular, so $\mathcal{O}$ admits a $G(\mathbb{R})$-invariant measure $\mu_{\mathcal{O}}$. By [Rao], this measure defines an invariant distribution, which obviously belongs to $\mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$.

Proposition 2.5. *If $\mathcal{O}$ is a nonzero $G(\mathbb{R})$-orbit in $\mathcal{N}$, then $\mu_{\mathcal{O}} \in \mathcal{D}^+(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$.*

Indeed, this is an immediate consequence of a (much more precise) description of the Θ_{u_i,Γ_i}-components of $\mu_{\mathcal{O}}$ that we shall now recall.

Fix a nilpotent $G(\mathbb{C})$-orbit $\mathcal{O}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, and denote by $\mathcal{O}_1, \ldots, \mathcal{O}_r$ its real forms, *i.e.* the nilpotent $G(\mathbb{R})$-orbits in $\mathfrak{g} \cap \mathcal{O}_{\mathbb{C}}$.

Using the Springer correspondence (Thm 3.6.9 in [ChG]), we get from the nilpotent orbit $\mathcal{O}_{\mathbb{C}}$ an irreducible representation $\chi_{\mathcal{O}}$ of the abstract Weyl group W_a of $\mathfrak{g}_{\mathbb{C}}$. The Springer correspondence is normalized so that the trivial orbit $\{0\}$ corresponds to the sign character ε of W_a. Of course, any choice of a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ identifies W_a with the Weyl group $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. To $\mathcal{O}_{\mathbb{C}}$ is also attached an integer $d_{\mathcal{O}}$ defined by

$$d_{\mathcal{O}} = \frac{1}{2}(\dim \mathfrak{g}_{\mathbb{C}} - \dim \mathfrak{h}_{\mathbb{C}} - \dim \mathcal{O}_{\mathbb{C}}).$$

The Weyl group $W(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ acts naturally on $S(\mathfrak{h}_\mathbb{C})$ and preserves the subspace $\mathcal{H}(\mathfrak{h}_\mathbb{C})$ of harmonic polynomials ([Ste]). The representation of $W(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ on $\mathcal{H}(\mathfrak{h}_\mathbb{C})$ is isomorphic to the regular representation. This space is graded by the degree of polynomials:

$$\mathcal{H}(\mathfrak{h}_\mathbb{C})=\bigoplus_{d\in\mathbb{N}}\mathcal{H}(\mathfrak{h}_\mathbb{C})_d$$

and this is a direct sum of representations of $W(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$. The representation $\chi_\mathcal{O}$ appears in $\mathcal{H}(\mathfrak{h}_\mathbb{C})_{d_\mathcal{O}}$ with multiplicity one. Let us denote by $\mathcal{H}(\mathfrak{h}_\mathbb{C})_{d_\mathcal{O},\chi_\mathcal{O}}$ the $\chi_\mathcal{O}$-isotypic component in $\mathcal{H}(\mathfrak{h}_\mathbb{C})_{d_\mathcal{O}}$. Let us remark also that the maximal degree appearing in the above decomposition of $\mathcal{H}(\mathfrak{h}_\mathbb{C})$ is $d_{\{0\}}$, and that

$$\mathcal{H}(\mathfrak{h}_\mathbb{C})_{d_{\{0\}}}$$

is the sign representation of $W(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$, generated by

$$\prod_{\alpha\in R^+}H_\alpha,$$

where R^+ is any positive root system in $R(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$.

Let us denote by $\mathcal{D}(\mathfrak{g})_{\mathcal{O}_\mathbb{C}}^{G(\mathbb{R})}$ the space of invariant distribution on $\mathfrak{g}$ with support in the nilpotent cone generated by the invariant measures on the nilpotent orbits $\mathcal{O}_1,\dots,\mathcal{O}_r$.

