EULER-POINCARÉ PAIRING, DIRAC INDEX AND ELLIPTIC PAIRING FOR HARISH-CHANDRA MODULES

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Abstract. Let G be a connected real reductive group with maximal compact subgroup K of equal rank, and let $\mathscr{M}$ be the category of Harish-Chandra modules for G. We relate three differentely defined pairings between two finite length modules X and Y in $\mathscr{M}$: the Euler-Poincaré pairing, the natural pairing between the Dirac indices of X and Y, and the elliptic pairing of [2]. (The Dirac index $\mathsf{I}_{\mathrm{Dir}}(X)$ is a virtual finite dimensional representation of $\widetilde{K}$, the spin double cover of K.)

Analogy with the case of Hecke algebras studied in [7] and [6] and a formal (but not rigorous) computation lead us to conjecture that the first two pairings coincide. In the second part of the paper, we show that they are both computed as the indices of Fredholm pairs (defined here in an algebraic sense) of operators acting on the same spaces.

We construct Euler-Poincaré functions f_X for any finite length Harish-Chandra module X. These functions are very cuspidal in the sense of Labesse, and their orbital integrals on elliptic elements coincide with the character of X. From this we deduce that the Dirac index pairing coincide with the elliptic pairing.

These results are the archimedean analog of results of Schneider-Stuhler [21] for p-adic groups.

1. Introduction

Our goal is to establish the archimedean analog of results of Schneider-Stuhler [21] for p-adic groups. Let us first describe these results. Let G be the group of rational points of a connected algebraic reductive group defined over a local non archimedean field F of characteristic 0. Assume that G has compact center. Let $\mathscr{M} = \mathscr{M}(G)$ be the category of smooth representations of G. This is also the category of non-degenerate modules over the Hecke algebra $\mathscr{H} = \mathscr{H}(G)$ of G. It is known from the work of J. Bernstein that $\mathscr{M}$ has finite cohomological dimension. Furthermore, for any finitely generated module (π, V) in $\mathscr{M}$, Schneider and Stuhler have constructed an explicit resolution of (π, V) by finitely generated projective modules. They also establish a general theory of Euler-Poincaré functions for modules of finite length, generalizing results of Kottwitz ([17]). Namely, for any finite length modules (π, V), (π', V') in $\mathscr{M}$, one can define their Euler-Poincaré pairing :

$$\mathbf{EP}(\pi, \pi') = \sum_i (-1)^i \ \dim \mathrm{Ext}^i_{\mathscr{M}}(\pi, \pi'). \tag{1.1}$$

They construct functions f_π (Euler-Poincaré functions) in $\mathscr{H}$ such that

$$\mathbf{EP}(\pi, \pi') = \Theta_{\pi'}(f_\pi) \tag{1.2}$$

where $\Theta_{\pi'}$ is the distribution-character of π' (a linear form on $\mathscr{H}$). Following [8] and [23], let us now give another point of view on these functions.

Let $\mathscr{K}(G)$ be the Grothendieck group of finitely generated projective modules in $\mathscr{M}$. Since $\mathscr{M}$ has finite cohomological dimension, this is also the Grothendieck group of all

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finitely generated modules in $\mathscr{M}$. Let $\mathscr{R} = \mathscr{R}(G)$ be the Grothendieck group of finite length modules in $\mathscr{M}$. If (π, V) is a finitely generated (resp. finite length) module in $\mathscr{M}$, we denote by $[\pi]$ its image in Grothendieck group $\mathscr{K}$ (resp. $\mathscr{R}$). Set $\mathscr{K}_{\mathbb{C}} = \mathscr{K} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathscr{R}_{\mathbb{C}} = \mathscr{R} \otimes_{\mathbb{Z}} \mathbb{C}$. Let (π, V) be a finite length module in $\mathscr{M}$, and

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow V \longrightarrow 0$$

be a resolution of π by finitely generated projective modules. Then

$$\text{EP} : \mathscr{R} \longrightarrow \mathscr{K}, \quad [\pi] \mapsto \sum_i (-1)^i \, [P_i] \tag{1.3}$$

is a well-defined map.

Let $\overline{\mathscr{H}} = \overline{\mathscr{H}(G)} = \mathscr{H}/[\mathscr{H}, \mathscr{H}]$ be the abelianization of $\mathscr{H}$. The Hattori rank map

$$\text{Rk} : \mathscr{K} \longrightarrow, \quad \overline{\mathscr{H}} \tag{1.4}$$

is defined as follows. Let P be a finitely generated projective module in $\mathscr{M}$. Write P as a direct factor of some $\mathscr{H}^n$ and let $e \in \text{End}_{\mathscr{H}}(\mathscr{H}^n)$ be the projector onto P. Then the trace of e is an element of $\mathscr{H}$, and its image in $\overline{\mathscr{H}}$ is well-defined. This defines $\text{Rk}([P])$. An alternative description of $\overline{\mathscr{H}}$ as the "cocenter" of the category $\mathscr{M}$ gives a natural definition of Rk for finitely generated projective modules (see [8], §1.3).

Let $\mathcal{D}'(G)$ be the space of distributions on G, and let $\mathcal{D}'(G)^G$ be the subspace of invariant distributions. Fix a Haar measure on G, so that $\mathscr{H}$ is identified with the convolution algebra of compactly supported smooth functions on G. The orthogonal of $\mathcal{D}'(G)^G$ in $\mathscr{H}$ for the natural pairing between $\mathcal{D}'(G)$ and $\mathscr{H}$ is exactly $[\mathscr{H}, \mathscr{H}]$, so there is an induced non-degenerate pairing :

$$\mathcal{D}'(G)^G \times \overline{\mathscr{H}} \to \mathbb{C}, \quad (T, f) \mapsto \langle T, f \rangle = T(f).$$

Let (π', V') be a finite length module in $\mathscr{M}$. Its distribution character $\Theta_{\pi'}$ is an element of $\mathcal{D}'(G)^G$. This defines a pairing

$$\mathscr{R}_{\mathbb{C}} \times \overline{\mathscr{H}} \to \mathbb{C}, \quad (\pi', f) \mapsto \langle \Theta_{\pi'}, f \rangle = \Theta_{\pi'}(f) \tag{1.5}$$

With the notation above, we have the following identity : for all finite length modules $(\pi, V), (\pi', V')$ in $\mathscr{M}$,

$$\mathbf{EP}(\pi, \pi') = \langle \Theta_{\pi'}, \text{Rk} \circ \text{EP}(\pi) \rangle. \tag{1.6}$$

Thus, the image in $\overline{\mathscr{H}}$ of the Euler-Poincaré function f_π constructed by Schneider and Stuhler is $\text{Rk} \circ \text{EP}(\pi)$ ([8], lemma 3.7).

There is a third way of seeing the space $\overline{\mathscr{H}}$, namely, as the space of orbital integrals on G. More precisely, recall that for a regular semisimple element x in G, one can define the orbital integral of $f \in \mathscr{H}$ at x as

$$\Phi(f, x) = \int_{G/T} f(gxg^{-1}) \, d\dot{g},$$

where T is the unique maximal torus containing x, and $d\dot{g}$ is an invariant measure on G/T. When f is fixed, $x \mapsto \Phi(f, x)$ is a smooth invariant function on G_{reg}, and we denote by $\mathscr{I}(G)$ the image of $\Phi : f \in \mathscr{H} \mapsto \Phi(f, .)$ in the space of smooth invariant functions on G_{reg}. This space can be explicitely described (by properties of orbital integrals, see [22]). Furthermore the kernel of Φ is exactly $[\mathscr{H}, \mathscr{H}]$ (this is called the geometric density theorem, *i.e.* the density of the space generated by the distributions $f \mapsto \Phi(f, x)$, $x \in G_{\text{reg}}$, in $\mathcal{D}'(G)^G$). Thus, we have an exact sequence

$$0 \longrightarrow [\mathscr{H}, \mathscr{H}] \longrightarrow \mathscr{H} \longrightarrow \mathscr{I}(G) \longrightarrow 0$$

and $\overline{\mathscr{H}}$ is identified with the space $\mathscr{I}(G)$ of orbital integrals.

Let us denote by $G_{\rm ell}$ the space of regular semisimple elliptic elements in G (*i.e.* elements whose centralizer is a compact maximal torus in G), and by $\mathscr{H}_c$ the subspace of functions $f \in \mathscr{H}$ such that $\Phi(f,x) = 0$ when x is not elliptic. Acccordingly, we write $\overline{\mathscr{H}}_c$ for the image of this subspace in $\overline{\mathscr{H}}$. This latter space is isomorphic to the subspace $\mathscr{I}(G)_c$ of orbital integrals which vanish outside $G_{\rm ell}$. The Selberg principle for Euler-Poincaré functions ([21], Rmk II.4.11) asserts that the functions f_π are in $\mathscr{H}_c$. In fact, it is a theorem of Brylinski-Blanc [3], (see also [8] and [9]) that the image Hattori rank map Rk is exactly $\overline{\mathscr{H}}_c$.

Recall that by Harish-Chandra regularity theorem, the character $\Theta_{\pi'}$ of a finite length module $(\pi', V') \in \mathscr{M}$ is given by a locally integrable function, denoted by $\theta_{\pi'}$, on G, which is smooth on the open dense subset of regular semisimple elements $G_{\rm reg}$, *i.e* for all $f \in \mathscr{H}$,

$$\langle \Theta_{\pi'}, f\rangle = \int_{G_{\rm reg}} \theta_{\pi'}(x)\, f(x)\; dx.$$

(Once again, we have chosen a Haar measure on G).

Following Kazhdan [14], let us denote by $C^{\rm ell}$ the set of regular semisimple elliptic conjugacy classes in G. Then any orbital integral $\Phi(f,.)$ or any character function $\theta_{\pi'}$ can be viewed as functions on $C^{\rm ell}$. By [14], §3, Lemma 1, there is a unique measure dc on $C^{\rm ell}$ such that for all $f \in \mathscr{H}$ with support in $G_{\rm ell}$,

$$\int_G f(x)\; dx = \int_{C^{\rm ell}} \Phi(f,c)\; dc.$$

The elliptic pairing between two finite length modules (π, V) and (π', V') in $\mathscr{M}$ is then defined by

$$\langle \pi, \pi'\rangle_{\rm ell} = \int_{C^{\rm ell}} \theta_\pi(c)\, \theta_{\pi'}(c^{-1})\; dc. \tag{1.7}$$

Let us now relate this elliptic pairing to the Euler-Poincaré pairing (1.1). On a regular semisimple elliptic element x, the orbital integrals of the Euler-Poincaré functions f_π at x coincide with the character θ_π of π at x^{-1} ([21] Thm III.4.16) :

$$(\forall x \in G_{\rm ell}), \qquad \Phi(f_\pi, x) = \theta_\pi(x^{-1}). \tag{1.8}$$

Therefore, the following formula, which was conjectured by Kazhdan [14] is established ([21] Thm III.4.21):

$$\mathbf{EP}(\pi,\pi') = \langle \Theta_{\pi'}, f_\pi\rangle = \int_{G_{\rm ell}} \theta_{\pi'}(x)\, f_\pi(x)\; dx = \int_{C^{\rm ell}} \theta_{\pi'}(c)\, \theta_\pi(c^{-1})\; dc = \langle \pi, \pi'\rangle_{\rm ell}. \tag{1.9}$$

Let us mention that it is explained in [8] how these results can be formulated and proved when the center of G is not compact.

Many of the objects and results described above make sense over an Archimedean field as well. So assume now that G be the group of rational points of a connected algebraic reductive group defined over $\mathbb{R}$ and, for simplicity, that G is connected (as a Lie group), with compact center. A little bit more generally, we may assume without introducing technical complications that G is actually a finite covering of such a group. Fix a maximal compact subgroup K of G. In section 1.8, also for simplicity, we will assume that G and K have same rank, *i.e.* G admits discrete series representations (this implies the compact center condition)[1]. The category of representations we are now considering is

[1]In a future version of this paper, we hope to get rid of some of these non necessary assumptions

the category of Harish-Chandra modules $\mathscr{M} = \mathscr{M}(\mathfrak{g}, K)$ ([15]). It is also known that $\mathscr{M}$ has finite cohomological dimension, and furthermore resolutions of finite length modules by explicit finitely generated projective modules can be constructed ([15], §II.7). Thus, the Euler-Poincaré pairing of two modules of finite length is well-defined by (1.1). The elliptic pairing (1.9) is also defined, and the purpose of this paper is to relate these two pairings by constructing Euler-Poincaré functions in that case, *i.e.* compactly supported smooth functions on G satisfying (1.2), but also (1.8), so that the formula (1.9) is valid. An interesting aspect is that this relation involves a third pairing defined through Dirac cohomology.

