Dirac operators in representation theory

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Introduction

Dirac operators in representation theory : Parthasarathy (1972), Atiyah-Schmid (1977) construction of discrete series representations of real semisimple groups.

Parthasarathy (1980) : criterion for unitarizability of representations (Parthasarathy Dirac-inequality).

Vogan (1990's) : introduce Dirac cohomology for Harish-Chandra modules + conjecture for the infinitesimal character of a module having non vanishing Dirac cohomology.

Proved by Huang-Pandzic (2002)

Kostant "cubic Dirac operator" (2000), more general setting, Huang-Pandzic result still holds.

Since then, vast literature on the subject...

Clifford algebras

(V, B) : finite dimensional complex vector space with a non-degenerate symmetric bilinear form.

$\mathrm{Cl}(V; B)$: Clifford algebra, with relations

$$v \otimes w + w \otimes v + 2B(v, w)\, 1, \qquad (v, w \in V).$$

- filtered algebra
- $\mathbb{Z}_2$-graded algebra (*i.e.* a super algebra):

$$\mathrm{Cl}(V; B) = \mathrm{Cl}^{\bar{0}}(V; B) \oplus \mathrm{Cl}^{\bar{1}}(V; B).$$

Chevalley isomorphism (graded symmetrization)

$$q : \bigwedge V \simeq \mathrm{Cl}(V; B)$$

$q(x)$, $x \in \bigwedge^2 V$ span a Lie subalgebra of $\mathrm{Cl}(V, B)$.

$$x \mapsto A_x, \quad A_x(v) = [q(x), v]_{\mathrm{Cl}}, \quad (v \in V)$$

defines A_x in $\mathfrak{so}(V; B)$, and

$$\bigwedge^2 V \longrightarrow \mathfrak{so}(V; B), \quad x \mapsto A_x \tag{1}$$

is a Lie algebra isomorphism, with inverse

$$\lambda : \ \mathfrak{so}(V; B) \longrightarrow \bigwedge^2 V$$

$$\lambda(A) = \frac{1}{4} \sum_i A(e_i) \wedge e^i \in \bigwedge^2 V, \qquad (A \in \mathfrak{o}(V; B)).$$

$(e_i)_i$ basis of V with dual basis $(e^i)_i$.

Theorem

(i) Suppose that $n = \dim_{\mathbb{C}}(V)$ *is even. Then there are :*

- *two isomorphism classes of irreducible* $\mathbb{Z}_2$*-graded* $\mathrm{Cl}(V, B)$*-modules,*
- *one isomorphism class of irreducible ungraded* $\mathrm{Cl}(V, B)$*-modules,*
- *two isomorphism classes of irreducible* $\mathrm{Cl}^{\bar{0}}(V, B)$*-modules.*

(ii) Suppose that $n = \dim_{\mathbb{C}}(V)$ *is odd. Then there are*

- *one isomorphism class of irreducible* $\mathbb{Z}_2$*-graded* $\mathrm{Cl}(V, B)$*-modules,*
- *two isomorphism classes of irreducible ungraded* $\mathrm{Cl}(V, B)$*-modules,*
- *one isomorphism class of irreducible* $\mathrm{Cl}^{\bar{0}}(V, B)$*-modules,*

fix S : irreducible $\mathbb{Z}_2$-graded $\mathrm{Cl}(V, B)$-module.

$\mathbf{Spin}(V, B)$: central extension of $\mathbf{SO}(V, B)$ realized in $\mathrm{Cl}(V; B)^{\times}$.

Dirac operator and $(\mathfrak{g}, K)$-modules

G : connected real reductive Lie group with Cartan involution θ

$K = G^\theta$: maximal compact subgroup of G.

$\mathfrak{g}_0$: Lie algebra of G, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$

$$\mathfrak{g}_0 = \mathfrak{k}_0 \overset{\theta}{\oplus} \mathfrak{p}_0, \qquad \mathfrak{g} = \mathfrak{k} \overset{\theta}{\oplus} \mathfrak{p}$$

Cartan decompositions

B : invariant nondegenerate symmetric bilinear form B on $\mathfrak{g}_0$,
$B_{|\mathfrak{p}_0}$: definite positive, $B_{|\mathfrak{k}_0}$ definite negative.
$\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{p}; B)$: Clifford algebra of $\mathfrak{p}$ with respect to B.

Dirac operator and $(\mathfrak{g}, K)$-modules

$$\begin{array}{ccccccc}
\widetilde{K} & \longrightarrow & \mathbf{Spin}(\mathfrak{p}_0) & \hookrightarrow & \mathbf{Spin}(\mathfrak{p}) & \hookrightarrow & \mathrm{Cl}^{\bar{0}}(\mathfrak{p})^\times \\
\downarrow & & \downarrow & & \downarrow & & \\
K & \xrightarrow{\mathrm{Ad}_{|\mathfrak{p}_0}} & \mathbf{SO}(\mathfrak{p}_0) & \hookrightarrow & \mathbf{SO}(\mathfrak{p}) & &
\end{array}$$

Also Lie algebra morphism

$$\mathrm{ad}_{|\mathfrak{p}} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}), \quad X \mapsto \mathrm{ad}(X)_{|\mathfrak{p}}$$

Recall $\lambda : \mathfrak{so}(\mathfrak{p}) \simeq \bigwedge^2 \mathfrak{p}$ (1) and inclusion $q : \bigwedge^2 \mathfrak{p} \hookrightarrow \mathrm{Cl}(\mathfrak{p})$.

$$\alpha : \mathfrak{k} \xrightarrow{\mathrm{Ad}_{|\mathfrak{p}_0}} \mathfrak{so}(\mathfrak{p}) \xrightarrow{\lambda} \bigwedge^2 \mathfrak{p} \overset{q}{\hookrightarrow} \mathrm{Cl}(\mathfrak{p}) \tag{2}$$

Dirac operator and $(\mathfrak{g}, K)$-modules

if $(Y_i)_i$ is a basis of $\mathfrak{p}$ with dual basis $(Z_i)_i$, then for any $X \in \mathfrak{k}$,

$$\alpha(X) = \frac{1}{4} \sum_{i,j} B([Z_i, Z_j], X) \; Y_i Y_j. \tag{3}$$

Main object : $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$:
$U(\mathfrak{g})$: envelopping algebra of $\mathfrak{g}$.