Theorem 2.6. *With the notations above, the map Φ restricts to an isomorphism*

$$\Phi:\ \bigoplus_i\mathcal{H}((\mathfrak{h}_i)_\mathbb{C})_{d_\mathcal{O},\chi_\mathcal{O}}^{\varepsilon_i^I}\xrightarrow{\sim}\mathcal{D}(\mathfrak{g})_{\mathcal{O}_\mathbb{C}}^{G(\mathbb{R})}$$
$$(u_i)_i\mapsto\sum_i\Theta_{u_i,\Gamma_i}$$

This result is due to Hotta-Kashiwara [HoKa] and Rossmann [Ross]. For a convenient reference, see [Ko3].

This gives in particular Rossmann's formula for the number of real forms of the complex orbit $\mathcal{O}_\mathbb{C}$

$$r=r_\mathcal{O}=\sum_i m(\varepsilon_i^I,\chi_\mathcal{O})$$

where $m(\varepsilon_i^I,\chi_\mathcal{O})$ is the multiplicity of the character ε_i^I of $W(G(\mathbb{R}),\mathfrak{h}_i)$ in $\mathcal{H}((\mathfrak{h}_i)_\mathbb{C})_{d_\mathcal{O},\chi_\mathcal{O}}$.

Let us now give a proof of Thm. 2.5. Write $\mu_\mathcal{O}$ as

$$\mu_\mathcal{O}=\sum_i\Theta_{u_i,\Gamma_i},\quad u_i\in\mathcal{H}((\mathfrak{h}_i)_\mathbb{C})_{d_\mathcal{O},\chi_\mathcal{O}}^{\varepsilon_i^I},$$

and set as above $\mathfrak{b}=\mathfrak{h}_{i_0}$ and $b=u_{i_0}$. Since $\mathcal{O}$ (hence $\mathcal{O}_\mathbb{C}$) is not the zero orbit, the character $\chi_\mathcal{O}$ is not ε, and thus, since b is in the $\chi_\mathcal{O}$-isotypic component of $S(\mathfrak{b}_\mathbb{C})$, its ε-projection is trivial. $\square$

2.6. **Main result.** To state the main result of this section, we need to recall some more definitions.

Definition 2.7. We say that a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *elliptic* if all the roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ are imaginary. Let $X \in \mathfrak{g}$ be semisimple and regular. We say that X is *elliptic* if the Cartan subalgebra $\mathfrak{g}^X$ is elliptic.

Definition 2.8. Let G be an algebraic connected reductive group defined over $\mathbb{R}$ and let X and Y be regular semisimple elements in $\mathfrak{g}$. Recall that X and Y are called *stably conjugate* if there exists $g \in G(\mathbb{C})$ such that $g \cdot X = Y$.

The goal of this section is to show the following

Theorem 2.9. *Let $f \in \mathcal{C}_c^\infty(\mathfrak{g})$ be a function such that for a neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$:*

1— $J_{\mathfrak{g}}(f)(X) = 0$ for all regular semisimple, non elliptic $X \in \mathcal{V}$.

2— $J_{\mathfrak{g}}(f)(X) = J_{\mathfrak{g}}(f)(Y)$ for all regular semisimple, elliptic and stably conjugate elements X, Y in a neighborhood of 0 in $\mathcal{V}$.

Let $\Theta \in \mathcal{D}(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$ and assume either that $\mathfrak{g}$ has no elliptic Cartan subalgebra or that $\Theta \in \mathcal{D}^+(\mathfrak{g})_{\mathcal{N}}^{G(\mathbb{R})}$. Then $\Theta(f) = 0$.

