DIRAC OPERATORS AND LIE ALGEBRA COHOMOLOGY

JING-SONG HUANG, PAVLE PANDŽIĆ, AND DAVID RENARD

Abstract. Dirac cohomology is a new tool to study unitary and admissible representations of semisimple Lie groups. It was introduced by Vogan and further studied by Kostant and ourselves [V2], [HP1], [K4]. The aim of this paper is to study the Dirac cohomology for the Kostant cubic Dirac operator and its relation to Lie algebra cohomology. We show that the Dirac cohomology coincides with the corresponding nilpotent Lie algebra cohomology in many cases, but in general it has better algebraic behavior and it is more accessible for calculation.

1. Introduction

In 1926 Dirac discovered a matrix valued first-order differential operator as a square root of the Laplacian operator in order to understand elementary particles. Since then this operator and its various analogues are called Dirac operators in the scientific community. Dirac made many astonishing discoveries using this operator, such as half-spin of the electron and the anti-matter positron. These discoveries laid the foundation for molecular physics and molecular chemistry, as well as for the application of nuclear magnetic resonance to medical imaging. The impact of the Dirac operator on the development of mathematics is also significant. The extension of the definition of Dirac operator to a differentiable manifold and a proof of the corresponding index theorem by Atiyah and Singer is one of the greatest achievements of mathematics in the twentieth century.

Kostant's 1962 paper [K1] proving the Bott-Borel-Weil theorem by employing Lie algebra cohomology (with respect to nilpotent Lie algebras) marks a new era of interaction of topology and representation theory. Lie algebra cohomology also plays a significant role in Schmid's proof [S] of the Kostant-Langlands conjecture that discrete series representations can be constructed from the Dolbeault cohomology of L^2-sections of certain holomorphic line bundles. Lie algebra cohomology is a fundamental tool in Vogan's 1976 MIT dissertation, which introduced an algebraic approach to the study of Harish-Chandra modules and revolutionized the field. Furthermore, Lie algebra cohomology is related to asymptotics of matrix coefficients (as shown by Casselman and Miličić), to embeddings into "standard modules" (both in real and in cohomological induction), and to geometric realization of representations via D-modules. Lie algebra cohomology is also the main tool throughout the classic book of Borel and Wallach [BW].

An analogue of the Dirac operator was defined and successfully used for geometric realization of most discrete series representations by Parthasarathy [P]. This approach was

1991 *Mathematics Subject Classification.* 22E47.

Key words and phrases. semisimple Lie group, unitary representation, admissible representation, Dirac operator, Lie algebra cohomology.

The research of the first author was partially supported by RGC-CERG grants of Hong Kong SAR and National Nature Science Foundation of China. The research of the second author was partially supported by a grant from the Ministry of Science and Technology of Republic of Croatia. Parts of this work were done during authors' visits to CNRS, University of Paris VII, Mathematisches Forschungsinstitut Oberwolfach, and Institute of Mathematical Sciences and Department of Mathematics at the National University of Singapore. The authors thank these institutions for their generous support and hospitality.

further developed by Atiyah and Schmid [AS], who constructed all discrete series representations as kernels of the Dirac operator acting on the associated spinor bundles. It has been clear for decades that the Dirac operators are formally similar to the differentials of the de Rham or Dolbeault cohomology. There have been two problems with using that analogy in representation theory. First, the Dirac operator on non-symmetric homogeneous spaces is not as nicely behaved. Second, the index of the Dirac operator behaves well only for some unitary representations, like the discrete series; so the algebraic tools of representation theory do not work well with it. The first problem was resolved by Kostant [K2], who introduced the modified cubic Dirac operator that works well also on non-symmetric homogeneous spaces. The second problem was resolved by Vogan [V2]. Vogan introduced the concept of Dirac cohomology which applies both to unitary and nonunitary representations, and made a conjecture on the infinitesimal character of irreducible representations with nonzero Dirac cohomology. This conjecture was proved in [HP1]. Kostant generalized this result to the case of his cubic Dirac operator and applied it to the topology of homogeneous spaces [K4]. Recently, Kumar [Ku] and Alexeev and Meinreken [AM] found further generalizations of the results in [HP1] and [K4]. They put these results into the broader setting of non-commutative equivariant cohomology.

The aim of this paper is to explore the relation between Dirac cohomology and (nilpotent) Lie algebra cohomology. Our results show that Dirac cohomology coincides with the corresponding Lie algebra cohomology for a large family of unitary representations, including the discrete series representations, and for all unitary representations in certain special cases. On the other hand, Dirac cohomology seems to have a better algebraic behavior.

We have also been able to use Dirac cohomology and the proved Vogan's conjecture to obtain improvements of some classical results. In [HP2] we describe how to simplify certain parts of Atiyah-Schmid's construction of discrete series representations [AS] and sharpen the Langlands' formula on automorphic forms [L], [HoP]. The details will appear in a forthcoming book by the first and second named authors. This suggests that Dirac cohomology may prove useful also in other situations, as a new tool for tackling problems not accessible to classical cohomology theories.

Let us now describe our main results on Dirac cohomology and its relation to (nilpotent) Lie algebra cohomology more precisely. Let G be a connected semisimple Lie group with finite center and complexified Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of G (and $\mathfrak{g}$), let $K = G^\theta$ be the corresponding maximal compact subgroup, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. As usual, the corresponding real forms will be denoted by $\mathfrak{g}_0$, $\mathfrak{k}_0$, etc. (One could also work in a more general setting of a reductive group G in the Harish-Chandra class.)

If G is of hermitian type, then $\mathfrak{p}$ decomposes as a sum of two abelian subalgebras, $\mathfrak{p}^+$ and $\mathfrak{p}^-$. We prove a Hodge decomposition for $\mathfrak{p}^-$-cohomology and $\mathfrak{p}^+$-homology for all unitary $(\mathfrak{g}, K)$-modules, and show that they are both isomorphic to the Dirac cohomology up to a twist by a modular character. We note that in this setting Enright [E] found an explicit formula for the $\mathfrak{p}^-$-cohomology of unitary highest weight modules.

For general $\mathfrak{g}$, and any two reductive subalgebras $\mathfrak{r}_1 \subset \mathfrak{r}_2 \subset \mathfrak{g}$ to which the Killing form restricts nondegenerately, we show that Kostant's cubic Dirac operator $D(\mathfrak{g}, \mathfrak{r}_1)$ can be decomposed as a sum of two anti-commuting Dirac operators $D(\mathfrak{g}, \mathfrak{r}_2) + D_\Delta(\mathfrak{r}_2, \mathfrak{r}_1)$. Here Δ denotes a certain "diagonal embedding".

In particular, we consider the case when $\mathfrak{r} \subset \mathfrak{k}$ is a reductive subalgebra, which is real, i.e., $\mathfrak{r}$ is the complexification of $\mathfrak{r}_0 = \mathfrak{r} \cap \mathfrak{g}_0$ Then we can show that for any admissible $(\mathfrak{g}, K)$-module, the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{r})$ is the same as the kernel of $D_\Delta(\mathfrak{k}, \mathfrak{r})$ on the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{k})$. Furthermore, the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{r})$ is also the same as the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{k})$ of the kernel of $D_\Delta(\mathfrak{k}, \mathfrak{r})$.

In particular, if $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, then $\mathfrak{r}$ can be a Levi subalgebra $\mathfrak{l}$ of a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, with $\mathfrak{l} \subset \mathfrak{k}$. It follows that the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{l})$, which is closely related to the $\bar{\mathfrak{u}}$-cohomology, has the advantage of being much easier to calculate. For instance, let $\mathfrak{t}$ be a compact Cartan subalgebra, and $\mathfrak{t} \oplus \mathfrak{n}$ be a Borel subalgebra. Then we show how to (easily) explicitly calculate the Dirac cohomology of the discrete series representations with respect to the Dirac operator $D(\mathfrak{g}, \mathfrak{t})$. Comparing the obtained result with Schmid's formula for $\bar{\mathfrak{n}}$-cohomology in [S], we see that they are the same up to an expected modular twist. In case G is of hermitian type, the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{l})$ coincides with the $\bar{\mathfrak{u}}$-cohomology or $\mathfrak{u}$-homology (up to a modular twist) for all unitary representations.

We now describe the organization of our paper. In Sections 2 and 3 we decompose the Kostant cubic Dirac operator as sum of two "half Dirac operators", which correspond to the differentials of $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology. Section 3 also contains a new proof of the Casselman-Osborne theorem on Lie algebra cohomology using an approach similar to that of [HP1]. This is aimed at explaining the formal similarity of the two results. In Section 4 we prove a Hodge decomposition for $\mathfrak{p}^-$-cohomology or $\mathfrak{p}^+$-homology for unitary representations in the Hermitian case, with the Dirac cohomology providing the "harmonic representatives" of both. The same proof applies to any finite-dimensional representation of an arbitrary semisimple group. In Section 5 we define the relative Dirac operators and show that in some cases Dirac cohomology can be calculated in stages. In Section 6 we develop this further and show how to use it to calculate Dirac cohomology explicitly. In particular, we compare the Dirac cohomology for a Levi subalgebra with the $\bar{\mathfrak{u}}$-cohomology in some cases. In Section 7 we obtain a Hodge decomposition and equality of Dirac and $\bar{\mathfrak{u}}$-cohomology for arbitrary unitary modules in the Hermitian case. We conclude the paper by showing that the homological properties of Dirac cohomology are quite different from those of Lie algebra cohomology. In fact, under certain conditions, we show that there is a six-term exact sequence of Dirac cohomology corresponding to a short exact sequence of $(\mathfrak{g}, K)$-modules. So Dirac cohomology resembles a K-theory, rather than a cohomology theory.

This work was initiated by David Vogan [V3]. We would like to thank him for many interesting and stimulating conversations. We believe that the results in this paper are not the end of the theory of Dirac cohomology in representation theory, but rather the beginning of further investigations and applications. For example, the results of this paper should be related to the results of Connes and Moscovici [CM] in a similar way as the results of [HP1] are related to [AS].

2. Construction of certain Dirac operators

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ a parabolic subalgebra, $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ the opposite parabolic subalgebra, and $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$. Then

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}.$$

Furthermore, the restrictions of the Killing form B to $\mathfrak{l}$ and $\mathfrak{s}$ are non-degenerate, and the above decomposition is orthogonal. Since $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are isotropic subspaces in perfect duality under B, we can identify $\bar{\mathfrak{u}}$ with $\mathfrak{u}^*$; this identification is $\mathfrak{l}$-invariant. Let $u_1, \dots, u_n$ be a basis of $\mathfrak{u}$, and let $u_1^*, \dots, u_n^*$ be the dual basis of $\bar{\mathfrak{u}}$.

Let $C(\mathfrak{s})$ be the Clifford algebra of $\mathfrak{s}$. Unlike in [HP1], we will use the same defining relations as Kostant, namely

$$vw + wv = 2B(v, w);$$

in particular, if $B(v,v)=1$, then $v^2=1$, and not -1 like in [HP1]. Of course, over $\mathbb{C}$ there is no substantial difference between the two conventions.

We are going to make use of the well known principle of constructing invariants by contracting dual indices. The aim is to construct a family of interesting $\mathfrak{l}$-invariants in $U(\mathfrak{g})\otimes C(\mathfrak{s})$. These will include Kostant's cubic Dirac operator D, but we will get D as a sum of four members of the family, and we will also be able to combine them in different ways, to get other operators with properties similar to the properties of D. For example, we will have nice expressions for their squares. The form of this principle we need is the statement of the following lemma; the proof is quite easy and essentially reduces to the fact that under the identification $\mathrm{Hom}(\mathfrak{u},\mathfrak{u})\cong\mathfrak{u}^*\otimes\mathfrak{u}$, the identity map corresponds to the sum $\sum_i u_i^*\otimes u_i$.

Lemma 2.1. *Let*

$$\psi:\,\mathfrak{s}^{\otimes 2k}\to U(\mathfrak{g})\otimes C(\mathfrak{s})$$

be a linear map which is $\mathfrak{l}$-equivariant with respect to the adjoint actions. Then

$$\sum_I \psi(u_I\otimes u_I^*)\in U(\mathfrak{g})\otimes C(\mathfrak{s})$$

is independent of the chosen basis u_i and $\mathfrak{l}$-invariant. Here $I=(i_1,\dots,i_k)$ ranges over all k-tuples of integers in $\{1,\dots,n\}$, $u_I=u_{i_1}\otimes\dots\otimes u_{i_k}$, and $u_I^*=u_{i_1}^*\otimes\dots\otimes u_{i_k}^*$. $\square$

For example, ψ can be composed of the obvious inclusions $\mathfrak{s}\hookrightarrow\mathfrak{g}\hookrightarrow U(\mathfrak{g})$ and $\mathfrak{s}\hookrightarrow C(\mathfrak{s})$, products, commutators in $\mathfrak{g}$ and the Killing form $B(.,.)$. Here are several examples of this kind which we will study in the following:

Examples 2.2.

$$\begin{aligned}
A&=\sum_i u_i^*\otimes u_i;\\
A^-&=\sum_i u_i\otimes u_i^*;\\
1\otimes a&=-\frac{1}{4}\sum_{i,j}1\otimes[u_i^*,u_j^*]u_iu_j;\\
1\otimes a^-&=-\frac{1}{4}\sum_{i,j}1\otimes[u_i,u_j]u_i^*u_j^*;\\
E=1\otimes e&=-\frac{1}{2}\sum_i 1\otimes u_i^*u_i.
\end{aligned}$$

Note the symmetry obtained by exchanging the roles of $\mathfrak{u}$ and $\bar{\mathfrak{u}}$. To see how the Dirac operator fits in here, note that one can build an orthonormal basis (Z_i) of $\mathfrak{s}$ from u_i and u_i^*, by putting

$$Z_j:=\frac{u_j+u_j^*}{\sqrt{2}},\quad Z_{n+j}:=\frac{i(u_j-u_j^*)}{\sqrt{2}},$$

for $j=1,\dots,n$. Then it is easy to check that

$$A+A^-=\sum_{i=1}^{2n}Z_i\otimes Z_i.$$

Also, we can rewrite a, a^- and e as follows:

Lemma 2.3.

$$a = -\frac{1}{2}\sum_{i<j}\sum_k B([u_i^*,u_j^*],u_k)\,u_i\wedge u_j\wedge u_k^* = -\frac{1}{4}\sum_{i,j} u_iu_j[u_i^*,u_j^*];$$

$$a^- = -\frac{1}{2}\sum_{i<j}\sum_k B([u_i,u_j],u_k^*)\,u_i^*\wedge u_j^*\wedge u_k = -\frac{1}{4}\sum_{i,j} u_i^*u_j^*[u_i,u_j];$$

$$e = -\frac{1}{2}\sum_i(-u_iu_i^*+2) = \frac{1}{2}\sum_i u_iu_i^* - n.$$

Proof. Since $[u_i^*,u_j^*]\in\bar{\mathfrak{u}}$, we can write it as

$$\sum_k B([u_i^*,u_j^*],u_k)\,u_k^*.$$

Also, the sum in the definition of a is clearly twice the same sum over only those i,j for which $i<j$. The only issue is thus to pass from the Clifford product to the wedge product. For this, we use (1.6) in [K2]. First, since $\mathfrak{u}$ is isotropic, $u_iu_j = u_i\wedge u_j$. Next, we calculate

$$u_k^*(u_i\wedge u_j) = u_k^*\wedge u_i\wedge u_j + B(u_k^*,u_i)\,u_j - B(u_k^*,u_j)\,u_i.$$

The second two terms here are clearly zero if k is different from i and j. For $k=i$, the second term is u_j while the third is zero. For $k=j$, the second term is zero while the third is $-u_i$. It follows that we will be done if we can show

$$\sum_{i,j}\big(B([u_i^*,u_j^*],u_i)\,u_j - B([u_i^*,u_j^*],u_j)\,u_i\big) = 0.$$

However, using Lemma 2.1, we see that $\sum_{i,j} B([u_i^*,u_j^*],u_i)\,u_j$ is an $\mathfrak{l}$-invariant element of $\mathfrak{u}$. Since there are no nonzero $\mathfrak{l}$-invariants in $\mathfrak{u}$, this sum must be 0. Analogously, $\sum_{i,j} B([u_i^*,u_j^*],u_j)\,u_i = 0$.

So we proved the first equality for a. Now in the form with wedge product, we can clearly commute u_k^* in front of u_i and u_j, and then we obtain the second equality by reversing the above argument.