Dirac cohomology of Harish-Chandra modules was introduced by D. Vogan (see [13] for an exposition of this theory). Dirac cohomology of a Harish-Chandra module X of finite length consists in a finite dimensional representation $\mathsf{H}_{\mathrm{Dir}}(X)$ of the compact group $\widetilde{K}$, the spin cover of K. In fact, $\mathsf{H}_{\mathrm{Dir}}(X)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, and a slightly more refined invariant, the Dirac index $\mathsf{I}_{\mathrm{Dir}}(X)$ can be defined as the formal difference $\mathsf{I}_{\mathrm{Dir}}(X) = \mathsf{H}_{\mathrm{Dir}}(X)^{\bar{0}} - \mathsf{H}_{\mathrm{Dir}}(X)^{\bar{1}}$ between the even and odd part of $\mathsf{H}_{\mathrm{Dir}}(X)$, a virtual finite dimensional representation of $\widetilde{K}$. If $[\sigma, \sigma']_{\widetilde{K}}$ is the usual pairing between two virtual finite dimensional representations σ, σ' of $\widetilde{K}$, then the third pairing between two Harish-Chandra modules X and Y of finite length that we introduce is

$$\langle X, Y\rangle_{\mathrm{Dir}} = [\mathsf{I}_{\mathrm{Dir}}(X), \mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}}\,. \tag{1.10}$$

A formal (but not rigorous) computation given in Section 3 of the paper lead us to conjecture that

$$\mathbf{EP}(X, Y) = \langle X, Y\rangle_{\mathrm{Dir}}, \tag{1.11}$$

for any Harish-Chandra modules X, Y of finite length. Interestingly enough, to give a complete proof is harder than it seems first, and we did not succeed so far. In Section 6 we show that both sides of (1.11) are indices of suitable Fredholm pairs defined on the same spaces. We found the definition of Fredholm pairs (and some properties of their indices) in [1] and we adapted it to our purely algebraic setting. Let us just say here that the index of a Fredholm pair is a generalization of the index of a Fredholm operator. The relevant material is exposed in an appendix. The first of these Fredholm pair is given by the complex computing the Ext groups and its differential, and its index is the Euler-Poincaré characteristic of the complex, i.e. the left-hand side of (1.11). The second Fredholm pair is given (on the same space) by actions of Dirac operators and its index is the right-hand side of (1.11). The conjecture thus boils down to the equality of these two indices. Of course this kind of result looks familiar. This gives some credit to the conjecture and points out a direction to look for a proof.

In section 4, we use Labesse "index" functions ([18]) to construct index functions [2] f_X for any finite length Harish-Chandra module X, *i.e.* f_X is a smooth compactly supported function on G satisfying

$$\langle X, Y\rangle_{\mathrm{Dir}}, = \Theta_Y(f_X)$$

for any finite length Harish-Chandra module Y. These functions are very cuspidal in the sense of Labesse, thus their orbital integrals $\Phi(f_X, x)$ vanish on non elliptic elements. In section 5, we show that when x is an elliptic element in G, the formula (1.8) is valid in this context. This easily implies that the elliptic pairing coincide with the Dirac pairing. The proof uses the density of tempered characters in the space of invariant distributions to reduce the problem to the case of limits of discrete series. In the case of discrete series,

[2]Assuming the conjecture, we could call these Euler-Poincaré functions.

the relevant results are well-known and their extension to the case of limits of discrete series is relatively easy.

The idea that the elliptic pairing for Harish-Chandra modules is related to Dirac index originates in the papers [7] and [6] where the result is established for Hecke algebra. The role of the group $\widetilde{K}$ is played in that context by the spin cover $\widetilde{W}$ of the relevant Weyl group. Since modules for the Hecke algebra are finite dimensional, the difficulties we met in proving (1.11) do not appear.

We learned from G. Zuckerman that he obtained results on Euler-Poincaré pairing for Harish-Chandra modules in the late 70's, but that these were never published. Pavle Pandžić and Peter Trapa informed us that they were also led to the identity (1.11) in a work in progress with Dan Barbasch.

2. Dirac cohomology and Dirac index of $(\mathfrak{g}, K)$-modules

2.1. **Notation and structural facts.** Let G be a connected real reductive Lie group with Cartan involution θ such that $K = G^\theta$ is a maximal compact subgroup of G. Let us denote by $\mathfrak{g}_0$ the Lie algebra of G, $\mathfrak{g}$ its complexification, with Cartan involutions also denoted by θ.

We fix an invariant nondegenerate symmetric bilinear form B on $\mathfrak{g}_0$, extending the Killing form on the semisimple part of $\mathfrak{g}_0$. Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \overset{\theta}{\oplus} \mathfrak{p}_0, \qquad \mathfrak{g} = \mathfrak{k} \overset{\theta}{\oplus} \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}_0$ and $\mathfrak{g}$ respectively. We assume furthermore that in extending the Killing form, we made sure that $B_{|\mathfrak{p}_0}$ remains definite positive and $B_{|\mathfrak{k}_0}$ definite negative. We assume for simplicity that the center of G is compact. This implies that the dimension of $\mathfrak{p}_0$ is even.

Let $\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{p}; B)$ be the Clifford algebra of $\mathfrak{p}$ with respect to B. The Clifford algebra $\mathrm{Cl}(\mathfrak{p})$ is the quotient of the tensor algebra $T(\mathfrak{p})$ by the two-sided ideal generated by elements of the form

$$v \otimes w + w \otimes v + 2B(v, w)\, 1, \qquad (v, w \in V).$$

Some authors use relations differing by a sign. We follow here the convention of [13]. We refer to [13], [16] or [19] for material on Clifford algebras and spinor modules.

Notice that all the terms in the above expression are of even degrees (2 or 0). Thus, the graded algebra structure on $T(\mathfrak{p})$ induces a filtered algebra structure on $\mathrm{Cl}(\mathfrak{p})$, but also a structure of $\mathbb{Z}/2\mathbb{Z}$-graded algebra (*i.e.* a super algebra structure). Simply put, the $\mathbb{Z}/2\mathbb{Z}$-grading and the filtration are defined by the condition that the generators $X \in \mathfrak{p}$ of $\mathrm{Cl}(\mathfrak{p})$ are odd, of filtration degree 1. We get a decomposition

$$\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}^{\bar{0}}(\mathfrak{p}) \oplus \mathrm{Cl}^{\bar{1}}(\mathfrak{p}).$$

Recall also that the Clifford algebra $\mathrm{Cl}(\mathfrak{p})$ is isomorphic as a $\mathbb{Z}/2\mathbb{Z}$-graded vector space to the exterior algebra $\bigwedge \mathfrak{p}$ by the Chevalley isomorphism. It is convenient to identify $\mathrm{Cl}(\mathfrak{p})$ to $\bigwedge \mathfrak{p}$, and to see the latter as a $\mathbb{Z}/2\mathbb{Z}$-graded vector space endowed with two different algebra products, the graded commutative wedge product (denoted $x \wedge y$) and the Clifford product (denoted xy). Let us denote be $\mathbf{SO}(\mathfrak{p}_0)$ (resp. $\mathbf{SO}(\mathfrak{p})$) the special orthogonal group of $(\mathfrak{p}_0, B)$ (resp. $(\mathfrak{p}, B)$). The subspace $\bigwedge^2 \mathfrak{p}$ is stable under the Clifford Lie bracket $[x, y]_{\mathrm{Cl}} = xy - yx$ on $\bigwedge \mathfrak{p} = \mathrm{Cl}(\mathfrak{p})$ and is isomorphic as a Lie algebra to the Lie algebra $\mathfrak{so}(\mathfrak{p})$ of $\mathbf{SO}(\mathfrak{p})$.

We denote by $\widetilde{K}$ the spin double cover of K, i.e., the pull-back of the covering map $\mathbf{Spin}(\mathfrak{p}_0) \to \mathbf{SO}(\mathfrak{p}_0)$ by the adjoint action map $\mathrm{Ad}_{|\mathfrak{p}_0} : K \to \mathbf{SO}(\mathfrak{p}_0)$. The compact groups $\mathbf{Spin}(\mathfrak{p}_0)$ and $\mathbf{SO}(\mathfrak{p}_0)$ embed in their complexification $\mathbf{Spin}(\mathfrak{p})$ and $\mathbf{SO}(\mathfrak{p})$, so we get the following diagram :

$$\begin{tikzcd}
\widetilde{K} \arrow[r] \arrow[d] & \mathbf{Spin}(\mathfrak{p}_0) \arrow[r, hook] \arrow[d] & \mathbf{Spin}(\mathfrak{p}) \arrow[r, hook] \arrow[d] & \mathrm{Cl}^{\bar{0}}(\mathfrak{p})^\times \\
K \arrow[r, "\mathrm{Ad}_{|\mathfrak{p}_0}"] & \mathbf{SO}(\mathfrak{p}_0) \arrow[r, hook] & \mathbf{SO}(\mathfrak{p}) &
\end{tikzcd}$$

The complexification of the differential at the identity of the Lie group morphism $\mathrm{Ad}_{|\mathfrak{p}_0}$: $K \to \mathbf{SO}(\mathfrak{p}_0)$, is the Lie algebra morphism

$$\mathrm{ad}_{|\mathfrak{p}} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}), \quad X \mapsto \mathrm{ad}(X)_{|\mathfrak{p}}$$

Let us denote by α the composition of this map with the identification between $\mathfrak{so}(\mathfrak{p})$ and $\bigwedge^2 \mathfrak{p}$ and the inclusion of $\bigwedge^2 \mathfrak{p}$ in $\bigwedge \mathfrak{p} = \mathrm{Cl}(\mathfrak{p})$:

$$\alpha : \mathfrak{k} \to \mathrm{Cl}(\mathfrak{p}). \tag{2.1}$$

A key role is played in the theory of Dirac cohomology of Harish-Chandra modules by the associative $\mathbb{Z}/2\mathbb{Z}$-graded superalgebra $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$. The $\mathbb{Z}/2\mathbb{Z}$-grading comes from the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(\mathfrak{p})$, *i.e.* elements in $U(\mathfrak{g})\otimes 1$ are even. The super Lie algebra structure on $\mathcal{A}$ is given by the (super)commutator bracket $[\,.\,,.]_{\mathcal{A}}$.

The group K acts on $U(\mathfrak{g})$ through $K \subset G$ by the adjoint action, and on $\mathrm{Cl}(\mathfrak{p})$ through the map $\widetilde{K} \to \mathrm{Cl}^{\bar{0}}(\mathfrak{p})^\times$ in the first row of the diagram above and conjugation in $\mathrm{Cl}(\mathfrak{p})$ (this action of $\widetilde{K}$ on $\mathrm{Cl}(\mathfrak{p})$ factors through K). Thus we get a linear action of K on $\mathcal{A}$. Differentiating this action at the identity, and taking the complexification, we get a Lie algebra representation of $\mathfrak{k}$ in $\mathcal{A}$. This represention can be described as follows. The map (2.1) is used to define a map

$$\Delta : \mathfrak{k} \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}), \quad \Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$$

which is a morphism of Lie algebra (it takes values in the even part of the super Lie algebra $\mathcal{A}$). Thus it extends to an algebra morphism

$$\Delta : U(\mathfrak{k}) \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}). \tag{2.2}$$

The action of an element $X \in \mathfrak{k}$ on $\mathcal{A}$ is then given by the adjoint action of $\Delta(X)$, *i.e.* $a \in \mathcal{A} \mapsto [\Delta(X), a]_{\mathcal{A}}$. We denote by $\mathcal{A}^K$ (resp. $\mathcal{A}^{\mathfrak{k}}$) the subalgebra of K-invariants (resp. $\mathfrak{k}$-invariants) in $\mathcal{A}$. Since K is assumed to be connected, $\mathcal{A}^K = \mathcal{A}^{\mathfrak{k}}$.

Let us now recall some facts about Clifford modules.

Theorem 2.1. *Suppose that* $n = \dim(\mathfrak{p})$ *is even. Then there are :*
- *two isomorphism classes of irreducible* $\mathbb{Z}/2\mathbb{Z}$*-graded* $\mathrm{Cl}(\mathfrak{p})$*-modules,*
- *one isomorphism class of irreducible ungraded* $\mathrm{Cl}(\mathfrak{p})$*-modules,*
- *two isomorphism classes of irreducible* $\mathrm{Cl}^{\bar{0}}(\mathfrak{p})$*-modules.*

In the case $\dim(\mathfrak{p})$ is even, an irreducible ungraded $\mathrm{Cl}(\mathfrak{p})$-module S can be realized as follows : choose a decomposition $\mathfrak{p} = U \oplus U^*$ into dual isotropic subspaces. As the notation indicates, U^* is identified to the dual of U by the bilinear form B. Set $S = \bigwedge U$. Let U act on S by wedging and U^* by contracting. The decomposition $\bigwedge U = \bigwedge^{\bar{0}} U \oplus \bigwedge^{\bar{1}} U$ give a decomposition $S = S^+ \oplus S^-$ into the two non-isomorphic simple $\mathrm{Cl}^{\bar{0}}(\mathfrak{p})$-modules S^+ and S^-. The dual of S is identified with $\bigwedge U^*$. The modules S, S^+ and S^- are finite dimensional representations of $\widetilde{K}$ and so are their duals and also $\mathrm{Cl}(\mathfrak{p}) = \bigwedge \mathfrak{p}$, $\mathrm{Cl}(\mathfrak{p})^{\bar{0}}$ and $\mathrm{Cl}(\mathfrak{p})^{\bar{1}}$.