Proof. Let Θ be as in the statement. From the first property of f, $\Theta_{u_i,\Gamma_i}(f) = 0$ whenever $\mathfrak{h}_i$ is a non-elliptic Cartan subalgebra and $u_i \in S(\mathfrak{h}_i)_{\mathbb{C}}$. By Theorem 2.2, we obtain that $\Theta(f) = 0$ unless $\mathfrak{g}$ admits an elliptic Cartan subalgebra $\mathfrak{b} = \mathfrak{h}_{i_0}$, which we assume from now on. We set for short $\Gamma = \Gamma_{i_0}$ and $\varepsilon^I = \varepsilon^I_{i_0}$. In this case, Theorem 2.2 only gives

$$\Theta(f) = \Theta_{b,\Gamma}(f),$$

where $b = b(\Theta) \in S(\mathfrak{b}_{\mathbb{C}})^{\varepsilon^I}$. Notice that since $\mathfrak{b}$ is elliptic, all its roots in $\mathfrak{g}_{\mathbb{C}}$ are imaginary and therefore ε^I is the restriction to $W(G(\mathbb{R}), \mathfrak{b})$ of the signature ε of the complex Weyl group $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$.

By assumption on Θ, $\tilde{b} = 0$ so it only remains to show that $\Theta_{b,\Gamma}(f) = \Theta_{\tilde{b},\Gamma}(f)$. More generally, we claim that under the assumptions on f

$$\forall u \in S(\mathfrak{b}_{\mathbb{C}}),\ \forall w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}),\quad \Theta_{w(u),\Gamma}(f) = \varepsilon(w)\Theta_{u,\Gamma}(f). \tag{2.6.1}$$

Indeed, it is enough to check (2.6.1) for any $u \in S(\mathfrak{b}_{\mathbb{C}})$ and any w of the form s_α where α is a root of $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\ker\alpha$ is a wall of Γ (those s_α generate $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ as $\mathfrak{b}_{\mathbb{C}}$ is elliptic).

Let α be such a root and fix $X \in \mathfrak{b}$ a semi-regular element with respect to α. In particular, $X \in \ker\alpha$ and $\ker\alpha$ is a wall of Γ. Let $H_\alpha \in \mathfrak{g}_{\mathbb{C}}^X$ be the coroot $\check{\alpha}$, so $iH_\alpha \in \mathfrak{b}$. Up to replacing α by $-\alpha$ if necessary, we may assume that

$$X + itH_\alpha \in \Gamma$$

for any small real $t > 0$. Note that the elements $X + itH_\alpha$ and $X - itH_\alpha$ of $\mathfrak{b}$ are regular for small $t \neq 0$ and stably conjugate under $s_\alpha = c_\alpha^2 \in G(\mathbb{C})$.

For $u \in S(\mathfrak{b}_\mathbb{C})$ and $\mu, t > 0$, consider

$$(\partial(s_\alpha(u))J_\mathfrak{g}(f)_{|\mathfrak{b}})(\mu X + tiH_\alpha) := (\partial(u)J_\mathfrak{g}(f)_{|\mathfrak{b}}^{s_\alpha})(\mu X - tiH_\alpha).$$

For $Y \in \mathfrak{b}$, $J_\mathfrak{g}(f)^{s_\alpha}(Y) := J_\mathfrak{g}(f)(s_\alpha(Y))$. By property (2) of f,

$$J_\mathfrak{g}(f)_{|\mathfrak{b}}^{s_\alpha} = J_\mathfrak{g}(f)_{|\mathfrak{b}}$$

in $\mathcal{V} \cap \mathfrak{b}$. Moreover, note that

$$\lim_{\mu,t\to 0^+} (\partial(u)J_\mathfrak{g}(f)_{|\mathfrak{b}})(\mu X + tiH_\alpha) = - \lim_{\mu,t\to 0^+} (\partial(u)J_\mathfrak{g}(f)_{|\mathfrak{b}})(\mu X - tiH_\alpha).$$