The formulas for e are obvious from the defining relations of $C(\mathfrak{s})$. □

Consider now the basis $(b_j)_{j=1,\dots,2n}$ of $\mathfrak{s}$, given by

$$b_1 = u_1,\dots,b_n = u_n,\ b_{n+1} = u_1^*,\dots,b_{2n} = u_n^*;$$

the dual basis is then

$$d_1 = u_1^*,\dots,d_n = u_n^*,\ d_{n+1} = u_1,\dots,d_{2n} = u_n.$$

Notice that for any i,j,k

$$B([u_i,u_j],u_k) = B([u_i^*,u_j^*],u_k^*) = 0.$$

Hence we can write Kostant's cubic element v as

$$v = -\frac{1}{2}\sum_{1\le i<j<k\le 2n}([d_i,d_j],d_k)\ b_i\wedge b_j\wedge b_k = a + a^-.$$

In particular, we obtain Kostant's cubic Dirac operator as

$$D = A + A^- + 1 \otimes (a + a^-).$$

We are also particularly interested in the elements

$$C = A + 1 \otimes a; \quad C^- = A^- + 1 \otimes a^-; \quad \text{and} \quad D^- = C - C^-.$$

Note that $D = C + C^-$. We will use the fact that commuting with E operates on C, C^-, D and D^- in the following way:

Proposition 2.4.

$$[E, C] = C; \quad [E, C^-] = -C^-; \quad [E, D] = D^- \quad \textit{and} \quad [E, D^-] = D.$$

Proof. The second two relations follow from the first two, and the first two are immediate from the following lemma. ☐

Lemma 2.5. *Commuting with e in the Clifford algebra $C(\mathfrak{s})$ acts as I on $\mathfrak{u}$ and as $-I$ on $\bar{\mathfrak{u}}$.*

Proof. Clearly, for $j \neq i$, $[u_j^* u_j, u_i] = 0$. For $j = i$, we calculate

$$u_i^* u_i^2 - u_i u_i^* u_i = -u_i(-u_i u_i^* + 2) = -2u_i.$$

Namely, since $\mathfrak{u}$ is isotropic, $u^2 = 0$ for any $u \in \mathfrak{u}$. The first claim now follows, and the second is analogous. ☐

Kostant [K2], Theorem 2.16, has calculated

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{l}_\Delta} + C.$$

Here $\Omega_{\mathfrak{g}}$ denotes the Casimir element of $Z(\mathfrak{g}) \subset U(\mathfrak{g})$. Further, $\Omega_{\mathfrak{l}_\Delta}$ is the Casimir element for the diagonal copy $\mathfrak{l}_\Delta$ of $\mathfrak{l}$, embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ via

$$X \longmapsto X \otimes 1 + 1 \otimes \alpha(X), \qquad X \in \mathfrak{l},$$

where $\alpha : \mathfrak{l} \to \mathfrak{so}(\mathfrak{s}) \to C(\mathfrak{s})$ is the action map followed by the standard inclusion of $\mathfrak{so}(\mathfrak{s})$ into $C(\mathfrak{s})$ using the identification $\mathfrak{so}(\mathfrak{s}) \cong \bigwedge^2 \mathfrak{s}$. Finally, C is the constant $||\rho||^2 - ||\rho_{\mathfrak{l}}||^2$.

Using this result and the above remarks, we can now quickly calculate the squares of C, C^- and D^-:

Proposition 2.6.

$$C^2 = (C^-)^2 = 0 \quad \textit{and} \quad (D^-)^2 = -D^2.$$

Proof. From Kostant's expression for D^2, it is clear that D^2 commutes with all $\mathfrak{l}$-invariant elements of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$. In particular, D^2 commutes with E, and using Proposition 2.4 we see

$$DD^- + D^- D = [E, D^2] = 0.$$

Since $D + D^- = 2C$ and $D - D^- = 2C^-$, it follows that

$$4C^2 = (D + D^-)^2 = D^2 + (D^-)^2 = (D - D^-)^2 = 4(C^-)^2,$$

so $C^2 = (C^-)^2$. On the other hand, Proposition 2.4 implies that

$$[E, C^2] = 2C^2 \quad \text{and} \quad [E, (C^-)^2] = -2(C^-)^2.$$

So we see that $C^2 = (C^-)^2 = 0$, and also $D^2 + (D^-)^2 = 4C^2 = 0$. ☐

Now we can completely describe the Lie superalgebra $\mathcal{D}$ spanned by E, C, C^- and $\Omega = D^2$ inside the superalgebra $U(\mathfrak{g}) \otimes C(\mathfrak{s})$. Here $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ is an associative superalgebra with $\mathbb{Z}_2$-grading of the Clifford factor; so it is also a Lie superalgebra in the usual way, with the supercommutator $[a, b] = ab - (-1)^{\deg a \deg b} ba$.

$\mathcal{D}$ is a subalgebra of the Lie superalgebra $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, with the commutation relations from Propositions 2.4 and 2.6. Namely, Ω is central, and

$$[E, C] = C, \quad [E, C^-] = -C^-, \quad [C, C] = [C^-, C^-] = 0 \quad \text{and} \quad [C, C^-] = \Omega.$$

The last relation is obtained as follows: $\Omega = D^2 = (C+C^-)^2 = CC^- + C^-C = [C, C^-]$. Note that we can regard $\mathcal{D}$ as a $\mathbb{Z}$-graded Lie superalgebra with, C^-, E, Ω and C of degrees -1, 0, 0 and 1 respectively. Note also that the subalgebra of $\mathcal{D}$ spanned by C, C^- and Ω is a Heisenberg superalgebra.

The Lie superalgebra $\mathcal{D}$ can be identified with $\mathfrak{gl}(1, 1)$, i.e., the endomorphisms of the superspace $\mathbb{C} \oplus \mathbb{C}$, with the first $\mathbb{C}$ even and the second $\mathbb{C}$ odd. In Kac's classification [Kac] it is denoted by $\mathfrak{l}(1, 1)$. It was extensively used by physicists under the name supersymmetric algebra. It is a completely solvable Lie superalgebra, and its irreducible finite-dimensional representations are described in [Kac].

To finish this section, let us note that D is independent not only of the choice of basis (u_i) but also of the choice of $\mathfrak{u} \subset \mathfrak{s}$. On the other hand, E, C and C^- do depend on the choice of $\mathfrak{u}$.

3. $\bar{\mathfrak{u}}$-COHOMOLOGY AND $\mathfrak{u}$-HOMOLOGY

We retain the notation from previous sections. Let V be an admissible $(\mathfrak{g}, K)$-module. For $p \in \mathbb{N}$, let

$$C^p(\bar{\mathfrak{u}}, V) := \mathrm{Hom}(\textstyle\bigwedge^p \bar{\mathfrak{u}}, V),$$

be the set of p-cochains of the complex defining the $\bar{\mathfrak{u}}$-cohomology of V. The differential $d : C^p(\bar{\mathfrak{u}}, V) \to C^{p+1}(\bar{\mathfrak{u}}, V)$ is given by the usual formula

$$\begin{aligned}(d\omega)(X_0 \wedge \ldots \wedge X_p) = & \sum_{i=0}^{p} (-1)^i \, X_i \cdot \omega(X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p) + \\ & \sum_{0 \le i < j \le p} (-1)^{i+j} \, \omega([X_i, X_j] \wedge X_0 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_p)\end{aligned}$$

We have the following identifications :

$$C^p(\bar{\mathfrak{u}}, V) = \mathrm{Hom}(\textstyle\bigwedge^p \bar{\mathfrak{u}}, V) \simeq \mathrm{Hom}(\textstyle\bigwedge^p (\mathfrak{u}^*), V) \simeq \mathrm{Hom}((\textstyle\bigwedge^p \mathfrak{u})^*, V) \simeq V \otimes \textstyle\bigwedge^p \mathfrak{u}$$

Like in the previous section, we fix a basis $(u_i)_{i=1,\ldots,n}$ of $\mathfrak{u}$ and denote the dual basis of $\bar{\mathfrak{u}}$ by $(u_i^*)_{i=1,\ldots,n}$.

Lemma 3.1. *Through the above identifications, the differential* $d : V \otimes \bigwedge^p \mathfrak{u} \to V \otimes \bigwedge^{p+1} \mathfrak{u}$ *is given by*

$$d(v \otimes Y_1 \wedge \ldots \wedge Y_p) = \sum_{i=1}^{n} u_i^* \cdot v \otimes u_i \wedge Y_1 \wedge \ldots \wedge Y_p$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{p} v \otimes u_i \wedge Y_1 \wedge \ldots \wedge [u_i^*, Y_j]_{\mathfrak{u}} \wedge \ldots \wedge Y_p$$

where $[u_i^*, Y_j]_{\mathfrak{u}}$ *denotes the projection of* $[u_i^*, Y_j]$ *on* $\mathfrak{u}$.

Proof. This is a straigtforward calculation, starting from the fact that the identification $\bigwedge^p(\mathfrak{u}^*) = (\bigwedge^p \mathfrak{u})^*$ is given via $(f_1 \wedge \cdots \wedge f_p)(X_1 \wedge \cdots \wedge X_p) = \det f_i(X_j)$. □

The space $V \otimes \bigwedge^p \mathfrak{u}$ is also the space of p-chains for the $\mathfrak{u}$-homology of V, with differential $\partial : V \otimes \bigwedge^p \mathfrak{u} \to V \otimes \bigwedge^{p-1} \mathfrak{u}$ given by :

$$\partial(v \otimes Y_1 \wedge \ldots \wedge Y_p) = \sum_{i=1}^{p} (-1)^i Y_i \cdot v \otimes Y_1 \wedge \ldots \hat{Y}_i \wedge \ldots \wedge Y_p +$$
$$\sum_{1 \leq i < j \leq p} (-1)^{i+j} v \otimes [Y_i, Y_j] \wedge Y_1 \wedge \ldots \wedge \hat{Y}_i \wedge \ldots \wedge \hat{Y}_j \wedge \ldots Y_p$$

Note that we are tensoring with $\bigwedge^{\cdot} \mathfrak{u}$ from the right and not from the left as usual; this is because we will have an action of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ on $V \otimes \bigwedge^{\cdot} \mathfrak{u}$ which will be more natural in this order.

For reasons that will become apparent later, we will instead of ∂ consider the operator $\delta = -2\partial$. Of course, δ defines the same homology as ∂.

To get our Dirac operators act, we need to consider the $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ - module $V \otimes S$, where S is the spin module for the Clifford algebra $C(\mathfrak{s})$. We will use the identification of $\bigwedge^{\cdot} \mathfrak{u}$ with S, given explicitly in [K4] and [K3]. Namely, one can construct S as the left ideal in $C(\mathfrak{s})$ generated by the element $u_{\text{top}}^* = u_1^* \ldots u_n^*$. One then has $S = (\bigwedge^{\cdot} \mathfrak{u}) u_{\text{top}}^*$, which is isomorphic to $\bigwedge^{\cdot} \mathfrak{u}$ as a vector space, and the action of $C(\mathfrak{s})$ is given by left Clifford multiplication. Explicitly, $u \in \mathfrak{u}$ and $u^* \in \bar{\mathfrak{u}}$ act on $Y = Y_1 \wedge \cdots \wedge Y_p \in \bigwedge^p \mathfrak{u}$ by

$$u \cdot Y = u \wedge Y$$

$$u^* \cdot Y = 2 \sum_{i=1}^{p} (-1)^{i+1} B(u^*, Y_i) Y_1 \wedge \ldots \hat{Y}_i \cdots \wedge Y_p.$$

Namely, since $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are isotropic, the Clifford and wedge products coincide on each of them; in particular, $u^* u_{\text{top}}^* = 0$.

The natural action of $\mathfrak{l}$ on $V \otimes S$ is the tensor product of the restriction of the $\mathfrak{g}$-action on V and the spin action on S. On the other hand, the usual $\mathfrak{l}$ action on $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology is given by the adjoint action on $\bigwedge^{\cdot} \bar{\mathfrak{u}}$ and $\bigwedge^{\cdot} \mathfrak{u}$. Thus, our identification of $\bigwedge^{\cdot} \mathfrak{u} \otimes V$ with $V \otimes S$ is not an $\mathfrak{l}$-isomorphism. However, as was proved in [K3], Prop.3.6, the two actions differ only by a twist with the one dimensional $\mathfrak{l}$-module $Z_{\rho(\bar{\mathfrak{u}})}$ of weight $\rho(\bar{\mathfrak{u}})$.

This means that, if we consider d and δ as operators on $V \otimes S$ via the above identification, then as an $\mathfrak{l}$-module, the cohomology of d gets identified with $H^{\cdot}(\bar{\mathfrak{u}}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}$, while the homology of δ gets identified with $H_{\cdot}(\mathfrak{u}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}$.

Proposition 3.2. *Under the action of $U(\mathfrak{g})\otimes C(\mathfrak{s})$ on $V\otimes S$, the operators $C=A+1\otimes a$ and $C^-=A^-+1\otimes a^-$ from Section 2 act as d and δ respectively. In particular, the cubic Dirac operator $D=C+C^-$ acts as $d+\delta=d-2\partial$.*

Proof. We are going to use the explicit formulas for the action of $C(\mathfrak{s})$ on S given above. From these formulas, it is immediate that the action of A coincides with the first (single) sum in the expression for d, while the action of A^- transforms $v\otimes Y_1\wedge\cdots\wedge Y_p\in V\otimes\bigwedge^p\mathfrak{u}$ into

$$\sum_{i=1}^n u_iv\otimes 2\sum_{k=1}^p(-1)^{k+1}B(u_i^*,Y_k)\,Y_1\wedge\dots\hat{Y}_k\cdots\wedge Y_p.$$

Since $\sum_i B(u_i^*,Y_k)\,u_i=Y_k$, we see that this is equal to minus twice the first (single) sum in the expression for the $\mathfrak{u}$-homology operator ∂.

It remains to identify the action of the cubic terms a and a^-.

We use the expression for a from Lemma 2.3, i.e., $a=-\frac{1}{4}\sum_{i,j}u_iu_j[u_i^*,u_j^*]$. This element transforms $v\otimes Y_1\wedge\cdots\wedge Y_p\in V\otimes\bigwedge^p\mathfrak{u}$ into

$$\begin{aligned}&-\frac{1}{4}v\otimes\sum_{i,j}u_iu_j\;2\sum_{k=1}^p(-1)^{k+1}B([u_i^*,u_j^*],Y_k)\,Y_1\wedge\dots\hat{Y}_k\cdots\wedge Y_p\\&=\frac{1}{2}v\otimes\sum_{i,j,k}(-1)^{k+1}B([u_i^*,Y_k],u_j^*)\,u_i\wedge u_j\wedge Y_1\wedge\dots\hat{Y}_k\cdots\wedge Y_p.\end{aligned}$$

Now we sum $\sum_j B([u_i^*,Y_k],u_j^*)\,u_j=[u_i^*,Y_k]_{\mathfrak{u}}$, and after commuting $[u_i^*,Y_k]_{\mathfrak{u}}$ into its proper place, we get the second (double) sum in the expression for d.

For a^- we use its definition from 2.2. To write the calculation nicely, we introduce the following notation: for $Y=Y_1\wedge\cdots\wedge Y_p$, let

$$\hat{Y}_{k,l}=Y_1\wedge\dots\hat{Y}_k\dots\hat{Y}_l\cdots\wedge Y_p,\qquad\text{if } k<l.$$

If $k>l$ then we change the sign and define

$$\hat{Y}_{k,l}=-Y_1\wedge\dots\hat{Y}_l\dots\hat{Y}_k\cdots\wedge Y_p,\qquad\text{if } k>l.$$

If $k=l$, we set $\hat{Y}_{k,l}=0$. This now allows us to write

$$u_i^*u_j^*\cdot Y_1\wedge\cdots\wedge Y_p=4\sum_{k,l}(-1)^{k+l}B(u_i^*,Y_k)B(u_j^*,Y_l)\hat{Y}_{k,l}.$$

It follows that $1\otimes a^-$ transforms $v\otimes Y_1\wedge\cdots\wedge Y_p$ into

$$-\frac{1}{4}\,4\,v\otimes\sum_{i,j,k,l}(-1)^{k+l}B(u_i^*,Y_k)B(u_j^*,Y_l)\,[u_i,u_j]\wedge\hat{Y}_{k,l}.$$

Upon summing up $\sum_i B(u_i^*,Y_k)\,u_i=Y_k$ and $\sum_j B(u_j^*,Y_l)u_j=Y_l$, we get that this is equal to

$$-v\otimes\sum_{k,l}(-1)^{k+l}[Y_k,Y_l]\wedge\hat{Y}_{k,l}.$$

This is now clearly invariant for exchanging the roles of k and l, hence it is twice the same sum extending just over $k<l$, i.e., minus twice the second (double) sum in the expression for ∂. $\square$

It is now clear why we considered $\delta = -2\partial$ instead of just ∂; in this way we have the action of D being equal to $d + \delta$.