Proposition 2.2. *As virtual $\widetilde{K}$ representations,*

$$\bigwedge^{\bar{\bullet}} \mathfrak{p} := \bigwedge^{\bar{1}} \mathfrak{p} - \bigwedge^{\bar{0}} \mathfrak{p} \simeq (S^+ - S^-)^* \otimes (S^+ - S^-) \tag{2.3}$$

Proof. As $\widetilde{K}$-modules :

$$\bigwedge \mathfrak{p} \simeq \mathrm{Cl}(\mathfrak{p}) \simeq \mathrm{End}(S) \simeq S^* \otimes S = (S^+ \oplus S^-)^* \otimes (S^+ \oplus S^-).$$

This can be obtained also using :

$$\bigwedge \mathfrak{p} = \bigwedge (U^* \oplus U) \simeq (\bigwedge U^*) \otimes (\bigwedge U) \simeq S^* \otimes S.$$

Writing

$$(S^+ \oplus S^-)^* \otimes (S^+ \oplus S^-) = ((S^+)^* \otimes S^+) \oplus ((S^-)^* \otimes S^+) \oplus ((S^+)^* \otimes S^-) \oplus ((S^-)^* \otimes S^-)$$

and identifying the even and odd parts of these $\mathbb{Z}/2\mathbb{Z}$-graded modules, we get

$$\bigwedge^{\bar{0}} \mathfrak{p} \simeq ((S^+)^* \otimes S^+) \oplus ((S^-)^* \otimes S^-),$$
$$\bigwedge^{\bar{1}} \mathfrak{p} \simeq ((S^+)^* \otimes S^-) \oplus ((S^-)^* \otimes S^+).$$

So, as virtual $\widetilde{K}$ representations,

$$\bigwedge^{\bar{\bullet}} \mathfrak{p} = (S^+ - S^-)^* \otimes (S^+ - S^-).$$

□

2.2. **Dirac cohomology of Harish-Chandra modules.** Let us now introduce the Dirac operator D :

Definition 2.3. If $(Y_i)_i$ is a basis of $\mathfrak{p}$ and $(Z_i)_i$ is the dual basis with respect to B, then

$$D = D(\mathfrak{g}, K) = \sum_i Y_i \otimes Z_i \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$$

is independent of the choice of basis $(Y_i)_i$ and K-invariant for the adjoint action on both factors. The Dirac operator D (for the pair $(\mathfrak{g}, K)$) is an element of $\mathcal{A}^K$ (see [13]).

The most important property of D is the formula

$$D^2 = -\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \Delta(\mathrm{Cas}_{\mathfrak{k}}) + (\|\rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2) 1 \otimes 1 \tag{2.4}$$

due to Parthasarathy [20] (see also [13]). Here $\mathrm{Cas}_{\mathfrak{g}}$ (respectively $\mathrm{Cas}_{\mathfrak{k}}$) denotes the Casimir element of $U(\mathfrak{g})$. The constant $(\|\rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2)$ is explained below. This formula has several important consequences for Harish-Chandra modules. To state them, we need more notation. Let us fix a maximal torus T in K, with Lie algebra $\mathfrak{t}_0$. Let $\mathfrak{a}_0$ denotes the centralizer of $\mathfrak{t}_0$ in $\mathfrak{p}_0$. Then

$$\mathfrak{h}_0 := \mathfrak{t}_0 \oplus \mathfrak{a}_0$$

is a fundamental Cartan subalgebra of $\mathfrak{g}_0$, and the above decomposition also gives an imbedding $\mathfrak{t}^* \to \mathfrak{h}^*$. Let $R = R(\mathfrak{g}, \mathfrak{h})$ denotes the root system of $\mathfrak{h}$ in $\mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$ its Weyl group. Let us also choose a positive root system R^+ in R. As usual, ρ denotes the half-sum of positive roots, an element in $\mathfrak{h}^*$. Similarly, we introduce the root system $R_{\mathfrak{k}} = R(\mathfrak{k}, \mathfrak{t})$, its Weyl group $W_{\mathfrak{k}}$, a positive root system $R^+_{\mathfrak{k}}$, compatible with R^+, and half-sum of positive roots $\rho_{\mathfrak{k}}$.

The bilinear form B on $\mathfrak{g}$ restricts to a non degenerate symmetric bilinear form on $\mathfrak{h}$, which is definite positive on the real form $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$. We denote by $\langle . , . \rangle$ the induced form on $i\mathfrak{t}_0^* \oplus \mathfrak{a}_0$ and in the same way its extension to $\mathfrak{h}^*$. The norm appearing in (2.4) is defined for any $\lambda \in \mathfrak{h}^*$ by $\|\lambda\|^2 = \langle \lambda, \lambda \rangle$. Notice that this is clearly an abuse of notation when Λ is not in $i\mathfrak{t}_0^* \oplus \mathfrak{a}_0$.

Recall the Harish-Chandra algebra isomorphism

$$\gamma_{\mathfrak{g}} : \mathfrak{Z}(\mathfrak{g}) \simeq S(\mathfrak{h})^W \tag{2.5}$$

between the center $\mathfrak{Z}(\mathfrak{g})$ of the envelopping algebra $U(\mathfrak{g})$ and the W-invariants in the symmetric algebra $S(\mathfrak{h})$ on $\mathfrak{h}$. Accordingly, a character χ of $\mathfrak{Z}(\mathfrak{h})$ is given by an element of $\mathfrak{h}^*$ (or rather its Weyl group orbit). If $\lambda \in \mathfrak{h}^*$, we denote by χ_λ the corresponding character of $\mathfrak{Z}(\mathfrak{g})$. Let X be a Harish-Chandra module. We say that X has infinitesimal character λ if any $z \in \mathfrak{Z}(\mathfrak{g})$ acts on X by the scalar operator $\chi_\lambda(z)\mathrm{Id}_X$.

Let $\mathscr{M} = \mathscr{M}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules for the pair $(\mathfrak{g}, K)$ (see [15] for details). If $X \in \mathscr{M}$, then $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ acts on $X \otimes S$. Then X decomposes as the direct sum of its K-isotypic components, these being finite dimensional if X is admissible. Accordingly, $X \otimes S$ decomposes as the direct sum of its (finite dimensional if X is admissible) $\widetilde{K}$-isotypic components. Let (γ, F_γ) be an irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$. We denote the corresponding $\widetilde{K}$-isotypic component of $X \otimes S$ by $(X \otimes S)(\gamma)$. Assume X is admissible and has infinitesimal character $\Lambda \in \mathfrak{h}^*$. Then D^2 acts on $(X \otimes S)(\gamma)$ by the scalar

$$-\|\Lambda\|^2 + \|\tau + \rho_{\mathfrak{k}}\|^2. \tag{2.6}$$

In particular, we see that in that case D^2 acts semi-simply on $X \otimes S$, and that the kernel of D^2 on $X \otimes S$ is a (finite) direct sum of full $\widetilde{K}$-isotypic components of $X \otimes S$: these are exactly those $(X \otimes S)(\gamma)$ for which

$$\|\tau + \rho_{\mathfrak{k}}\|^2 = \|\Lambda\|^2. \tag{2.7}$$

Another important fact is that the action of D preserves $\widetilde{K}$-isotypic components of $X \otimes S$. If X is unitary (resp. finite dimensional), one can put an positive definite hermitian form on $X \otimes S$ and one sees that D is symmetric (resp. skew-symmetric).

Let us now review Vogan's definition of Dirac cohomology.

Definition 2.4. Let $X \in \mathscr{M}$. The Dirac operator D acts on $X \otimes S$ with kernel $\ker D$ and image $\mathrm{Im}\, D$. Vogan's Dirac cohomology of X is the quotient

$$\mathsf{H}_{\mathrm{Dir}}(X) = \ker D / (\ker D \cap \mathrm{Im}\, D).$$

Since $D \in \mathcal{A}^K$, $\widetilde{K}$ acts on $\ker D$, $\mathrm{Im}\, D$ and $\mathsf{H}_{\mathrm{Dir}}(X)$. Also, assume that X is admissible and has infinitesimal character $\Lambda \in \mathfrak{h}^*$. Then, since $\ker D \subset \ker D^2$ and since we have seen that the latter is the sum the full $\widetilde{K}$-isotypic components of $X \otimes S$ satisfying (2.7) (these are obviously in finite number), we see that $\mathsf{H}_{\mathrm{Dir}}(X)$ is a finite dimensional representation of $\widetilde{K}$. This is particularly helpful if X is unitary, admissible and has infinitesimal character $\Lambda \in \mathfrak{h}^*$. Then it follows that D acts semisimply on $X \otimes S$ and so

$$\ker D^2 = \ker D = \mathsf{H}_{\mathrm{Dir}}(X). \tag{2.8}$$

In this case, the Dirac cohomology of X is a sum the full isotypic components $(X \otimes S)(\gamma)$ such that (2.7) holds. For general X, (2.8) does not hold, but note that D is always a differential on $\ker D^2$, and $\mathsf{H}_{\mathrm{Dir}}(X)$ is the usual cohomology of this differential.

Let us state the main result of [12], which gives a strong condition on the infinitesimal character of an admissible Harish-Chandra module X with non zero Dirac cohomology.

Proposition 2.5. *Let $X \in \mathscr{M}$ be an admissible Harish-Chandra module with infinitesimal character $\Lambda \in \mathfrak{h}^*$. Assume that (γ, F_γ) is an irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$ such that $(X \otimes S)(\gamma)$ contributes to $\mathsf{H}_{\mathrm{Dir}}(X)$. Then*

$$\Lambda = \tau + \rho_{\mathfrak{k}} \quad \textit{up to conjugacy by the Weyl group } W. \tag{2.9}$$

Thus for unitary X, (2.7) is equivalent to the stronger condition (2.9), provided that γ appears in $X \otimes S$.

2.3. **Dirac index.** The Dirac index of Harish-Chandra modules is a refinement of Dirac cohomology. It uses the decomposition $S = S^+ \oplus S^-$ of the spinor module as a $\mathrm{Cl}(\mathfrak{p})^{\bar{0}}$-module (and thus also as a representation of $\widetilde{K}$). Since D is an odd element in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ its action on $X \otimes S$, for any Harish-Chandra module X exchanges $X \otimes S^+$ and $X \otimes S^-$:

$$D : X \otimes S^+ \longleftrightarrow X \otimes S^-$$

Accordingly, the Dirac cohomology of X decomposes as

$$\mathsf{H}_{\mathrm{Dir}}(X) = \mathsf{H}_{\mathrm{Dir}}(X)^+ \oplus \mathsf{H}_{\mathrm{Dir}}(X)^-.$$

The **index of the Dirac operator** acting on $X \otimes S$ is the virtual representation

$$\mathsf{I}_{\mathrm{Dir}}(X) = \mathsf{H}_{\mathrm{Dir}}(X)^+ - \mathsf{H}_{\mathrm{Dir}}(X)^-$$

of $\widetilde{K}$. The following proposition is interpreted as an Euler-Poincaré principle.

Proposition 2.6. *Let X be an admissible Harish-Chandra module with infinitesimal character. Then*

$$\mathsf{I}_{\mathrm{Dir}}(X) = X \otimes S^+ - X \otimes S^-$$

as virtual $\widetilde{K}$-representations.

Proof. A virtual $\widetilde{K}$-representation is by definition an element of the Grothendieck group $\mathscr{R}(\widetilde{K})$ of the category of finite dimensional representations of $\widetilde{K}$. This is the free $\mathbb{Z}$-module generated by equivalence classes of irreducible representations, *i.e.* one can write

$$\mathscr{R}(\widetilde{K}) = \bigoplus_{\gamma \in (\widetilde{K})^\wedge} \mathbb{Z}.$$

The right-hand side of the equation in the proposition cannot *a priori* be interpreted as an element of $\mathscr{R}(\widetilde{K})$, but only of the larger group $\prod_{\gamma \in (\widetilde{K})^\wedge} \mathbb{Z}$.

Let us now prove the equality. We have seen that D^2 acts semisimply on $X \otimes S$. Furthermore, each eigenspace of D^2 in $X \otimes S$ is a sum of full $\widetilde{K}$-isotypic components and that these are preserved by the action of D. Each of these $\widetilde{K}$-isotypic components $(X \otimes S)(\gamma)$ decomposes also as

$$(X \otimes S)(\gamma) = (X \otimes S)(\gamma)^+ \oplus (X \otimes S)(\gamma)^-$$

where $(X \otimes S)(\gamma)^\pm := (X \otimes S)(\gamma) \cap (X \otimes S^\pm)$.