Indeed, this identity is exactly property I_2 of orbital integrals if α is a compact root. If α is non compact, we may consider a root datum $(\mu X, \mathfrak{b}, \mathfrak{a}, c_\alpha)$ and apply to it property I_3 of orbital integrals. We obtain that the same identity holds as well, as the Cartan subalgebra $\mathfrak{a}$ is not elliptic and by property (1) of f. As a consequence,

$$\forall u \in S(\mathfrak{b}_\mathbb{C}), \;\; \Theta_{s_\alpha(u),\Gamma}(f) = -\Theta_{u,\Gamma}(f),$$

and we are done. □

As an immediate consequence of the theorem and of Proposition 2.5, we obtain the

Corollary 2.10. *If f is as in the statement of the theorem, then $\mu_\mathcal{O}(f) = 0$ for any nonzero $G(\mathbb{R})$-orbit $\mathcal{O} \subset \mathcal{N}$. Moreover, $f(0) = 0$ if $\mathfrak{g}$ has no elliptic Cartan subalgebra.*

3. Proof of Theorem A

3.1. **The setting.** Let G be a linear algebraic connected reductive group over $\mathbb{R}$ and θ an automorphism of finite order of G defined over $\mathbb{R}$ (see §1.2).

Let $\gamma \in G(\mathbb{R})\theta$ and denote by I_γ the identity component of its centralizer in G. Choose any $G(\mathbb{R})$-invariant measure μ on the $G(\mathbb{R})$-conjugacy class of γ, or equivalently on the quotient $G(\mathbb{R})/I_\gamma(\mathbb{R})$ (it is well-known that $I_\gamma(\mathbb{R})$ is unimodular). For $f \in \mathcal{C}_c^\infty(G(\mathbb{R})\theta)$, define the orbital integral of f at γ as:

$$O_\gamma(f) = \int_{G(\mathbb{R})/I_\gamma(\mathbb{R})} f(g\gamma g^{-1})\, d\mu.$$

When γ is semisimple, we assume that μ is normalized as in our convention 1.3. Recall that a semisimple element $\gamma \in G(\mathbb{R})\theta$ is elliptic if the split component of $Z(I_\gamma)$ coincides with the split component of $Z(G)^\theta$ (the θ-invariants in $Z(G)$).

Definition 3.1. Let γ, γ' be strongly regular elements of $G(\mathbb{R})\theta$. We say that γ and γ' are stably conjugate if there exists $g \in G(\mathbb{C})$ such that $g\gamma g^{-1} = \gamma'$ and $\sigma(g)^{-1}g \in T_\gamma(\mathbb{C})$, where T_γ is the maximal torus of G defined in Prop. 1.2 (ii).

Notice that when $\theta = 1$ the condition on $\sigma(g)^{-1}g$ is superfluous.

Theorem 3.2. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R})\theta)$. Assume that:*

1 – $O_\gamma(f) = 0$ if $\gamma \in G(\mathbb{R})\theta$ is semisimple, strongly regular, and non-elliptic,

2 – $O_\gamma(f) = O_{\gamma'}(f)$ if $\gamma, \gamma' \in G(\mathbb{R})\theta$ are semisimple, strongly regular, and stably conjugate.

Then $O_\gamma(f) = 0$ for all $\gamma \in G(\mathbb{R})\theta$ which is not semisimple elliptic.

3.2. **A special case.** We shall first prove a special instance of this theorem, namely the case where θ is trivial and γ is unipotent in $G(\mathbb{R})$, which is the group theoretic analogue of Thm. 2.9. Assume $\theta = 1$.

Theorem 3.3. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$. Assume that in a neighborhood $\mathcal{U}$ of 1 in $G(\mathbb{R})$:*

1 – $O_\gamma(f) = 0$ if $\gamma \in \mathcal{U}$ is semisimple, strongly regular, and non-elliptic,

2 – $O_\gamma(f) = O_{\gamma'}(f)$ if $\gamma, \gamma' \in \mathcal{U}$ are semisimple, strongly regular, and stably conjugate.

Then $O_u(f) = 0$ for all unipotent elements $u \neq 1$ in $G(\mathbb{R})$. Moreover, $f(1) = 0$ if $\mathfrak{g}$ has no elliptic Cartan subalgebra.