Before we go on, let us note how the element E of Section 2 acts on $V \otimes S$; it is in fact a degree operator up to a shift. This means E can be used to identify the degrees in which the cohomology (homology) is appearing.

Proposition 3.3. *The element E of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ from 2.2 acts on $V \otimes \bigwedge^k \mathfrak{u}$ as multiplication by the scalar $k - n$. Consequently, it preserves the kernel and image of both d and δ, and hence acts on the k-th cohomology of d and the k-th homology of δ, by the same scalar $k - n$. In particular, $E + n$ is the degree operator.*

Proof. Using Lemma 2.5 we see that for any $u \in \mathfrak{u}$, $eu = ue + u$. On the other hand, by Lemma 2.3, $e = \frac{1}{2}\sum_i u_i u_i^* - n$, hence $eu_{\mathrm{top}}^* = -nu_{\mathrm{top}}^*$. It now immediately follows that in $C(\mathfrak{s})$ we have $e(Y_1 \wedge \cdots \wedge Y_k \ u_{\mathrm{top}}^*) = (k-n)Y_1 \wedge \cdots \wedge Y_k \ u_{\mathrm{top}}^*$, so the action of E on $V \otimes \bigwedge^k \mathfrak{u}$ is indeed multiplication by the scalar $k - n$.

It now immediately follows that E preserves the kernel and image of d and δ, as these are homogeneous operators (of degree 1 and -1 respectively). (Note that this last assertion can also be obtained from the commutation relations of Proposition 2.4.) $\square$

We will now state a result for the operators C and C^- analogous to the one obtained for D in [HP1] and [K4]. A corollary will be the Casselman-Osborne Theorem. Our goal here is not to give a new proof of the Casselman-Osborne Theorem, the existing ones being completely satisfactory, but to shed some light on the formal similarity between the Casselman-Osborne Theorem and the main result of [HP1].

Define $d_D, d_C, d_{C^-} : U(\mathfrak{g}) \otimes C(\mathfrak{s}) \to U(\mathfrak{g}) \otimes C(\mathfrak{s})$ by

$$\begin{aligned} d_D(x) &= Dx - \epsilon_x x D \\ d_C(x) &= Cx - \epsilon_x x C \\ d_{C^-}(x) &= C^- x - \epsilon_x x C^- \end{aligned}$$

where ϵ_x is 1 for even x and -1 for odd x. In the following we fix a compact group L with complexified Lie algebra $\mathfrak{l}$.

Theorem 3.4. *d_D, d_C and d_{C^-} are L-equivariant. They induce maps from $(U(\mathfrak{g}) \otimes C(\mathfrak{s}))^L$ into itself and $d_D^2 = d_C^2 = d_{C^-}^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{s}))^L$.*

Furthermore $Z(\mathfrak{l}_\Delta) \subset \operatorname{Ker} d_D$ (resp. $\operatorname{Ker} d_C$, $\operatorname{Ker} d_{C^-}$), and one has $\operatorname{Ker} d_D = Z(\mathfrak{l}_\Delta) \oplus \operatorname{Im} d_D$, (resp. $\operatorname{Ker} d_C = Z(\mathfrak{l}_\Delta) \oplus \operatorname{Im} d_C$, $\operatorname{Ker} d_{C^-} = Z(\mathfrak{l}_\Delta) \oplus \operatorname{Im} d_{C^-}$).

Proof. The result for d_D is due to Kostant [K4]. The proof is the same as the proof of the main result of [HP1]. We give details for d_C, the proof for d_{C^-} being entirely similar. As in [HP1], we use the standard filtration on $U(\mathfrak{g})$, which induces a filtration $(F_n A)_{n \in \mathbb{N}}$ on $A := U(\mathfrak{g}) \otimes C(\mathfrak{s})$. This filtration being L-invariant, it induces in turn a filtration on A^L. Clearly,

$$C = \sum_{j=1}^{n} u_j^* \otimes u_j + (1 \otimes a) \in F_1 A^L.$$

The $\mathbb{Z}_2$-gradation on the Clifford algebra $C(\mathfrak{s})$ induces a $\mathbb{Z}_2$-gradation on A. We set $A^0 = U(\mathfrak{g}) \otimes C(\mathfrak{s})^0$ and $A^1 = U(\mathfrak{g}) \otimes C(\mathfrak{s})^1$. Then $A = A^0 \bigoplus A^1$ and this $\mathbb{Z}_2$-gradation is compatible with the filtration $(F_n A)_{n \in \mathbb{N}}$

If $x \in F_n A^0$, then $d_C(x) = d_C^0(x) = Cx - xC \in F_{n+1} A^1$. If $x \in F_n A^1$, then $d_C(x) = d_C^1(x) = Cx + xC \in F_{n+1} A^0$. Thus

$$d_C^0 : \ F_n A^0 \to F_{n+1} A^1 \quad \text{ and } \quad d_C^1 : \ F_n A^1 \to F_{n+1} A^0$$

induce

$$\bar{d}_C^0 : \ \mathrm{Gr}_n \ A^0 \to \mathrm{Gr}_{n+1} A^1 \quad \text{ and } \quad \bar{d}_C^1 : ; \ \mathrm{Gr}_n \ A^1 \to \mathrm{Gr}_{n+1} \ A^0.$$

Let $\bar{C} = \sum_{j=1}^n u_j^* \otimes u_j \in S^1(\mathfrak{g}) \otimes C(\mathfrak{s})$ be the image of C in $\left(\mathrm{Gr}_1 \ A^1\right)^L$ (notice that the cubic term disappears since it is in $F_0 A^1$). If $x \in F_n A^0$,

$$\bar{d}_C^0(\bar{x}) = \overline{Cx - xC} = \bar{C}\bar{x} - \bar{x}\bar{C}$$

and if $x \in F_n A^1$,

$$\bar{d}_C^1(\bar{x}) = \overline{Cx + xC} = \bar{C}\bar{x} + \bar{x}\bar{C}.$$

Therefore $\bar{d}_C = \bar{d}_C^0 \oplus \bar{d}_C^1 : \mathrm{Gr}\, A \to \mathrm{Gr}\, A$. Note also that unlike d_D, d_C is a differential on all of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, because $C^2 = 0$. Hence $\bar{d}_C$ is a differential on $S(\mathfrak{g}) \otimes C(\mathfrak{s})$.

Let us compute $\bar{d}_C(\bar{x})$ for $\bar{x} = y \otimes u_{i_1} \cdots u_{i_k} u_{j_1}^* \cdots u_{j_l}^* \in S(\mathfrak{g}) \otimes C(\mathfrak{s})$. We can assume i_r are different from each other and likewise for j_s.

$$\begin{aligned}
\bar{d}_C(\bar{x}) = & \left(\sum_{j=1}^n u_j^* \otimes u_j \right) \left(y \otimes u_{i_1} \cdots u_{i_k} u_{j_1}^* \cdots u_{j_l}^* \right) - \\
& \qquad (-1)^{k+l} \left(y \otimes u_{i_1} \cdots u_{i_k} u_{j_1}^* \cdots u_{j_l}^* \right) \left(\sum_{j=1}^n u_j^* \otimes u_j \right) \\
= & \sum_{j=1}^n u_j^* y \otimes \left((-1)^k u_{i_1} \cdots u_{i_k} u_j u_{j_1}^* \cdots u_{j_l}^* - (-1)^{k+l} u_{i_1} \cdots u_{i_k} u_{j_1}^* \cdots u_{j_l}^* u_j \right)
\end{aligned}$$

If $j \neq j_s$ for all s, then the contribution to the sum is zero. If $j = j_s$, then

$$\begin{aligned}
u_{j_s} u_{j_1}^* \cdots u_{j_l}^* - (-1)^l u_{j_1}^* \cdots u_{j_l}^* u_{j_s} &= (-1)^{s-1} u_{j_1}^* \cdots u_{j_{s-1}}^* (u_{j_s} u_{j_s}^* + u_{j_s}^* u_{j_s}) u_{j_{s+1}}^* \cdots u_{j_l}^* \\
&= 2(-1)^{s-1} u_{j_1}^* \cdots \widehat{u_{j_s}^*} \cdots u_{j_l}^*.
\end{aligned}$$

So we see

$$\bar{d}_C(\bar{x}) = \sum_{s=1}^l -2(-1)^{k+s} u_{j_s}^* y \otimes u_{i_1} \cdots u_{i_k} u_{j_1}^* \cdots \widehat{u_{j_s}^*} \cdots u_{j_l}^*.$$

Since $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u} \oplus \bar{\mathfrak{u}}$ and $C(\mathfrak{s}) \simeq \bigwedge^{\cdot} \mathfrak{s} = \bigwedge^{\cdot} \mathfrak{u} \otimes \bigwedge^{\cdot} \bar{\mathfrak{u}}$, one has

$$S(\mathfrak{g}) \otimes C(\mathfrak{s}) = S(\mathfrak{l}) \otimes S(\mathfrak{u}) \otimes \textstyle\bigwedge^{\cdot} \mathfrak{u} \otimes S(\bar{\mathfrak{u}}) \otimes \bigwedge^{\cdot} \bar{\mathfrak{u}}.$$

It follows that $\bar{d}_C$ is (up to a sign depending on the $\bigwedge^{\cdot} \mathfrak{u}$-degree) equal to $-2\,\mathrm{Id} \otimes d_{\bar{\mathfrak{u}}}$, where $d_{\bar{\mathfrak{u}}}$ is the Koszul differential for the vector space $\bar{\mathfrak{u}}$. It is well known that $d_{\bar{\mathfrak{u}}}$ is exact except at degree zero, where the cohomology is $\mathbb{C}$, embedded as the constants. It follows that $\bar{d}_C$ is exact except at degree zero, where the cohomology is $S(\mathfrak{l}) \otimes S(\mathfrak{u}) \otimes \bigwedge^{\cdot} \mathfrak{u}$ embedded in the obvious way. It remains to pass to the invariants:

Lemma 3.5. *The differential $\bar{d}_C$ on $(S(\mathfrak{g})\otimes C(\mathfrak{s}))^L$ is exact except at degree zero. The zeroth cohomology is $(S(\mathfrak{l})\otimes S(\mathfrak{u})\otimes \bigwedge^{\cdot}\mathfrak{u})^L = S(\mathfrak{l})^L\otimes 1$, embedded in the obvious way. More precisely*

$$\operatorname{Ker}\bar{d}_C = \big(S(\mathfrak{l})^L\otimes 1\big)\oplus \operatorname{Im}\bar{d}_C$$

To prove the lemma, we need to show that $(S(\mathfrak{l})\otimes S(\mathfrak{u})\otimes \bigwedge^{\cdot}\mathfrak{u})^L = S(\mathfrak{l})^L\otimes 1\otimes 1$. To see this, we may choose some element h in the center of $\mathfrak{l}$, such that $\mathfrak{l}$ is the centralizer of h in $\mathfrak{g}$, $\operatorname{ad} h$ has real eiganvalues and $\mathfrak{u}$ is the sum of the positive eigenspaces of $\operatorname{ad} h$. Making h act on an element in $(S(\mathfrak{l})\otimes S(\mathfrak{u})\otimes \bigwedge^{\cdot}\mathfrak{u})^L$, we see that this element has to be in $S(\mathfrak{l})^L\otimes 1\otimes 1$.

We can now finish the proof of the theorem. We proceed as in [HP1] by induction on the degree of the filtration. □

Let $z\in Z(\mathfrak{g})$. Since $z\otimes 1\in \operatorname{Ker} d_C$, we can write

$$z\otimes 1 = \eta_{\mathfrak{l}}(z) + Ca + aC$$

for some $a\in (U(\mathfrak{g})\otimes C(\mathfrak{s}))^L$ and $\eta_{\mathfrak{l}}(z)\in Z(\mathfrak{l}_\Delta)$.

Our goal is now to compute $\eta_{\mathfrak{l}}(z)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$, and let us denote respectively by $W_{\mathfrak{l}}$ and $W_{\mathfrak{g}}$ the Weyl groups of $\mathfrak{h}$ in $\mathfrak{l}$ and $\mathfrak{g}$.

We have Harish-Chandra isomophisms

$$\mu_{\mathfrak{l}/\mathfrak{h}} : Z(\mathfrak{l}) \to S(\mathfrak{h})^{W_{\mathfrak{l}}}, \qquad \mu_{\mathfrak{g}/\mathfrak{h}} : Z(\mathfrak{g}) \to S(\mathfrak{h})^{W_{\mathfrak{g}}},$$

and an obvious inclusion $i : S(\mathfrak{h})^{W_{\mathfrak{g}}} \to S(\mathfrak{h})^{W_{\mathfrak{l}}}$. Set $\mu_{\mathfrak{g}/\mathfrak{l}} := \mu_{\mathfrak{l}/\mathfrak{h}}^{-1}\circ i\circ \mu_{\mathfrak{g}/\mathfrak{h}}$. With this notation we have:

Lemma 3.6. *For all $z\in Z(\mathfrak{g})$, $\eta_{\mathfrak{l}}(z) = \mu_{\mathfrak{g}/\mathfrak{l}}(z)$.*

Proof. The proof is similar to the proof of Theorem 4.2 in [K4], but much simpler. We give only a sketch. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\bar{\mathfrak{u}}$, and suppose the Cartan algebra $\mathfrak{h}$ has been chosen to lie in $\mathfrak{b}$.

Let V_λ be the irreducible finite dimensional representation with highest weight λ (relative to $\mathfrak{b}$), and let v_λ be a non-zero highest weight vector in V_λ. Recall the element u^*_{top} in $C(\mathfrak{s})$ used to define the spin module S.

One can see easily that $C\cdot(v_\lambda\otimes u^*_{\text{top}}) = 0$ and that $v_\lambda\otimes u^*_{\text{top}}\in V_\lambda\otimes S$ defines a non-zero cohomology class in $H^0(\bar{\mathfrak{u}}, V_\lambda)$. Since the infinitesimal character of V_λ is given by $\lambda+\rho$, and any $q\in Z(\mathfrak{l}_\Delta)$ acts by the scalar $\mu_{\mathfrak{l}/\mathfrak{h}}(q)(\lambda+\rho)$ on $v_\lambda\otimes u^*_{\text{top}}$ (see [K4], Theorem 4.1), we get

$$\mu_{\mathfrak{g}/\mathfrak{h}}(z)(\lambda+\rho) = \mu_{\mathfrak{l}/\mathfrak{h}}(\eta_{\mathfrak{l}}(z))(\lambda+\rho)$$

for all $z\in Z(\mathfrak{g})$.

Since this is true for all dominant weights λ, and since these form a Zariski dense set in $\mathfrak{h}^*$, we conclude that indeed $\mu_{\mathfrak{g}/\mathfrak{h}}(z) = \mu_{\mathfrak{l}/\mathfrak{h}}(\eta_{\mathfrak{l}}(z))$. □

Let V be a representation of $\mathfrak{g}$ admitting an infinitesimal character χ_V and let $z\in Z(\mathfrak{g})$. Then z acts on $H^{\cdot}(\bar{\mathfrak{u}}, V)$ by the scalar $\chi_V(z)$.

Notice that $z\otimes 1$ acts on $\operatorname{Ker} C/\operatorname{Im} C$ as $\eta_{\mathfrak{l}}(z)$. Namely, $Ca+aC$ leaves $\operatorname{Ker} C$ and $\operatorname{Im} C$ stable, and induces the zero action on $\operatorname{Ker} C/\operatorname{Im} C$.

Thus, the induced action of $\eta_{\mathfrak{l}}(z)$ on $\operatorname{Ker} C/\operatorname{Im} C \cong H^{\cdot}(\bar{\mathfrak{u}}, V)\otimes Z_{\rho(\bar{\mathfrak{u}})}$ is equal to the scalar multiplication by $\chi_V(z)$. This is exactly the statement of the Casselman-Osborne

Theorem (see [CO], or [V1], Theorem 3.1.5). Namely, our definition of $\mu_{\mathfrak{g}/\mathfrak{l}} = \eta_{\mathfrak{l}}$ differs from the map $Z(\mathfrak{g}) \to Z(\mathfrak{l})$ from the Casselman-Osborne Theorem exactly by the above ρ-shift.

4. Hodge decomposition for $\mathfrak{p}^-$ - cohomology

In this section we will study hermitian forms on $V \otimes S$ for unitary modules V. Ideally we would like our d and δ to be adjoints or minus adjoints of each other. This will in general be possible only for an indefinite form. It is however much easier to obtain Hodge-type decompositions in the presence of a positive definite form. In this section we will mostly study a special case when $\mathfrak{l}$ is equal to $\mathfrak{k}$; this can happen only when the pair $(\mathfrak{g}, \mathfrak{k})$ is hermitian symmetric. In this case, d and δ are minus adjoints of each other with respect to a positive definite form, and we can use a variant of standard Hodge decomposition to conclude that the Dirac cohomology, $\mathfrak{p}^-$-cohomology and $\mathfrak{p}^+$-homology are all isomorphic to the space of "harmonics". In Section 7 we will do something similar when $\mathfrak{l}$ is only contained in $\mathfrak{k}$.

Assume that V is unitary, i.e., V posseses a positive definite hermitian form invariant with respect to the real form $\mathfrak{g}_0$. We denote by $^-$ the conjugation with respect to $\mathfrak{g}_0$ and by θ the Cartan involution of $\mathfrak{g}_0$ extended to $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition.

Note that if our parabolic $\mathfrak{l} \oplus \mathfrak{u}$ is θ-stable, then $\bar{\mathfrak{u}}$ is indeed the complex conjugate of $\mathfrak{u}$, and thus, our notation is coherent. Furthermore, we can choose the basis u_i so that each u_i is either in $\mathfrak{k}$ or in $\mathfrak{p}$. Then, after suitable normalization of the u_i's, we can take $u_i^* = -\theta\bar{u}_i$; so $u_i^* = \bar{u}_i$ for $u_i \in \mathfrak{p}$, and $u_i^* = -\bar{u}_i$ for $u_i \in \mathfrak{k}$. We denote by ϵ_i the sign in these formulas; so $u_i^* = \epsilon_i \bar{u}_i$ for all i. In the following we assume that $\mathfrak{q}$ is θ-stable.

We consider the positive definite form on $S \cong \bigwedge^{\cdot} \mathfrak{u}$, given by $\langle X, Y \rangle_{pos} := -2B(X, \theta\bar{Y})$ on $\mathfrak{u}$, and extended to all of S in the usual way, using the determinant. Notice that we have $\langle u_i, u_j \rangle_{pos} = 2\delta_{ij}$.

This form is in general not $\mathfrak{l}$-invariant (but it will be $\mathfrak{l}$-invariant in the special cases we study below).

Lemma 4.1. *With respect to the form $\langle X, Y \rangle_{pos}$ on S, the adjoint of the operator $u_i \in C(\mathfrak{s})$ is u_i^*.*

Proof. We need to show that

$$\langle u_i \wedge X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_{k+1} \rangle_{pos} = \langle X_1 \wedge \cdots \wedge X_k, u_i^* \cdot (Y_1 \wedge \cdots \wedge Y_{k+1}) \rangle_{pos}.$$

We can check this assuming that $X_a = u_{s_a}$ and $Y_b = u_{t_b}$ are basis elements, with s_a and t_b increasing. The left hand side of the equality we have to prove is nonzero if and only if $u_i = Y_j$ for some j, and X_a's equal the rest of Y_b's. In that case, the left hand side is

$$(-1)^{j-1} 2^{k+1}.$$

The right hand side is again nonzero if and only if $u_i = Y_j$ for some j, and X_a's equal the rest of Y_b's. In that case, the right hand side is exactly the same as the left hand side. □

Lemma 4.2. *With respect to the form $\langle X, Y \rangle_{pos}$ on S, the adjoint of the operator $u_i u_j [u_i^*, u_j^*]$ is the operator $[u_i, u_j] u_i^* u_j^*$. Consequently, the adjoint of a is a^-.*

Proof. We can write $[u_i^*, u_j^*] = \sum_k B([u_i^*, u_j^*], u_k)\, u_k^*$ and hence by Lemma 4.1 the adjoint of $[u_i^*, u_j^*]$ is

$$\sum_k \overline{B([u_i^*, u_j^*], u_k)}\, u_k = \sum_k B([\bar{u}_i^*, \bar{u}_j^*], \bar{u}_k)\, u_k = \sum_k \epsilon_i \epsilon_j \epsilon_k B([u_i, u_j], u_k^*)\, u_k.$$

If u_i and u_j are both in $\mathfrak{k}$, then so is $[u_i, u_j]$, and in the above sum we can assume $u_k \in \mathfrak{k}$. Then ϵ_i, ϵ_j and ϵ_k are all -1, and their product is -1. If $u_i \in \mathfrak{k}$, $u_j \in \mathfrak{p}$, then $[u_i, u_j] \in \mathfrak{p}$, we can assume $u_k \in \mathfrak{p}$ and again $\epsilon_i \epsilon_j \epsilon_k = -1$. If $u_i, u_j \in \mathfrak{p}$, then $[u_i, u_j] \in \mathfrak{k}$, we can assume $u_k \in \mathfrak{k}$ and again $\epsilon_i \epsilon_j \epsilon_k = -1$. So in all cases the above sum is

$$-\sum_k B([u_i, u_j], u_k^*)\, u_k = -[u_i, u_j],$$

i.e., the adjoint of $[u_i^*, u_j^*]$ is $-[u_i, u_j]$. Thus the adjoint of $u_i u_j [u_i^*, u_j^*]$ is $-[u_i, u_j] u_j^* u_i^* = [u_i, u_j] u_i^* u_j^*$, as claimed.

The adjointness of a and a^- is now clear from the definitions in 2.2. $\square$

Let now V be a unitary $(\mathfrak{g}, K)$-module. Then the adjoint of the operator u_i on V is $-\bar{u}_i = -\epsilon_i u_i^*$. So the adjoint of $u_i \otimes u_i^*$ on $V \otimes S$ is $-\epsilon_i u_i^* \otimes u_i$. Here we consider the tensor product hermitian form on $V \otimes S$; this form will again be denoted by $\langle ., . \rangle_{pos}$. In other words, if we denote by $A_\mathfrak{k}$ and $A_\mathfrak{p}$ the $\mathfrak{k}$ respectively $\mathfrak{p}$ - parts of A, then we see:

Corollary 4.3. *With respect to the form $\langle ., . \rangle_{pos}$ on $V \otimes S$, the adjoint of $A_\mathfrak{k}$ is $A_\mathfrak{k}^-$ while the adjoint of $A_\mathfrak{p}$ is $-A_\mathfrak{p}^-$. Hence the adjoint of $C = A_\mathfrak{k} + A_\mathfrak{p} + 1 \otimes a$ is $A_\mathfrak{k}^- + 1 \otimes a^- - A_\mathfrak{p}^-$ and the adjoint of $C^- = A_\mathfrak{k}^- + A_\mathfrak{p}^- + 1 \otimes a^-$ is $A_\mathfrak{k} + 1 \otimes a - A_\mathfrak{p}$.* $\square$

We will use this corollary in Sections 5 and 7. Now we turn our attention to the case when $A_\mathfrak{k}$ and $1 \otimes a$ do not appear in C. This is the already mentioned case, when $\mathfrak{l} = \mathfrak{k}$, and $\mathfrak{u}$ is contained in $\mathfrak{p}$. Then $\mathfrak{u}$ is forced to be abelian and we denote as usual $\mathfrak{u} = \mathfrak{p}^+$, $\bar{\mathfrak{u}} = \mathfrak{p}^-$. In this case, the Dirac operator $D = D(\mathfrak{g}, \mathfrak{l})$ is the "ordinary" Dirac operator corresponding to $\mathfrak{k}$, there is no cubic part, and we conclude

Corollary 4.4. *Let $(\mathfrak{g}, \mathfrak{k})$ be a hermitian symmetric pair and set $\mathfrak{l} = \mathfrak{k}$. Let V be a unitary $(\mathfrak{g}, K)$-module and consider the form $\langle ., . \rangle_{pos}$ on $V \otimes S$. Then the operators $C = d$ and $C^- = \delta$ are minus adjoints of each other. Hence the Dirac operator $D = D(\mathfrak{g}, \mathfrak{l}) = D(\mathfrak{g}, \mathfrak{k})$ is anti-self-adjoint.*

More generally, if $\mathfrak{l}$ contains $\mathfrak{k}$ (this can happen when $\mathfrak{g}$ is not simple), then $D(\mathfrak{g}, \mathfrak{l})$ is anti-self-adjoint with respect to $\langle ., . \rangle_{pos}$. $\square$

Remark 4.5. There is another form one can put on $S \cong \bigwedge^\cdot \mathfrak{u}$, and then combine it with the unitary form on V. This form is simply induced by B: on $\mathfrak{u}$, it is given by $\langle X, Y \rangle_{inv} = 2B(X, \bar{Y})$. Unlike $\langle X, Y \rangle_{pos}$, this form is always $\mathfrak{l}$-invariant. On the other hand it is rarely positive definite. (In the situation of Corollary 4.4, it is however the same as $\langle X, Y \rangle_{pos}$.)

One can make calculations very similar to the above ones, and conclude that the adjoint of u_i on S is now $\epsilon_i u_i^*$. It follows that the adjoint of a is $-a^-$; furthermore, the adjoint of A is $-A^-$ and hence the adjoint of C is $-C^-$. So D is now anti-self-adjoint, in a completely general situation, but with respect to a typically indefinite form. We have been unable to use this to get a Hodge decomposition like the one below. $\square$

For the rest of this section we assume that $\mathfrak{l} = \mathfrak{k}$. So D is anti-self-adjoint with respect to the positive definite form $\langle ., . \rangle_{pos}$ on $V \otimes S$. In particular, the operators D and D^2 have the same kernel on $V \otimes S$.

By [P], Proposition 3.2 (or more generally, [K2], Theorem 2.16), we know that

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{k}_\Delta} + C,$$

where $\Omega_{\mathfrak{g}}$ and $\Omega_{\mathfrak{k}_\Delta}$ are the Casimir operators for $\mathfrak{g}$ and diagonally embedded $\mathfrak{k}$, and C is the constant $||\rho_{\mathfrak{g}}||^2 - ||\rho_{\mathfrak{k}}||^2$. It follows that if $\Omega_{\mathfrak{g}}$ acts on V by a constant, then $\Omega_{\mathfrak{k}_\Delta}$ is up to a constant equal to D^2 on $V \otimes S$. Since $\Omega_{\mathfrak{k}_\Delta}$ acts by a scalar on each $\tilde{K}$-type in $V \otimes S$, the same is true for D^2. So D^2 is a semisimple operator, i.e., $V \otimes S$ is a direct sum of eigenspaces for D^2. In particular:

Corollary 4.6. *If the $(\mathfrak{g}, K)$-module V has infinitesimal character, then* $V \otimes S = \operatorname{Ker}(D^2) \oplus \operatorname{Im}(D^2)$

Proof. We have seen that $V \otimes S$ is a direct sum of eigenspaces for D^2. Clearly, the zero eigenspace is $\operatorname{Ker} D^2$, and the sum of the nonzero eigenspaces is $\operatorname{Im} D^2$. □

It is now easy to obtain a variant of the usual Hodge decomposition. The following arguments are well known; see e.g. [W], Scholium 9.4.4. We first have

Lemma 4.7. *(a)* $\operatorname{Ker} D = \operatorname{Ker} d \cap \operatorname{Ker} \delta$*;*

(b) $\operatorname{Im} \delta$ *is orthogonal to* $\operatorname{Ker} d$ *and* $\operatorname{Im} d$ *is orthogonal to* $\operatorname{Ker} \delta$*.*

Proof. (a) Since $D = d + \delta$, it is clear that $\operatorname{Ker} d \cap \operatorname{Ker} \delta$ is contained in $\operatorname{Ker} D$. On the other hand, if $Dx = 0$, then $dx = -\delta x$, hence $\delta dx = -\delta^2 x = 0$. So $\langle dx, dx \rangle_{pos} = \langle -\delta dx, x \rangle_{pos} = 0$, hence $dx = 0$. Now $Dx = 0$ implies that also $\delta x = 0$.

(b) is obvious since d and δ are minus adjoint to each other. □

Combining Corollary 4.6, Lemma 4.7 and the fact $\operatorname{Ker} D = \operatorname{Ker} D^2$, we get

Theorem 4.8. *Let $(\mathfrak{g}, \mathfrak{k})$ be a hermitian symmetric pair and set $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$. Let V be an irreducible unitary $(\mathfrak{g}, K)$-module. Then:*

(a) $V \otimes S = \operatorname{Ker} D \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta$*;*

(b) $\operatorname{Ker} d = \operatorname{Ker} D \oplus \operatorname{Im} d$*;*

(c) $\operatorname{Ker} \delta = \operatorname{Ker} D \oplus \operatorname{Im} \delta$*.*

In particular, The Dirac cohomology of V is equal to $\mathfrak{p}^-$-cohomology and to $\mathfrak{p}^+$-homology, up to modular twists:

$$\operatorname{Ker} D \cong H^{\cdot}(\mathfrak{p}^-, V) \otimes Z_{\rho(\mathfrak{p}^-)} \cong H_{\cdot}(\mathfrak{p}^+, V) \otimes Z_{\rho(\mathfrak{p}^-)}.$$

More precisely, (up to modular twists) the Dirac cohomology $\operatorname{Ker} D$ *is the space of harmonic representatives for both $\mathfrak{p}^-$-cohomology and $\mathfrak{p}^+$-homology.* □

Remark 4.9. There is a variant of the above results for a finite dimensional module V. In this case one puts a positive definite form on V invariant under the compact form $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of $\mathfrak{g}$; this is sometimes called an admissible form. In this way the adjoint of u_i on V will be $\bar{u}_i$ if $u_i \in \mathfrak{p}$ and $-\bar{u}_i$ if $u_i \in \mathfrak{k}$. Hence the adjoint of u_i on V is u_i^* for all i.

So this form combines with $\langle \, , \rangle_{pos}$ on S better than a unitary form. In particular, d and δ are now adjoint to each other with respect to $\langle \, , \rangle_{pos}$, D is self-adjoint, and the above arguments, including Theorem 4.8, apply without change. There is no need here to assume that $\mathfrak{l} = \mathfrak{k}$; $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ can be any θ-stable parabolic subalgebra of (any) $\mathfrak{g}$.

This case was however already known; it is implicit in [K3] and it is explicitly mentioned in [V3]. □

Remark 4.10. For a finite-dimensional module V, one can generalize the above remark and prove self-adjointness of D for the more general setting when $\mathfrak{r} \subset \mathfrak{g}$ is not necessarily a Levi subalgebra of a parabolic subalgebra, but any real and θ-stable reductive subalgebra (to which B then restricts nondegenerately). Namely, let $\mathfrak{s}$ be the orthocomplement of $\mathfrak{r}$

and let $\mathfrak{s}^+$ be a maximal isotropic subspace of $\mathfrak{s}$. Let $S = \bigwedge^{\cdot} \mathfrak{s}^+$ be a spin module for $C(\mathfrak{s})$. Define the form $\langle\, ,\rangle_{pos}$ on S and $V \otimes S$ in the same way as above, using the admissible form on V. One shows as in Lemma 4.1 that the adjoint of any $X \in \mathfrak{s}$ on S with respect to $\langle\, ,\rangle_{pos}$ is $-\theta\bar{X}$. In particular, if we take bases Z_i of $\mathfrak{s}_0 \cap \mathfrak{k}_0$ and Z'_i of $\mathfrak{s}_0 \cap \mathfrak{p}_0$ orthonormal with respect to $\langle\, ,\rangle_{pos}$, then the adjoint of Z_i is $-Z_i$ and the adjoint of Z'_i is Z'_i. So the linear term $\sum Z_i \otimes Z_i + \sum Z'_i \otimes Z'_i$ of D is self adjoint with respect to $\langle\, ,\rangle_{pos}$ on $V \otimes S$. Moreover, the dual bases of Z_i, Z'_i with respect to B are $-Z_i$, Z'_i, and so the cubic term of D can be written as

$$v = -\frac{1}{2}\left(\sum_{i<j<k} B([-Z_i,-Z_j],-Z_k)Z_iZ_jZ_k + \sum_{i;j<k} B([-Z_i,Z'_j],Z'_k)Z_iZ'_jZ'_k\right)$$

$$= \frac{1}{2}\left(\sum_{i<j<k} B([Z_i,Z_j],Z_k)Z_iZ_jZ_k + \sum_{i;j<k} B([Z_i,Z'_j],Z'_k)Z_iZ'_jZ'_k\right)$$

(the other terms are zero). Since the adjoint of $Z_iZ_jZ_k$ is $(-Z_k)(-Z_j)(-Z_i) = Z_iZ_jZ_k$, the adjoint of $Z_iZ'_jZ'_k$ is $Z'_kZ'_j(-Z_i) = Z_iZ'_jZ'_k$, and the coefficients are real, we see that v is also self-adjoint. Hence D is self-adjoint.