For $\widetilde{K}$-isotypic components corresponding to a non-zero eigenvalue of D^2, we thus get that D is a bijective intertwining operator (for the $\widetilde{K}$-action) between $(X \otimes S)(\gamma)^+$ and $(X \otimes S)(\gamma)^-$. Thus the contribution of these $\widetilde{K}$-isotypic components to $X \otimes S^+ - X \otimes S^-$ is zero. So only $\ker D^2$ will contribute, i.e.

$$X \otimes S^+ - X \otimes S^- = (\ker D^2 \cap (X \otimes S^+)) - (\ker D^2 \cap (X \otimes S^-)).$$

Let us write

$$\ker D^2 \cap (X \otimes S^\pm) = \ker D \cap (X \otimes S^\pm) \oplus W^\pm$$

for some $\widetilde{K}$-invariant complementary subspaces $W^{\pm}$. Then, as above, D is bijective intertwining operator for the $\widetilde{K}$-action between $W^{\pm}$ and $D(W^{\pm}) \subset \ker D^2 \cap (X \otimes S^{\mp})$. So these contributions also cancel, and what remains is exactly the virtual $\widetilde{K}$-representation $\mathsf{H}_{\mathrm{Dir}}(X)^+ - \mathsf{H}_{\mathrm{Dir}}(X)^-$. □

2.4. **Dirac index of limits of discrete series.** We assume that G has discrete series, so that G and K have same rank, and T is a compact Cartan subgroup. Harish-Chandra modules of limits of discrete series of G are obtained as cohomologically induced $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$-modules (see [15]), where $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ is a Borel subalgebra containing $\mathfrak{t}$ with nilpotent radical $\mathfrak{u}$, and $\mathbb{C}_\lambda$ is the one-dimensional representation of T with weight $\lambda \in i\mathfrak{t}_0^*$. Some positivity conditions on λ are required, that we now describe. The Borel subalgebra $\mathfrak{b}$ determines a set of positive roots $R_{\mathfrak{b}}^+$ of $R = R(\mathfrak{t}, \mathfrak{g})$ (the roots of $\mathfrak{t}$ in $\mathfrak{u}$). Let us denote by $\rho(\mathfrak{b})$, (resp. $\rho_c(\mathfrak{b})$, resp. $\rho_n(\mathfrak{b})$) the half-sum of (resp. compact , resp.non-compact) roots in $R_{\mathfrak{b}}^+$. The positivity condition on λ is that

$$\langle \lambda + \rho(\mathfrak{b}), \alpha \rangle \geq 0. \tag{2.10}$$

Then, $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ is a discrete series modules if the inequalities in (2.10) are strict, and otherwise a limit of discrete series ($\chi = \lambda + \rho(\mathfrak{b})$ is the infinitesimal character of $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$), or 0 (but we are not interested in this case). The lowest K-type of $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ has multiplicity one and highest weight $\Lambda = \lambda + 2\rho_n(\mathfrak{b}) = \chi + \rho_n(\mathfrak{b}) - \rho_c(\mathfrak{b})$, and all other K-types have highest weights of the form $\Lambda + \sum_j \beta_j$ for some positive roots β_j.

In [11] these facts are used together with Prop 2.5 to show that the Dirac cohomology of $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ consists in the multiplicity-one $\widetilde{K}$-type F_μ with highest weight

$$\mu = \Lambda - \rho_n(\mathfrak{b}) = \lambda + \rho_n(\mathfrak{b}) = \chi - \rho_c(\mathfrak{b}).$$

(In [11], it is assumed that $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ is a discrete series, but inspection of the proof easily shows that it works also for limits of discrete series.) Thus the Dirac index of $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ is $\pm F_\mu$. To determine the sign, recall that the spinor module S, as a $\widetilde{K}$-representation, doesn't depend on any choice, nor does the set $\{S^+, S^-\}$, in particular not on the way we realized this module, but the distinction between S^+ and S^- does (resulting on a sign change in the Dirac index). So suppose we have fixed once for all a Borel subalgebra $\mathfrak{b}_1 = \mathfrak{t} \oplus \mathfrak{u}_1$ and choose $\mathfrak{u}_1 \cap \mathfrak{p}$ as the isotropic subspace U of $\mathfrak{p}$ used to construct the spinor modules in section 2.1. Then S^+ is the $\widetilde{K}$-representation containing the weight $-\rho_n(\mathfrak{b}_1)$ and S^- is the one containing the weight $\rho_n(\mathfrak{b}_1)$. With this choice, it is easy to see that $\mathsf{I}_{\mathrm{Dir}}(A_{\mathfrak{b}}(\mathbb{C}_\lambda)) = \mathrm{sgn}(w) F_\mu$, where $w \in W$ is the Weyl group element sending $\mathfrak{b}_1$ to $\mathfrak{b}$ and sgn is the sign character.

Let us now determine the Dirac index of the virtual modules X which are the linear combinations of limits of discrete series with the same infinitesimal character, whose characters are the supertempered distributions constructed by Harish-Chandra (see [10] and [4], §7). Namely take an integral but non necessarily regular weight χ in $i\mathfrak{t}_0^*$ and consider the limits of discrete series $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ as above with $\chi = \lambda + \rho(\mathfrak{b})$ satisfying (2.10). Notice that χ being fixed, $\mathfrak{b}$ determines $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ and that Borel subalgebras $\mathfrak{b}$ which occurs are the one such that the corresponding Weyl chamber $C_{\mathfrak{b}}$ has χ in its closure. So we can forget the λ in the notation, and the set of limits of discrete series we are considering is $\mathcal{L} = \{A_{\mathfrak{b}} \,|\, \chi \in \overline{C_{\mathfrak{b}}}\}$. We should also take care of the fact that the $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ could be 0, but the important remark here is that if one of them in the set above is non zero, then all are (χ is not on a wall in the kernel of a simple compact root). Choose one of them as a base point, say $A_{\mathfrak{b}_2}$. Then the linear combination introduced by Harish- Chandra is

$$X_{\chi,\mathfrak{b}_2} = \frac{1}{|W_\chi|} \sum_{A_\mathfrak{b} \in \mathcal{L}} \epsilon(\mathfrak{b}) A_\mathfrak{b}$$

where $\epsilon(\mathfrak{b}) = \operatorname{sgn}(w)$, and $w \in W$ is the Weyl group element sending $\mathfrak{b}_2$ to $\mathfrak{b}$ and $W_\chi = \{w \in W | \, w \cdot \chi = \chi\}$. This construction is made because all the the $\epsilon(\mathfrak{b})A_\mathfrak{b}$ have the same character formula on T and this is also the formula for the character of $X_{\chi,\mathfrak{b}_2}$. Notice that a different choice of $\mathfrak{b}_2$ as base point would result in at most a sign change.

The fact that the $A_\mathfrak{b}$ are non-zero implies that the corresponding Weyl chambers $C_\mathfrak{b}$ are all included in a single Weyl chamber for $R(\mathfrak{k},\mathfrak{t})$. In particular, the various $\rho_c(\mathfrak{b})$ are all equal (let say to ρ_c). This shows that the Dirac cohomology of all the $A_\mathfrak{b} \in \mathcal{L}$ is the same, namely the multiplicity-one $\widetilde{K}$-type F_μ with highest weight $\mu = \chi - \rho_c$. Taking signs into account, we see that all the $\epsilon(\mathfrak{b})A_\mathfrak{b}$ have same Dirac index $\operatorname{sgn}(w)F_\mu$ where w is the Weyl group element sending $\mathfrak{b}_1$ to $\mathfrak{b}_2$. This is thus also the Dirac index of $X_{\chi,\mathfrak{b}_2}$.

For discrete series, χ is regular, so $\mathcal{L}$ contains only one element $A_\mathfrak{b}(\lambda)$ with $\lambda = \chi - \rho(\mathfrak{b})$.

Proposition 2.7. *If $\mathsf{I}_{\mathrm{Dir}}(X_{\chi,\mathfrak{b}}) = \pm I(X_{\chi',\mathfrak{b}'})$ then χ and χ' are conjugate by a element of $W(\mathfrak{k},\mathfrak{t})$, and thus by an element in K. Therefore $X_{\chi,\mathfrak{b}} = X_{\chi',\mathfrak{b}'}$.*

Proof. Say that $\mathsf{I}_{\mathrm{Dir}}(X_{\chi,\mathfrak{b}}) = \pm F_\mu = \pm I(X_{\chi',\mathfrak{b}'})$. For any $g \in G$ normalizing T, $X_{g\cdot\chi,g\cdot\mathfrak{b}} = X_{\chi,\mathfrak{b}}$, so we can assume that $C_\mathfrak{b}$ and $C_{\mathfrak{b}'}$ are in the same Weyl chamber for $R(\mathfrak{k},\mathfrak{t})$. We have then $\mu = \chi - \rho_c(\mathfrak{b}) = \chi' - \rho_c(\mathfrak{b}')$ and $\rho_c(\mathfrak{b}) = \rho_c(\mathfrak{b}')$ so that $\chi = \chi'$. □

3. Euler-Poincaré pairing and Dirac pairing

For two finite length Harish-Chandra modules X and Y, one can define their Euler-Poincaré pairing as the alternate sum of dimensions of Ext functors as in (1.1). Recall that $\mathscr{M}$ has finite cohomological dimension, so this sum has finite support. More precisely, an explicit projective resolution of X in $\mathcal{M}(\mathfrak{g},K)$ is given by

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with $P_i = (U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^i \mathfrak{p}) \otimes X$ ([15], §II.7). Set

$$C^i = \operatorname{Hom}_{\mathscr{M}}(P_i, X) \simeq \operatorname{Hom}_K(\textstyle\bigwedge^i \mathfrak{p} \otimes X, Y).$$

Thus $\operatorname{Ext}^i(X,Y)$ is given by the i-th cohomology group of the complex $\mathbf{C} = (C^i)_i$, with differential d^i given explicitely in *loc. cit.*

If γ, σ are virtual finite dimensional representations of $\widetilde{K}$, and χ_γ, χ_σ are their characters, we denote by

$$[\gamma,\sigma]_{\widetilde{K}} = \int_{\widetilde{K}} \overline{\chi_\gamma(k)} \chi_\sigma(k) \, dk$$

the usual (hermitian) pairing between these virtual representations. Since virtual finite dimensional representations of K are also virtual representations of $\widetilde{K}$, we use also the notation for their pairing. Notice that when γ, σ are actual finite dimensional representations, then

$$[\gamma,\sigma]_{\widetilde{K}} = \dim \operatorname{Hom}_{\widetilde{K}}(\gamma,\sigma).$$

We can now compute, assuming that X and Y have infinitesimal character [3] :

[3] A technical remark is in order here : recall that a Harish-Chandra module with infinitesimal character is finite length if and only if it is admissible.

$$
\begin{aligned}
\mathbf{EP}(X,Y) &= \sum_i (-1)^i \ \dim \operatorname{Ext}^i(X,Y) \\
&= \sum_i (-1)^i \ \dim \operatorname{Hom}_K \left(\textstyle\bigwedge^i \mathfrak{p} \otimes X, Y \right) \quad \text{(Euler-Poincaré principle)} \\
&= \sum_i (-1)^i \ \left[\textstyle\bigwedge^i \mathfrak{p} \otimes X, Y \right]_{\widetilde{K}} = \left[\displaystyle\sum_i (-1)^i \textstyle\bigwedge^i \mathfrak{p} \otimes X, Y \right]_{\widetilde{K}} = \left[\textstyle\bigwedge^{\bar{\bullet}} \mathfrak{p} \otimes X, Y \right]_{\widetilde{K}} \\
&= \left[(S^+ - S^-)^* \otimes (S^+ - S^-) \otimes X, Y \right]_{\widetilde{K}} \qquad \text{(see (2.3))} \\
&= \left[(S^+ - S^-) \otimes X, (S^+ - S^-) \otimes Y \right]_{\widetilde{K}} \\
&= \left[\mathsf{H}_{\mathrm{Dir}}(X)^+ - \mathsf{H}_{\mathrm{Dir}}(X)^-, \mathsf{H}_{\mathrm{Dir}}(Y)^+ - \mathsf{H}_{\mathrm{Dir}}(Y)^- \right]_{\widetilde{K}} \qquad \text{(Prop. 2.6.)} \\
&= [\mathsf{I}_{\mathrm{Dir}}(X), \mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}} \, .
\end{aligned}
$$

The attentive reader probably noticed a small problem with the above computation, namely the application of the Euler-Poincaré principle is not justified since the terms $C^i = \operatorname{Hom}_K \left(\bigwedge^i \mathfrak{p} \otimes X, Y \right)$ could be infinite dimensional. As explained in the introduction, we were not able to find a proof of the equality of the two extreme terms so far, so let us state this as a conjecture :

Connjecture 3.1. *Let X and Y be two finite length Harish-Chandra modules with infinitesimal character. Then*

$$\mathbf{EP}(X,Y) = [\mathsf{I}_{\mathrm{Dir}}(X), \mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}} = \langle X, Y \rangle_{\mathrm{Dir}}.$$

The last equality is the definition of the Dirac pairing mentioned in the introduction. The main content of this conjecture is that the **EP** pairing of two finite length Harish-Chandra modules with infinitesimal character factors through their Dirac indices, and thus through their Dirac cohomology. In particular, the results on Dirac cohomology and Dirac index recalled in Section 2.2 put severe conditions on modules X and Y for their Dirac pairing to be non-zero. For many interesting modules, the Dirac index is explicitly known, and then so is the Dirac pairing between these modules.