We fix once and for all an f as above. The following lemma is well known:

Lemma 3.4. *There exists:*

— a $G(\mathbb{R})$-invariant neighborhood $\mathcal{V}_0$ of 0 in $\mathfrak{g}$,

— a $G(\mathbb{R})$-invariant neighborhood $\mathcal{U}_0$ of 1 in $G(\mathbb{R})$,

such that $\exp : \mathcal{V}_0 \longrightarrow \mathcal{U}_0$ is a diffeomorphism.

Note that for $X \in \mathcal{V}_0$, $G^X = G^{\exp(X)}$. Thus, if X is regular semisimple, then $\exp(X)$ is strongly regular. Furthermore, for such an X, $\exp(X)$ is elliptic if and only if X is elliptic.

Lemma 3.5. *There exists $h \in \mathcal{C}_c^\infty(\mathfrak{g})$ and a neighborhood $\mathcal{V} \subset \mathcal{V}_0$ of 0 in $\mathfrak{g}$ such that for any $X \in \mathcal{V}$,*

$$O_{\exp(X)}(f) = O_X(h).$$

Proof. By [Bou], Cor. 2.3.2, there exists a $G(\mathbb{R})$-invariant $\chi \in \mathcal{C}_c^\infty(\mathcal{V}_0)$ such that χ equals 1 in a $G(\mathbb{R})$-invariant neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$. As already noticed, for any $X \in \mathcal{V} \subset \mathcal{V}_0$, G^X coincides with the centralizer in G of $\exp(X)$, it is now obvious that $h(X) := f(\exp(X))\chi(X)$ satisfies the statement. $\square$

Let us now prove the theorem. By the corollary of Theorem 2.9 and the fact that any nilpotent orbit in $\mathfrak{g}$ meets $\mathcal{V}$, it is enough to show that h satisfies conditions 1 and 2 of that theorem. We have already remarked that if $X \in \mathcal{V}_0$ is semisimple regular and non-elliptic, then $\exp(X)$ is strongly regular and non-elliptic, thus

$$O_X(h) = O_{\exp(X)}(f) = 0,$$

so condition 1 is satisfied. Let now $X, Y \in \mathcal{V}$ be semisimple regular and stably conjugate (so $\exp(X)$ and $\exp(Y)$ are strongly regular). Let $g \in G(\mathbb{C})$ be such that $g.X = Y$. We obviously have $g\exp(X)g^{-1} = \exp(g.X) = \exp(Y)$, so $\exp(X)$ and $\exp(Y)$ are stably conjugate in $G(\mathbb{R})$, hence

$$O_X(h) = O_{\exp(X)}(f) = O_{\exp(Y)}(f) = O_Y(h),$$

and we are done. $\square$

3.3. **The general case.** We return to the proof of Theorem 3.2. Let $\gamma \in G(\mathbb{R})\theta$ and $\gamma = su = us$ its Jordan decomposition, with $s \in G(\mathbb{R})\theta$ semisimple and $u \in G(\mathbb{R})$ nilpotent. Let $M = I_s$ be the identity component of the centralizer of s in G. The following lemma is an application of Harish-Chandra's descent method.

Lemma 3.6. *There is a function $h \in \mathcal{C}_c^\infty(M(\mathbb{R}))$, and a $M(\mathbb{R})$-invariant neighborhood $\mathcal{U}$ of 1 in $M(\mathbb{R})$ such that $\forall m \in \mathcal{U}$, $(G^{sm})^0 = (M^m)^0$ and $O_{sm}(f) = O_m(h)$.*

Proof. The proof can essentially be found in [CCl, Prop. 3.11] which is a similar statement in the non archimedean case. There is a small number of minor changes that we now indicate. We first remark that the compactness lemma, which is an essential argument in the proof, is established in the twisted case by Arthur in [A2] Lemma 2.1, and that it is valid also for groups defined over $\mathbb{R}$. Then, we need the existence of an $M(\mathbb{R})$-invariant open neighborhood $\mathcal{U}$ of 1 in $M(\mathbb{R})$ such that if $g \in G(\mathbb{R})$ satisfies $gs\exp\mathcal{U}g^{-1} \cap \exp\mathcal{U} \neq \emptyset$, then $g \in G^s(\mathbb{R})$. This is proved in [R], Section 6. Finally, the function χ introduced in *loc.cit.* has to be replaced by any function in $\mathcal{C}_c^\infty(G(\mathbb{R})/M(\mathbb{R}))$ which equals 1 on the compact ω introduced in *loc.cit.* $\square$