5. Relative Dirac operators

In this section we compare various Dirac operators arising from two compatible decompositions

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s} = \mathfrak{r}' \oplus \mathfrak{s}'.$$

Here both decompositions are like in the Kostant's setting for the cubic Dirac operator: $\mathfrak{r}$ and $\mathfrak{r}'$ are reductive subalgebras of $\mathfrak{g}$ such that the Killing form B restricts to a nondegenerate form on each of them, while $\mathfrak{s}$ respectively $\mathfrak{s}'$ are the orthogonals of $\mathfrak{r}$ respectively $\mathfrak{r}'$. Compatibility of the two decompositions means

$$\mathfrak{r} = \mathfrak{r}\cap\mathfrak{r}' \oplus \mathfrak{r}\cap\mathfrak{s}'; \qquad \mathfrak{s} = \mathfrak{s}\cap\mathfrak{r}' \oplus \mathfrak{s}\cap\mathfrak{s}'; \qquad \mathfrak{r}' = \mathfrak{r}'\cap\mathfrak{r} \oplus \mathfrak{r}'\cap\mathfrak{s}; \qquad \mathfrak{s}' = \mathfrak{s}'\cap\mathfrak{r} \oplus \mathfrak{s}'\cap\mathfrak{s}.$$

Clearly, all these decompositions are orthogonal.

Example 5.1. If $\mathfrak{l}$ is a θ-stable Levi subalgebra of $\mathfrak{g}$, then $\mathfrak{r} = \mathfrak{l}$ and $\mathfrak{r}' = \mathfrak{k}$ satisfy the above conditions.

Now $\mathfrak{r}\cap\mathfrak{r}'$ is also a reductive subalgebra of $\mathfrak{g}$ to which B restricts nondegenerately. The corresponding decomposition is

$$\mathfrak{g} = \mathfrak{r}\cap\mathfrak{r}' \oplus (\mathfrak{s} \oplus \mathfrak{r}\cap\mathfrak{s}').$$

The corresponding Dirac operator will be denoted by $D(\mathfrak{g}, \mathfrak{r}\cap\mathfrak{r}')$. To write it down, take orthonormal bases Z_i for $\mathfrak{s}$, and Z'_i for $\mathfrak{r}\cap\mathfrak{s}'$. Identify

$$U(\mathfrak{g}) \otimes C(\mathfrak{s} \oplus \mathfrak{r}\cap\mathfrak{s}') = U(\mathfrak{g}) \otimes C(\mathfrak{s})\bar{\otimes}C(\mathfrak{r}\cap\mathfrak{s}'), \tag{1}$$

where $\bar{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product. Now if W_j is the union of the bases Z_i and Z'_i, then by Kostant's definition, $D(\mathfrak{g}, \mathfrak{r}\cap\mathfrak{r}')$ is the element

$$D(\mathfrak{g}, \mathfrak{r}\cap\mathfrak{r}') = \sum_j W_j \otimes W_j - \frac{1}{2}\sum_{i<j<k} B([W_i,W_j],W_k) \otimes W_iW_jW_k$$

of $U(\mathfrak{g}) \otimes C(\mathfrak{s} \oplus \mathfrak{r}\cap\mathfrak{s}')$.

Kostant's original definition uses exterior multiplication instead of Clifford multiplication in the second sum (with the Clifford and exterior algebras identified as vector spaces via Chevalley identification). For orthogonal vectors, there is however no difference between exterior and Clifford multiplication, so the above definition is the same as Kostant's.

Taking into account the decomposition (1), we see

(2)
$$D(\mathfrak{g},\mathfrak{r}\cap\mathfrak{r}')=\sum_i Z_i\otimes Z_i\otimes 1+\sum_j Z_j'\otimes 1\otimes Z_j'-\frac{1}{2}\sum_{i<j<k}B([Z_i,Z_j],Z_k)\otimes Z_iZ_jZ_k\otimes 1-$$
$$\frac{1}{2}\sum_{i<j}\sum_k B([Z_i,Z_j],Z_k')\otimes Z_iZ_j\otimes Z_k'-\frac{1}{2}\sum_{i<j<k}B([Z_i',Z_j'],Z_k')\otimes 1\otimes Z_i'Z_j'Z_k'.$$

Note that the terms with Z_i, Z_j' and Z_k' do not appear, because $B([Z_i,Z_j'],Z_k')=B(Z_i,[Z_j',Z_k'])=0$, as $[Z_j',Z_k']\in\mathfrak{r}$ is orthogonal to $\mathfrak{s}$.

We consider $U(\mathfrak{g})\otimes C(\mathfrak{s})$ as the subalgebra $U(\mathfrak{g})\otimes C(\mathfrak{s})\otimes 1$ of $U(\mathfrak{g})\otimes C(\mathfrak{s})\bar{\otimes}C(\mathfrak{r}\cap\mathfrak{s}')$. In view of this, we see that the first and third sum in (2) combine to give $D(\mathfrak{g},\mathfrak{r})$, the Kostant's cubic Dirac operator corresponding to $\mathfrak{r}\subset\mathfrak{g}$.

The remaining three sums come from the cubic Dirac operator corresponding to $\mathfrak{r}\cap\mathfrak{r}'\subset\mathfrak{r}$. However, this is an element of the algebra $U(\mathfrak{r})\otimes C(\mathfrak{r}\cap\mathfrak{s}')$, and this algebra has to be embedded into $U(\mathfrak{g})\otimes C(\mathfrak{s})\bar{\otimes}C(\mathfrak{r}\cap\mathfrak{s}')$ diagonally, not as $U(\mathfrak{r})\otimes 1\otimes C(\mathfrak{r}\cap\mathfrak{s}')$. Namely, we use the diagonal embedding $U(\mathfrak{r})\cong U(\mathfrak{r}_\Delta)\subset U(\mathfrak{g})\otimes C(\mathfrak{s})$ from the setting $\mathfrak{g}=\mathfrak{r}\oplus\mathfrak{s}$; the definition is the same as for $\mathfrak{r}=\mathfrak{l}$ above Proposition 2.6. Thus we embed

$$\Delta: U(\mathfrak{r})\otimes C(\mathfrak{r}\cap\mathfrak{s}')\cong U(\mathfrak{r}_\Delta)\bar{\otimes}C(\mathfrak{r}\cap\mathfrak{s}')\subset U(\mathfrak{g})\otimes C(\mathfrak{s})\bar{\otimes}C(\mathfrak{r}\cap\mathfrak{s}').$$

We will denote $\Delta(D(\mathfrak{r},\mathfrak{r}\cap\mathfrak{r}'))$ by $D_\Delta(\mathfrak{r},\mathfrak{r}\cap\mathfrak{r}')$ and call it a relative Dirac operator.

Theorem 5.2. *With notation as above,*
(i) $D(\mathfrak{g},\mathfrak{r}\cap\mathfrak{r}')=D(\mathfrak{g},\mathfrak{r})+D_\Delta(\mathfrak{r},\mathfrak{r}\cap\mathfrak{r}')$;
(ii) The summands $D(\mathfrak{g},\mathfrak{r})$ *and* $D_\Delta(\mathfrak{r},\mathfrak{r}\cap\mathfrak{r}')$ *anticommute.*

Proof. To prove (i), we need to describe the image under Δ of

$$D(\mathfrak{r},\mathfrak{r}\cap\mathfrak{r}')=\sum_i Z_i'\otimes Z_i'-\frac{1}{2}\sum_{i<j<k}B([Z_i',Z_j'],Z_k')\otimes Z_i'Z_j'Z_k'\in U(\mathfrak{r})\otimes C(\mathfrak{r}\cap\mathfrak{s}'),\tag{3}$$

and see that it matches the second, fourth and fifth sum in (2). In fact, it is obvious that the image under Δ of the second sum in (3) equals the fifth sum in (2), and it remains to identify

$$\sum_i\Delta(Z_i'\otimes Z_i')=\sum_i Z_i'\otimes 1\otimes Z_i'+1\otimes\alpha(Z_i')\otimes Z_i'.\tag{4}$$

Namely, $\Delta(Z\otimes Z')=Z\otimes 1\otimes Z'+1\otimes\alpha(Z)\otimes Z'$, where $\alpha:\mathfrak{r}\to\mathfrak{so}(\mathfrak{s})\hookrightarrow C(\mathfrak{s})$ is the action map of $\mathfrak{r}$ on $\mathfrak{s}$, followed by the standard inclusion of $\mathfrak{so}(\mathfrak{s})$ into $C(\mathfrak{s})$, given by

$$E_{ij}-E_{ji}\mapsto\frac{1}{2}Z_iZ_j,$$

where E_{ij} are the matrix units relative to the basis Z_i.

Thus we are left with showing that the second sum in (4) equals the third sum in (2), i.e., that

$$\sum_k 1\otimes\alpha(Z_k')\otimes Z_k'=-\frac{1}{2}\sum_{i<j}\sum_k B([Z_i,Z_j],Z_k')\otimes Z_iZ_j\otimes Z_k'.$$

This will follow if we see that

$$\alpha(Z'_k) = -\frac{1}{2}\sum_{i<j} B([Z_i, Z_j], Z'_k) Z_i Z_j$$

for any k. But this last equality is clear from the definition of α. Namely, the matrix of $\operatorname{ad} Z'_k$ on $\mathfrak{s}$ in the basis Z_i is $\sum_{i<j} B([Z'_k, Z_j], Z_i)(E_{ij} - E_{ji})$.

To prove (ii), we use the fact that $D(\mathfrak{g}, \mathfrak{r})$ commutes with $\mathfrak{r}_\Delta$, which is one of the most basic properties of $D(\mathfrak{g}, \mathfrak{r})$. It follows that the anticommutator

$$\left[D(\mathfrak{g}, \mathfrak{r}) \otimes 1, (Z'_i \otimes 1 + 1 \otimes \alpha(Z'_i)) \otimes Z'_i\right] = \left[D(\mathfrak{g}, \mathfrak{r}), Z'_i \otimes 1 + 1 \otimes \alpha(Z'_i)\right] \otimes Z'_i$$

is zero for any i. Hence $\left[D(\mathfrak{g}, \mathfrak{r}) \otimes 1, \Delta(\sum_i Z'_i \otimes Z'_i)\right] = 0$. It remains to see that also

$$\left[D(\mathfrak{g}, \mathfrak{r}) \otimes 1, 1 \otimes 1 \otimes (-\frac{1}{2}) \sum_{i<j<k} B([Z'_i, Z'_j], Z'_k) Z'_i Z'_j Z'_k\right] = 0.$$

This follows from the definition of $\bar{\otimes}$, since all the $C(\mathfrak{s})$-parts of the monomial terms of $D(\mathfrak{g}, \mathfrak{r})$, and also all $Z'_i Z'_j Z'_k \in C(\mathfrak{r} \cap \mathfrak{s}')$, are odd. $\square$

Example 5.3. In the setting of Example 5.1, we obtain

$$D(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}) = D(\mathfrak{g}, \mathfrak{l}) + D_\Delta(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k}) = D(\mathfrak{g}, \mathfrak{k}) + D_\Delta(\mathfrak{k}, \mathfrak{l} \cap \mathfrak{k}),$$

and both decompositions have anticommuting summands. If $\mathfrak{l}$ is contained in $\mathfrak{k}$, which is possible if and only if $\mathfrak{g}$ and $\mathfrak{k}$ have equal ranks, then we get only one nontrivial decomposition, $D(\mathfrak{g}, \mathfrak{l}) = D(\mathfrak{g}, \mathfrak{k}) + D_\Delta(\mathfrak{k}, \mathfrak{l})$. This case will be of special interest below.

We now want to use Theorem 5.2 to relate the Dirac cohomology of the various Dirac operators involved. In some cases one can apply the following easy fact from linear algebra. We define the cohomology of any linear operator T on a vector space V to be the vector space $H(T) = \operatorname{Ker} T / (\operatorname{Im} T \cap \operatorname{Ker} T)$.

Lemma 5.4. *Let A and B be anticommuting linear operators on a vector space V. Assume that A^2 diagonalizes on V, i.e. $V = \bigoplus_\lambda V_\lambda$, with $A^2 = \lambda$ on V_λ. Then the cohomology $H(A+B)$ of $A+B$ on V is the same as the cohomology of the restriction of $A+B$ to $V_0 = \operatorname{Ker} A^2$.*

Proof. Since $A+B$ commutes with A^2, its kernel, image and cohomology decompose accordingly to eigenspaces V_λ. We thus have to prove that $A+B$ has no cohomology on V_λ for $\lambda \neq 0$. In other words, we are to prove that $\operatorname{Ker}(A+B) \subset \operatorname{Im}(A+B)$ on V_λ.

Let $v \in V_\lambda$ be such that $(A+B)v = 0$, i.e., $Av = -Bv$. Then

$$(A+B)Av = A^2 v + BAv = A^2 v - ABv = 2A^2 v = 2\lambda v,$$

and hence $v = \frac{1}{2\lambda}(A+B)Av$ is in the image of $A+B$. $\square$

Corollary 5.5. *In the setting of Lemma 5.4, assume further that* $\operatorname{Ker} A^2 = \operatorname{Ker} A = H(A)$*; so* $\operatorname{Ker} A \cap \operatorname{Im} A = 0$*. Then $H(A+B)$ is equal to the cohomology of B restricted to the cohomology (i.e., kernel) of A.*

Proof. By Lemma 5.4, $H(A+B)$ is the cohomology of $A+B$ on $\operatorname{Ker} A$. But on $\operatorname{Ker} A$, $A+B = B$. $\square$

To apply this to Dirac cohomology, denote by $H_D(\mathfrak{g}, \mathfrak{r}; V)$ the Dirac cohomology of a $(\mathfrak{g}, K)$-module V with respect to $D(\mathfrak{g}, \mathfrak{r})$; analogous notation will be used for other Dirac operators. The reader should bear in mind that $H_D(\mathfrak{g}, \mathfrak{r}; V)$ is in fact the cohomology of the operator $D(\mathfrak{g}, \mathfrak{r})$ on the space $V \otimes S$.

Corollary 5.6. *Let $\mathfrak{r} \subset \mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$, with $B|_{\mathfrak{r}\times\mathfrak{r}}$ nondegenerate. As usual, let $\mathfrak{s}$ be the orthocomplement of $\mathfrak{r}$. Assume that either $\dim \mathfrak{p}$ is even, or $\dim \mathfrak{s} \cap \mathfrak{k}$ is even*[1]*. Let V be an irreducible unitary $(\mathfrak{g}, K)$-module. Then*

$$H_D(\mathfrak{g}, \mathfrak{r}; V) = H_D(\mathfrak{k}, \mathfrak{r}; H_D(\mathfrak{g}, \mathfrak{k}; V)),$$

i.e., the Dirac cohomology can be calculated "in stages", as the $D(\mathfrak{k}, \mathfrak{r})$-cohomology of the $D(\mathfrak{g}, \mathfrak{k})$-cohomology.

Proof. Since V is unitary, we can consider the form $\langle\,,\rangle_{pos}$ on $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$ introduced in Section 4. Here $\mathfrak{p}^+$ is just a maximal isotropic subspace of $\mathfrak{p}$, it is not a subalgebra in general. We can extend this form to all of $V \otimes S$, by choosing any positive definite form on the spin module S_1 for $C(\mathfrak{s} \cap \mathfrak{k})$. Here we identify $S = \bigwedge^{\cdot} \mathfrak{p}^+ \otimes S_1$, which can be done by the assumption on dimensions. Let $A = D(\mathfrak{g}, \mathfrak{k})$ and $B = D_\Delta(\mathfrak{k}, \mathfrak{r})$.