Remark 3.2. In the case where $X = F$ is a finite-dimensional irreducible Harish-Chandra module, $\operatorname{Ext}^i(F,Y)$ is also the i-th $(\mathfrak{g},K)$-cohomology group of $Y \otimes F^*$. Thus, in that case,

$$\sum_i (-1)^i \dim H^i(\mathfrak{g}, K; Y \otimes F^*) = [\mathsf{I}_{\mathrm{Dir}}(F), \mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}} \, .$$

In the case where Y is unitary, we have much stronger results : the differential d on $\operatorname{Hom}_K(\bigwedge \mathfrak{p} \otimes F, Y)$ is 0 so

$$\bigoplus_i H^i(\mathfrak{g}, K; Y \otimes F^*) = \bigoplus_i \operatorname{Hom}_K(\textstyle\bigwedge^i \mathfrak{p} \otimes F, Y) = \operatorname{Hom}_K(\bigwedge \mathfrak{p} \otimes F, Y) \simeq \operatorname{Hom}_{\widetilde{K}}(F \otimes S, Y \otimes S)$$

and the only common $\widetilde{K}$-types between $F \otimes S$ and $Y \otimes S$ have their isotypic components in $\ker(D_{|F\otimes S}) = \ker(D^2_{|F\otimes S}) = \mathsf{H}_{\mathrm{Dir}}(F)$ and $\ker(D_{|X\otimes S}) = \ker(D^2_{|X\otimes S}) = \mathsf{H}_{\mathrm{Dir}}(X)$ respectively. See [24], §9.4 and [13], §8.3.4.

4. Labesse index functions and Euler-Poincaré functions

Let us continue the computation of the Dirac pairing (and thus conjecturally of the **EP** pairing) for two finite length Harish-Chandra modules X and Y with infinitesimal character. We have

$$\langle X, Y\rangle_{\mathrm{Dir}} = [\mathsf{I}_{\mathrm{Dir}}(X), \mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}} = \sum_{\gamma\in(\widetilde{K})^\wedge} [\gamma, \mathsf{I}_{\mathrm{Dir}}(X)]_{\widetilde{K}} \times [\gamma, \mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}}\,.$$

Let us now introduce some material from [18]. If γ is a genuine virtual finite dimensional representation of $\widetilde{K}$, let us denote by $I_K(\gamma)$ the virtual finite dimensional representation

$$\gamma\otimes(S^+ - S^-)^*$$

of K and by $\tilde{\chi}_\gamma$ of the character of $I_K(\gamma)$ (a conjugation invariant function on K). If (π, X) is an admissible Harish-Chandra module, with π denoting the action of K on X, the operator

$$\pi(\overline{\tilde{\chi}_\gamma}) = \int_K \pi(k)\,\overline{\tilde{\chi}_\gamma(k)}\,dk$$

is of finite rank, and so is a trace operator (dk is the normalized Haar measure on K). Let us denote its trace by $I(X,\gamma)$. If furthermore X has infinitesimal character, we have

$$I(X,\gamma) = \left[\gamma\otimes(S^+ - S^-)^*, X\right]_{\widetilde{K}} = \left[\gamma, X\otimes(S^+ - S^-)\right]_{\widetilde{K}} = [\gamma, \mathsf{I}_{\mathrm{Dir}}(X)]_{\widetilde{K}}\,.$$

Labesse main result is the existence for all $\gamma\in(\widetilde{K})^\wedge$ of a smooth, compactly supported and bi-K-finite function f_γ which satisfies, for all finite length Harish-Chandra module X

$$\Theta_X(f_\gamma) = I(X,\gamma),$$

where Θ_X denotes the distribution-character of X. The main ingredient in Labesse's construction is the Paley-Wiener theorem of Arthur ([2]).

From this, we get

$$\langle X, Y\rangle_{\mathrm{Dir}} = \sum_{\gamma\in(\widetilde{K})^\wedge} I(X,\gamma)\times I(Y,\gamma) = \sum_{\gamma\in(\widetilde{K})^\wedge} \Theta_X(f_\gamma)\Theta_Y(f_\gamma).$$

Let us call

$$f_Y = \sum_{\gamma\in(\widetilde{K})^\wedge} \Theta_Y(f_\gamma)\; f_\gamma$$

an index (or Euler-Poincaré, if we believe in Conjecture 3.1) function for Y. Note that the sum above has finite support, so that f_Y is smooth, compactly supported and bi-K-finite. Furthermore, Labesse shows ([18], Prop. 7) that the functions f_γ are "very cuspidal", *i.e.* that their constant terms for all proper parabolic subgroups of G vanish. Thus the same property holds for f_Y. This implies the vanishing of the orbital integrals $\Phi(f_Y, x)$ on regular non-elliptic element x in G. We have obtained :

Theorem 4.1. *For any finite length Harish-Chandra module Y with infinitesimal character, there exists a smooth, compactly supported, bi-K-finite and very cuspidal function f_Y on G such that for any finite length Harish-Chandra module X with infinitesimal character,*

$$\langle X, Y\rangle_{\mathrm{Dir}} = \Theta_X(f_Y).$$

5. Integral orbital of f_Y as the character of Y on elliptic elements

The goal of this section is to show that the value of the orbital integral $\Phi(f_Y, x)$ of f_Y at an elliptic regular element x coincide with the value of the character θ_Y of Y at x^{-1}. The character Θ_Y of Y is a distribution on G, but recall that according to Harish-Chandra regularity theorem, there is an analytic, conjugation invariant function that we will denote by θ_Y on the set G_{reg} such that for all $f \in \mathcal{C}_c^\infty(G)$,

$$\Theta_Y(f) = \int_G \theta_Y(x)\; f(x)\; dx.$$

Using Weyl integration formula, this could be written as

$$\Theta_Y(f) = \sum_{[H]} \frac{1}{|W(G,H)|} \int_H |D_G(h)|\, \theta_Y(h)\; \Phi(f,h)\; dh$$

where the first sum is on a system of representative of conjugacy classes of Cartan subgroups H of G, $W(G,H) = N_G(H)/H$ is the real Weyl group of H and $|D_G|$ is the usual Jacobian. The assertion is thus that :

$$\theta_Y(x^{-1}) = \Phi(f_Y, x), \qquad (x \in G_{\mathrm{ell}}). \tag{5.1}$$

The characterization of orbital integrals due to A. Bouaziz [4] shows that there indeed exists a function ψ_Y in $\mathcal{C}_c^\infty(G)$ such that

$$\Phi(\psi_Y, x) = \begin{cases} \theta_Y(x^{-1}) \text{ if } x \in G_{\mathrm{ell}} \\ 0 \text{ if } x \in G_{\mathrm{reg}} \setminus G_{\mathrm{ell}} \end{cases}.$$

Let $\mathcal{F}$ be a family of elements X in $\mathscr{R}_{\mathbb{C}}$ such that the space generated by the Θ_X is dense in the space of invariant distributions $\mathcal{D}'(G)^G$ on G. For instance, $\mathcal{F}$ could be the set of characters of all irreducible tempered representations, but we will rather take $\mathcal{F}$ to be the family of virtual representations with characters Θ_{h^*} defined in [4], §7. The density of this family of invariant distributions is a consequence of the inversion formula of orbital integrals [5]. Elements in $\mathcal{F}$ are generically irreducible tempered representations, but in general, they are linear combinations of some of these with same infinitesimal character. The distributions Θ_{h^*} are supertempered in the sense of Harish-Chandra [10]. In any case, by density of the family $\mathcal{F}$, to prove (5.1), it is enough to show that for all $X \in \mathcal{F}$, we have :

$$\Theta_X(f_Y) = \Theta_X(\psi_Y). \tag{5.2}$$

If X is a linear combination of parabolically induced representations, then both side are 0 since Θ_X vanishes on elliptic elements. Thus, it is sufficient to prove (5.2) for X corresponding to the Θ_{h^*} of [4] attached to the fundamental Cartan subgroup T. For simplicity, we assume now that T is compact, *i.e.* that G and K have same rank. Then X is either a discrete series or a linear combination of limits of discrete series (with same infinitesimal character) described in section 2.4.

Assume that (5.1) is established for all such X. The left-hand side of (5.2) then also equals

$$\begin{aligned}\Theta_Y(f_X) &= \frac{1}{|W(G,T)|} \int_T |D_G(t)|\, \theta_Y(t)\; \Phi(f_X, t)\; dt \\ &= \frac{1}{|W(G,T)|} \int_T |D_G(t)|\, \theta_Y(t)\; \theta_X(t^{-1})\; dt\end{aligned}$$

and equals the right-hand side, by using the definition of ψ_Y and the Weyl integration formula again. By definition of the measure dc on the set C^{ell} of regular semisimple elliptic conjugacy classes in G and of the elliptic pairing recalled in the introduction, we have also

$$\Theta_Y(f_X) = \int_{C^{\mathrm{ell}}} \theta_Y(c)\, \theta_X(c^{-1})\, dc = \langle X, Y\rangle_{\mathrm{ell}}.$$

Thus we have reduced the proof of (5.1) for all Y to the case when Y is either a discrete series or a linear combination of limits of discrete series as described above. In turns, it is enough to show (5.2) when both X and Y are of this kind. In case Y corresponds to a parameter h^* of [5], ψ_Y is exactly the function denoted ψ_{h^*} there, and in particular :

$$\Theta_X(\psi_Y) = 1 \text{ if } X = Y, \quad 0 \text{ otherwise.}$$

With the notation of section 2.4, we can take $X = X_{\chi,\mathfrak{b}}$ and $Y = X_{\chi',\mathfrak{b}'}$. Then $f_X = \sum_{\gamma\in(\widetilde{K})^\wedge} \Theta_X(f_\gamma) f_\gamma = \sum_{\gamma\in(\widetilde{K})^\wedge} [\gamma, I(X,\gamma)]_{\widetilde{K}}\ f_\gamma$ and by the results in section 2.4, we get $f_X = \epsilon(\mathfrak{b}) f_{\chi-\rho_c(\mathfrak{b})}$. Similarly we get $f_Y = \epsilon(\mathfrak{b}') f_{\chi'-\rho_c(\mathfrak{b}')}$ and

$$\Theta_X(f_Y) = [\mathsf{I}_{\mathrm{Dir}}(X), \mathsf{I}_{\mathrm{Dir}}(Y)]_K = \left[\epsilon(\mathfrak{b})\, F_{\chi-\rho(\mathfrak{b})}, \epsilon(\mathfrak{b}')\, F_{\chi'-\rho(\mathfrak{b}')}\right]_K.$$

By Proposition 2.7, we see that this is 0 if $X \neq Y$ and 1 if $X = Y$. This finishes the proof of (5.2) in the case under consideration. We have proved :

Theorem 5.1. *Let X and Y be finite length Harish-Chandra modules with infinitesimal character in $\mathscr{M}$ and let f_X, f_Y be the Euler-Poincaré functions for X and Y respectively, constructed in section 4. Then, the orbital integral $\Phi(f_X, x)$ at a regular element x of G is 0 if x is not elliptic, and equals $\theta_X(x^{-1})$ if x is elliptic. Furthermore :*

$$\langle X, Y\rangle_{\mathrm{Dir}} = \Theta_X(f_Y) = \Theta_Y(f_X) = \int_{C^{\mathrm{ell}}} \theta_Y(c)\, \theta_X(c^{-1})\, dc = \langle X, Y\rangle_{\mathrm{ell}}.$$

6. About Conjecture 3.1

As explained in the introduction, the goal of this section is to provide evidence for Conjecture 3.1, as well as a possible direction of investigation for a proof. In what follows, X and Y are finite length Harish-Chandra modules with infinitesimal characters. We may assume that the infinitesimal characters of X and Y are the same, since otherwise, both side of the identity we aim to prove are 0.

Let $\mathscr{C} = \mathrm{Hom}_{\widetilde{K}}(X\otimes S, Y\otimes S)$. Then $\mathscr{C} = \mathscr{C}^{\bar{0}} \oplus \mathscr{C}^{\bar{1}}$, with

$$\mathscr{C}^{\bar{0}} = \mathrm{Hom}_{\widetilde{K}}(X\otimes S^+, Y\otimes S^+) \oplus \mathrm{Hom}_{\widetilde{K}}(X\otimes S^-, Y\otimes S^-),$$

$$\mathscr{C}^{\bar{1}} = \mathrm{Hom}_{\widetilde{K}}(X\otimes S^+, Y\otimes S^-) \oplus \mathrm{Hom}_{\widetilde{K}}(X\otimes S^-, Y\otimes S^+).$$

Let us consider the following various actions of the Dirac operator :

$$D_X^{+-} : X\otimes S^+ \longrightarrow X\otimes S^-, \qquad D_X^{-+} : X\otimes S^- \longrightarrow X\otimes S^+,$$

$$D_Y^{+-} : Y\otimes S^+ \longrightarrow Y\otimes S^-, \qquad D_Y^{-+} : Y\otimes S^- \longrightarrow Y\otimes S^+,$$

For $\phi^{++} \in \mathrm{Hom}_{\widetilde{K}}(X\otimes S^+, Y\otimes S^+)$, and $\phi^{--} \in \mathrm{Hom}_{\widetilde{K}}(X\otimes S^-, Y\otimes S^-)$ set

$$S\phi^{++} = -\phi^{++}\circ D_X^{-+} + D_Y^{+-}\circ \phi^{++}.$$

$$S\phi^{--} = -\phi^{--}\circ D_X^{+-} - D_Y^{-+}\circ \phi^{--}.$$

This defines a linear map $S : \mathscr{C}^{\bar{0}} \to \mathscr{C}^{\bar{1}}$.

For $\psi^{+-} \in \mathrm{Hom}_{\widetilde{K}}(X \otimes S^+, Y \otimes S^-)$, and $\psi^{-+} \in \mathrm{Hom}_{\widetilde{K}}(X \otimes S^-, Y \otimes S^+)$, set

$$T\psi^{+-} = -\psi^{+-} \circ D_X^{-+} + D_Y^{-+} \circ \psi^{+-}.$$

$$T\psi^{-+} = -\psi^{-+} \circ D_X^{+-} - D_Y^{+-} \circ \psi^{-+}.$$

This defines a linear map $T : \mathscr{C}^{\bar{1}} \to \mathscr{C}^{\bar{0}}$.

Let us take $\phi^{++} + \phi^{--}$ in $\ker S$. Thus

$$\phi^{++} \circ D_X^{-+} + D_Y^{-+} \circ \phi^{--} = 0 \text{ and } -\phi^{--} \circ D_X^{+-} + D_Y^{+-} \circ \phi^{++} = 0. \tag{6.1}$$

From this we see that

$$\phi^{++}(\ker(D_X^{+-})) \subset \ker(D_Y^{+-}) \text{ and } \phi^{--}(\ker(D_X^{-+})) \subset \ker(D_Y^{-+}),$$

and also that

$$\phi^{++}(\mathrm{Im}\,(D_X^{-+})) \subset \mathrm{Im}\,(D_Y^{-+}) \text{ and } \phi^{--}(\mathrm{Im}\,(D_X^{+-})) \subset \mathrm{Im}\,(D_Y^{+-}).$$

Therefore ϕ^{++} induces

$$\bar{\phi}^{++} : \mathsf{H}_{\mathrm{Dir}}(X)^+ = \frac{\ker(D_X^{+-})}{\ker(D_X^{+-}) \cap \mathrm{Im}\,(D_X^{-+})} \longrightarrow \mathsf{H}_{\mathrm{Dir}}(Y)^+ = \frac{\ker(D_Y^{+-})}{\ker(D_Y^{+-}) \cap \mathrm{Im}\,(D_Y^{-+})},$$

and ϕ^{--} induces

$$\bar{\phi}^{--} : \mathsf{H}_{\mathrm{Dir}}(X)^- = \frac{\ker(D_X^{-+})}{\ker(D_X^{-+}) \cap \mathrm{Im}\,(D_X^{+-})} \longrightarrow \mathsf{H}_{\mathrm{Dir}}(Y)^- = \frac{\ker(D_Y^{-+})}{\ker(D_Y^{-+}) \cap \mathrm{Im}\,(D_Y^{+-})}.$$

Let us now show that if $\phi^{++} + \phi^{--} \in \ker S \cap \mathrm{Im}\, T$, then $(\bar{\phi}^{++}, \bar{\phi}^{--}) = (0, 0)$. Write

$$\phi^{++} = -\psi^{-+} \circ D_X^{+-} + D_Y^{-+} \circ \psi^{+-}, \quad \phi^{--} = -\psi^{+-} \circ D_X^{-+} - D_Y^{+-} \circ \psi^{-+}.$$

This implies that

$$\phi^{++}(\ker(D_X^{+-})) \subset \mathrm{Im}\,(D_Y^{-+}) \cap \ker(D_Y^{-+}),\ \phi^{--}(\ker(D_X^{-+})) \subset \mathrm{Im}\,(D_Y^{+-}) \cap \ker(D_Y^{+-}), \tag{6.2}$$

which proves the assertion. Therefore, we have shown that there is a well-defined morphism $\phi^{++} + \phi^{--} \mapsto \bar{\phi}^{++} + \bar{\phi}^{--}$ from $\dfrac{\ker S}{\ker S \cap \mathrm{Im}\, T}$ to

$$\mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^+, \mathsf{H}_{\mathrm{Dir}}(Y)^+\right) \oplus \mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^-, \mathsf{H}_{\mathrm{Dir}}(Y)^-\right).$$

Similarly, there is a well-defined morphism $\psi^{+-} + \psi^{-+} \mapsto \bar{\psi}^{+-} + \bar{\psi}^{-+}$ from $\dfrac{\ker T}{\ker T \cap \mathrm{Im}\, S}$ to

$$\mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^+, \mathsf{H}_{\mathrm{Dir}}(Y)^-\right) \oplus \mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^-, \mathsf{H}_{\mathrm{Dir}}(Y)^+\right).$$

Lemma 6.1. *The two morphisms defined above are isomophisms :*

$$\frac{\ker S}{\ker S \cap \mathrm{Im}\, T} \simeq \mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^+, \mathsf{H}_{\mathrm{Dir}}(Y)^+\right) \oplus \mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^-, \mathsf{H}_{\mathrm{Dir}}(Y)^-\right),$$

$$\frac{\ker T}{\ker T \cap \mathrm{Im}\, S} \simeq \mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^+, \mathsf{H}_{\mathrm{Dir}}(Y)^-\right) \oplus \mathrm{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^-, \mathsf{H}_{\mathrm{Dir}}(Y)^+\right).$$

Proof. Let us compute TS. If $\phi^{++} \in \mathrm{Hom}_{\widetilde{K}}(X \otimes S^+, Y \otimes S^+)$, we get

$$\begin{aligned} TS\phi^{++} &= T(-\phi^{++} \circ D_X^{-+} + D_Y^{+-} \circ \phi^{++}) \\ &= \phi^{++} \circ D_X^{-+} \circ D_X^{+-} + D_Y^{+-} \circ \phi^{++} \circ D_X^{-+} - D_Y^{+-} \circ \phi^{++} \circ D_X^{-+} + D_Y^{-+} \circ D_Y^{+-} \circ \phi^{++} \\ &= \phi^{++} \circ D_X^{-+} \circ D_X^{-+} + D_Y^{-+} \circ D_Y^{+-} \circ \phi^{++}, \end{aligned}$$

and if $\phi^{--} \in \mathrm{Hom}_{\widetilde{K}}(X \otimes S^-, Y \otimes S^-)$,

$$\begin{aligned} TS\phi^{--} &= T(-\phi^{--} \circ D_X^{+-} - D_Y^{-+} \circ \phi^{--}) \\ &= \phi^{--} \circ D_X^{+-} \circ D_X^{-+} - D_Y^{-+} \circ \phi^{--} \circ D_X^{+-} + D_Y^{-+} \circ \phi^{--} \circ D_X^{+-} + D_Y^{+-} \circ D_Y^{-+} \circ \phi^{--} \\ &= \phi^{--} \circ D_X^{+-} \circ D_X^{+-} + D_Y^{+-} \circ D_Y^{-+} \circ \phi^{--}. \end{aligned}$$

Similar computations for ST show that in fact, for any $\phi \in \mathscr{C}^{\bar{0}}$, and any $\psi \in \mathscr{C}^{\bar{1}}$

$$TS\phi = D^2 \circ \phi + \phi \circ D^2, \quad ST\psi = D^2 \circ \psi + \psi \circ D^2 \tag{6.3}$$

Let us now use the fact that

$$\mathscr{C} = \mathrm{Hom}_{\widetilde{K}}(X \otimes S, Y \otimes S) = \prod_{\gamma \in (\widetilde{K})\hat{}} \mathrm{Hom}_{\widetilde{K}}((X \otimes S)(\gamma), (Y \otimes S)(\gamma)).$$

Since X and Y have same infinitesimal character, by Equation (2.6), D^2 acts on $(X \otimes S)(\gamma)$ and $(Y \otimes S)(\gamma)$ by the same scalar, let us call it α_γ for short, so we see that on each $\mathbb{Z}/2\mathbb{Z}$-graded component of $\mathrm{Hom}_{\widetilde{K}}((X \otimes S)(\gamma), (Y \otimes S)(\gamma))$, TS or ST (depending on the component) is multiplication by α_γ^2. From this, we deduce two facts that are worth recording.

Lemma 6.2. $\ker(TS)$ *(resp.* $\ker ST$*) is finite dimensional, and*

$$ker(TS) \oplus \mathrm{Im}\,(TS) = \mathscr{C}^{\bar{0}}, \quad \ker(ST) \oplus \mathrm{Im}\,(ST) = \mathscr{C}^{\bar{1}}.$$

Indeed, we see that $\ker(TS)$ (or ST) is

$$\prod_{\gamma \in (\widetilde{K})\hat{}\ |\ (X \otimes S)(\gamma) \subset \ker D^2} \mathrm{Hom}_{\widetilde{K}}((X \otimes S)(\gamma), (Y \otimes S)(\gamma)).$$

Notice that $\mathscr{C}^{\bar{0}}$ may not be written as a direct sum of eigenspaces for TS, but is isomorphic to a direct product for which each factor is an eigenspace for TS.

Let us show now that the morphisms in Lemma 6.1 are injective and surjective. We will do it only for the first one, so suppose that $\phi = \phi^{++} + \phi^{--} \in \ker S$ is such that $(\bar{\phi}^{++}, \bar{\phi}^{--}) = (0, 0)$. This is equivalent to (6.2). We want to find $\psi = \psi^{+-} + \psi^{-+}$ such that $\phi = T\psi$, *i.e.*

$$\phi^{++} = -\psi^{-+} \circ D_X^{+-} + D_Y^{-+} \circ \psi^{+-}, \quad \phi^{--} = -\psi^{+-} \circ D_X^{-+} - D_Y^{+-} \circ \psi^{-+}. \tag{6.4}$$

Equations (6.2) imply that there exists ψ^{+-} defined on $\ker(D_X^{+-})$ and ψ^{-+} defined on $\ker(D_X^{+-})$ such that $\phi^{++} = D_Y^{-+} \circ \psi^{+-}$ and $\phi^{--} = -D_Y^{+-} \circ \psi^{-+}$. Therefore, equations (6.4) are satisfied on $\ker(D_X^{+-})$ and $\ker(D_X^{+-})$ respectively.

Remark 6.3. Notice that any choice of ψ^{+-} on $\ker(D_X^{+-})$ and ψ^{-+} on $\ker(D_X^{+-})$ satisfying $\phi^{++} = D_Y^{-+} \circ \psi^{+-}$ and $\phi^{--} = -D_Y^{+-} \circ \psi^{-+}$ can be modified by adding to ψ^{+-} any morphism from $\ker(D_X^{+-})$ to $\ker(D_Y^{-+})$ and to ψ^{-+} any morphism from $\ker(D_X^{-+})$ to $\ker(D_Y^{+-})$.

The problem is to extend ψ^{+-} and ψ^{-+} to $X\otimes S^+$ and $X\otimes S^-$ respectively. Since $\ker(S)\subset\ker(TS)$, we see from the description of $\ker(TS)$ given above that ϕ is 0 on the $\widetilde{K}$-isotypic components which are not in the kernel of D^2. Therefore, we may set ψ^{+-} and ψ^{-+} to be 0 on these components, and (6.4) will be satisfied on them. So it remains to define ψ^{+-} (resp. ψ^{-+}) on some complement W^+ of $\ker(D_X^{+-})$ in $\ker(D_X^{-+}\circ D_X^{+-})$ (resp. some complement W^- of $\ker(D_X^{-+})$ in $\ker(D_X^{+-}\circ D_X^{-+})$).

Let $x\in W^+$. Then $D_X^{+-}(x)\in\ker D_X^{-+}$, so $\psi^{-+}(D_X^{+-}(x))$ is defined, and

$$\phi^{--}(D_X^{+-}(x))=-D_Y^{-+}(\psi^{-+}(D_X^{+-}(x))).$$

But since $\phi\in\ker S$, Equations (6.1) hold, and thus

$$\phi^{--}(D_X^{+-}(x))=D_Y^{+-}(\phi^{++}(x))=-D_Y^{-+}(\psi^{-+}(D_X^{+-}(x))),$$

and $\phi^{++}(x)+\psi^{-+}(D_X^{+-}(x))\in\ker(D_Y^{-+})$. Notice that D_X^{+-} induces an isomorphism from W^- to $\ker(D_X^{-+})\cap\operatorname{Im}(D_X^{+-})$. But by the remark above, we may assume that in fact $\phi^{++}+\psi^{-+}\circ D_X^{+-}$ is 0 on W^+. Setting ψ^{+-} to be 0 on W^+, we see that the first equation of (6.4) is satisfied on W^+. Similarly, we extend ψ^{-+}. This finishes the proof of the injectivity of the first morphism in Lemma 6.1.

We now prove the surjectivity. Let :

$$\alpha:\ \frac{\ker(D_X^{+-})}{\ker(D_X^{+-})\cap\operatorname{Im}(D_X^{-+})}\longrightarrow\frac{\ker(D_Y^{+-})}{\ker(D_Y^{+-})\cap\operatorname{Im}(D_Y^{-+})},$$

and

$$\beta:\ \frac{\ker(D_X^{-+})}{\ker(D_X^{-+})\cap\operatorname{Im}(D_X^{+-})}\longrightarrow\frac{\ker(D_Y^{-+})}{\ker(D_Y^{-+})\cap\operatorname{Im}(D_Y^{+-})}.$$

We would like to find $\phi=\phi^{++}+\phi^{--}$ such that $(\bar{\phi}^{++},\bar{\phi}^{--})=(\alpha,\beta)$. First, lift α to a morphism $\tilde{\alpha}:\ \ker(D_X^{+-})\to\ker(D_Y^{-+})$ which is identically 0 on $\ker(D_X^{+-})\cap\operatorname{Im}(D_X^{-+})$ and set $\phi^{++}=\tilde{\alpha}$ on $\ker(D_X^{+-})$. Since ϕ should be in $\ker S$, we set ϕ to be 0 on all $\widetilde{K}$-isotypic components not in the kernel of D^2. Let W^+, W^- be as above. Then, it remains to define ϕ^{++} on W^+ and we do this by setting $\phi^{++}=0$ on W^+. Similarly, extend ϕ^{--}. It is then immediately clear that Equations (6.1) are satisfied, so we have constructed ϕ as we wished. This finishes the proof of Lemma 6.1. $\square$

Let us introduce now the index $\operatorname{ind}(S,T)$ of the Fredholm pair (S,T). The material about Fredholm pairs and their indices is exposed in the appendix, and by definition

$$\operatorname{ind}(S,T)=\dim\left(\frac{\ker S}{\ker S\cap\operatorname{Im}T}\right)-\dim\left(\frac{\ker T}{\ker T\cap\operatorname{Im}S}\right).$$

Corollary 6.4.

$$\operatorname{ind}(S,T)=[\mathsf{I}_{\mathrm{Dir}}(X),\mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}}$$

Proof.

$$\begin{aligned}\operatorname{ind}(S,T)&=\dim\left(\frac{\ker S}{\ker S\cap\operatorname{Im}T}\right)-\dim\left(\frac{\ker T}{\ker T\cap\operatorname{Im}S}\right)\\&=\dim\left(\operatorname{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^+,\mathsf{H}_{\mathrm{Dir}}(Y)^+\right)\right)+\dim\left(\operatorname{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^-,\mathsf{H}_{\mathrm{Dir}}(Y)^-\right)\right)\\&\quad-\dim\left(\operatorname{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^+,\mathsf{H}_{\mathrm{Dir}}(Y)^-\right)\right)-\dim\left(\operatorname{Hom}_{\widetilde{K}}\left(\mathsf{H}_{\mathrm{Dir}}(X)^-,\mathsf{H}_{\mathrm{Dir}}(Y)^+\right)\right).\\&=[\mathsf{I}_{\mathrm{Dir}}(X),\mathsf{I}_{\mathrm{Dir}}(Y)]_{\widetilde{K}}\end{aligned}$$

Conjecture 3.1 is then equivalent to the fact that $\mathrm{ind}(S,T)$ is equal to the Euler-Poincaré characteristic of the complex computing the Ext groups of X and Y, namely $\mathbf{C} = \sum_{i\in\mathbb{N}} C^i$, where $C^i = \mathrm{Hom}_K(\bigwedge^i \mathfrak{p} \otimes X, Y)$. By example 7.3, this Euler-Poincaré characteristic is also the index of the Fredholm pair given by the differentials between the even and odd part of the complex. More precisely, with

$$d^+ : \mathbf{C}^{\bar{0}} = \bigoplus_{i\in\mathbb{N}} C^{2i} \longrightarrow \mathbf{C}^{\bar{1}} = \bigoplus_{i\in\mathbb{N}} C^{2i+1},$$

and $d^- : \mathbf{C}^{\bar{1}} \to \mathbf{C}^{\bar{0}}$, we have $\mathbf{EP}(X,Y) = \mathrm{ind}(d^+, d^-)$. So we would like to show that $\mathrm{ind}(S,T) = \mathrm{ind}(d^+,d^-)$. To facilitate the comparison, first notice that

$$\mathbf{C} = \bigoplus_{i\in\mathbb{N}} C^i = \bigoplus_{i\in\mathbb{N}} \mathrm{Hom}_K(\textstyle\bigwedge^i \mathfrak{p}\otimes X, Y) \simeq \mathrm{Hom}_K(\bigwedge \mathfrak{p}\otimes X, Y) \simeq \mathrm{Hom}_{\widetilde{K}}(X\otimes S, Y\otimes S) = \mathscr{C}$$

Then transport (S,T) to $(\mathscr{D}^+, \mathscr{D}^-)$ via this isomorphism. Thus

$$\mathscr{D}^+ : \mathbf{C}^{\bar{0}} \longrightarrow \mathbf{C}^{\bar{1}}, \quad \mathscr{D}^- : \mathbf{C}^{\bar{1}} \longrightarrow \mathbf{C}^{\bar{0}},$$

and set $\mathscr{D} = \mathscr{D}^+ \oplus \mathscr{D}^-$, an operator on $\mathbf{C}$. The conjecture is now that

$$\mathrm{ind}(d^+, d^-) = \mathrm{ind}(\mathscr{D}^+, \mathscr{D}^-). \tag{6.5}$$

We give now explicit formulas for d and $\mathscr{D}$. The differential d on $\mathbf{C}$ is the sum of the $d^i : C^i \to C^{i+1}$

$$d^i\phi(X_0 \wedge \ldots \wedge X_i \otimes x) = \tag{6.6}$$
$$\sum_{j=0}^{i} (-1)^j \big(X_j \cdot \phi(X_0 \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_i \otimes x) - \phi(X_0 \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_i \otimes X_j \cdot x) \big)$$

Recall that $\mathfrak{p} = U \oplus U^*$, where U and U^* are maximal isotropic subspaces for the bilinear form B, and U^*, as the notation suggests, is identified with the dual of U via B. We have realized the Clifford module S as $\bigwedge U$, and thus S^* as $\bigwedge U^*$. Let us denote respectively γ and γ^* the Clifford actions of $\mathrm{Cl}(\mathfrak{p})$ on S and S^*. Explicitely, for $u \in U$, $u^* \in U^*$, $\lambda_1 \wedge \ldots \wedge \lambda_r \in \bigwedge^r U \subset S$, and $\mu_1 \wedge \ldots \wedge \mu_s \in \bigwedge^s U^* \subset S^*$,

$$\gamma(u)(\lambda_1 \wedge \ldots \wedge \lambda_r) = u \wedge \lambda_1 \wedge \ldots \wedge \lambda_r \tag{6.7}$$

$$\gamma(u^*)(\lambda_1 \wedge \ldots \wedge \lambda_r) = 2\sum_{j=1}^{r} (-1)^j B(u^*, \lambda_j)\, \lambda_1 \wedge \ldots \wedge \hat{\lambda}_j \wedge \ldots \wedge \lambda_r \tag{6.8}$$

$$\gamma^*(u^*)(\mu_1 \wedge \ldots \wedge \mu_s) = u^* \wedge \mu_1 \wedge \ldots \wedge \mu_s \tag{6.9}$$

$$\gamma^*(u)(\lambda_1 \wedge \ldots \wedge \lambda_r) = 2\sum_{j=1}^{r} (-1)^j B(u, \mu_j)\, \mu_1 \wedge \ldots \wedge \hat{\mu}_j \wedge \ldots \wedge \mu_s \tag{6.10}$$

Fix a basis $u_1, \ldots, u_m$ of U, with dual basis $u_1^*, \ldots, u_m^*$. Decompose the Dirac operator as

$$D = A + B \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}), \quad A = \sum_{i=1}^{m} u_i \otimes u_i^*, \quad B = \sum_{i=1}^{m} u_i^* \otimes u_i$$

Then A acts on $\mathbf{C} = \mathrm{Hom}_K(\bigwedge \mathfrak{p} \otimes X, Y) \simeq \mathrm{Hom}_K(S \otimes S^* \otimes X, Y)$ as follows : for $\lambda_1 \wedge \ldots \wedge \lambda_r \in \bigwedge^r U \subset S$, $\mu_1 \wedge \ldots \wedge \mu_s \in \bigwedge^s U^* \subset S^*$, $x \in X$ and $\phi \in \mathrm{Hom}_K(S \otimes S^* \otimes X, Y)$,

$$
\begin{aligned}
(6.11) \qquad & (A \cdot \phi)(\lambda_1 \wedge \ldots \wedge \lambda_r \wedge \mu_1 \wedge \ldots \wedge \mu_s \otimes x) \\
&= \sum_{i=1}^{m} \big[u_i \cdot \phi(\big(\gamma(u_i^*)(\lambda_1 \wedge \ldots \wedge \lambda_r)\big) \wedge (\mu_1 \wedge \ldots \wedge \mu_s) \otimes x) \\
&\qquad + (-1)^r \, u_i \cdot \phi((\lambda_1 \wedge \ldots \wedge \lambda_r) \wedge (\gamma^*(u_i^*)(\mu_1 \wedge \ldots \wedge \mu_s)) \otimes x) \\
&\qquad - \phi((\gamma(u_i^*)(\lambda_1 \wedge \ldots \wedge \lambda_r)) \wedge (\mu_1 \wedge \ldots \wedge \mu_s) \otimes u_i \cdot x) \\
&\qquad - (-1)^r \, \phi((\lambda_1 \wedge \ldots \wedge \lambda_r) \wedge (\gamma^*(u_i^*)(\mu_1 \wedge \ldots \wedge \mu_s)) \otimes u_i \cdot x) \big] \\
&= 2 \sum_{i=1}^{m} \sum_{j=1}^{r} (-1)^j \, B(u_i^*, \lambda_j) \, u_i \cdot \phi(\lambda_1 \wedge \ldots \wedge \hat{\lambda}_j \wedge \ldots \wedge \lambda_r \wedge (\mu_1 \wedge \ldots \wedge \mu_s) \otimes x) \\
&\quad + \sum_{i=1}^{m} (-1)^r \, u_i \cdot \phi((\lambda_1 \wedge \ldots \wedge \lambda_r) \wedge (u_i^* \wedge \mu_1 \wedge \ldots \wedge \mu_s) \otimes x) \\
&\quad -2 \sum_{i=1}^{m} \sum_{j=1}^{r} (-1)^j \, B(u_i^*, \lambda_j) \, \phi(\lambda_1 \wedge \ldots \wedge \hat{\lambda}_j \wedge \ldots \wedge \lambda_r \wedge (\mu_1 \wedge \ldots \wedge \mu_s) \otimes u_i \cdot x) \\
&\quad - \sum_{i=1}^{m} (-1)^r \, \phi((\lambda_1 \wedge \ldots \wedge \lambda_r) \wedge (u_i^* \wedge \mu_1 \wedge \ldots \wedge \mu_s) \otimes u_i \cdot x) \\
&= 2 \sum_{j=1}^{r} (-1)^j \; \lambda_j \cdot \phi(\lambda_1 \wedge \ldots \wedge \hat{\lambda}_j \wedge \ldots \wedge \lambda_r \wedge (\mu_1 \wedge \ldots \wedge \mu_s) \otimes x) \\
&\quad + \sum_{i=1}^{m} (-1)^r \, u_i \cdot \phi((\lambda_1 \wedge \ldots \wedge \lambda_r) \wedge (u_i^* \wedge \mu_1 \wedge \ldots \wedge \mu_s) \otimes x) \\
&\quad -2 \sum_{j=1}^{r} (-1)^j \, \phi(\lambda_1 \wedge \ldots \wedge \hat{\lambda}_j \wedge \ldots \wedge \lambda_r \wedge (\mu_1 \wedge \ldots \wedge \mu_s) \otimes \lambda_j \cdot x) \\
&\quad - \sum_{i=1}^{m} (-1)^r \, \phi((\lambda_1 \wedge \ldots \wedge \lambda_r) \wedge (u_i^* \wedge \mu_1 \wedge \ldots \wedge \mu_s) \otimes u_i \cdot x).
\end{aligned}
$$

In this computation, notice that when we identify $\bigwedge \mathfrak{p}$ with $S \otimes S^*$, the latter is endowed with the super tensor product, and accordingly for the Clifford action. This explain the $(-1)^r$ factors in the formulas above. Similarly, the action of B is given by

$$(6.12) \qquad (B\cdot\phi)(\lambda_1\wedge\ldots\wedge\lambda_r\wedge\mu_1\wedge\ldots\wedge\mu_s\otimes x)$$

$$\begin{aligned}
&= \sum_{i=1}^{m}\big[u_i^*\cdot\phi\big(\big(\gamma(u_i)(\lambda_1\wedge\ldots\wedge\lambda_r)\big)\wedge(\mu_1\wedge\ldots\wedge\mu_s)\otimes x\big)\\
&\qquad + (-1)^r\ u_i^*\cdot\phi((\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\gamma^*(u_i)(\mu_1\wedge\ldots\wedge\mu_s))\otimes x)\\
&\qquad - \phi((\gamma(u_i)(\lambda_1\wedge\ldots\wedge\lambda_r))\wedge(\mu_1\wedge\ldots\wedge\mu_s)\otimes u_i^*\cdot x)\\
&\qquad - (-1)^r\ \phi((\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\gamma(u_i)(\mu_1\wedge\ldots\wedge\mu_s))\otimes u_i^*\cdot x)\big]\\
&= \sum_{i=1}^{m}u_i^*\cdot\phi((u_i\wedge\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\mu_s)\otimes x)\\
&+2(-1)^r\sum_{i=1}^{m}\sum_{j=1}^{s}(-1)^j\ B(u_i,\mu_j)\ u_i^*\cdot\phi((\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\hat\mu_j\wedge\ldots\wedge\mu_s)\otimes x)\\
&- \sum_{i=1}^{m}\phi((u_i\wedge\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\mu_s)\otimes u_i^*\cdot x)\\
&-2(-1)^r\sum_{i=1}^{m}\sum_{j=1}^{s}(-1)^j\ B(u_i,\mu_j)\ \phi((\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\hat\mu_j\wedge\ldots\wedge\mu_s)\otimes u_i^*\cdot x)\\
&= \sum_{i=1}^{m}u_i^*\cdot\phi((u_i\wedge\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\mu_s)\otimes x)\\
&+2(-1)^r\sum_{j=1}^{s}(-1)^j\ \mu_j^*\cdot\phi((\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\hat\mu_j\wedge\ldots\wedge\mu_s)\otimes x)\\
&- \sum_{i=1}^{m}\phi((u_i\wedge\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\mu_s)\otimes u_i^*\cdot x)\\
&-2(-1)^r\sum_{j=1}^{s}(-1)^j\ \phi((\lambda_1\wedge\ldots\wedge\lambda_r)\wedge(\mu_1\wedge\ldots\wedge\hat\mu_j\wedge\ldots\wedge\mu_s)\otimes \mu_j^*\cdot x).
\end{aligned}$$

The action of D is the sum of the actions of A and B. Consider the four operators defined by the last four lines of (6.11) and denote them respectively by

$$2d_1 = \mathscr{E}_1,\quad \delta_1 = \mathscr{D}_1,\quad 2d_2 = \mathscr{D}_2,\quad \delta_2 = \mathscr{E}_2$$

and similarly for the four operators defined by the last four lines of (6.12)

$$\delta_3 = \mathscr{E}_3,\quad 2d_3 = \mathscr{D}_3,\quad \delta_4 = \mathscr{D}_4,\quad 2d_4 = \mathscr{E}_4.$$

Then we have

$$d = d_1 + d_2 + d_3 + d_4,\qquad \mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3 + \mathscr{D}_4.$$

(actually, the operators S and T of $\mathscr{C}$ at the begining of this section where defined to make this true).

We get easily now that $\partial = \partial_1 + \partial_2 + \partial_3 + \partial_4$ satisfies $\partial^2 = 0$. It is likely that the cohomology of ∂ in degree i comptutes $\mathrm{Ext}^i(Y\check{}, X\check{})$ $Y\check{}$ and $X\check{}$ being the contragredient Harish-Chandra modules of Y and X respectively, but we haven't checked this. The operator $\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3 + \mathscr{E}_4$ is very similar to $\mathscr{D}$. It is obtained as $\mathscr{D}$ from the transport of (T,S), but using a different isomorphism

$$\mathbf{C} = \mathrm{Hom}_K(\bigwedge \mathfrak{p} \otimes X, Y) \simeq \mathrm{Hom}_{\widetilde{K}}(X \otimes S, Y \otimes S) = \mathscr{C}$$

obrtained by exchanging the role of the two copies of S after identifying it with its dual. Thus d and $\mathscr{D}$ are related by

$$2d + \delta = \mathscr{D} + \mathscr{E}. \tag{6.13}$$

A possible way to prove (6.5) would be to show that both indices are equal to the index of the operator (6.13).

7. Appendix : Fredholm pairs

In this section, we adapt from [1] the definition of the index of a Fredholm pair, as well as some properties of this invariant which enable one to calculate it in pratical applications. We do this in a purely algebraic setting, while the theory is developed for Banach spaces in [1].

Definition 7.1. Let X, Y be complex vector spaces and let $S \in \mathcal{L}(X;Y)$, $T \in \mathcal{L}(Y;X)$. Then (S,T) is called a Fredholm pair if the following dimensions are finite :

$$a := \dim \ker(S)/\ker(S) \cap \mathrm{Im}\,(T); \quad b := \dim \ker(T)/\ker(T) \cap \mathrm{Im}\,(S).$$

In this case, the number

$$\mathrm{ind}(S,T) := a - b$$

is called the index of (S,T).

Example 7.2. Take $T = 0$. Then $(S,0)$ is a Fredholm pair if and only if S is a Fredholm operator and

$$\mathrm{ind}(S,0) = \mathrm{ind}(S) = \dim \ker(S) - \dim \mathrm{coker}(S)$$

Example 7.3. Consider a differential complex

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots$$

Suppose the cohomology groups $H^i := \ker d^i / \mathrm{Im}\, d^{i-1}$ of this complex are finite dimensional, and non zero only for a finite number of them. Put $X = \bigoplus_{i\in\mathbb{Z}} C^{2i}$, $Y = \bigoplus_{i\in\mathbb{Z}} C^{2i+1}$, $S = \oplus_{i\in 2\mathbb{Z}} d^{2i}$, $T = \oplus_{i\in 2\mathbb{Z}} d^{2i+1}$. Then (S,T) is a Fredholm pair and its index is equal to the Euler-Poincaré characteristic of the complex :

$$\mathrm{ind}(S,T) = \sum_{i\in\mathbb{Z}} (-1)^i \dim H^i.$$

With the notation of the definition, notice that T induces an isomorphism

$$\widetilde{T} : \frac{\mathrm{Im}\,(S)}{\mathrm{Im}\,(S) \cap \ker(T)} \longrightarrow \mathrm{Im}\,(TS)$$

and that S induces an isomorphism

$$\widetilde{S} : \frac{\mathrm{Im}\,(T)}{\mathrm{Im}\,(T) \cap \ker(S)} \longrightarrow \mathrm{Im}\,(ST).$$

Since $S(\mathrm{Im}\,(TS)) = \mathrm{Im}\,(STS) \subset \mathrm{Im}\,(ST)$, we see also that S induces a morphism

$$\bar{S} : X/\mathrm{Im}\,(TS) \longrightarrow Y/\mathrm{Im}\,(ST).$$

Similarly T induces a morphism

$$\bar{T} : Y/\mathrm{Im}\,(ST) \longrightarrow X/\mathrm{Im}\,(TS).$$

Set $\bar{X} = X/\mathrm{Im}\,(TS)$, $\bar{Y} = Y/\mathrm{Im}\,(ST)$.

Lemma 7.4. *Suppose that (S, T) is a Fredholm pair. Then $(\bar{S}, \bar{T})$ is also a Fredholm pair and $\mathrm{ind}(S, T) = \mathrm{ind}(\bar{S}, \bar{T})$. Suppose furthermore than $\bar{X}$ and $\bar{Y}$ are finite dimensional, then $\mathrm{ind}(\bar{S}, \bar{T}) = \dim \bar{X} - \dim(\bar{Y})$.*

We leave the easy proof to the reader. Notice that $\bar{T}\bar{S} = 0$ and that $\bar{S}\bar{T} = 0$, so that $\mathrm{Im}\,(\bar{S}) \subset \ker(\bar{T})$, $\mathrm{Im}\,(\bar{T}) \subset \ker(\bar{S})$.

Lemma 7.5. *Suppose (S_j, T_j) are pairs of linear maps between spaces X_j and Y_j with $S_jT_j = 0$, $T_jS_j = 0$, $j = 1, 2, 3$. Suppose that the following diagram is commutative, with vertical exact sequences :*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
X_1 \arrow[r, "S_1"] \arrow[d, "\alpha"] & Y_1 \arrow[r, "T_1"] \arrow[d, "\gamma"] & X_1 \arrow[d, "\alpha"] \\
X_2 \arrow[r, "S_2"] \arrow[d, "\beta"] & Y_2 \arrow[r, "T_2"] \arrow[d, "\delta"] & X_2 \arrow[d, "\beta"] \\
X_3 \arrow[r, "S_3"] \arrow[d] & Y_3 \arrow[r, "T_3"] \arrow[d] & X_3 \arrow[d] \\
0 & 0 & 0
\end{tikzcd}.$$

Then if two of the pairs $(S_j; T_j)$ are Fredholm, the third is, and

$$\mathrm{ind}(S_1, T_1) - \mathrm{ind}(S_2, T_2) + \mathrm{ind}(S_3, T_3) = 0.$$

This is [1], Lemma 2.2.

Consider now the following situation. Suppose $V = V^{\bar{0}} \oplus V^{\bar{1}}$ is a super vector space. Suppose that d, ∂ are odd operators on V, satisfying $d^2 = \partial^2 = 0$. Set $d^+, \partial^+ : V^{\bar{0}} \to V^{\bar{1}}$ and $d^-, \partial^- : V^{\bar{1}} \to V^{\bar{0}}$ for the restriction of d, ∂ to the even and odd part respectively. Suppose that the cohomology groups for d, $H_d^+ = \ker d^+/\mathrm{Im}\,(d^-)$ and $H_d^- = \ker d^-/\mathrm{Im}\,(d^+)$ are finite dimensional. Then according to example 7.3, $(d^+; d^-)$ is a Fredholm pair and $\mathrm{ind}(d^+, d^-) = \dim H_d^+ - H_d^-$.

Set $\mathscr{F} = d + \partial$. This is an odd operator on V and similarly to d and ∂, we denote by $\mathscr{F}^+$ and $\mathscr{F}^-$ its restriction to the even and odd part of V respectively. The following result could be useful in proving (6.5) via (6.13).

Proposition 7.6. *Suppose $\mathscr{F}$ has the following properties : $\ker(\mathscr{F}^2)$ is finite dimensional and $\ker(\mathscr{F}^2) \oplus \mathrm{Im}\,(\mathscr{F}^2) = V$.*

Then $(\mathscr{F}^+, \mathscr{F}^-)$ is a Fredholm pair and $\mathrm{ind}(\mathscr{F}^+, \mathscr{F}^-) = \mathrm{ind}(d^+, d^-)$.

Proof. The first assertion is obvious. From Lemma 7.4, we have furthermore than $\mathrm{ind}(\mathscr{F}^+, \mathscr{F}^-) = \dim((\ker \mathscr{F}^2)^{\bar{0}}) - \dim((\ker \mathscr{F}^2)^{\bar{1}})$, where we have set

$$(\ker \mathscr{F}^2)^{\bar{0}} = \ker(\mathscr{F}^2) \cap V^{\bar{0}} = \ker(\mathscr{F}^-\mathscr{F}^+), \quad (\ker \mathscr{F}^2)^{\bar{1}} = \ker(\mathscr{F}^2) \cap V^{\bar{1}} = \ker(\mathscr{F}^+\mathscr{F}^-).$$

Let us now compute the index of the pair (d^+, d^-). First note that $\mathrm{Im}\,\mathscr{F}^2$ is stable under d. Indeed, if $x = \mathscr{F}^2(y) = (d\partial + \partial d)(y) \in \mathrm{Im}\,(\mathscr{F}^2)$, then $d(x) = d^2\partial(y) + d\partial d(y) = d\partial d(y) = (d\partial + \partial d)(d(y)) = \mathscr{F}^2(dy) \in \mathrm{Im}\,(\mathscr{F}^2)$. Thus d induces operators, that we will still denote by d, on the spaces $\mathrm{Im}\,(\mathscr{F}^2)$ and $V/\mathrm{Im}\,(\mathscr{F}^2)$. These spaces inherits from V

the structure of super vector spaces, and thus operators d^+, d^- are well defined on them with the obvious meaning. Let us apply Lemma 7.5 to the exact sequence

$$0 \longrightarrow \mathrm{Im}\,(\mathscr{F}^2) \longrightarrow V \longrightarrow V/\mathrm{Im}\,(\mathscr{F}^2) \longrightarrow 0.$$

Set $W = \mathrm{Im}\,(\mathscr{F}^2)$, and let us compute the index of (d^+, d^-) acting on $W = W^{\bar{0}} \oplus W^{\bar{1}}$. We claim that :

$$W = \ker \partial + \mathrm{Im}\, d \tag{7.1}$$

Indeed, let $x = \mathscr{F}^2(y) = (d\partial + \partial d)(y) \in \mathrm{Im}\,(\mathscr{F}^2)$. Then $\partial(x) = \partial d \partial(y)$, so $x - d\partial(y) \in \ker \partial$. This proves the claim, since $d\partial(y) \in \mathrm{Im}\, d$.

Now, suppose that $x \in W$ is in $\ker d$. Write $x = z + d(y)$, with $z \in \ker \partial$, as we have just shown possible. Then $0 = d(x) = d(z)$ so $z \in \ker d$. But $\mathscr{F}^2 = d\partial + \partial d$ implies that $\ker d \cap \ker \partial \subset \ker(\mathscr{F}^2)$. But by assumption, $\mathrm{Im}\,(\mathscr{F}^2) \cap \ker(\mathscr{F}^2) = \{0\}$, and thus $z = 0$. This shows that $x = d(y) \in \mathrm{Im}\, d$. Thus, the index of (d^+, d^-) acting on W is 0. The Lemma 7.5 then implies that the index of (d^+, d^-) acting on W is equal to the index of (d^+, d^-) acting on V/W. But by the semisimplicity of $\mathscr{F}^2$, V/W is isomorphic to $\ker(\mathscr{F}^2)$ as super vectors spaces, and therefore finite dimensional. Another application of Lemma 7.4 then tells us that the index of (d^+, d^-) acting on V/W equals $\dim((\ker \mathscr{F}^2)^{\bar{0}}) - \dim((\ker \mathscr{F}^2)^{\bar{1}})$. □

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