We may also assume, up to replacing $\mathcal{U}$ by a smaller open neighbohrood of 1 in $M(\mathbb{R})$, that there exists a $M(\mathbb{R})$-invariant neighborhood $\mathcal{V}$ of 0 in $\mathfrak{m}$ such that exp is a diffeomorphism from $\mathcal{V}$ onto $\mathcal{U}$ and satisfying Proposition 1.2 (iv).

We shall now apply Theorem 3.3 to the connected group M and the function h. We have to check that h satisfies conditions 1 and 2 of that theorem. If $m \in \mathcal{U}$ is semisimple strongly regular in M, then

$m = \exp X$ with $X \in \mathcal{V}$ regular in $\mathfrak{m}$, so sm is strongly regular in $G(\mathbb{R})\theta$ by Prop. 1.2. Moreover, $(G^{sm})^0 = M^m$ and

$$Z(G)^\theta \subset Z(M) \subset M^m,$$

thus m non-elliptic in $M(\mathbb{R}) \Rightarrow sm$ non-elliptic in $G(\mathbb{R})\theta$, and for those m we have

$$O_m(h) = O_{sm}(f) = 0,$$

so condition 1 is satisfied. Let now $m_1, m_2 \in \mathcal{U}$ be strongly regular and stably conjugate (so sm_1 and sm_2 are strongly regular). Let $g \in M(\mathbb{C})$ be such that $gm_1g^{-1} = m_2$. We have

$$\sigma(g)^{-1}g \in M^{m_1}(\mathbb{C}) \subset T_{sm_1}(\mathbb{C})$$

by Prop. 1.2 (ii), and obviously $gsm_1g^{-1} = gsg^{-1}gm_1g^{-1} = sm_2$, so sm_1 and sm_2 are stably conjugate in $G(\mathbb{R})$ and

$$O_{m_1}(h) = O_{sm_1}(f) = O_{sm_2}(f) = O_{m_2}(h),$$

so h satisfies condition 2.

As the unipotent part u of γ lies in $\mathcal{U}$, we obtain that $O_u(h) = O_\gamma(f) = 0$ if $u \neq 1$, *i.e.* if γ is non semisimple. It only remains to show that $O_\gamma(f)$ vanishes for semisimple but non elliptic elements $\gamma \in G(\mathbb{R})\theta$.

Lemma 3.7. *Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R})\theta)$ be any function such that $O_\gamma(f) = 0$ for all semisimple $\gamma \in G(\mathbb{R})\theta$ which are strongly regular and non-elliptic. Then $O_\gamma(f) = 0$ for all non-elliptic semisimple $\gamma \in G(\mathbb{R})\theta$.*

Proof. Let $\gamma \in G(\mathbb{R})\theta$ be semisimple. Applying the descent argument above to the identity component M of the centralizer of γ, and then a descent to the Lie algebra as in §3.2, we may find an $M(\mathbb{R})$-invariant neighborghood $\mathcal{V}$ of 0 in $\mathfrak{m}$ (with $\exp : \mathcal{V} \to \exp(\mathcal{V})$ an onto diffeormorphism) and a function $h \in \mathcal{C}_c^\infty(\mathcal{V})$ such that

(3.3.1) $$\forall X \in \mathcal{V},\ M^X = M^{\exp X} \text{ and } (M^{\exp X})^0 = (G^{\gamma \exp X})^0,$$