By Corollary 4.3, A is anti-self-adjoint, and consequently the conditions of Corollary 5.5 are satisfied. So the cohomology with respect to $D(\mathfrak{g}, \mathfrak{r})$ is the cohomology with respect to B of $\operatorname{Ker} A = H_D(\mathfrak{g}, \mathfrak{k}; V) \otimes S_1$.

Now $H_D(\mathfrak{g}, \mathfrak{k}; V) \subset V \otimes \bigwedge^{\cdot} \mathfrak{p}^+ \subset V \otimes S$ is a $\tilde{K}$-module, with Lie algebra $\mathfrak{k}$ acting through $\mathfrak{k}_\Delta$. The Dirac cohomology of this module with respect to $D(\mathfrak{k}, \mathfrak{r})$ is thus identified with the cohomology with respect to $B = D_\Delta(\mathfrak{k}, \mathfrak{r})$. □

6. The case of compact Levi subalgebra

In this section we first consider a reductive subalgebra $\mathfrak{r}_0$ of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. Later on we will specialize to the case when $\mathfrak{r} = \mathfrak{l}$ is a Levi subalgebra of a θ-stable parabolic subalgebra of $\mathfrak{g}$.

The first thing we will do in this situation is generalize Corollary 5.6 to nonunitary modules. Like there, we assume for simplicity that either $\dim \mathfrak{p}$ is even or $\dim \mathfrak{s} \cap \mathfrak{k}$ is even, so that we can write the spin module as $S = \bigwedge^{\cdot} \mathfrak{p}^+ \otimes S_1$. The idea is to reverse the roles of $D(\mathfrak{g}, \mathfrak{k})$ and $D_\Delta(\mathfrak{k}, \mathfrak{r})$. Namely, for any admissible $(\mathfrak{g}, K)$-module V, we can decompose $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$ into a direct sum of finite dimensional (unitary) modules for the spin double cover $\tilde{K}$ of K. Hence, by Remark 4.10, there is a positive definite form $\langle\,,\rangle$ on $V \otimes S = V \otimes \bigwedge^{\cdot} \mathfrak{p}^+ \otimes S_1$, such that $D_\Delta(\mathfrak{k}, \mathfrak{r})$ is self-adjoint with respect to $\langle\,,\rangle$.

It follows that $B = D_\Delta(\mathfrak{k}, \mathfrak{r})$ is a semisimple operator, while for $A = D(\mathfrak{g}, \mathfrak{k})$ we still have that A^2 is semisimple. In this situation, we have the following lemma which complements Corollary 5.5:

Lemma 6.1. *Let A and B be anticommuting linear operators on a vector space V, such that A^2 and B are semisimple (i.e., can be diagonalized). Then $H(A+B)$ is the cohomology (i.e., the kernel) of B acting on $H(A)$.*

Proof. Applying Lemma 5.4, we can replace V by $\operatorname{Ker} A^2$, i.e., assume $A^2 = 0$. On the other hand, by Corollary 5.5, $H(A + B)$ is the cohomology of A acting on $\operatorname{Ker} B$.

Since B is semisimple, we can decompose

$$V = \operatorname{Ker} B \oplus \bigoplus_\lambda V_\lambda \oplus V_{-\lambda}$$

into the (discrete) sum of eigenspaces for B. Here if both λ and $-\lambda$ are eigenvalues, we choose one of them to represent the pair. Since A anticommutes with B, it preserves $\operatorname{Ker} B$, and maps V_λ to $V_{-\lambda}$ and vice versa. Therefore, $H(A)$ decomposes into a $\operatorname{Ker} B$-part and $V_\lambda \oplus V_{-\lambda}$-parts. The $\operatorname{Ker} B$-part is equal to $H(A + B)$ and we will be done if

[1]It should be possible to eliminate this assumption by using the graded version of the spin module, as explained at the end of Section 8.

we show that B has no kernel on $H(A|_{V_\lambda\oplus V_{-\lambda}})$. Let $v = v_1 + v_2 \in V_\lambda \oplus V_{-\lambda}$ be in $\operatorname{Ker} A$, and assume that $Bv \in \operatorname{Im} A$. This implies $\lambda v_1 - \lambda v_2$ is in $\operatorname{Im} A$, so $v_1 - v_2 \in \operatorname{Im} A$. This however can only happen if both v_1 and v_2 are in $\operatorname{Im} A$, again because A exchanges V_λ and $V_{-\lambda}$. But then also $v = v_1 + v_2$ is in $\operatorname{Im} A$, so v is zero in cohomology and we are done.

□

This now immediately implies the following theorem which says that Dirac cohomology with respect to a subalgebra $\mathfrak{r} \subset \mathfrak{k}$ as above can be calculated "in stages".

Theorem 6.2. *Let $\mathfrak{r}_0$ be a reductive subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. Let V be an admissible $(\mathfrak{g}, K)$-module. Then the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{r})$ can be calculated as the Dirac cohomology with respect to $D(\mathfrak{k}, \mathfrak{r})$ of the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{k})$ of V. In other words:*

$$H_D(\mathfrak{g}, \mathfrak{r}; V) = H_D(\mathfrak{k}, \mathfrak{r}; H_D(\mathfrak{g}, \mathfrak{k}; V)).$$

Also, we can reverse the order of taking Dirac cohomology, i.e.,

$$H_D(\mathfrak{g}, \mathfrak{r}; V) = H(D(\mathfrak{g}, \mathfrak{k})\big|_{H_D(\mathfrak{k},\mathfrak{r};V)}).$$

Proof. The first formula was explained above, and the second is a direct application of Corollary 5.5, with $A = D_\Delta(\mathfrak{k}, \mathfrak{r})$ and $B = D(\mathfrak{g}, \mathfrak{k})$ (opposite from Corollary 5.6). □

For the rest of this section we consider a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, with the Levi subalgebra $\mathfrak{l}$ contained in $\mathfrak{k}$. In particular, there is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{l} \subset \mathfrak{k}$; so $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank. The opposite parabolic subalgebra is $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$. As before, we denote $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$, so $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}$.

We apply the above considerations to $\mathfrak{r} = \mathfrak{l}$. Since $H_D(\mathfrak{g}, \mathfrak{k}; V)$ is a finite dimensional $\tilde{K}$-module, and $\mathfrak{k}$ and $\mathfrak{l}$ have equal rank, $H_D(\mathfrak{k}, \mathfrak{l}; H_D(\mathfrak{g}, \mathfrak{k}; V))$ is given by [K4], Theorem 5.1. (This can also be read off from Remark 4.9, by using Poincaré duality to pass to $\mathfrak{u}$-cohomology and then the better known Kostant's formula for $\mathfrak{u}$-cohomology; in fact, this is how Kostant proves it.) This gives $H_D(\mathfrak{g}, \mathfrak{l}; V)$ very explicitly provided we know $H_D(\mathfrak{g}, \mathfrak{k}; V)$ explicitly. For example, one can in this way calculate the Dirac cohomology of the discrete series representations with respect to the (compact) Cartan subalgebra $\mathfrak{t}$:

Example 6.3. Let $V = A_{\mathfrak{b}}(\lambda)$ be a discrete series representation; here $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$ containing a compact Cartan subalgebra $\mathfrak{t}$. The infinitesimal character of V is $\lambda + \rho$. Then the Dirac cohomology of V with respect to $D(\mathfrak{g}, \mathfrak{k})$ consists of a single $\tilde{K}$-type $V(\mu)$, whose highest weight is $\mu = \lambda + \rho_n$, where $\rho_n = \rho(\mathfrak{u} \cap \mathfrak{p})$. This is obtained from the highest weight of the lowest K-type of V, $\lambda + 2\rho_n$, by shifting by $-\rho_n$ (the lowest weight of S).

Namely, it is shown in [HP1], Proposition 5.4, that this $\tilde{K}$-type is contained in the Dirac cohomology. Since V has a unique lowest K-type, and since $-\rho_n$ is the lowest weight of the spin module, with multiplicity one, it follows that any other $\tilde{K}$-type has strictly larger highest weight, and thus can not contribute to the Dirac cohomology.

We now apply the above mentioned Kostant's formula (Theorem 5.1 of [K4]) to calculate the Dirac cohomology with respect to $D(\mathfrak{k}, \mathfrak{t})$ (we again stress that $\mathfrak{k}$ and $\mathfrak{t}$ have equal rank):

$$H_D(\mathfrak{k}, \mathfrak{t}; V(\mu)) = \operatorname{Ker} D(\mathfrak{k}, \mathfrak{t}) = \oplus_{w\in W_{\mathfrak{k}}} \mathbb{C}_{w(\mu+\rho_c)}.$$

It follows from $\mu + \rho_c = \lambda + \rho$ that

$$H_D(\mathfrak{k}, \mathfrak{t}; V(\mu)) = \oplus_{w\in W_{\mathfrak{k}}} \mathbb{C}_{w(\lambda+\rho)}.$$

Remark 6.4. Comparing with Schmid's formula in Theorem 4.1 of [S], we have

$$H^*(\bar{\mathfrak{n}}, A_{\mathfrak{b}}(\lambda)) = H_D(\mathfrak{g}, \mathfrak{t}; A_{\mathfrak{b}}(\lambda)) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{n}})}.$$

(note that Schmid's $\mathfrak{n}$ is our $\bar{\mathfrak{n}}$, and his λ is our $\lambda + \rho$.)

In other words, $\mathfrak{n}$-cohomology of a discrete series representation coincides with the Dirac cohomology up to a ρ-shift. This fact is however not covered by our results in Sections 4 and 7. This indicates that it should be possible to generalize our results, probably under some conditions on the representations involved. □

We now want to similarly analyze the "half-Dirac" operators C and C^-. Let $u_1, \dots, u_k$ be a basis for $\mathfrak{u} \cap \mathfrak{k}$ and let $v_1, \dots, v_p$ be a basis for $\mathfrak{u} \cap \mathfrak{p}$. These can be taken to be the root vectors corresponding to compact, respectively noncompact positive roots, with respect to some $\Delta^+(\mathfrak{g}, \mathfrak{t})$ compatible with $\mathfrak{u}$. We normalize these bases so that the dual bases for $\bar{\mathfrak{u}} \cap \mathfrak{k}$ respectively $\bar{\mathfrak{u}} \cap \mathfrak{p}$ with respect to the Killing form are $u_i^* = -\bar{u}_i$ respectively $v_i^* = \bar{v}_i$.

As before, $D = D(\mathfrak{g}, \mathfrak{l})$, $C = C(\mathfrak{g}, \mathfrak{l}) = A + 1 \otimes a$ and $C^- = C^-(\mathfrak{g}, \mathfrak{l}) = A^- + 1 \otimes a^-$ denote the Dirac operator for the pair $(\mathfrak{g}, \mathfrak{l})$ and its parts. We can now further decompose these parts and write

$$A = A_{\mathfrak{k}} + A_{\mathfrak{p}}; \quad a = a_{\mathfrak{k}} + a_{\mathfrak{k}\mathfrak{p}} + a_{\mathfrak{p}}$$

and analogously for A^- and a^-. Here

$$A_{\mathfrak{k}} = \sum_i u_i^* \otimes u_i, \quad A_{\mathfrak{p}} = \sum_i v_i^* \otimes v_i;$$
$$a_{\mathfrak{k}} = -\frac{1}{4} \sum_{i,j} [u_i^*, u_j^*] u_i u_j, \quad a_{\mathfrak{k}\mathfrak{p}} = -\frac{1}{2} \sum_{i,j} [u_i^*, v_j^*] u_i v_j, \quad a_{\mathfrak{p}} = -\frac{1}{4} \sum_{i,j} [v_i^*, v_j^*] v_i v_j.$$

The expressions for $A_{\mathfrak{k}}^-$, $A_{\mathfrak{p}}^-$, $a_{\mathfrak{k}}^-$, $a_{\mathfrak{k}\mathfrak{p}}^-$ and $a_{\mathfrak{p}}^-$ are obtained by exchanging u_i with u_i^* and v_i with v_i^*.

In the following, we will consider the Clifford algebra $C(\mathfrak{s})$ as a subalgebra of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, embedded as $1 \otimes C(\mathfrak{s})$. In particular, $1 \otimes a_{\mathfrak{k}}$ gets identified with $a_{\mathfrak{k}}$, $1 \otimes a_{\mathfrak{k}\mathfrak{p}}$ with $a_{\mathfrak{k}\mathfrak{p}}$ and so on.

Recall that by Theorem 5.2, $D(\mathfrak{g}, \mathfrak{l}) = D(\mathfrak{g}, \mathfrak{k}) + D_\Delta(\mathfrak{k}, \mathfrak{l})$, where $D_\Delta(\mathfrak{k}, \mathfrak{l})$ is the image of $D(\mathfrak{k}, \mathfrak{l})$ under the diagonal embedding $\Delta : U(\mathfrak{k}) \otimes C(\mathfrak{s} \cap \mathfrak{k}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p}) \bar{\otimes} C(\mathfrak{s} \cap \mathfrak{k})$. Here Δ sends $1 \otimes C(\mathfrak{s} \cap \mathfrak{k})$ identically onto $1 \otimes 1 \otimes C(\mathfrak{s} \cap \mathfrak{k})$, and for $X \in \mathfrak{k}$,

$$\Delta(X \otimes 1) = X \otimes 1 \otimes 1 + 1 \otimes \alpha(X) \otimes 1,$$

with $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ given by the action of $\mathfrak{k}$ on $\mathfrak{p}$ as before.

Clearly, $D(\mathfrak{g}, \mathfrak{k}) = A_{\mathfrak{p}} + A_{\mathfrak{p}}^-$, while $D_\Delta(\mathfrak{k}, \mathfrak{l})$ is the sum of all other of the above parts. We want to make this more precise; namely, in the obvious notation, $D(\mathfrak{k}, \mathfrak{l}) = C(\mathfrak{k}, \mathfrak{l}) + C^-(\mathfrak{k}, \mathfrak{l})$, and we want to identify the images of these summands under Δ. Denote these images by $C_\Delta(\mathfrak{k}, \mathfrak{l})$ and $C_\Delta^-(\mathfrak{k}, \mathfrak{l})$.

To do this, we use an expression for $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ in terms of a basis and a dual basis: if w_r is a basis of $\mathfrak{p}$ with dual basis w_r^*, then

$$\alpha(X) = -\frac{1}{4} \sum_{r,s} B([w_r^*, w_s^*], X) \otimes w_r w_s.$$

(This was already used in Section 5, for an orthonormal basis. The proof of this slightly more general version is the same.) Applying this to the basis $v_1, \dots, v_p, v_1^*, \dots, v_p^*$ and the dual basis $v_1^*, \dots, v_p^*, v_1, \dots, v_p$, we get

$$\alpha(X) = -\frac{1}{4}\sum_{j,k} B([v_j^*, v_k^*], X) \otimes v_j v_k \\ - \frac{1}{2}\sum_{j,k} B([v_j, v_k^*], X) \otimes v_j^* v_k - \frac{1}{4}\sum_{j,k} B([v_j, v_k], X) \otimes v_j^* v_k^*.$$

Since $C(\mathfrak{k}, \mathfrak{l}) = \sum_i u_i^* \otimes u_i - \frac{1}{4} \otimes \sum_{i,j}[u_i^*, u_j^*]u_i u_j$, we see that $\Delta(C(\mathfrak{k}, \mathfrak{l})) = A_\mathfrak{k} + \sum_i 1 \otimes \alpha(u_i^*) \otimes u_i + a_\mathfrak{k}$. We need to calculate the middle term, $\sum_i 1 \otimes \alpha(u_i^*) \otimes u_i$. Applying the above expression for α, we get three sums over i, j and k.

We first notice that the first of these three sums is 0, since B is 0 on $\bar{\mathfrak{u}}$. To calculate the second sum, write $B([v_j, v_k^*], u_i^*) = B(v_j, [v_k^*, u_i^*])$, and observe that since $[v_k^*, u_i^*] \in \bar{\mathfrak{u}} \cap \mathfrak{p}$, $\sum_j B(v_j, [v_k^*, u_i^*])v_j^* = [v_k^*, u_i^*]$. Therefore the second of the three sums is

$$-\frac{1}{2} \otimes \sum_{i,k}[v_k^*, u_i^*]v_k u_i = a_{\mathfrak{k}\mathfrak{p}}.$$

Finally, the third sum is calculated by noting that since $[v_j, v_k] \in \mathfrak{u} \cap \mathfrak{k}$, $\sum_i B([v_j, v_k], u_i^*)u_i = [v_j, v_k]$. It follows that the third sum is

$$-\frac{1}{4} \otimes \sum_{j,k} v_j^* v_k^* [v_j, v_k] = a_\mathfrak{p}^-.$$

A completely analogous calculation applies to $C^-(\mathfrak{k}, \mathfrak{l})$, so we proved:

Proposition 6.5. *Under the diagonal map $\Delta : U(\mathfrak{k}) \otimes C(\mathfrak{s} \cap \mathfrak{k}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p}) \bar{\otimes} C(\mathfrak{s} \cap \mathfrak{k})$, $C(\mathfrak{k}, \mathfrak{l})$ and $C^-(\mathfrak{k}, \mathfrak{l})$ correspond to*

$$C_\Delta(\mathfrak{k}, \mathfrak{l}) = A_\mathfrak{k} + a_\mathfrak{k} + a_{\mathfrak{k}\mathfrak{p}} + a_\mathfrak{p}^- \qquad \text{and} \qquad C_\Delta^-(\mathfrak{k}, \mathfrak{l}) = A_\mathfrak{k}^- + a_\mathfrak{k}^- + a_{\mathfrak{k}\mathfrak{p}}^- + a_\mathfrak{p}. \quad \square$$

Note the unexpected feature of this result, the mixing of the positive and negative parts under the diagonal embedding. Namely, $a_\mathfrak{p}$ and $a_\mathfrak{p}^-$ have opposite positions from the ones one would expect. So we do not have an analogue of Theorem 5.2 for C and C^-, unless $a_\mathfrak{p} = a_\mathfrak{p}^- = 0$. This last thing happens precisely when the pair $(\mathfrak{g}, \mathfrak{k})$ is hermitian symmetric. This is the reason why we are able to obtain results about $\mathfrak{u}$-cohomology only in hermitian case. Maybe this peculiar behavior has something to do with the fact that some of the most concrete results about $\mathfrak{n}$-cohomology, like [E], [C] or [A], are also obtained in hermitian situation only.

Another difficulty with the non-hermitian case is the fact that while we can write $D(\mathfrak{g}, \mathfrak{k}) = A_\mathfrak{p} + A_\mathfrak{p}^-$, the two summands here are not differentials (they are also not K-invariant). So there is no hope to get a Hodge decomposition like the one in Section 4. Of course, there is also no $\mathfrak{u} \cap \mathfrak{p}$-homology or cohomology, since $\mathfrak{u} \cap \mathfrak{p}$ is not a Lie algebra. Yet there is perfectly well defined Dirac cohomology for $D(\mathfrak{g}, \mathfrak{k})$, and one can hope that it will somehow replace the nonexistent $\mathfrak{p}^-$-cohomology.

What we do get without the hermitian assumption, is a copy of $\mathfrak{gl}(1,1)$ inside $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, spanned by $C_\Delta(\mathfrak{k}, \mathfrak{l})$, $C_\Delta^-(\mathfrak{k}, \mathfrak{l})$, $E_\mathfrak{k} = \Delta E(\mathfrak{k}, \mathfrak{l}) = -\frac{1}{2} \otimes \sum_i u_i^* u_i$ and $\Delta D(\mathfrak{k}, \mathfrak{l})^2$. This follows immediately from the fact that Δ is (obviously) a morphism of superalgebras.

If the pair $(\mathfrak{g}, \mathfrak{k})$ is hermitian symmetric, as we will assume in the following, then there is another copy of $\mathfrak{gl}(1,1)$ inside $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, supercommuting with the first one. It is spanned by $A_\mathfrak{p} = C(\mathfrak{g}, \mathfrak{k})$, $A_\mathfrak{p}^- = C^-(\mathfrak{g}, \mathfrak{k})$, $E_\mathfrak{p} = E(\mathfrak{g}, \mathfrak{k}) = -\frac{1}{2} \otimes \sum_i v_i^* v_i$, and $D(\mathfrak{g}, \mathfrak{k})^2$. The fact that these two copies of $\mathfrak{gl}(1,1)$ supercommute is completely analogous to Theorem 5.2.(ii). (Without the hermitian assumption, one could try to replace this second $\mathfrak{gl}(1,1)$ by a smaller superalgebra, spanned just by $D(\mathfrak{g}, \mathfrak{k})$ and $D(\mathfrak{g}, \mathfrak{k})^2$.)

Note that $C(\mathfrak{g},\mathfrak{l}) = C_\Delta(\mathfrak{k},\mathfrak{l}) + C(\mathfrak{g},\mathfrak{k})$ and $C^-(\mathfrak{g},\mathfrak{l}) = C^-_\Delta(\mathfrak{k},\mathfrak{l}) + C^-(\mathfrak{g},\mathfrak{k})$. Clearly, an analogous decomposition holds for $E(\mathfrak{g},\mathfrak{l})$; it also holds for $D(\mathfrak{g},\mathfrak{l})^2$. Namely, since the two copies of $\mathfrak{gl}(1,1)$ supercommute, we have

$$D(\mathfrak{g},\mathfrak{l})^2 = [C(\mathfrak{g},\mathfrak{l}), C^-(\mathfrak{g},\mathfrak{l})] = [C_\Delta(\mathfrak{k},\mathfrak{l}), C^-_\Delta(\mathfrak{k},\mathfrak{l})] + [C(\mathfrak{g},\mathfrak{k}), C^-(\mathfrak{g},\mathfrak{k})] = D_\Delta(\mathfrak{k},\mathfrak{l})^2 + D(\mathfrak{g},\mathfrak{k})^2.$$

This last equality can also be obtained from Kostant's formula for D^2:

$$\begin{aligned} D_\Delta(\mathfrak{k},\mathfrak{l})^2 + D(\mathfrak{g},\mathfrak{k})^2 = \\ \Delta(\Omega_\mathfrak{k}\otimes 1 - \Omega_{\mathfrak{l}_\Delta} + ||\rho_\mathfrak{k}^2|| - ||\rho_\mathfrak{l}^2||) + (\Omega_\mathfrak{g}\otimes 1 - \Omega_{\mathfrak{k}_\Delta} + ||\rho_\mathfrak{g}^2|| - ||\rho_\mathfrak{k}^2||) = \\ \Omega_\mathfrak{g}\otimes 1 - \Omega_{\mathfrak{l}_\Delta} + ||\rho_\mathfrak{g}^2|| - ||\rho_\mathfrak{l}^2|| = D(\mathfrak{g},\mathfrak{l})^2. \end{aligned}$$

(Note that this notation is not very precise, as in the equality $\Delta(\Omega_{\mathfrak{l}_\Delta}) = \Omega_{\mathfrak{l}_\Delta}$, the two $\Omega_{\mathfrak{l}_\Delta}$'s have different meanings. Still, there should be no confusion if one keeps track just where the elements are.)

In other words, we see that the $\mathfrak{gl}(1,1)$ corresponding to the pair $(\mathfrak{g},\mathfrak{l})$ sits diagonally in the direct product of the two copies of $\mathfrak{gl}(1,1)$ described above. Let us summarize the above discussion:

Corollary 6.6. *Assume $(\mathfrak{g},\mathfrak{k})$ is a hermitian symmetric pair, let $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, and assume that $\mathfrak{l}\subset\mathfrak{k}$ and $\mathfrak{u}\supset\mathfrak{p}^+$. Then there are two supercommuting copies of $\mathfrak{gl}(1,1)$ inside $U(\mathfrak{g})\otimes C(\mathfrak{s})$. One is spanned by $C_\Delta(\mathfrak{k},\mathfrak{l})$, $C^-_\Delta(\mathfrak{k},\mathfrak{l})$, $\Delta E(\mathfrak{k},\mathfrak{l})$ and $\Delta D(\mathfrak{k},\mathfrak{l})^2$ and the other is spanned by $C(\mathfrak{g},\mathfrak{k})$, $C^-(\mathfrak{g},\mathfrak{k})$, $E(\mathfrak{g},\mathfrak{k})$, and $D(\mathfrak{g},\mathfrak{k})^2$. The diagonal of the product of these two super subalgebras is the copy of $\mathfrak{gl}(1,1)$ spanned by C, C^-, E and D^2 from the end of Section 2.* □

7. Hodge decomposition for $\bar{\mathfrak{u}}$-cohomology in hermitian case

In this section, $(\mathfrak{g},\mathfrak{k})$ is a hermitian symmetric pair, $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, and we assume $\mathfrak{l}\subset\mathfrak{k}$ and $\mathfrak{u}\supset\mathfrak{p}^+$.

Let V be a unitary $(\mathfrak{g},K)$-module, and consider the form $\langle\ ,\rangle_{pos}$ on $V\otimes S$ introduced in Section 4. To apply the results of Section 4, we decompose

$$V\otimes S = V\otimes \textstyle\bigwedge^\cdot\mathfrak{p}^+\otimes\bigwedge^\cdot\mathfrak{u}\cap\mathfrak{k},$$

and embed $V\otimes\bigwedge^\cdot\mathfrak{p}^+$ as $V\otimes\bigwedge^\cdot\mathfrak{p}^+\otimes 1$. The form $\langle\ ,\rangle_{pos}$ restricts to the analogous definite form on $V\otimes\bigwedge^\cdot\mathfrak{p}^+$.

Denote as before by $D = D(\mathfrak{g},\mathfrak{l})$ the Dirac operator for the pair $(\mathfrak{g},\mathfrak{l})$ and by $C = C(\mathfrak{g},\mathfrak{l})$ and $C^- = C^-(\mathfrak{g},\mathfrak{l})$ its parts coresponding to $\mathfrak{u}$ and $\bar{\mathfrak{u}}$. By Corollary 6.6, $C = C_\Delta(\mathfrak{k},\mathfrak{l}) + C(\mathfrak{g},\mathfrak{k})$, and similarly for C^-. Moreover, the copy of $\mathfrak{gl}(1,1)$ corresponding to the pair $(\mathfrak{g},\mathfrak{k})$ supercommutes with the copy of $\mathfrak{gl}(1,1)$ corresponding to the pair $(\mathfrak{k},\mathfrak{l})$.

By Corollary 4.3, the adjoints of $C_\Delta(\mathfrak{k},\mathfrak{l})$ and $C(\mathfrak{g},\mathfrak{k})$ are respectively $C^-_\Delta(\mathfrak{k},\mathfrak{l})$ and $-C^-(\mathfrak{g},\mathfrak{k})$. So the adjoint of C is $C^{adj} = C^-_\Delta(\mathfrak{k},\mathfrak{l}) - C^-(\mathfrak{g},\mathfrak{k})$.

We consider the positive semidefinite operator $\Delta = CC^{adj} + C^{adj}C = [C, C^{adj}]$. By the above remarks we have

$$\begin{aligned} \Delta = [C_\Delta(\mathfrak{k},\mathfrak{l}) + C(\mathfrak{g},\mathfrak{k}), C^-_\Delta(\mathfrak{k},\mathfrak{l}) - C^-(\mathfrak{g},\mathfrak{k})] = \\ [C_\Delta(\mathfrak{k},\mathfrak{l}), C^-_\Delta(\mathfrak{k},\mathfrak{l})] - [C(\mathfrak{g},\mathfrak{k}), C^-(\mathfrak{g},\mathfrak{k})] = D_\Delta(\mathfrak{k},\mathfrak{l})^2 - D(\mathfrak{g},\mathfrak{k})^2. \end{aligned}$$

We know from Section 4 that $V\otimes\bigwedge^\cdot\mathfrak{p}^+$ decomposes into eigenspaces of $D(\mathfrak{g},\mathfrak{k})^2$ for eigenvalues $\lambda\le 0$. Each eigenspace is $\tilde{K}$-invariant, and each $\tilde{K}$-isotypic component of

$V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$ is contained in an eigenspace. We assume V is admissible, so the eigenspaces are finite-dimensional.

Passing from $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$ to $V \otimes S$ is tensoring with the finite-dimensional $\mathfrak{l}$-module $\bigwedge^{\cdot} \mathfrak{u} \cap \mathfrak{k}$. On this last space, there is no action of $U(\mathfrak{g})$ or $U(\mathfrak{k}_\Delta)$. So every eigenspace of $D(\mathfrak{g},\mathfrak{k})^2$ on $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$ just gets tensored with $\bigwedge^{\cdot} \mathfrak{u} \cap \mathfrak{k}$, and this gives the eigenspace on $V \otimes S$ for the same eigenvalue.

Since $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ commutes with $D(\mathfrak{g},\mathfrak{k})^2$, it preserves these eigenspaces. Moreover, the Levi subgroup $L \subset K$ corresponding to $\mathfrak{l}$ is compact. So is then the double cover $\tilde{L}$, which acts on $V \otimes S$. Since $\tilde{L}$ commutes with $D(\mathfrak{g},\mathfrak{k})^2$, it also preserves its eigenspaces and hence these eigenspaces decompose into $\tilde{L}$-irreducibles. Since $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ is up to a constant equal to the Casimir element of $\mathfrak{l}_\Delta$, it follows that $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ diagonalizes on each eigenspace of $D(\mathfrak{g},\mathfrak{k})^2$. To conclude:

Lemma 7.1. *$V \otimes S$ is a direct sum of eigenspaces for Δ. In particular, $V \otimes S = \operatorname{Ker}\Delta \oplus \operatorname{Im}\Delta$.* □

This is an analogue of Corollary 4.6 for Δ in place of D^2. Now the arguments proving Lemma 4.7 and Theorem 4.8 work without change, and we obtain

Theorem 7.2. *(a)* $\operatorname{Ker}\Delta = \operatorname{Ker} C \cap \operatorname{Ker} C^{adj}$*;*

(b) $V \otimes S = \operatorname{Ker}\Delta \oplus \operatorname{Im} C \oplus \operatorname{Im} C^{adj}$*;*

(c) $\operatorname{Ker} C = \operatorname{Ker}\Delta \oplus \operatorname{Im} C$*;*

(d) $\operatorname{Ker} C^{adj} = \operatorname{Ker}\Delta \oplus \operatorname{Im} C^{adj}$*.*

In other words, we have obtained a Hodge theorem for $\bar{\mathfrak{u}}$-cohomology.

To obtain it also for $\mathfrak{u}$-homology, we note that $(C^-)^{adj} = (C^-_\Delta(\mathfrak{k},\mathfrak{l}) + C^-(\mathfrak{g},\mathfrak{k}))^{adj} = C_\Delta(\mathfrak{k},\mathfrak{l}) - C(\mathfrak{g},\mathfrak{k})$, and so

$$[C^-,(C^-)^{adj}] = [C^-_\Delta(\mathfrak{k},\mathfrak{l}) + C^-(\mathfrak{g},\mathfrak{k}), C_\Delta(\mathfrak{k},\mathfrak{l}) - C(\mathfrak{g},\mathfrak{k})] = [C^-_\Delta(\mathfrak{k},\mathfrak{l}), C_\Delta(\mathfrak{k},\mathfrak{l})] - [C^-(\mathfrak{g},\mathfrak{k}), C(\mathfrak{g},\mathfrak{k})] = \Delta.$$

So the situation for C^- is exactly the same as for C and we conclude

Theorem 7.3. *(a)* $\operatorname{Ker}\Delta = \operatorname{Ker} C^- \cap \operatorname{Ker}(C^-)^{adj}$*;*

(b) $V \otimes S = \operatorname{Ker}\Delta \oplus \operatorname{Im} C^- \oplus \operatorname{Im}(C^-)^{adj}$*;*

(c) $\operatorname{Ker} C^- = \operatorname{Ker}\Delta \oplus \operatorname{Im} C^-$*;*

(d) $\operatorname{Ker}(C^-)^{adj} = \operatorname{Ker}\Delta \oplus \operatorname{Im}(C^-)^{adj}$*.*

In other words, Hodge decomposition also holds for $\mathfrak{u}$-homology. Moreover, we see that $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology have the same set of harmonic representatives, $\operatorname{Ker}\Delta$. In particular they are isomorphic.

We now want to relate $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology to Dirac cohomology with respect to $D(\mathfrak{g},\mathfrak{l})$. The main observation here is

Lemma 7.4. $\operatorname{Ker}\Delta = \operatorname{Ker} D_\Delta(\mathfrak{k},\mathfrak{l})^2 \cap \operatorname{Ker} D(\mathfrak{g},\mathfrak{k})^2 = \operatorname{Ker} D_\Delta(\mathfrak{k},\mathfrak{l}) \cap \operatorname{Ker} D(\mathfrak{g},\mathfrak{k})$*.*

Proof. The operators $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ and $-D(\mathfrak{g},\mathfrak{k})^2$ are both positive semidefinite and their sum is Δ. This immediately implies the first equality. The second follows from $\operatorname{Ker} D_\Delta(\mathfrak{k},\mathfrak{l})^2 = \operatorname{Ker} D_\Delta(\mathfrak{k},\mathfrak{l})$ (since $D_\Delta(\mathfrak{k},\mathfrak{l})$ is self-adjoint) and $\operatorname{Ker} D(\mathfrak{g},\mathfrak{k})^2 = \operatorname{Ker} D(\mathfrak{g},\mathfrak{k})$ (since $D(\mathfrak{g},\mathfrak{k})$ is anti-self-adjoint). □

We can now combine Theorems 7.2 and 7.3 with Lemmas 7.4 and 4.7 to conclude

Corollary 7.5. $\operatorname{Ker}\Delta = \operatorname{Ker} C_\Delta(\mathfrak{k},\mathfrak{l}) \cap \operatorname{Ker} C^-_\Delta(\mathfrak{k},\mathfrak{l}) \cap \operatorname{Ker} C(\mathfrak{g},\mathfrak{k}) \cap \operatorname{Ker} C^-(\mathfrak{g},\mathfrak{k})$*.*

Proof. It is obvious that the left hand side contains the right hand side. Conversely, if $x \in \operatorname{Ker} \Delta$, then $Cx = 0$ by Theorem 7.2, $D(\mathfrak{g}, \mathfrak{k})x = 0$ by Lemma 7.4, so $C(\mathfrak{g}, \mathfrak{k})x = 0$ by Lemma 4.7 and so also $Cx - C(\mathfrak{g}, \mathfrak{k})x = C_\Delta(\mathfrak{k}, \mathfrak{l})x = 0$. Analogously, $C^-(\mathfrak{g}, \mathfrak{k})x = 0$ and $C^-_\Delta(\mathfrak{k}, \mathfrak{l})x = 0$. □

Since $\operatorname{Ker} C_\Delta(\mathfrak{k}, \mathfrak{l}) \cap \operatorname{Ker} C(\mathfrak{g}, \mathfrak{k})$ can be thought of as the kernel of $C_\Delta(\mathfrak{k}, \mathfrak{l})$ acting on the kernel of $C(\mathfrak{g}, \mathfrak{k})$, and similarly for the C^--operators, in view of Theorem 4.8 and Remark 4.9 we can reinterprete Corollary 7.5 as follows:

Corollary 7.6. *To calculate the $\bar{\mathfrak{u}}$-cohomology of V, one can first calculate the $\mathfrak{p}^-$-cohomology of V to obtain a $\tilde{K}$-module, and then calculate the $\bar{\mathfrak{u}} \cap \mathfrak{k}$-cohomology of this module. Analogously, to calculate the $\mathfrak{u}$-homology of V, one can first calculate the $\mathfrak{p}^+$-homology of V, and then the $\mathfrak{u} \cap \mathfrak{k}$-homology of the resulting $\tilde{K}$-module.* □

Remark 7.7. Note that this is in fact the Hochschild-Serre spectral sequence for the ideal $\mathfrak{p}^-$ of $\bar{\mathfrak{u}}$ respectively the ideal $\mathfrak{p}^+$ of $\mathfrak{u}$. What we have obtained is that these Hochschild-Serre spectral sequences are always degenerate for a unitary $(\mathfrak{g}, K)$-module V. □

We now turn our attention to the Dirac cohomology of $D = D(\mathfrak{g}, \mathfrak{l})$. In addition to the above considerations, we bring in Corollary 5.6, and note that for both $D_\Delta(\mathfrak{k}, \mathfrak{l})$ and $D(\mathfrak{g}, \mathfrak{k})$ the cohomology is the same as the kernel or the kernel of the square. Thus we obtain:

Theorem 7.8. *The Dirac cohomology $H_D(\mathfrak{g}, \mathfrak{l}; V)$ of a unitary $(\mathfrak{g}, K)$-module V is isomorphic to the $\bar{\mathfrak{u}}$-cohomology of V and the $\mathfrak{u}$-homology of V up to appropriate modular twists. Moreover, all three cohomologies have the same set of harmonic representatives,* $\operatorname{Ker} \Delta$. □

8. Homological properties of Dirac cohomology

Let us start by showing that although we proved that in some cases Dirac cohomology of a unitary $(\mathfrak{g}, K)$-module with respect to $D(\mathfrak{g}, \mathfrak{l})$ can be identified with $\bar{\mathfrak{u}}$-cohomology or $\mathfrak{u}$-homology, one should by no means expect that these notions agree for general $(\mathfrak{g}, K)$-modules. Let us see that this is not the case even for $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-modules.

Consider the module V which is a nontrivial extension of the discrete series representation W of highest weight -2 by the trivial module $\mathbb{C}$:

$$0 \to \mathbb{C} \to V \to W \to 0.$$

(In other words, V is a dual Verma module.) The $\mathfrak{k}$-weights of V (for the basis element $\left[\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right]$ of $\mathfrak{k}$) are $0, -2, -4, \dots$. We are considering the case $\mathfrak{l} = \mathfrak{k}$, $\mathfrak{u}$ is spanned by $u = \frac{1}{2}\left[\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right]$ and $\bar{\mathfrak{u}}$ is spanned by $u^* = \frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right]$.

For any $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-module X we have

$$X \otimes S = X \otimes 1 \quad \oplus \quad X \otimes u,$$

with $d : X \otimes 1 \to X \otimes u$ given by $d(v \otimes 1) = u^* \cdot v \otimes u$, $\partial : X \otimes u \to X \otimes 1$ given by $\partial(v \otimes u) = u \cdot v \otimes 1$, and $D = d - 2\partial$. By an easy direct calculation, we see that

$$H_0(\mathfrak{u}; V) = 0; \qquad H_1(\mathfrak{u}; V) = \mathbb{C}v_0 \otimes u;$$
$$H^0(\bar{\mathfrak{u}}; V) = \mathbb{C}v_0 \otimes 1; \qquad H^1(\bar{\mathfrak{u}}; V) = \mathbb{C}v_0 \otimes u \oplus \mathbb{C}v_{-2} \otimes u;$$
$$H_D(V) = \operatorname{Ker} D = \mathbb{C}v_0 \otimes u.$$

as vector spaces. Here v_i denotes a vector in V of $\mathfrak{k}$-weight i. So we see

$$H_D(V) = H_\cdot(\mathfrak{u}; V) \neq H^\cdot(\bar{\mathfrak{u}}; V)$$

On the other hand, for $\mathbb{C}$ and W another easy calculation (which can be shortened by using Theorem 4.8, as both $\mathbb{C}$ and W are unitary) implies

$$\begin{gathered}
H^0(\bar{\mathfrak{u}};\mathbb{C}) = H_0(\mathfrak{u};\mathbb{C}) = H_D^0(\mathbb{C}) = \mathbb{C}1\otimes 1;\\
H^1(\bar{\mathfrak{u}};\mathbb{C}) = H_1(\mathfrak{u};\mathbb{C}) = H_D^1(\mathbb{C}) = \mathbb{C}1\otimes u;\\
H^0(\bar{\mathfrak{u}};W) = H_0(\mathfrak{u};W) = H_D^0(W) = 0;\\
H^1(\bar{\mathfrak{u}};W) = H_1(\mathfrak{u};W) = H_D^1(W) = \mathbb{C}v_{-2}\otimes u.
\end{gathered}$$

To explain why Dirac cohomology of V differs from the $\bar{\mathfrak{u}}$-cohomology of V, we will examine their behavior with respect to extensions.

Recall the well known long exact sequences for Lie algebra homology and cohomology corresponding to our short exact sequence

$$0\to\mathbb{C}\to V\to W\to 0.$$

They are

$$0\to\mathbb{C}1\otimes u\to\mathbb{C}v_0\otimes u\xrightarrow{0}\mathbb{C}v_{-2}\otimes u\to\mathbb{C}1\otimes 1\to 0\to 0\to 0$$

for $\mathfrak{u}$-homology and

$$0\to\mathbb{C}1\otimes 1\to\mathbb{C}v_0\otimes 1\to 0\to\mathbb{C}1\otimes u\to\mathbb{C}v_{-2}\otimes u\oplus\mathbb{C}v_0\otimes u\to\mathbb{C}v_{-2}\otimes u\to 0$$

for $\bar{\mathfrak{u}}$-cohomology. Here all arrows are the obvious ones except for the one labelled by 0.

For Dirac cohomology, instead of a long exact sequence (which clearly does not makes sense in general, as Dirac cohomology is not $\mathbb{Z}$-graded), there is a six-term exact sequence. In the above example, this sequence is

$$\begin{tikzcd}
\mathbb{C}1\otimes 1 \arrow[r] & 0 \arrow[r] & 0 \arrow[d]\\
\mathbb{C}v_{-2}\otimes u \arrow[u] & \mathbb{C}v_0\otimes u \arrow[l, "0"'] & \mathbb{C}1\otimes u \arrow[l]
\end{tikzcd}$$

We see that in this example the six-term sequence agrees with the $\mathfrak{u}$-homology long exact sequence only because of the presence of zeros; in general, all three sequences are different.

To define the six-term sequence in a more general situation, let us assume that the Dirac cohomology is $\mathbb{Z}_2$-graded. This happens whenever we are starting from $\mathfrak{g}=\mathfrak{r}\oplus\mathfrak{s}$ with $\mathfrak{s}$ even-dimensional; this is automatic when $\mathfrak{r}=\mathfrak{l}$ is a Levi subalgebra. Let

$$0\to X\xrightarrow{i} Y\xrightarrow{p} Z\to 0$$

be a short exact sequence of $(\mathfrak{g},K)$-modules. Tensor this sequence by S, and denote the arrows still by i and p (they get tensored by the identity on S). Assuming that D^2 is a semisimple operator for each of the three modules, we can construct a six-term exact sequence

$$\begin{tikzcd}
H_D^0(X) \arrow[r] & H_D^0(Y) \arrow[r] & H_D^0(Z) \arrow[d]\\
H_D^1(Z) \arrow[u] & H_D^1(Y) \arrow[l] & H_D^1(X) \arrow[l]
\end{tikzcd}$$

The horizontal arrows are induced by i and p. The vertical arrows are the connecting homomorphisms, defined as follows. Let $z\in Z\otimes S$ represent a Dirac cohomology class, so $Dz=0$. Choose $y\in Y\otimes S$ such that $py=z$. Since D^2 is semisimple, we can assume $D^2y=0$. Since $pDy=Dpy=Dz=0$, we see that $Dy=ix$ for some $x\in X$. Since $D^2y=0$, we see that $Dx=0$, so x defines a cohomology class. This class is by definition the image of the class of z under the connecting homomorphism. Clearly, we changed parity when we applied D, and this defines both vertical arrows at once.

It is easy to see that this map is well defined, and that the obtained six-term sequence is exact. To conclude:

Theorem 8.1. *Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be an orthogonal decomposition, with $\mathfrak{r}$ a reductive subalgebra and $\mathfrak{s}$ even-dimensional. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of $(\mathfrak{g}, K)$-modules and assume that the square of the Dirac operator $D(\mathfrak{g}, \mathfrak{r})$ is a semisimple operator for X, Y and Z. Then there is a six-term exact sequence corresponding to this short exact sequence, as described above.* □

Finally, let us comment on what can be done when $\mathfrak{s}$ is odd dimensional, say $\dim \mathfrak{s} = 2n+1$. The usual spin modules S_1 and S_2 are not $\mathbb{Z}_2$-graded and thus it seems the above construction does not make sense. Recall that S_1 and S_2 are defined by writing $\mathfrak{s} = \mathbb{C}^{2n} \oplus \mathbb{C}$, considering the spin module for $C(\mathbb{C}^{2n})$, and letting the last basis element of $\mathfrak{s}$ act in two different ways (preserving the even and odd subspace instead of exchanging them).

We can instead consider the unique irreducible graded module $\tilde{S}$ of $C(\mathfrak{s})$. It can be constructed as the restriction of the (unique) spin module for $C(\mathbb{C}^{2n+2})$ to $C(\mathfrak{s}) \subset C(\mathbb{C}^{2n+2})$. As a non-graded module, $\tilde{S}$ decomposes as $S_1 \oplus S_2$. If we define Dirac cohomology using $\tilde{S}$ in place of S_1 or S_2, we double it, but we do get a $\mathbb{Z}_2$-grading. Then the above construction works also in the odd case. Thus, this is probably a more natural definition of Dirac cohomology in the odd case.

It remains to see what can be done if D^2 is not a semisimple operator. One possibility might be to consider a more general definition of Dirac cohomology in that setting.

References

[A] J. Adams, *Nilpotent cohomology of the oscillator representation*, J. Reine Angew. Math. **449** (1994), 1–7.

[AM] A. Alexeev, E. Meinrenken, *Chern-Weil constructions in Lie theory*, in preparation.

[AS] M. Atiyah, W. Schmid, *A geometric construction of the discrete series for semisimple Lie groups*, Invent. Math. **42** (1977), 1–62.

[BW] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000.

[CO] W. Casselman, M. S. Osborne, *The $\mathfrak{n}$-cohomology of representations with an infinitesimal character*, Comp. Math. **31** (1975), 219–227.

[Ch] C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, 1954.

[C] D. H. Collingwood, *The $\mathfrak{n}$-homology of Harish-Chandra modules: generalizing a theorem of Kostant*, Math. Ann. **272** (1985), no. 2, 161–187.

[CM] A. Connes, H. Moscovici, *The L^2-index theorem for homogeneous spaces of Lie groups*, Ann. of Math. **115** (1982), no. 2, 291–330.

[E] T. J. Enright, *Analogues of Kostant's $\mathfrak{u}$-cohomology formulas for unitary highest weight modules*, J. Reine Angew. Math. **392** (1988), 27–36.

[HoP] R. Hotta, R. Parthasarathy, *A geometric meaning of the multiplicities of integrable discrete classes in $L^2(\Gamma\backslash G)$*, Osaka J. Math. **10** (1973), 211–234.

[HP1] J.-S. Huang, P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15** (2002), 185–202.

[HP2] J.-S. Huang, P. Pandžić, *Dirac operators in representation theory*, to appear in Lecture Notes Series, vol. 3, Institute for Mathematical Sciences, National University of Singapore

[Kac] V. Kac, *Lie superalgebras*, Adv. in Math. **26** (1977), 8–96.

[K1] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. **74** (1961), 329–387.

[K2] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. Jour. **100** (1999), 447–501.

[K3] B. Kostant, *A generalization of the Bott-Borel-Weil theorem and Euler number multiplets of representations*, Lett. Math. Phys. **52** (2000), 61–78.

[K4] B. Kostant, *Dirac cohomology for the cubic Dirac operator*, Studies in memory of I. Schur, in Progress of Math. vol. **210** (2003), 69–93.

[Ku] S. Kumar, *Induction functor in non-commutative equivariant cohomology and Dirac cohomology*, preprint, University of North Carolina, 2003.

[L] R. Langlands, *The dimension of spaces of automorphic forms*, Amer. J. of Math. **85** (1963), 99–125.

[P] R. Parthasarathy *Dirac operator and the discrete series*, Ann. of Math. **96** (1972), 1–30.

[S] W. Schmid, *L^2-cohomology and the discrete series*, Ann. of Math. **103** (1976), 375–394.

[V1] D.A. Vogan, *Representations of real reductive Lie groups*, Birkhäuser, Boston, 1981.

[V2] D.A. Vogan, *Dirac operators and unitary representations*, 3 talks at MIT Lie groups seminar, Fall 1997.

[V3] D.A. Vogan, $\mathfrak{n}$*-cohomology in representation theory*, a talk at "Functional Analysis VII", Dubrovnik, Croatia, September 2001.

[W] N. R. Wallach, *Real Reductive Groups, Volume I*, Academic Press, 1988.

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR, China

E-mail address: mahuang@ust.hk

Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: pandzic@math.hr

Department of Mathematics, University of Poitiers, Téléport 2, Boulevard Marie et Pierre Curie, BP 30179, 86962 Futuroscope Cedex, France

E-mail address: David.Renard@mathlabo.univ-poitiers.fr