- $\mathcal{A}$ associative $\mathbb{Z}_2$-graded superalgebra (elements in $U(\mathfrak{g})\otimes 1$ are even)).
- Linear action of K on $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$, gives Lie algebra representation of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$.

Dirac operator and $(\mathfrak{g}, K)$-modules

The map (17) is used to define

$$\Delta : \mathfrak{k} \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}), \quad \Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$$

Lie algebra morphism.

Extends to an algebra morphism

$$\Delta : U(\mathfrak{k}) \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}). \tag{4}$$

$X \in \mathfrak{k}$ acts on $\mathcal{A}$ by adjoint action of $\Delta(X)$, *i.e.* $a \in \mathcal{A} \mapsto [\Delta(X), a]$.

$\mathcal{A}^K$ the subalgebra of K-invariant in $\mathcal{A}$.

Dirac operator and $(\mathfrak{g}, K)$-modules

We can now introduce the Dirac operator D :

Definition

if $(Y_i)_i$ *is a basis of* $\mathfrak{p}$ *and* $(Z_i)_i$ *is the dual basis with respect to* B, *then*

$$D = \sum_i Y_i \otimes Z_i \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$$

is independent of the choice of basis $(Y_i)_i$ *and* K*-invariant for the adjoint action on both factors :*

$$D \in \mathcal{A}^K.$$

Categories of representations

$\mathcal{M}(\mathfrak{g}, K)$, the category of Harish-Chandra modules (*i.e.* $(\mathfrak{g}, K)$-modules).

Main idea : study $X \in \mathcal{M}(\mathfrak{g}, K)$ by considering the natural action of D on $X \otimes S$,
S : a module of spinors for $C(\mathfrak{p})$.

Modules in $\mathcal{M}(\mathfrak{g}, K)$ are complex vector spaces with action of $U(\mathfrak{g})$, action of K + compatibility conditions.

Formalized in the notion of **Harish-Chandra pair** $(\mathscr{A}, \mathrm{K})$ and Harish-Chandra modules for $(\mathscr{A}, \mathrm{K})$ (category $\mathcal{M}(\mathscr{A}, \mathrm{K})$.

Examples : $(\mathscr{A}, \mathrm{K}) = (U(\mathfrak{g}), K)$, $(\mathscr{A}, \mathrm{K}) = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}), \widetilde{K})$,

Categories of representations

If X $(\mathfrak{g}, K)$-module, then $X \otimes S$ is a $(\mathcal{A}, \widetilde{K})$-module :
This defines a functor :

$$X \mapsto X \otimes S, \qquad \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathcal{A}, \widetilde{K}).$$

which is an equivalence of categories.

Inverse $M \mapsto \mathrm{Hom}_{\mathrm{Cl}(\mathfrak{p})}(S, M)$ (resp. $M \mapsto \mathrm{Hom}_{\mathrm{Cl}^{\bar{0}}(\mathfrak{p})}(S, M)$) if $\dim \mathfrak{p}$ is even (resp. odd).

Let us now put this principle into perspective by discussing a theorem of Harish-Chandra.

On a theorem of Harish-Chandra

Harish-Chandra : an irreducible $(\mathfrak{g}, K)$-module is characterized by the action of $U(\mathfrak{g})^K$ on any non-trivial K-isotypic component. (simplified algebraic proof by Lepowsky-McCollum, alternative proof below)

Idea : use this to study and classify irreducible $(\mathfrak{g}, K)$-modules.

success : HC subquotient theorem, spherical representations, Vogan's classification by lowest K-types.

Problem : $U(\mathfrak{g})^K$ highly non commutative and very little is known about its structure and representation theory.

On a theorem of Harish-Chandra

Harish-Chandra result still holds in $\mathcal{M}(\mathcal{A}, \widetilde{K})$.

$\mathcal{A}^K = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K$ is slightly better $U(\mathfrak{g})^K$:

- It contains a non-trivial interesting elements : the Dirac operator D.
- $\mathcal{A}^K$: differential superalgebra, $d = \mathrm{ad} D$.

On a theorem of Harish-Chandra

Harish-Chandra thm is in fact a consequence of a general result about algebras with idempotents.

$(\mathscr{A}, \mathrm{K})$: generalized Harish-Chandra pair

$\mathcal{M}(\mathscr{A}, \mathrm{K})$ category of Harish-Chandra modules

equivalent to the category of non-degenerate modules over the Hecke algebra $R(\mathscr{A}, \mathrm{K})$, an algebra with idempotents (or algebra with an approximate identity)

As vector spaces

$$R(\mathscr{A}, \mathrm{K}) \simeq \mathscr{A} \otimes_{U(\mathfrak{k})} R(\mathrm{K}),$$

$R(\mathrm{K})$ is the convolution algebra of K-finite distributions on K

J. Bernstein (theory of reductive p-adic groups) proved the following

A : algebra with idempotents

$\mathcal{M}(A)$: category of non-degenerate left A-modules

$e \in A$ idempotent.

$$M \in \mathcal{M}(A), \qquad M = e \cdot M \oplus (1-e) \cdot M \tag{5}$$

eAe : algebra with unit e,
$e \cdot M$: unital eAe-module.

$$j_e : \mathcal{M}(A) \to \mathcal{M}(eAe), \quad M \mapsto e \cdot M.$$

The functor j_e is exact.