(3.3.2) $$O_X(h) = O_{\gamma \exp X}(f).$$

Assume that γ is not elliptic in $G(\mathbb{R})\theta$, which means that $Z(M)$ contains a non trivial split torus T with $Z(G)^\theta \cap T = 1$. Then for any semisimple $X \in \mathcal{V}$, $\gamma \exp(X)$ is not elliptic either, since by (3.3.1)

$$T \subset Z(M) \subset Z((G^{\gamma \exp X})^0) = Z((M^{\exp X})^0).$$

If furthermore X is regular in $\mathfrak{m}$, then $\gamma \exp X$ is strongly regular in $G(\mathbb{R})\theta$ by (3.3.1) and Prop.1.2. Therefore the assumption on f and (3.3.2) show that $O_X(h) = 0$ for all regular $X \in \mathcal{V}$. In particular, going back to Harish-Chandra's normalization of integral orbital used in §2, $J_\mathfrak{m}(h)$ is identically zero on $\mathcal{V} \cap \mathfrak{m}_{reg}$. By Harish-Chandra's limit formula recalled in §2.5, we obtain that $h(0) = 0$, and we are done. □

References

[A1] J. Arthur, *The invariant trace formula II. Global theory*, J. Amer. Math. Soc. **1** (1988), 501–554.

[A2] J. Arthur, *The local behaviour of weighted orbital integrals*, Duke Math. J. **56** (1988), 223–293.

[A3] J. Arthur, *The L^2 Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), 257–290.

[BeC] J. Bellaïche & G. Chenevier, *The sign of Galois representations attached to automorphic forms for unitary groups*, preprint (2008).

[BLS] A. Borel, J.-P. Labesse & J. Schwermer, *On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields*, Compositio Math. **102** (1996), 1–40.

[Bou] A. Bouaziz, *Intégrales orbitales sur les algèbres de Lie réductives*, Inv. Math. **115** (1994), 163-207.

[CCl] G. Chenevier & L. Clozel, *Corps de nombres peu ramifiés et formes automorphes autoduales*, Journal of the A.M.S **22** vol 2, 467–519 (2009).

[ChG] N. Chriss & V. Ginzburg, Representation theory and complex geometry. Birkhuser Boston, Inc., Boston, MA, 1997.

[CL] L. Clozel & J.-P. Labesse , *Orbital integrals and distributions*, to appear in volume in the honor of F. Shahidi.

[DKV] P. Deligne, D. Kazhdan, M-F. Vignéras, *Représentations des groupes réductifs sur un corps local*, Hermann (1984).

[HT] M. Harris & R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Math. Studies **151** (2001).

[HoKa] R. Hotta & M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Inv. Math. **75** (1984).

[Ko1] R. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), 785–806.

[Ko2] R. Kottwitz, *Tamagawa numbers*, Ann. of Math. **127** (1988), 629–646.

[Ko3] R. Kottwitz, *Stable nilpotent orbital integrals on real reeductive Lie algebras*, Representation Theory. (2000), 16–31 .

[KoS] R. Kottwitz & D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255** (1999).

[L] J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque **257** (1999).

[Rao] R. Ranga Rao, *Orbital Integrals in reductive groups*, Ann. of Math. (2), **96** (1972), 505–510.

[R] D. Renard, *Intégrales orbitales tordues sur les groupes de Lie réductifs réels*, Journal of Fuctional Analysis **145** (1997), 374–454.

[Ross] W. Rossmann, *Nilpotent orbital integrals in a real semisimple Lie algebra and representations of Weyl groups*, Operator Algebras, Unitary Representations, Envelopping Algebras and Invariant Theory, Progress in Math. **309**, Birkhäuser (1990), 215–229.

[Ste] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. AMS. **112** (1964), 392–400.

[T] R. Taylor, *Galois representations*, Annales de la Faculté des Sciences de Toulouse **13** (2004), 73–119.

[TY] R. Taylor & T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), 467–493.

[Var] V.S. Varadarajan, *Harmonic Analysis on Real Reductive Groups*, LMN **576**, Springer (1977).

Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique