TWISTED ENDOSCOPY FOR REAL GROUPS

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1. Introduction

The transfer factors for twisted endoscopy were defined by Kottwitz and Shelstad in [KS]. In this paper, we show that these factors define a transfer of (twisted) orbital integrals between a real reductive group G and a reductive quasi-split real group H_1 associated to an endoscopic datum (Theorem 1 below). More precisely, we reduce the proof of the transfer to a list of properties of transfer factors. These are established in a forthcoming paper of D. Shelstad [Sh6]. We consider here only real groups. Even in the non-twisted case, the transfer is still conjectural for p-adic groups.

Let G be a connected reductive algebraic group defined over $\mathbb{R}$, $\hat{G}$ its complex dual group and ${}^LG = \hat{G} \rtimes_{\rho_G} W_{\mathbb{R}}$ a realization of its L-group ($W_{\mathbb{R}}$ is the Weil group of $\mathbb{C}/\mathbb{R}$ and ρ_G an L-action of $W_{\mathbb{R}}$ on $\hat{G}$). A Langlands parameter is an L-homomorphism:

$$\phi : W_{\mathbb{R}} \to {}^LG.$$

Two Langlands parameters are equivalent if they are conjugate by an inner automorphism $\operatorname{Int} g$, $g \in \hat{G}$. An equivalence class of Langlands parameters is associated to a packet of irreducible admissible representations of $G(\mathbb{R})$ (see [L1]). These L-packets are finite, and all the representations inside a L-packet have same infinitesimal character. Langlands parameters with a bounded image are associated to L-packets of tempered representations. Temperedness is respected by L-packets, but not unitarity (i.e. there might be unitary and non-unitary representations in the same L-packet), and this complication leads to very interesting developments in the theory of unitary representations (see [Art] and [ABV]).

Suppose that we have two groups H and G (algebraic, connected, reductive, defined over $\mathbb{R}$), and that we are given an L-homomorphism:

$$\epsilon : {}^LH \to {}^LG.$$

This gives a map from (equivalence classes of) Langlands parameters for H to (equivalence classes of) Langlands parameters for G. The Langlands functoriality principle asserts that there should be a map:

$$\epsilon_* : K\Pi(H(\mathbb{R})) \to K\Pi(G(\mathbb{R})).$$

from the Grothendieck group of virtual representations of $H(\mathbb{R})$ to the Grothendieck group of virtual representations of $G(\mathbb{R})$, compatible with L-packets (in the sense that

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the image of a L-packet for $H(\mathbb{R})$ should be contained in the subgroup of $K\Pi(G(\mathbb{R}))$ generated by the representations in the corresponding L-packet for $G(\mathbb{R})$).

This is a very rough formulation, which has to be refined when dealing with non-tempered L-packets. This was carried out in [ABV], solving some conjectures of J. Arthur ([Art]). The map ϵ_* is defined and proved to be compatible with tempered L-packets. In an endoscopic setting, this result is generalized by introducing Arthur parameters and Arthur packets instead of Langlands ones. In this paper, we shall restrict our attention to tempered representations, and deal only with Langlands parameters, although some generalization might be possible.

The motivations for studying Langlands functoriality are connected with automorphic representation theory and trace formula (see [L2]), from a global perspective. From a local point of view (here it means over $\mathbb{R}$), the problem may be expressed as follows. It is known ([Sh1]) that the sum of characters of representations in a tempered L-packet is a stable distribution (for a discussion of stable conjugacy, see [L2]). Let Θ_ϕ be the stable distribution associated to a parameter ϕ. Now, given an irreducible admissible tempered representation π of $G(\mathbb{R})$, one would like to write down a 'character identity' relating the character Θ_π of π and some stable distributions Θ_{ϕ_H} on quasi-split groups H of dimension smaller than G. These groups are such that there exist an L-homomorphism $\epsilon : {}^L H \to {}^L G$, and the parameters ϕ and ϕ_H are related by $\phi = \epsilon \circ \phi_H$.

This program has been carried out by D. Shelstad in a series of papers (see [Sh1], [Sh2], [Sh3], [Sh4]). Only a finite number of groups H are necessary to decompose all irreducible tempered characters of $G(\mathbb{R})$. They are called endoscopic groups because they enable us to 'look inside' L-packets.

What we mean by 'character identity' is somehow stronger than what is provided by the maps ϵ_* and the Langlands functoriality principle as described above, in the sense that what is needed here is also a geometric correspondence between conjugacy classes in $G(\mathbb{R})$ and its endoscopic groups, in order to compute explicitly the characters Θ_π. We will be more precise about this after having introduced our more general setting of 'twisted endoscopy'.

Let G be as above, and θ a finite order automorphism of G. A basic example is when θ is the automorphism associated with a base change $\mathbb{C}/\mathbb{R}$ of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G$. In that case, we recover the results of Shelstad ([Sh4]). Endoscopy for (G, θ) concerns the (tempered) representations π of $G(\mathbb{R})$ such that $\pi \circ \theta$ is equivalent to π, or more generally the (tempered) L-packets Π such that $\Pi \circ \theta = \{\pi \circ \theta \,|\, \pi \in \Pi\} = \Pi$. The automorphism θ preserves L-packets. From θ, we construct an automorphism $\hat{\theta}$ of $\hat{G}$ and an automorphism ${}^L\theta$ of ${}^L G$. Usual notions of invariant harmonic analysis on G, $G(\mathbb{R})$ or $\hat{G}$ can be suitably generalized to θ-twisted invariant harmonic analysis, and we will speak of θ-semi-simple or θ-regular elements, θ-twisted conjugacy classes, θ-twisted orbital integrals and so on (see [R]). If Π has parameter $\phi : W_{\mathbb{R}} \to {}^L G$, then $\Pi \circ \theta$ has parameter ${}^L\theta \circ \phi$, so $\Pi \circ \theta = \Pi$ if and only if

$$S_\phi = \{s \in \hat{G} \mid \mathrm{Int}\, s \circ {}^L\theta \circ \phi = \phi\}$$

is non-empty. If it is so, then S_ϕ contains $\hat\theta$-semi-simple elements. Assume that s is one of them, and let $\hat H$ be the connected component of the identity of $\mathrm{Cent}_{\hat\theta}(s,\hat G) := \{g \in \hat G \mid gs\hat\theta(g)^{-1} = s\}$. This is a reductive subgroup of $\hat G$. Let $\mathcal{H}$ be the subgroup of ${}^LG$ generated by the image of ϕ and $\hat H$. We have then a split exact sequence:

$$1 \to \hat H \to \mathcal{H} \to W_{\mathbb{R}} \to 1$$

from which we obtain a L-action $\rho_{\mathcal{H}}$ of $W_{\mathbb{R}}$ on $\hat H$. Let H be a quasi-split connected algebraic group over $\mathbb{R}$ such that ${}^LH = \hat H \rtimes_{\rho_{\mathcal{H}}} W_{\mathbb{R}}$ is a realization of its L-group. The tuple $(H, \mathcal{H}, s, \xi)$, where ξ is the inclusion of $\mathcal{H}$ in ${}^LG$ constitutes what is called an endoscopic datum for (G,θ). It turns out that $\mathcal{H}$ is not necessarily isomorphic to ${}^LH$, which causes some complications and introduces new features in the theory. From [KS] we recall the definition of a z-pair (H_1, ξ_{H_1}) for $\mathcal{H}$, where H_1 is a central extension

$$1 \to Z_1 \to H_1 \to H \to 1$$

of H and ξ_{H_1} is an embedding of $\mathcal{H}$ into ${}^LH_1$.

By definition, the parameter ϕ has its image in $\mathcal{H}$, so $\xi_{H_1} \circ \phi$ defines a parameter for H_1. In line with the Langlands functoriality principle, we expect to find a 'character identity' involving θ-twisted characters of representations in Π and the stable character on $H_1(\mathbb{R})$ associated to $\xi_{H_1} \circ \phi$. As we said above, this character identity relies on a 'geometric correspondence' between G and H_1. But note that the relation between H and G is usually defined on the dual group side (i.e. H is not a subgroup of G). Nevertheless, there is a map (defined in [KS]):

$$\mathcal{A}_{H/G} : \mathrm{Cl}_{ss}(H) \to \mathrm{Cl}(\theta, G)$$

between semi-simple conjugacy classes in H and θ-semi-simple conjugacy classes in G. This map is not always defined over $\mathbb{R}$, but we will assume it is, i.e. that the cocycle defined in Lemma 3.1.A of [KS] is trivial. Let $\gamma \in H(\mathbb{R})$ be a sufficiently regular element and $\mathcal{O}_\gamma$ its conjugacy class in H. Then $\mathcal{A}_{H/G}(\mathcal{O}_\gamma) \cap G(\mathbb{R})$ is either empty or constitutes a stable θ-conjugacy class in $G(\mathbb{R})$. In this case we say that γ is a 'norm' of any $\delta \in \mathcal{O}_\gamma \cap G(\mathbb{R})$.

We turn now to the problem of transfer of orbital integrals. Let us denote by J_G^θ the map which sends a function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ to its θ-twisted orbital integral (J_G^θ is a smooth function on the open dense subset of θ-regular elements of $G(\mathbb{R})$, invariant under twisted conjugacy). The image of this map is denoted by $\mathcal{I}^\theta(G(\mathbb{R}))$, and is characterized by a set of properties. Furthermore, it is endowed with a topology of an inductive limit of Fréchet spaces. The transpose map ${}^tJ_G^\theta$ realizes an isomorphism from the dual space $\mathcal{I}^\theta(G(\mathbb{R}))'$ onto the space of distributions invariant under θ-conjugacy ([R]).

There are of course similar results for ordinary or stable orbital integrals on $H_1(\mathbb{R})$ ([B1]). We have to consider here a slightly different situation, where we don't deal with orbital integrals of compactly supported functions, but rather of smooth functions f compactly supported modulo $Z_1(\mathbb{R})$ and satisfying:

$$f(zh) = \lambda_{H_1}(z)^{-1} f(h)$$

for all $z \in Z_1(\mathbb{R})$ and all $h \in H_1(\mathbb{R})$. Here, λ_{H_1} is a quasi-character on $Z_1(\mathbb{R})$ obtained from the endoscopic data $(H, \mathcal{H}, s, \xi)$ and the z-pair (H_1, ξ_{H_1}). We denote by $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$ the space of stable orbital integrals of these functions just described.

The principal result of this paper is

Theorem 1. *There is a continuous map:*

$$\begin{aligned} \mathit{Trans} : \mathcal{I}^\theta(G(\mathbb{R})) &\to \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1}) \\ \psi &\mapsto \mathit{Trans}(\psi) \end{aligned}$$

The orbital function Trans(ψ) *is given for sufficiently regular elements* γ_1 *of* $H_1(\mathbb{R})$ *by a formula:*

$$\mathit{Trans}(\psi)(\gamma_1) = \sum_{\delta \in \Sigma_\gamma} \Delta(\gamma_1, \delta)\psi(\delta),$$

where γ *is the projection of* γ_1 *on* $H(\mathbb{R})$ *and the sum (which might be empty, in which case the right-hand side is* 0*) is taken over a set of representative of* θ*-conjugacy classes under* $G(\mathbb{R})$ *of elements* $\delta \in G(\mathbb{R})$ *for which* γ *is a norm.*

The factors $\Delta(\gamma_1, \delta)$ are the transfer factors defined in [KS].

In order to establish this result, we need two kinds of ingredients. Firstly, we will use the properties of transfer factors as found in [KS], mainly to show that Trans(ψ) is well-defined. Secondly, we will study the behavior of the transfer factors when moving transversally to (θ-twisted) conjugacy classes (i.e. along 'Cartan subspaces'). For this, we use Diana Shelstad's results [Sh6] concerning transfer factors. This will enable us to deduce that the properties of ψ transfer to the corresponding properties of Trans(ψ), and then prove that Trans(ψ) is indeed in $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$.

In a forthcoming paper, we will use these results to establish characters identities as described above, using the transpose map ^{t}Trans from stable invariant distributions on $H_1(\mathbb{R})$ to θ-twisted invariant distributions on $G(\mathbb{R})$.

Let us conclude this introduction with a survey of the contents of this paper. After introducing notation, we recall basic results on twisted conjugacy in Section 2, and the definition of endoscopic datum in Section 3. Section 4 is devoted to the geometric correspondence between the group G and its endoscopic group H, and its properties. In Section 5, we recall results on orbital integrals (θ-twisted orbital integrals on G, stable orbital integrals on H) and their characterizing properties. We are then in position to state the transfer theorem (Theorem 5.4.1). In section 6, we reduce the proof of Theorem 5.4.1 to various lemmas and one proposition (Proposition 6.3.1). Some of these lemmas are of geometric nature, and involve Cayley transforms, Hirai order... They are established in Section 7. The remaining assertions in Proposition 6.3.1 and Lemma 6.6.4 concern fine properties of transfer factors. In fact, guided by available examples, these were defined in [KS] in order to satisfy these properties. Some of them will be established in [Sh6], the rest is in Section 8.

Finally, let us point out that we have not considered here the most general case of twisted endoscopy. Firstly, we suppose θ to be of finite order, where Kottwitz and Shelstad consider semi-simple automorphisms. This restriction shouldn't be too important in applications, however. Secondly, Kottwitz and Shelstad introduce a character ω on $G(\mathbb{R})$, in order study representations π such that $\pi \circ \theta \simeq \pi \otimes \omega$. We limit ourselves to the case where ω is trivial, because we don't know how to generalize the results in Section 5 to the (θ, ω)-twisted setting. Thirdly, there is the technical assumption that the cocycle z_σ of [KS], Lemma 3.1.A is trivial (see begining of Section 4). We notice that this is frequently the case in the applications, and that following [KS], Section 5.4, we could relax this assumption.