Induction functor i:

$$i : \mathcal{M}(eAe) \to \mathcal{M}(A), \qquad Z \mapsto A \otimes_{eAe} Z.$$

Proposition

$M \mapsto e \cdot M$ bijection from irreducible modules M in $\mathcal{M}(A)$ such that $e \cdot M \neq 0$ and irreducible modules in $\mathcal{M}(eAe)$

On a theorem of Harish-Chandra

$\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$, $\mathcal{M}(\mathcal{A}, \widetilde{K}) \simeq \mathcal{M}(R(\mathcal{A}, \widetilde{K}))$

(γ, F_γ) : irreducible finite-dimensional representation of $\widetilde{K}$, χ_γ : character of the contragredient.

$1 \otimes \chi_\gamma$: idempotent of $R(\mathcal{A}, \widetilde{K})$.

Theorem

The algebra

$$(1 \otimes \chi_\gamma) \cdot R(\mathcal{A}, \widetilde{K}) \cdot (1 \otimes \chi_\gamma)$$

is isomorphic to

$$\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \mathrm{End}(F_\gamma).$$

On a theorem of Harish-Chandra

Bernstein result in this case gives :

$V \in \mathcal{M}(\mathcal{A}, \widetilde{K})$-module V,

$$(1 \otimes \chi_\gamma) \cdot V = V(\gamma)$$

$\widetilde{K}$-isotypic component in V.

If V is irreducible, and $V(\gamma) \neq 0$, it is characterized by the action of $\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \mathrm{End}(V_\gamma)$ on $V(\gamma)$.

On a theorem of Harish-Chandra

To study an $(\mathfrak{g}, K)$-module X, one would like to study the action of $U(\mathfrak{g})^K$ on a (non-zero) K-isotypic component of X,

but since a little is known about $U(\mathfrak{g})^K$, we will instead study the action of $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\widetilde{K}}$ a (non-zero) K-isotypic component of $X \otimes S$.

The structure of $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\widetilde{K}}$ is better (but not ompletely) understood than the he structure of $U(\mathfrak{g})^K$.

We will now see what can be said from that fact that it contains the Dirac operator D.

The square of the Dirac operator

$$D^2 = -\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \Delta(\mathrm{Cas}_{\mathfrak{k}}) + (\|\rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2) 1 \otimes 1 \qquad (6)$$

$\mathrm{Cas}_{\mathfrak{g}}$: Casimir element of $U(\mathfrak{g})$
$\mathrm{Cas}_{\mathfrak{k}}$: Casimir element of $U(\mathfrak{k})$.

$$\Delta : U(\mathfrak{k}) \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}).$$

$$D^2 \text{ is in the center of the algebra } \mathcal{A}^K. \qquad (7)$$

The square of the Dirac operator

T imaximal torus n K, $\mathfrak{t}_0 = \mathrm{Lie}(T)$, $\mathfrak{t} = \mathfrak{t}_0 \otimes_{\mathbb{R}} \mathbb{C}$.
$\mathfrak{a} = \mathfrak{p}^{\mathfrak{t}}$, $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$: fundamental Cartan subalgebra of $\mathfrak{g}$,

$R = R(\mathfrak{g}, \mathfrak{h})$: root system of $\mathfrak{h}$ in $\mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$: Weyl group. Let us also choose a positive root system
$R^+ \subset R$: positive root system, $\rho = \frac{1}{2} \sum_{R^+} \alpha \in \mathfrak{h}^*$

$R_{\mathfrak{k}} = R(\mathfrak{k}, \mathfrak{t})$, $W_{\mathfrak{k}}$, $R_{\mathfrak{k}}^+ \subset R_{\mathfrak{k}}$ compatible with R^+, $\rho_{\mathfrak{k}}$.
The bilinear form B on $\mathfrak{g}$ restricts to a positive definite form on $i\mathfrak{t}_0 \oplus \mathfrak{a}$.

$\langle . , . \rangle$: induced form on $i\mathfrak{t}_0^* \oplus \mathfrak{a}$ and $\mathfrak{h}^*$. The norm appearing in (6) is defined for any $\lambda \in \mathfrak{h}^*$ by $\|\lambda\|^2 = \langle \lambda, \lambda \rangle$.

The square of the Dirac operator

Harish-Chandra algebra isomorphism

$$\gamma_{\mathfrak{g}} : \mathfrak{Z}(\mathfrak{g}) \simeq S(\mathfrak{h})^W \tag{8}$$

$\lambda \in \mathfrak{h}^*$, χ_λ : character of $\mathfrak{Z}(\mathfrak{g})$.

$X \in \mathcal{M}(\mathfrak{g}, K)$ with infinitesimal character $\Lambda \in \mathfrak{h}^*$.
(γ, F_γ) : irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$.

Then D^2 acts on $(X \otimes S)(\gamma)$ by the scalar

$$-\|\Lambda\|^2 + \|\tau + \rho_{\mathfrak{k}}\|^2. \tag{9}$$

$\ker(D^2)$ on $X \otimes S$: direct sum of full $\widetilde{K}$-isotypic components of $X \otimes S$: these are exactly those $(X \otimes S)(\gamma)$ for which

$$\|\tau + \rho_{\mathfrak{k}}\|^2 = \|\Lambda\|^2. \tag{10}$$

Dirac operator and unitarizable of $(\mathfrak{g}, K)$-modules

$X \in \mathcal{M}(\mathfrak{g}, K)$ unitarizable :
definite positive invariant Hermitian product $\langle . , . \rangle_X$ on X.

$X \in \mathfrak{g}_0$ act as skew-symmetric operator on X, *i.e.*

$$\langle X.v, w \rangle_X = -\langle v, X \cdot w \rangle_X, \qquad (v, w \in X), (X \in \mathfrak{g}_0).$$

Also definite positive Hermitian product $\langle . , . \rangle_S$ on S so that the elements of $\mathfrak{p}_0 \subset \mathrm{Cl}(\mathfrak{p})$ act as skew-symmetric operators

$X \otimes S$ definite positive Hermitian product tensor product of $\langle . , . \rangle_X$ and $\langle . , . \rangle_S$, denoted by $\langle . , . \rangle_{X \otimes S}$.

Dirac operator and unitarizable of $(\mathfrak{g}, K)$-modules

D is symmetric with respect to $\langle . , . \rangle_{X \otimes S}$.

D^2 is a positive symmetric operator on $X \otimes S$.

From (9) we get :

Proposition (Parthasarathy-Dirac inequality)

Assume that the unitarizable $(\mathfrak{g}, K)$-module X has infinitesimal character $\Lambda \in \mathfrak{h}^*$. Let (γ, F_γ) be an irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$ such that $(X \otimes S)(\tau) \neq 0$. Then

$$\|\tau + \rho_{\mathfrak{k}}\|^2 \geq \|\Lambda\|^2.$$

Dirac operator and unitarizable of $(\mathfrak{g}, K)$-modules

Rmk : If the $(\mathfrak{g}, K)$-module X is unitarizable and has an infinitesimal character, D acts semisimply on $X \otimes S$. In particular

$$\ker D^2 = \ker D. \tag{11}$$

If X is finite-dimensional, inner product on $X \otimes S$ such that D is skew-symmetric with respect to this inner product : D acts semisimply on $X \otimes S$ and $\ker D^2 = \ker D$.

Spherical principal series of $\mathbf{SL}(2, \mathbb{R}) \times \mathbf{SL}(2, \mathbb{R})$

Example

Spherical principal series of $\mathbf{SL}(2, \mathbb{R})$
Spherical principal series of $\mathbf{SL}(2, \mathbb{R}) \times \mathbf{SL}(2, \mathbb{R})$

Definition

Let $X \in \mathcal{M}(\mathfrak{g}, K)$. The Dirac operator D acts on $X \otimes S$. Vogan's Dirac cohomology of X is the quotient

$$H^D_{\mathrm{V}}(X) = \ker D / (\ker D \cap \mathrm{Im} D).$$

Since $D \in \mathcal{A}^K$, $\widetilde{K}$ acts on $\ker D$, $\mathrm{Im} D$ and $H^D_{\mathrm{V}}(X)$.

If X is unitary, D acts semisimply on $X \otimes S$

$$\ker D^2 = \ker D = H^D_{\mathrm{V}}(X). \tag{12}$$

In this case, the Dirac cohomology of X is a sum the full isotypic components $X \otimes S(\gamma)$ such that (10) holds.

For general X,this does not hold, but note that D is always a differential on $\ker D^2$, and $H^D_{\mathrm{V}}(X)$ is the usual cohomology of this differential.

The theorem of Huang-Pandzic gives a strong condition on the infinitesimal character of a $(\mathfrak{g}, K)$-module X with non zero Dirac cohomology.

Proposition

Let $X \in \mathcal{M}(\mathfrak{g}, K)$ be a Harish-Chandra module with infinitesimal character $\Lambda \in \mathfrak{h}^*$. Assume that (γ, F_γ) is an irreducible representation of $\widetilde{K}$ with highest weight $\tau = \tau_\gamma \in \mathfrak{t}^*$ such that $(X \otimes S)(\gamma)$ contibutes to $H^D_{\mathrm{V}}(X)$. Then

$$\Lambda = \tau + \rho_{\mathfrak{k}} \quad \textit{up to conjugacy by the Weyl group } W. \tag{13}$$

Thus for unitary X, (10) is equivalent to the stronger condition (13), provided that γ appears in $X \otimes S$.

Why is Dirac cohomology an interesting invariant

Many interesting modules have non-vanishing Dirac cohomology :

- Finite dimensional representations (Kostant).
- Discrete series, and more generally Vogan-Zuckerman $A_{\mathfrak{q}}(\lambda)$-modules
- Highest weight modules
- Unipotent representations

Dirac cohomology is related to other kinds of cohomological invariants :

- $\mathfrak{n}$-cohomology for highest weight-modules
- $(\mathfrak{g}, K)$-cohomology for $A_{\mathfrak{q}}(\lambda)$-modules

Dirac cohomology and $(\mathfrak{g}, K)$-cohomology

An important problem in the theory of automorphic forms is to compute cohomology of locally symmetric spaces. Matsushima's formula relates this to computation of $(\mathfrak{g}, K)$-cohomology of irreducible unitary Harish-Chandra modules for the corresponding semisimplegroup G.

Vogan and Zuckerman : classification of irreducible unitary Harish-Chandra modules X such that $H^*(\mathfrak{g}, K, X \otimes F^*) \neq 0$ where F is a finite-dimensional representation of G.

Cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ with the same infinitesimal character as F. VZ have explicitely computed the cohomology.

Dirac cohomology and $(\mathfrak{g}, K)$-cohomology

$X \in \mathcal{M}(\mathfrak{g}, K)$: irreducible unitary Harish-Chandra module with infinitesimal character as finite dimensional representation F (this is an obvious necessary condition for $H^*(\mathfrak{g}, K, X \otimes F^*)$ to be non zero).

If $\dim \mathfrak{p}$ is even :

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)),$$

and if $\dim \mathfrak{p}$ is odd :

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)) \oplus \mathrm{Hom}_{\widetilde{K}}(H_D(F), H_D(X)).$$

Dirac cohomology of some $(\mathfrak{g}, K)$-modules

- Finite dimensional representations (K, HKP)
- $A_{\mathfrak{q}}(\lambda)$ (HKP)
- Unipotent representations of $\mathbf{Sp}(n\mathbb{R})$, $\mathbf{U}(p, q)$
- Wallach's representations (HPP)
- Complex groups (C-P Dong)

Kostant's cubic Dirac operator

$(\mathfrak{g}, B)$ as before

$\mathfrak{r} \subset \mathfrak{g}$ s.t. $B_{\mathfrak{r}}$ non-degenerate.

$$\mathfrak{g} = \mathfrak{r} \overset{\perp}{\oplus} \mathfrak{s}.$$

$B_{\mathfrak{s}}$ non degenerate

$\mathrm{Cl}(\mathfrak{s})$: Clifford algebra of $\mathfrak{s}$.

Chevalley isomorphism $q : \bigwedge \mathfrak{s} \simeq \mathrm{Cl}(\mathfrak{s})$

Kostant's cubic Dirac operator

The restriction of the fundamental 3-form of $\mathfrak{g}$ gives an element $\nu \in \bigwedge^3 \mathfrak{s}$ characterized by the identity

$$B(\nu, X \wedge Y \wedge Z) = \frac{1}{2} B(X, [Y, Z]), \qquad (X, Y, Z \in \mathfrak{s}) \tag{14}$$

(If $\mathfrak{r} = \mathfrak{k}$ as before, $\nu = 0$ because $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.)

Let $(X_i)_i$ be an orthonormal basis of $\mathfrak{s}$.

$$D(\mathfrak{g}, \mathfrak{r}) = \sum_i X_i \otimes X_i + 1 \otimes \nu \tag{15}$$

of $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$.

Kostant's cubic Dirac operator

$$\nu = \frac{1}{2} \sum_{i<j<k} B([X_i, X_j], X_k)\; X_i X_j X_k \qquad (16)$$

As before

$$\alpha : \mathfrak{r} \xrightarrow{\mathrm{Ad}_{|\mathfrak{s}}} \mathfrak{so}(\mathfrak{s}) \xrightarrow{\lambda} \bigwedge^2 \mathfrak{s} \overset{q}{\hookrightarrow} \mathrm{Cl}(\mathfrak{s}) \qquad (17)$$

$$\Delta : \mathfrak{r} \longrightarrow U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}), \qquad X \longmapsto X \otimes 1 + 1 \otimes \alpha(X)$$

$$\Delta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}). \qquad (18)$$

Kostant's cubic Dirac operator

Lemma

The cubic Dirac operator $D(\mathfrak{g}, \mathfrak{r})$ is $\mathfrak{r}$-invariant, i.e. it (super)commutes with the image of $U(\mathfrak{r})$ by Δ. We write $D(\mathfrak{g}, \mathfrak{r}) \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$.

Theorem

$$D(\mathfrak{g}, \mathfrak{r})^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Delta(\Omega_{\mathfrak{r}}) + (||\rho_{\mathfrak{r}}||^2 - ||\rho||^2) 1 \otimes 1,$$

where $\Omega_{\mathfrak{g}}$ (resp. $\Omega_{\mathfrak{r}}$) denotes the Casimir element in $\mathfrak{Z}(\mathfrak{g})$ (resp. $\mathfrak{Z}(\mathfrak{r})$).

Kostant's cubic Dirac operator

Simplified proof due to N. Prudhon :
two subalgebras $\mathfrak{r}$ and $\mathfrak{l}$ of $\mathfrak{g}$ with

$$\mathfrak{g} \supset \mathfrak{r} \supset \mathfrak{l}$$

$B_{\mathfrak{r}}$ and $B_{\mathfrak{l}}$ non degenerate.

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}, \quad \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{s} \oplus \mathfrak{m}_{\mathfrak{r}}, \quad \mathfrak{r} = \mathfrak{l} \oplus \mathfrak{m}_{\mathfrak{r}}.$$

$$\mathrm{Cl}(\mathfrak{m}) = \mathrm{Cl}(\mathfrak{s}) \bar{\otimes} \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}) \quad g\textit{radedtensorproduct}$$

Kostant's cubic Dirac operator

$$D(\mathfrak{g},\mathfrak{l}) \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{m}), \quad D(\mathfrak{g},\mathfrak{r}) \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}), \quad D(\mathfrak{r},\mathfrak{l}) \in U(\mathfrak{r}) \otimes \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}).$$

$$\Delta : \ U(\mathfrak{r}) \otimes \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}) \bar{\otimes} \mathrm{Cl}(\mathfrak{m}_{\mathfrak{r}}) \simeq U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{m}). \quad (19)$$

Proposition

(i) $D(\mathfrak{g},\mathfrak{l}) = D(\mathfrak{g},\mathfrak{r}) + \Delta(D(\mathfrak{r},\mathfrak{l}))$
(ii) The components $D(\mathfrak{g},\mathfrak{r})$ *et* $\Delta(D(\mathfrak{r},\mathfrak{l}))$ *(super)commute.*

Use the formula with $\mathfrak{l} = 0$ to compute $D(\mathfrak{g},\mathfrak{r})^2$ Computation of the square of $D(\mathfrak{g},0)$ and $D(\mathfrak{r},0)$ is easier.

Huang-Pandzic theorem

Consider

$$d = \mathrm{ad}D : a \mapsto [D, a]$$

on the superalgebra $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$ (superbracket).
Super-Jacobi identity gives $d^2 = (\mathrm{ad}D)^2 = \mathrm{ad}(D^2) = 0$ on $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$

Cohomology of d : $\ker d / \mathrm{Im} d$ on $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$.

The theorem of Huang and Pandzic computes this cohomology.
Remark that $\Delta(\mathfrak{Z}(\mathfrak{r}))$ is in the kernel of D.

Huang-Pandzic theorem

Theorem

(Huang-Pandzic) On $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$, we have

$$\ker d = \Delta(\mathfrak{Z}(\mathfrak{r})) \oplus \mathrm{Im} d.$$

Proof : filtration on $U(\mathfrak{g})$,

$$\mathbf{Gr}(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})) \simeq S(\mathfrak{g}) \otimes \bigwedge \mathfrak{s} \simeq S(\mathfrak{r}) \otimes S(\mathfrak{s}) \otimes \bigwedge \mathfrak{s}.$$

Exactness of Koszul complex.

Huang-Pandzic theorem

Corollary

$z \in \mathfrak{Z}(\mathfrak{g})$, $z \otimes 1 \in \ker d \subset (U(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{r}}$,
$z \otimes 1$ can be written as

$$z \otimes 1 = \Delta(z_1) + Da + aD$$

for some $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{r}}$ (in the odd part of the superalgebra), and some $z_1 \in \mathfrak{Z}(\mathfrak{r})$.

Let us now identify z_1 explicitely.

$\mathfrak{h}_{\mathfrak{r}} \subset \mathfrak{h}$: Cartan subalgebras of $\mathfrak{r}$ and $\mathfrak{g}$.
$R_{\mathfrak{r}} = R(\mathfrak{r}, \mathfrak{h}_{\mathfrak{r}})$, $R = R(\mathfrak{g}, \mathfrak{h})$: root systems
$W_{\mathfrak{r}} = W(\mathfrak{r}, \mathfrak{h}_{\mathfrak{r}})$, $W = W(\mathfrak{g}, \mathfrak{h})$: Weyl groups.

$$\mathfrak{Z}(\mathfrak{g}) \xrightarrow{\gamma_{\mathfrak{g}}} S(\mathfrak{h})^{W_{\mathfrak{g}}}, \quad \mathfrak{Z}(\mathfrak{r}) \xrightarrow{\gamma_{\mathfrak{r}}} S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}}.$$

Harish-Chandra isomorphisms

Huang-Pandzic theorem

Restriction of functions from $\mathfrak{h}^*$ to $\mathfrak{h}_{\mathfrak{r}}$ induces a morphism

$$\mathrm{res} : S(\mathfrak{h})^{W_{\mathfrak{g}}} \to S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}}$$

Proposition

There is a unique algebra morphism $\eta_{\mathfrak{r}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{r})$ such that

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \xrightarrow{\eta_{\mathfrak{r}}} & \mathfrak{Z}(\mathfrak{r}) \\ \downarrow{\gamma_{\mathfrak{g}}} & & \downarrow{\gamma_{\mathfrak{r}}} \\ S(\mathfrak{h})^{W_{\mathfrak{g}}} & \xrightarrow{\mathrm{res}} & S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}} \end{array}$$

commutes.

$$(\forall z \in \mathfrak{Z}(\mathfrak{g})), \qquad z \otimes 1 = \Delta(\eta_{\mathfrak{r}}(z)) + Da + aD$$

for some $a \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{r}}$

Huang-Pandzic theorem

V: $\mathfrak{g}$-module, S : Spinors for $\mathrm{Cl}(\mathfrak{s})$.

Definition

The Dirac cohomology $H_D(\mathfrak{g}, \mathfrak{r}; V) = \ker D / \ker D \cap \mathrm{Im} D$ *on* $V \otimes S$*.*
Since D *is* $\mathfrak{r}$*-invariant,* $H_D(\mathfrak{g}, \mathfrak{r}; V)$ *is naturally a* $\mathfrak{r}$*-modules.*

Proposition

The action of an element $z \otimes 1$ *in* $\mathfrak{Z}(\mathfrak{g}) \otimes 1$ *on* $H_D(\mathfrak{g}, \mathfrak{r}; V)$ *coincide with the action of* $\eta_{\mathfrak{r}}(z) \in U(\mathfrak{r})$ *(ie. with the action of* $\Delta(\eta_{\mathfrak{r}}(z))$*).*
If V *has infinitesimal character* $\Lambda \in \mathfrak{h}^*$*, and if* (γ, F_γ) *is a finite dimensional* $\mathfrak{r}$*-module with highest weight* $\tau_\gamma \in \mathfrak{h}_{\mathfrak{r}}^*$ *in* $H_D(\mathfrak{g}, \mathfrak{r}; V)$*, then* $\Lambda \in W \cdot (\tau_\gamma + \rho_{\mathfrak{r}})$*.*

Cubic Dirac operators for Levi subalgebras

$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$: parabolic subalgebra of $\mathfrak{g}$
$\mathfrak{q}^- = \mathfrak{l} \oplus \mathfrak{u}^-$ opposite parabolic subalgebra
$\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{u}^-$.

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}. \tag{20}$$

$D = D(\mathfrak{g}, \mathfrak{l}) \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s}))^{\mathfrak{l}}$ as above.

Cubic Dirac operators for Levi subalgebras

Convenient basis of $\mathfrak{s}$ to express this operator.
$\mathfrak{u}$ and $\mathfrak{u}^-$: isotropic subspaces in perfect duality under B, $\mathfrak{u}^* \simeq \mathfrak{u}^-$.
$(u_1, \dots, u_n)$: basis of $\mathfrak{u}$,
$u_1^-, \dots, u_n^-$ dual basis in $\mathfrak{u}^-$.

$$D = A + A^- + 1 \otimes a + 1 \otimes a^- = C + C^-$$

$$A = \sum_{i=1}^n u_i^- \otimes u_i, \quad A^- \sum_{i=1}^n u_i \otimes u_i^-$$

$$a = -\frac{1}{2} \sum_{i<j} \sum_k B([u_i^-, u_j^-], u_k)\, u_i \wedge u_j \wedge u_k^-,$$

$$a^- = -\frac{1}{2} \sum_{i<j} \sum_k B([u_i, u_j], u_k^-)\, u_i^- \wedge u_j^- \wedge u_k$$

Cubic Dirac operators for Levi subalgebras

We are interested in the action of these elements on the $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ - module $V \otimes S$,
V : $\mathfrak{g}$-module,
S : spin module for $\mathrm{Cl}(\mathfrak{s})$.
identification $S \simeq \bigwedge^{\cdot} \mathfrak{u}$

$$\cdots \to V \otimes \bigwedge^{p-1} \xrightarrow{\delta} V \otimes \bigwedge^{p} \xrightarrow{\delta} V \otimes \bigwedge^{p+1} \xrightarrow{\delta} \cdots$$

Complex for $\mathfrak{u}$-homology
action of C^- on $V \otimes S \simeq V \otimes \bigwedge \mathfrak{u}$ is 2δ.

Cubic Dirac operators for Levi subalgebras

Make the following identifications:

$$V \otimes \bigwedge^p \mathfrak{u} \cong \mathrm{Hom}((\bigwedge^p \mathfrak{u})^*, V) \cong \mathrm{Hom}(\bigwedge^p (\mathfrak{u}^*), V) \cong \mathrm{Hom}(\bigwedge^p \mathfrak{u}^-, V).$$

The last space is the space of p-cochains for the $\mathfrak{u}^-$-cohomology complex differential with d,

Through the above identifications, C acts on $V \otimes S \simeq V \otimes \bigwedge \mathfrak{u} \simeq \mathrm{Hom}(\bigwedge^p \mathfrak{u}^-, V)$ as d.

Thus $D = C + C^-$ acts on $V \otimes S$ as $2\delta + d$

Goal : Relate Dirac cohomology $H_D(\mathfrak{g}, \mathfrak{l}; V)$ with Lie algebra homology $H_\bullet(\mathfrak{u}; V)$ and cohomology $H^\bullet(\mathfrak{u}^-; V)$
Need some "Hodge decomposition" for some invariant hermitian product.

Hodge decomposition for $\mathfrak{p}^-$-cohomology (Hermitian symmetric case)

Back to $(\mathfrak{g}, K)$-modules of the first lecture, with $(\mathfrak{g}, \mathfrak{k})$ hermitian symmetric.

$\mathfrak{k}$: Levi subalgebra of $\mathfrak{g}$, $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$
previous setting with $\mathfrak{l} = \mathfrak{k}$, $\mathfrak{u} = \mathfrak{p}^+$ and $\mathfrak{u}^- = \mathfrak{p}^-$.

V : unitary $(\mathfrak{g}, K)$-module.
Then d and 2δ are minus adjoints of each other with respect to a positive definite form $\langle .,. \rangle_{pos}$ on $V \otimes S$

Corollary

With respect to the form $\langle .,. \rangle_{pos}$ on $V \otimes S$, the adjoint of C is C^-. Therefore D is self-adjoint on $V \otimes S$.

Hodge decomposition for $\mathfrak{p}^-$-cohomology (Hermitian symmetric case)

Variant of the usual Hodge decomposition.

Lemma

(a) $\ker D = \ker d \cap \ker \delta$;
(b) $\mathrm{Im}\delta$ *is orthogonal to* $\ker d$ *and* $\mathrm{Im} d$ *is orthogonal to* $\ker \delta$.

Combining this and the fact $\ker D = \ker D^2 = H_V^D(V)$, we get

Theorem

(a) $V \otimes S = \ker D \oplus \mathrm{Im} d \oplus \mathrm{Im}\delta$;
(b) $\ker d = \ker D \oplus \mathrm{Im} d$;
(c) $\ker \delta = \ker D \oplus \mathrm{Im}\delta$.

$$\ker D \cong H_V^D(V) \cong H^{\cdot}(\mathfrak{p}^-, V) \otimes Z_{\rho(\mathfrak{p}^-)} \cong H_{\cdot}(\mathfrak{p}^+, V) \otimes Z_{\rho(\mathfrak{p}^-)}.$$

(up to modular twists) the Dirac cohomology $\ker D$ *is the space of "harmonic representatives" for* $\mathfrak{p}^-$*-cohomology and* $\mathfrak{p}^+$*-homology.*

Dirac cohomology of finite dimensional modules

$\mathfrak{h}$: Cartan subalgebra of $\mathfrak{g}$,
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$: Borel subalgebra
$\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ the opposite Borel subalgebra.

$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{n}^-$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$.

$\mathfrak{u}$: compact form of $\mathfrak{g}$, U : compact adjoint group.
$X \mapsto \bar{X}$ the complex conjugation in $\mathfrak{g}$ with respect to the real form $\mathfrak{u}$.

Then

$$\langle X, Y \rangle = -2B(X, \bar{Y}), \quad (X, Y \in \mathfrak{g})$$

positive definite U-invariant hermitian form on $\mathfrak{g}$. This hermitian form restricts to $\mathfrak{n}$, and can be extended to $S \simeq \bigwedge \mathfrak{n}$.

V : finite dimensional $\mathfrak{g}$-module.
$\langle . , . \rangle_V$: U-invariant positive definite hermitian form on V
$\langle . , . \rangle_{V \otimes S}$: positive definite hermitian form on $V \otimes S$.

$D = C + C^-$ as before. The adjoint of C acting on $V \otimes S$ is $-C^-$, thus D is anti-self-adjoint with respect to $\langle . , . \rangle_{V\otimes S}$.

The operator $D = D(\mathfrak{g}, \mathfrak{h})$ acting on $V \otimes S$ is semi-simple,

$$H_D(\mathfrak{g}, \mathfrak{h}; V) = \ker D = \ker D^2 = H^{\cdot}(\mathfrak{n}-, V) \otimes Z_{\rho(\mathfrak{n}^-)} \cong H_{\cdot}(\mathfrak{n}, V) \otimes Z_{\rho(\mathfrak{n})}.$$

Theorem

Let V be the irreducible finite dimensional representation of V with highest weight μ. Then, as a $\mathfrak{h}$-module

$$H_D(\mathfrak{g}, \mathfrak{h}; V) = \bigoplus_{w \in W} \mathbb{C}_{w\cdot(\mu+\rho)}$$

Proof : the weights $\mathbb{C}_{w\cdot(\mu+\rho)}$ occur in $V \otimes S$ with multiplicity one and the corresponding weight spaces are in the kernel of D^2 (this can be checked directly from the formula for D^2). No other weights can occur in the Dirac cohomology (HP thm).

Dirac cohomology of finite dimensional modules

$$S = S^+ \oplus S^-$$

$S \simeq \bigwedge \mathfrak{n}$, S^+ (resp. S^-) corresponds to the even (resp. odd) part in $\bigwedge \mathfrak{n}$.

$$V \otimes S^+ \overset{D}{leftrightarrow} V \otimes S^-$$

The **index of the Dirac operator** acting on $V \otimes S$ is the virtual representation

$$V \otimes S^+ - V \otimes S^-$$

of $\mathfrak{h}$.

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Dirac cohomology of finite dimensional modules

Theorem

Let V be the irreducible finite dimensional representation of V with highest weight μ. Then, as virtual representations of $\mathfrak{h}$

$$V \otimes S^{+} - V \otimes S^{-} = \sum_{w \in W} (-1)^{l(w)} \mathbb{C}_{w\cdot(\mu+\rho)}.$$

Corollary

(Weyl character formula) The character of the finite dimensional representation of V with highest weight μ is given by

$$\mathrm{ch}(V) = \frac{\sum_{w \in W} (-1)^{l(w)} \mathbb{C}_{w\cdot(\mu+\rho)}}{\mathrm{ch}(S^{+} - S^{-})}.$$

The character $\mathrm{ch}(S^{+} - S^{-}) = \sum_{w \in W} (-1)^{l(w)} \mathbb{C}_{w\cdot\rho}$ is the usual Weyl denominator.

Kostant proves this in the general situation

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}.$$

Realization of finite dimensional modules

$\mathfrak{u}$: compact real form of $\mathfrak{g}$
U : connected, simply-connected compact semi-simple group with Lie algebra $\mathfrak{u}$.
T : maximal torus of U
$\mathfrak{h}$: complexification of $\mathfrak{t} = \mathrm{Lie}(T)$.

representation $\mathbb{C}_\lambda$: one dimensional representation of T, $\lambda \in i\mathfrak{t}^*$ integral weight.

$\mathcal{L}_\lambda$: line bundle on U/T with fiber the representation $\mathbb{C}_\lambda \otimes S$ of T

$\Gamma_{L^2}(U/T, \mathcal{L}_\lambda)$: space of L^2 sections of this vector bundle.

$$\Gamma_{L^2}(U/T, \mathcal{L}_\lambda) \simeq L^2(U) \otimes_T (\mathbb{C}_\lambda \otimes S)$$

$\mathfrak{g}$-module, $X \in \mathfrak{g}$ acting by (right) differentiation on the $L^2(U)$ factor.

$\Gamma_{L^2}(U/T, \mathcal{L}_\lambda)$ is a $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$-module. In particular, the Dirac operator $D = D(\mathfrak{g}, \mathfrak{h})$ acts on this space.

Realization of finite dimensional modules

$$\Gamma_{L^2}(U/T, \mathcal{L}_\lambda) \simeq L^2(U) \otimes_T (\mathbb{C}_\lambda \otimes S) \simeq \mathrm{Hom}_T(\mathbb{C}_{-\lambda}, L^2(U) \otimes S).$$

By Peter-Weyl theorem, one has

$$L^2(U) = \bigoplus_{\nu \in \mathcal{P}^+} V_\nu \otimes V_\nu^*$$

D is self-adjoint, so

$$\ker D = \bigoplus_{\nu \in \mathcal{P}^+} V_\nu \otimes \ker\{D \text{ on } \mathrm{Hom}_T(\mathbb{C}_{-\lambda}, V_\nu^* \otimes S).\}$$

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Realization of finite dimensional modules

The contragredient representation V_ν^* has lowest weight $-\nu$. Thus HP theorem implies that $\ker D \neq 0$ iff $-\nu - \rho$ is conjugate to $-\lambda$, *ie.* $\nu + \rho$ is conjugate to λ. In fact, :

Theorem

(Landweber) One has $\ker D = V_{w\cdot\lambda+\rho}$ *if there exists* $w \in W$ *such that* $w \cdot \lambda - \rho$ *is dominant, and* $\ker D = 0$ *otherwise.*
One has $\mathrm{Index}(D) = (-1)^{l(w)} V_{w\cdot\lambda-\rho}$ *if there exists* $w \in W$ *such that* $w \cdot \lambda + \rho$ *is dominant, and* $\mathrm{Index}(D) = 0$ *otherwise.*

This realization of irreducible finite dimensional representation is essentially equivalent to the Borel-Weil-Bott theorem, Dirac operators and Dirac cohomology playing the role of $\mathfrak{n}$-cohomology.