We thank D. Shelstad for making the preliminary version of [Sh6] available to us.

2. Twisted conjugacy

2.1. **Notation.** We begin by introducing some notation. Let A be a Lie group, and B a set on which A acts. Let:

$$\mathrm{Cent}(B, A) = \{a \in A \mid \forall b \in B,\ a \cdot b = b\}$$
$$\mathrm{Norm}(B, A) = \{a \in A \mid \forall b \in B,\ a \cdot b \in B\}.$$

Different actions of a group on itself will be considered, so unless otherwise stated, the above notation will refer to the usual action by conjugation. The inner automorphism of A given by an element $a \in A$ will be denoted by $\mathrm{Int}\, a$ and $A^a := \{b \in A \mid \mathrm{Int}\, a(b) = aba^{-1} = b\}$. The connected component of the trivial element in A will be denoted by A_0.

Let H be a connected reductive algebraic group defined over $\mathbb{R}$. The group H is identified with the group of its complex points. Let σ be the non trivial element of $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We will denote by σ_H the action of σ on H, by $H(\mathbb{R})$ the group of real points of H, and by $\mathfrak{h}$ (resp. $\mathfrak{h}_\mathbb{R}$) the Lie algebra of H (resp. $H(\mathbb{R})$).

Following [KS], by a pair in H, we mean a couple (B, T) where B is a Borel subgroup of H and T a maximal torus in B, and by a splitting of H, we mean a triple $\mathbf{spl}_H = (B, T, \{X\})$ where (B, T) is a pair in H and $\{X\}$ a collection of non-zero root vectors, one for each simple root of T in B. The group H is quasi-split if and only if it has an $\mathbb{R}$-splitting, i.e. one preserved by σ_H. It will be convenient to construct L-groups as follows. Suppose that H is quasi-split and let $\mathbf{spl}_H = (B, T, \{X\})$ be an $\mathbb{R}$-splitting. Let $\hat{H}$ be the dual group of H and $\mathbf{spl}_{\hat{H}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ a splitting of $\hat{H}$. The dual of the based root datum $\Psi(B, T)$ obtained from (B, T) is isomorphic to the based root datum $\Psi(\mathcal{B}, \mathcal{T})$ obtained from $(\mathcal{B}, \mathcal{T})$. The action σ_H induces an action on $\Psi(B, T)$, which transfers to an action $\sigma_{\hat{H}}$ on $\Psi(\mathcal{B}, \mathcal{T})$ by this isomorphism. Let us denote also by $\sigma_{\hat{H}}$ the algebraic action on $\hat{H}$ which induces $\sigma_{\hat{H}}$ on $\Psi(\mathcal{B}, \mathcal{T})$ and preserves the splitting $\mathbf{spl}_{\hat{H}}$. Let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$ (see [Bo]). As usual, we define the L-group ${}^L H$ of H to be the semi-direct product $\hat{H} \rtimes_{\rho_H} W_\mathbb{R}$, where the action ρ_H of $W_\mathbb{R}$ on $\hat{H}$ factors through $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, with $\rho_H(1 \times \sigma)$ acting by $\sigma_{\hat{H}}$. Note that such a realization of the L-group depends on the choice of the splitting $\mathbf{spl}_{\hat{H}}$ but not of the splitting $\mathbf{spl}_H$. If H is not quasi-split, we choose a quasi-split inner form H^* of H, and a L-group for H will be one constructed from H^*. We denote by H_{sc} the universal covering of the derived group H_{der} of H, and if T is a maximal torus in H, T_{sc} denote the inverse image of $T \cap H_{der}$ in H_{sc} under the natural projection. We will often identify elements in H_{sc} or T_{sc} with their image in H or T without comments.

2.2. **Basic results on twisted conjugacy.** Let G be a connected reductive algebraic group defined over $\mathbb{R}$ and θ an $\mathbb{R}$-automorphism of G of finite order. We denote also by θ its differential which is an isomorphism of $\mathfrak{g}$ =Lie(G). The group G acts upon itself by θ-twisted conjugacy:

$$(g,x) \mapsto gx\theta(g)^{-1}.$$

We begin by recalling some material about θ-twisted invariant harmonic analysis on G (and $G(\mathbb{R})$).

For any $x \in G$, let us denote $G^{x\theta}$ the twisted centralizer of x, i.e. $G^{x\theta} = \{g \in G |\, gx\theta(g)^{-1} = x\}$ and by $\mathfrak{g}^{x\theta}$ the centralizer in of $\mathfrak{g}$ of $\mathrm{Ad}\, x \circ \theta$.

Definition 2.2.1. - *An element $x \in G$ is θ-semi-simple if $Ad\, x \circ \theta$ is a semi-simple automorphism of $\mathfrak{g}$.*

- *An element $x \in G$ is θ-regular if the multiplicity of the eigenvalue* 1 *of $Ad\, x \circ \theta$ is minimal.*

- *An element $x \in G$ is strongly θ-regular if $G^{x\theta}$ is abelian.*

We denote by $G_{\theta-reg}$ (resp. $G(\mathbb{R})_{\theta-reg}$) the (dense, open) subset of θ-regular elements in G (resp. $G(\mathbb{R})$). More generally, if A is a subset of G, we denote by $A_{\theta-reg}$ the set of θ-regular elements in A.

Proposition 2.2.2. *We have the following results:*

(i) If $g \in G$ is θ-regular, then g is θ-semi-simple and $\mathfrak{g}^{g\theta}$ is an abelian subalgebra of $\mathfrak{g}$ whose elements are semi-simple. Moreover, $\mathfrak{g}^{g\theta}$ contains regular elements of $\mathfrak{g}$, i.e. $\mathrm{Cent}(\mathfrak{g}^{g\theta}, \mathfrak{g})$ *is a Cartan subalgebra of $\mathfrak{g}$.*

(ii) A strongly θ-regular element is θ-regular.

(iii) Let $x \in G$ be θ-semi-simple, $\mathfrak{z} = \mathfrak{g}^{x\theta}$ and $Z = (G^{x\theta})_0$. Then there exists a θ-regular element $y \in (\exp \mathfrak{z})\, x$ such that $\mathfrak{a} = \mathfrak{g}^{y\theta}$ is a Cartan subalgebra of $\mathfrak{z}$ and $\exp \mathfrak{a}$ *is a maximal torus in Z.*

Proof. For (i)and (iii), see [R], propositions 2.1 and 2.4. The second assertion is obvious as soon as we see that $g \in G$ is regular iff $(G^{\theta g})_0$ is a torus, which is a consequence of (i). □

Let x be a θ-regular element of G, $\mathfrak{a} = \mathfrak{g}^{x\theta}$, and $\mathfrak{h}$ =Cent($\mathfrak{a}, \mathfrak{g}$). We have a decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\beta \in R(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^{\beta}$$

where $R(\mathfrak{g}, \mathfrak{a})$ is a (non-reduced) root system and $\mathfrak{g}^{\beta}$ is the root space for the root β. The root system $R(\mathfrak{g}, \mathfrak{a})$ is the set of restrictions to $\mathfrak{a}$ of the roots in $R(\mathfrak{g}, \mathfrak{h})$.

Since θ is of finite order, it preserves a pair (B, T) of G, i.e. we have $\theta(B) = B$ and $\theta(T) = T$ ([KS](1.1)). For such a pair, we have the following results:

Proposition 2.2.3. *(i) There exist θ-regular elements in T. More precisely, there exist θ-regular elements in each subset of the form $t.T^{\theta}$ where $t \in T$ and T^{θ} denotes the fixed points of θ in T.*

(ii) Let $t \in T$ be θ-regular. Then $\mathfrak{g}^{t\theta} = \mathfrak{t}^{\theta}$, $(G^{t\theta})_0 = (T^{\theta})_0$, and $\mathrm{Cent}((T^{\theta})_0, G) = T$. *Furthermore $G^{t\theta} \subset \mathrm{Norm}((T^{\theta})_0, G)$ and if t is strongly θ-regular, then $G^{t\theta} = T^{\theta}$.*

Proof. These are easy consequences of [R], §2, and [KS] theorem 1.1. □

Definition 2.2.4. *A Cartan subspace of $G(\mathbb{R})$ is a subset of the form $A = (\exp \mathfrak{g}_{\mathbb{R}}^{x\theta}) \cdot x$ where x is a θ-regular element of $G(\mathbb{R})$.*

Proposition 2.2.5. *The number of θ-twisted conjugacy classes of Cartan subspaces in $G(\mathbb{R})$ is finite.*

Proof. See [R] proposition 7.4. □

The interest of introducing these subspaces is that the twisted orbital integrals of compactly supported functions on $G(\mathbb{R})$ are determined by their restrictions to a system of representatives of θ-twisted conjugacy classes of Cartan subspaces. They will replace Cartan subgroups in the point correspondences, since in twisted endoscopy Cartan subgroups of G and its endoscopic groups are not isomorphic for non trivial θ (see section 4 below).

We fix a quasi-split group G^* in the inner class of real forms of G with an inner twisting $\psi : G \to G^*$. The group G^* being quasi-split, it has an $\mathbb{R}$-splitting $\mathbf{spl}_{G^*} = (B^*, T^*, \{X^*\})$. We may choose $g_\theta \in G^*$ such that the automorphism

$$\theta^* = \mathrm{Int}\, g_\theta \circ \psi \circ \theta \circ \psi^{-1} \tag{2.1}$$

preserves $\mathbf{spl}_{G^*}$. Furthermore $\sigma_{G^*}(\theta^*) = \sigma_{G^*} \circ \theta^* \circ \sigma_{G^*} = \theta^*$ because θ^* is uniquely determined by θ and $\mathbf{spl}_{G^*}$, and thus θ^* is defined over $\mathbb{R}$.. Suppose we have a L-group data for G given by a splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$. Then θ induces bijections of the based root data $\Psi(B^*, T^*)$ and $\Psi(\mathcal{B}, \mathcal{T})$. We denote by $\hat{\theta}$ the automorphism of $\hat{G}$ which induces this bijection of $\Psi(\mathcal{B}, \mathcal{T})$ and preserves $\mathbf{spl}_{\hat{G}}$. We denote by ${}^L\theta$ the automorphism $\hat{\theta} \rtimes 1_{W_{\mathbb{R}}}$ of ${}^LG$.

3. Endoscopic data

3.1. Following [KS], we call the tuple $(H, \mathcal{H}, s, \xi)$ an endoscopic data for (G, θ) if :

(3.1.1) H is a quasi-split connected reductive algebraic group defined over $\mathbb{R}$,

(3.1.2) $\mathcal{H}$ is a split extension of $W_{\mathbb{R}}$ by $\hat{H}$ such that the L-action of $W_{\mathbb{R}}$ on $\hat{H}$ defined by this extension coincide with ρ_H,

(3.1.3) s is a $\hat{\theta}$-semi-simple element in $\hat{G}$.

(3.1.4) $\xi : \mathcal{H} \to {}^LG$ is a L-homomorphism satisfying:

  (3.1.4.a) $\mathrm{Int}\, s \circ {}^L\theta \circ \xi = a \cdot \xi$ where a is a 1-coboundary of $W_{\mathbb{R}}$ in $Z(\hat{G})$.

  (3.1.4.b) ξ maps isomorphically $\hat{H}$ into $(\hat{G}^{\hat{\theta} s})_0$

Recall that in we have chosen a realization of the L-group of H, i.e. we have fixed a splitting $\mathbf{spl}_{\hat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ to define ρ_H ($\mathbf{spl}_{\hat{H}}$ is preserved by ρ_H). The action of σ on $\hat{H}$ given by the split extension $\mathcal{H}$ can be modified by an inner automorphism of $\hat{H}$ to preserve $\mathbf{spl}_{\hat{H}}$ and (3.1.2) means that it coincides with ρ_H.

There is a notion of isomorphism of endoscopic data: $(H, \mathcal{H}, s, \xi)$ is isomorphic to $(H', \mathcal{H}', s', \xi')$ if there exists $g \in \hat{G}$ such that:

$$g\xi(\mathcal{H})g^{-1} = \xi'(\mathcal{H}') \tag{3.2}$$

$$gs\hat{\theta}(g)^{-1} = s' \quad \text{modulo} \quad Z(\hat{G}) \tag{3.3}$$

Recall the splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ used to form ${}^LG$. We may replace (s, ξ) by $(gs\hat{\theta}(g)^{-1}, \operatorname{Int} g \circ \xi)$ for any $g \in \hat{G}$ and therefore assume that $s \in \mathcal{T}$. Then $(\mathcal{B}, \mathcal{T})$ is an $\operatorname{Int} s \circ \hat{\theta}$-stable pair and by making another such replacement with a $g \in (\hat{G}^{s\hat{\theta}})_0$, we may assume that:

$$\xi(\mathcal{B}_H) = \mathcal{B} \cap (\hat{G}^{s\hat{\theta}})_0 \ \text{ and } \ \xi(\mathcal{T}_H) = \mathcal{T} \cap (\hat{G}^{s\hat{\theta}})_0 = (\mathcal{T}^{\hat{\theta}})_0. \tag{3.4}$$

3.2. **z-pair.** Since $\mathcal{H}$ is not necessarily an L-group for H (see [KS], §2.1) we need to introduce a z-extension H_1 of H. We recall the definition ([K]):

Definition 3.2.1. *A z-extension of a connected reductive algebraic quasi-split real group H is a central extension H_1 of H:*

$$1 \to Z_1 \to H_1 \to H \to 1$$

where H_1 is a connected reductive algebraic quasi-split real group whose derived group is simply-connected and Z_1 is a central torus in H_1, isomorphic to a product of $Res_{\mathbb{C}/\mathbb{R}}\mathbb{R}^\times$.

Note that since $H^1(\Gamma, Z_1) = \{1\}$, we have also:

$$1 \to Z_1(\mathbb{R}) \to H_1(\mathbb{R}) \to H(\mathbb{R}) \to 1.$$

Dual to the exact sequence in the above definition is

$$1 \to \hat{H} \to \hat{H}_1 \to \hat{Z}_1 \to 1$$

so we regard $\hat{H}$ as a subgroup of $\hat{H}_1$. This inclusion can be extended to a L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^LH_1$ (see [KS] lemma 2.2.A).

Definition 3.2.2. *By a z-pair for $\mathcal{H}$, we mean a pair (H_1, ξ_{H_1}) where H_1 is a z-extension of H and $\xi_{H_1} : \mathcal{H} \to {}^LH_1$ a L-homomorphism that extends $\hat{H} \hookrightarrow \hat{H}_1$.*

Observe that ξ_{H_1} determines a character λ_{H_1} of $Z_1(\mathbb{R})$. This character has Langlands parameter:

$$W_{\mathbb{R}} \xrightarrow{c} \mathcal{H} \xrightarrow{\xi_{H_1}} {}^LH_1 \to {}^LZ_1,$$

where c is any section of $\mathcal{H} \to W_{\mathbb{R}}$. For a discussion of the significance of λ_{H_1} in terms of Langlands functoriality principle, see [KS], end of section 2.2. Let us just say here that if $\mathcal{H}$ is not a L-group, then there is no matching between compactly supported smooth functions on $G(\mathbb{R})$ and compactly supported smooth functions on $H(\mathbb{R})$. Instead, we will establish a matching between compactly supported smooth functions on $G(\mathbb{R})$ and smooth functions on $H_1(\mathbb{R})$, compactly supported modulo $Z_1(\mathbb{R})$ and transforming under translations by elements of $Z_1(\mathbb{R})$ according to λ_{H_1}.

4. Norm mappings

4.1. Recall that $\theta^* = \operatorname{Int} g_\theta \circ \psi \circ \theta \circ \psi^{-1}$ preserves the $\mathbb{R}$- splitting $\mathbf{spl}_{G^*}$. Let us choose $u_\sigma \in G^*_{sc}$ such that

$$\psi\sigma(\psi)^{-1} = \operatorname{Int} u_\sigma. \tag{4.1}$$

Then the map:

$$m: G \to G^*$$
$$\delta \mapsto \psi(\delta) g_\theta^{-1}$$

is easily checked to preserve θ-conjugacy, so it induces a bijection:

$$m: \operatorname{Cl}(G,\theta) \to \operatorname{Cl}(G^*,\theta^*)$$

where $\operatorname{Cl}(G,\theta)$ (resp. $\operatorname{Cl}(G^*,\theta^*)$) is the set of θ (resp. θ^*)-conjugacy classes of G (resp. G^*). This map may not respect the action of σ. In the rest of this paper we consider only the case when it does, i.e. we suppose that the class of the 1-cocycle z_σ of [KS], lemma 3.1.A is trivial in $H^1(\Gamma, Z_\theta^{sc})$ (for example, this holds for standard (non-twisted) endoscopy, when θ arise from base change $\mathbb{C}/\mathbb{R}$, or when $G(\mathbb{R})$ is quasi-split). We may then choose g_θ and u_σ such that $g_\theta u_\sigma = \theta^*(u_\sigma)\sigma_{G^*}(g_\theta)$. Then m is defined over $\mathbb{R}$ (on classes), and for all $\delta \in G$:

$$\sigma(m)(\delta) = u_\sigma^{-1} m(\delta)\theta^*(u_\sigma). \tag{4.2}$$

Suppose that (B,T) is a θ^*-stable pair in G^*, let $\Omega = \Omega(G^*,T)$ be the Weyl group of T in G^* and $\Omega^{\theta^*} = \{w \in \Omega \,|\, w \circ \theta^* = \theta^* \circ \omega\}$. Note that each element of Ω^{θ^*} is represented by an element of $G^1 := (G^{\theta^*})_0$ ([KS], Section 1.1). Let us denote $(1-\theta^*)T = \{u\theta^*(u)^{-1} \,|\, u \in T\}$, $T_{\theta^*} = T/(1-\theta^*)T$ and N_{θ^*} the projection from T onto T_{θ^*}.

Lemma 4.1.1. *(*[KS] *3.2.A)*
(i) Each $\mathcal{O} \in Cl_{ss}(G^*,\theta^*)$ meets T.
(ii) The image of $\mathcal{O} \cap T$ in T_{θ^*} is a single Ω^{θ^*}-orbit

The subscript 'ss' stands for 'semi-simple'. Thus, we have a bijection:

$$\operatorname{Cl}_{ss}(G^*,\theta^*) \to T_{\theta^*}/\Omega^{\theta^*}.$$

between θ semi-simple twisted conjugacy classes in G and θ^* semi-simple twisted conjugacy classes in G^*. The composition:

$$N_\theta: \operatorname{Cl}_{ss}(G,\theta) \to \operatorname{Cl}_{ss}(G^*,\theta^*) \to T_{\theta^*}/\Omega^{\theta^*}$$

is called an abstract norm map. If T is defined over $\mathbb{R}$, then N_θ is also defined over $\mathbb{R}$.

Let us consider the endoscopic data $(H,\mathcal{H},s,\xi)$. We recall now the point correspondences between H and G.

Theorem 4.1.2. *(*[KS] *3.3.A) There is a canonical map defined over $\mathbb{R}$:*

$$\mathcal{A}_{H/G}: Cl_{ss}(H) \to Cl_{ss}(G,\theta) \tag{4.3}$$

between semi-simple conjugacy classes in H and θ-semi-simple twisted conjugacy classes in G.

This map is obtained in the following way: suppose (B_H, T_H) is a pair in H and that (B, T) is a θ^*-stable pair in G^*. Attached to (B_H, T_H) and $(\mathcal{B}_H, \mathcal{T}_H)$ is an isomorphism $\hat{T}_H \simeq \mathcal{T}_H$ and attached to (B, T) and $(\mathcal{B}, \mathcal{T})$ is an isomorphism $\hat{T} \simeq \mathcal{T}$. By θ-stability, the latter induces:

$$\widehat{T}_{\theta^*} = (\widehat{T}^{\hat{\theta}})_0 \simeq (\mathcal{T}^{\hat{\theta}})_0.$$

We have therefore a chain of isomorphism:

$$\widehat{T}_H \simeq \mathcal{T}_H \xrightarrow{\xi} (\mathcal{T}^{\hat{\theta}})_0 \simeq \widehat{T}_{\theta^*}$$

which yields $T_H \simeq T_{\theta^*}$ and $\Omega_H := \Omega(H, T_H) \hookrightarrow \Omega^{\theta^*}$, and so induces:

$$T_H/\Omega_H \to T_{\theta^*}/\Omega^{\theta^*}.$$

Therefore we have:

$$\mathrm{Cl}_{ss}(H) \to T_H/\Omega_H \to T_{\theta^*}/\Omega^{\theta^*} \to \mathrm{Cl}_{ss}(G^*, \theta^*) \to \mathrm{Cl}_{ss}(G, \theta)$$

yielding the map 4.3

If T_H is defined over $\mathbb{R}$, we may choose (B, T) and B_H such that both $\mathbb{T}$ and the isomorphism $T_H \simeq T_{\theta^*}$ are defined over $\mathbb{R}$ (see [KS], 3.3.B). An $\mathbb{R}$-isomorphism $T_H \simeq T_{\theta^*}$ as above will be called an admissible embedding of T_H, and T_H will be called a norm group for T.

4.2. **Norms.** We define norms in $H(\mathbb{R})$ in the following way. Let $\delta \in G$ be θ-regular and let $T_\delta = \mathrm{Cent}((G^{\delta\theta})_0, G)$. We know that T_δ is a maximal torus of G, stable under $\mathrm{Int}\,\delta \circ \theta$ and that

$$T_\delta^{\delta\theta} \subset G^{\delta\theta}$$

with equality if and only if δ is strongly regular.

Let us denote by $\mathcal{O}_\gamma$ the conjugacy class of an element $\gamma \in H$, and by $\mathcal{O}_{\theta-\delta}$ the θ-twisted conjugacy class of $\delta \in G$.

Definition 4.2.1. *An element $\gamma \in H$ is G-regular (resp. strongly G-regular if $\mathcal{A}_{H/G}(\mathcal{O}_\gamma)$ is a θ-regular (resp. strongly θ-regular) θ-conjugacy class in G.*

Lemma 4.2.2. *(*[KS]*,3.3.C)*

(i) G-regular implies regular

(ii) strongly G-regular implies strongly regular.

Corollary 4.2.3. *The stable conjugacy class of a strongly G-regular element $\gamma \in H(\mathbb{R})$ is $\mathcal{O}_\gamma \cap H(\mathbb{R})$.*

Proof. This is merely the definition of stable conjugacy class for strongly regular elements. □

Definition 4.2.4. *The stable θ-conjugacy class of a strongly θ-regular element $\delta \in G(\mathbb{R})$ is $\mathcal{O}_{\theta-\delta} \cap G(\mathbb{R})$.*

We extend this definition for θ-regular elements. Let $\delta \in G$ be such a element. Suppose that $\delta' = g\delta\theta(g)^{-1} \in G(\mathbb{R})$. Then we have $\sigma_G(g)^{-1}g \in G^{\delta\theta}$. Let us denote $T_\delta = \mathrm{Cent}((G^{\delta\theta})_0, G)$. Since δ is θ-regular, T_δ is a torus.

Definition 4.2.5. *In the above setting, we say that δ' is in the stable θ-conjugacy class of δ if and only if $\sigma_G(g)^{-1}g \in I_\delta := G^{\delta\theta} \cap T_\delta$.*

We will see below the reason for this requirement. Note that if δ is strongly θ-regular, then $G^{\delta\theta} \subset T_\delta$ and the two definitions agree.

Definition 4.2.6. *Let $\gamma \in H(\mathbb{R})$ be G-regular, and let T_H be the maximal torus of H containing it. Fix an admissible embedding $T_H \xrightarrow{\eta} T_{\theta^*}$ of T_H in G^* defined over $\mathbb{R}$. We say that γ is a norm of $\delta \in G(\mathbb{R})$ if:*

(i) δ lies in the image of $\mathcal{O}_\gamma$ under $\mathcal{A}_{H/G}$.

Then, by definition, there exist $x \in G^*_{sc}$ and $\delta^* \in T$ such that $\delta^* = xm(\delta)\theta^*(x)^{-1}$ and $N_{\theta^*}(\delta^*) = \eta(\gamma)$. An easy computation shows that $\mathrm{Int}\, x \circ \psi$ maps $G^{\delta\theta}$ bijectively onto $(G^*)^{\delta^*\theta^*}$ and $T_\delta^{\delta\theta} = T_\delta \cap G^{\delta\theta}$ onto $T^{\theta^*} = T \cap (G^*)^{\delta^*\theta^*}$.

(ii) $\mathrm{Int}\, x \circ \psi : T_\delta^{\delta\theta} \to T^{\theta^*}$ is defined over $\mathbb{R}$.

We will see in the proof of the next theorem that if γ is strongly G-regular, then the condition (ii) is automatically fulfilled. When it is not possible to find such an element δ, we say that γ is not a norm.

Theorem 4.2.7. *Let $\gamma \in H(\mathbb{R})$ be G-regular. Then γ is a norm of exactly one stable θ-conjugacy class in $G(\mathbb{R})$ or is not a norm.*

Proof. In the setting as above, $\mathrm{Int}\, x \circ \psi : T_\delta^{\delta\theta} \to T^{\theta^*}$ is defined over $\mathbb{R}$ if and only if $\mathrm{Int}(v_\sigma)$ is trivial on T^{θ^*}, where $v_\sigma = xu_\sigma\sigma_{G^*}(x)^{-1}$. To see this, recall that we have from (4.1)

$$\psi = \mathrm{Int}\, u_\sigma \circ \sigma_{G^*} \circ \psi \circ \sigma_G$$

Thus

$$\sigma(\mathrm{Int}\, x \circ \psi) = \sigma_{G^*} \circ (\mathrm{Int}\, x \circ \psi) \circ \sigma_G = \mathrm{Int}(\sigma_{G^*}(x)u_\sigma^{-1}) \circ \psi,$$

and $\sigma(\mathrm{Int}\, x \circ \psi) = \mathrm{Int}\, x \circ \psi$ if and only if $\mathrm{Int}(v_\sigma)$ is trivial, proving the assertion.

Now, we deduce from (4.2):

$$\begin{aligned} \sigma_{G^*}(\delta^*) &= \sigma_{G^*}(xm(\delta)\theta^*(x)^{-1}) = \sigma_{G^*}(x)\sigma(m)(\sigma_G(\delta))\sigma_{G^*}(\theta^*(x))^{-1} \\ &= \sigma_{G^*}(x)u_\sigma^{-1}m(\delta)\theta^*(u_\sigma)\theta^*(\sigma_{G^*}(x))^{-1} \\ &= (\sigma_{G^*}(x)u_\sigma^{-1}x^{-1})\,(xm(\delta)\theta^*(x)^{-1})\,\theta^*(xu_\sigma\sigma_{G^*}(x)^{-1}) \\ & v_\sigma^{-1}\delta^*\theta^*(v_\sigma) \end{aligned}$$

and, since η is defined over $\mathbb{R}$, $\sigma_{G^*}(\delta^*) = \delta^*$ modulo $(1-\theta^*)T$, it exists $t \in T$ such that $\sigma_{g^*}(\delta^*) = t\delta^*\theta^*(t)^{-1}$ and thus

$$(v_\sigma t)^{-1}\delta^*\theta^*(v_\sigma t) = \delta^*.$$

Therefore, we have:

$$v_\sigma t \in (G^*)^{\delta^*\theta^*} \qquad \text{with } \ v_\sigma \in G^*_{sc}.$$

Note that $(G^*)^{\delta^*\theta^*} \subset \mathrm{Norm}(T^{\theta^*}, G^*)$, so $\mathrm{Int}\, v_\sigma$ normalizes T^{θ^*} and T, and if γ is strongly G-regular, then $(G^*)^{\delta^*\theta^*} = T^{\theta^*}$ (see (2.2.3)).

Lemma 4.2.8. *$\mathrm{Int}\, v_\sigma$ is trivial on T^{θ^*} if and only if $v_\sigma t \in T^{\theta^*}$.*

Proof. Suppose that $\mathrm{Int}\, v_\sigma$ is trivial on T^{θ^*}. Then it is trivial on T, because T^{θ^*} contains strongly θ^*-regular elements. We conclude that $v_\sigma \in T_{sc}$ and that $v_\sigma t \in T^{\theta^*}$. The other implication is immediate.

From this discussion, we see that if γ is strongly G-regular, then $\operatorname{Int} x\circ\psi : T_\delta^{\delta\theta} \to T^{\theta^*}$ is defined over $\mathbb{R}$.

We now come back to the proof of the theorem. Suppose that γ is a norm of δ, and let x, v_σ be as above. In particular, $\operatorname{Int} v_\sigma$ is trivial on T^{θ^*}. Suppose that δ_1 is stably θ-conjugate to δ, i.e. there exists $g \in G$, with $\delta_1 = g\delta\theta(g)^{-1} \in G(\mathbb{R})$ and $\sigma_G(g)^{-1}g \in G^{\delta\theta} \cap T_\delta$. Then we compute

$$\delta^* = x\psi(g)^{-1}m(\delta_1)\theta^*(x\psi(g)^{-1})^{-1}.$$

We want to prove that γ is a norm for δ_1, so we have to check that $\operatorname{Int}(x\psi(g^{-1}) \circ \psi : T_{\delta_1}^{\delta_1\theta} \to T^{\theta^*}$ is defined over $\mathbb{R}$. From the discussion above, this is the case if and only if $\operatorname{Int}(x\psi(g)^{-1}u_\sigma\sigma_{G^*}(x\psi(g)^{-1})^{-1})$ is trivial on T^{θ^*}. We compute:

$$\begin{aligned} x\psi(g)^{-1}u_\sigma\sigma_{G^*}(x\psi(g)^{-1})^{-1} &= \left(x\psi(g)^{-1}u_\sigma\right)\,\left(u_\sigma^{-1}\psi(\sigma_G(g))u_\sigma\right)\,\sigma_{G^*}(x)^{-1} \\ &= x\,\psi(g^{-1}\sigma_G(g))\,u_\sigma\sigma_{G^*}(x)^{-1} \\ &= x\,\psi(g^{-1}\sigma_G(g))\,x^{-1}v_\sigma \end{aligned}$$

Hence, we want to check that $\operatorname{Int}(x\psi(g^{-1}\sigma_G(g))x^{-1})$ is trivial on T^{θ^*}, i.e. $\operatorname{Int}(g^{-1}\sigma_G(g))$ is trivial on $T^{\delta\theta} \cap T_\delta$. This is now obvious by the definition of stable θ-conjugacy and the requirement that $g^{-1}\sigma_G(g)) \in G^{\delta\theta} \cap T_\delta$.

Let us now prove the other inclusion, and so suppose that γ is a norm for δ, $\delta_1 \in G(\mathbb{R})$, i.e. there exist δ^*, δ_1^* in T and x, $x_1 \in G_{sc}^*$ such that

$$\begin{cases} \delta^* = xm(\delta)\theta^*(x)^{-1} \\ \delta_1^* = x_1m(\delta_1)\theta^*(x_1)^{-1} \\ N_{\theta^*}(\delta_1^*) = N_{\theta^*}(\delta^*) = \eta(\gamma). \end{cases}$$

Therefore, there is a $t \in T$ such that :

$$\delta_1^* = t\delta^*\theta^*(t)^{-1}.$$

From this last equation, easy computations lead to :

$$\delta_1 = \psi^{-1}(x_1^{-1}tx)\delta\theta(\psi^{-1}(x_1^{-1}tx))^{-1}.$$

We have then to show that $w^{-1}\sigma_G(w) \in G^{\delta\theta} \cap T_\delta$, where $w = \psi^{-1}(x_1^{-1}tx)$. This is equivalent to : $\operatorname{Int} x \circ \psi(w^{-1}\sigma_G(w)) \in (G^*)^{\delta^*\theta^*} \cap T = T^{\theta^*}$, but:

$$\operatorname{Int} x \circ \psi(w^{-1}\sigma_G(w)) = t^{-1}\sigma_{G^*}(v_1)^{-1}\sigma_{G^*}(t)\sigma_{G^*}(v),$$

where $v = v_\sigma$ as above and $v_1 = x_1u_\sigma\sigma_{G^*}(x_1)^{-1}$. This is a product of elements in T, hence $\operatorname{Int} x \circ \psi(w^{-1}\sigma_G(w)) \in T$, and it's easy to check that it fixes δ^*, and it is in fact in T^{θ^*}. The proof of the theorem is now complete. □

4.3. **Admissible coverings.** Let T_H be a maximal torus of H defined over $\mathbb{R}$, and let A_H be a connected component of $T_H(\mathbb{R})$. If there exist a G-regular element $\gamma \in A_H$ which is the norm of an element $\delta \in G(\mathbb{R})$, we say that A_H originates in $G(\mathbb{R})$. Recall this means that given an admissible embedding $T_H \xrightarrow{\eta} T_{\theta^*}$ of T_H in G^* defined over $\mathbb{R}$, there exist $x \in G_{sc}^*$ and $\delta^* \in T$ such that $\delta^* = xm(\delta)\theta^*(x)^{-1}$, $N_{\theta^*}(\delta^*) = \eta(\gamma)$ and

$$\operatorname{Int} x \circ \psi : T_\delta^{\delta\theta} \to T^{\theta^*}$$

is defined over $\mathbb{R}$.

We want to prove now that every G-regular element in $A_H := (\exp(\mathfrak{t}_H)_{\mathbb{R}})\,\gamma$ is a norm of an element in the Cartan subspace $A = (\exp \mathfrak{g}_{\mathbb{R}}^{\delta\theta})\,\delta$ of $G(\mathbb{R})$.

Lemma 4.3.1. *The restriction of N_{θ^*} to $(T^{\theta^*}(\mathbb{R}))_0 = \exp \mathfrak{t}_{\mathbb{R}}^{\theta^*}$ is a finite covering of $(T_{\theta^*}(\mathbb{R}))_0$.*

Proof. The surjectivity comes from the decomposition

$$\mathfrak{t}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}^{\theta^*} \oplus (1-\theta^*)\mathfrak{t}_{\mathbb{R}}.$$

Let us show now that θ^* has finite order on T. Recall that θ has finite order on G, say r. From 2.1, we compute easily that θ^{*r} is a inner automorphism of G^*, given by the element $\theta^{*r-1}(g_\theta)\theta^{*r-2}(g_\theta)\ldots\theta^*(g_\theta)g_\theta$. Thus its restriction to the θ^*-stable torus T is of finite order. Let l be the order of θ^* on T.

Suppose that $t \in (T^{\theta^*}(\mathbb{R}))_0$ such that there exist $u \in T$ with $t = u\theta^*(u)^{-1}$. We compute :

$$t\theta^*(t)\ldots\theta^{*l-1}(t) = t^l = 1.$$

Since in a torus there are only a finite number of elements of a given finite order, we see that this covering is indeed finite. □

Let $X \in (\mathfrak{t}_H)_{\mathbb{R}}$ such that $(\exp X)\gamma$ is G-regular. Choose $Y \in \mathfrak{t}_{\mathbb{R}}^{\theta^*}$ such that $N_{\theta^*}(\exp Y) = \eta(\exp X)$ and $U \in \mathfrak{g}_{\mathbb{R}}^{\delta\theta}$ such that $\operatorname{Int} x \circ \psi(\exp U) = \exp Y$. We have then :

$$xm((\exp U)\delta)\theta^*(x)^{-1} = (x\psi(\exp U)\,x^{-1})xm(\delta)\theta^*(x)^{-1} = (\exp Y)\delta^*$$

Thus, since

$$(G^{(\exp U)\delta\theta})_0 = (G^{\delta\theta})_0 \quad\text{and}\quad ((G^*)^{(\exp Y)\delta^*\theta^*})_0 = ((G^*)^{\delta^*\theta^*})_0 = (T^{\theta^*})_0,$$

$(\exp X)\gamma$ is a norm of $(\exp U)\delta$. □

Notice that in the previous lemma we define a map from A to A_H, depending a priori on the initial choice of γ and δ. It is easy to check that this map depends in fact only on the admissible isomorphism $T_H \xrightarrow{\eta} T_{\theta^*}$ and the choice of x. We summarize this discussion.

Proposition 4.3.2. *Let A_H be a Cartan subspace of $H(\mathbb{R})$ and let $\gamma \in A_H$ be a G-regular element, norm of $\delta \in G(\mathbb{R})$. Then there is a covering map:*

$$\eta_{(A,A_H)} : A = \exp(\mathfrak{g}_{\mathbb{R}}^{\delta\theta})\delta \to A_H$$

such that $\eta_{(A,A_H)}(\delta')$ is a norm of δ' for all regular element $\delta' \in A$.

We call these maps admissible coverings. They provide local isomorphisms between Cartan subspaces of $G(\mathbb{R})$ and Cartan subspaces (i.e. connected components of Cartan subgroups) of $H(\mathbb{R})$. The following lemma might seem obvious but it could also help for a better understanding of what is going on. Consider the following situation, where $\gamma \in A_H$ is G-regular and a norm for a stable θ-conjugacy class of regular elements in $G(\mathbb{R})$. Let Σ_γ be a set of representatives of θ-conjugacy classes in $G(\mathbb{R})$ in this given stable θ-conjugacy class. For each $\delta_i \in \Sigma_\gamma$, there is a admissible covering $\eta_{(A_i,A_H)}$ constructed as above (the construction depends on some choices, but note we may take the same admissible embedding $T_H \to T_{\theta^*}$ for the various $\eta_{(A_i,A_H)}$). Now let

$\gamma' \in A_H$ another G-regular element. Then for every admissible covering $\eta_{(A_i,A_H)}$, we can pick a δ'_i in A_i such that $\eta_{(A_i,A_H)}(\delta'_i) = \gamma'$ (in particular γ' is a norm for δ'_i, so they are all stably θ-conjugate). We have then :

Lemma 4.3.3. *The set of δ'_i described above is set of representatives of θ-conjugacy classes in $G(\mathbb{R})$ in the stable θ-conjugacy class of elements for which γ' is a norm.*

Proof. It is not difficult to check from the constructions that the δ'_i are indeed representatives of different θ-conjugacy classes in the stable θ-conjugacy class of elements for which γ' is a norm. We can conclude by a counting argument that we have a complete set of representatives. It comes from the parametrization of θ-conjugacy classes in $G(\mathbb{R})$ in the stable θ-conjugacy class of a θ-regular element $\delta \in G(\mathbb{R})$ found in [KS], p.54, or in [La], p. 54 in terms of

$$\mathrm{Ker}[H^1(\Gamma, G^{\delta\theta} \cap T_\delta) \to H^1(\Gamma, G)].$$

□

In the situation described above, we say that $\{\eta_{(A_1,A_H)}, \ldots, \eta_{(A_r,A_H)}\}$ is a complete system of (admissible) coverings, if for any G-regular $\gamma \in A_H$, we obtain a system of representatives of θ-conjugacy classes in $G(\mathbb{R})$ in the stable θ-conjugacy class of elements for which γ is a norm by taking for each i an element $\delta_i \in A_i$ in the fiber above γ of $\eta_{(A_i,A_H)}$.

5. MATCHING FUNCTIONS

5.1. **Normalization of measures.** In order to define the transfer of orbital integrals we have to normalize invariant measures on the various groups in a consistent way. We chose Duflo-Vergne's normalization, defined as follows: let A be a reductive group (complex or real), and pick an A-invariant symmetric, non-degenerate bilinear form κ on $\mathfrak{a}$. Then $\mathfrak{a}$ will be endowed with the Lebesgue measure dX such that the volume of a parallelotope supported by a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{a}$ is equal to $|\det(\kappa(X_i, X_j))|^{\frac{1}{2}}$ and A will be endowed with the Haar measure tangent to dX. If M is a closed subgroup of A, such that κ is non-degenerate on $\mathfrak{m}$, we endow M with the Haar measure determined by κ as above. If $M' \subset M$ are two closed subgroups of A such that κ is non-degenerate on their respective Lie algebras, we endow M/M' with the M-invariant measure, which is the quotient of the Haar measures on M and M' defined as above. We will denote it by $d\dot{m}$.

5.2. **Twisted orbital integrals on $G(\mathbb{R})$.** Let $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$. Its θ-twisted orbital integral is the function defined on $G(\mathbb{R})_{\theta-reg}$ by:

$$J_G^\theta(f)(x) = |\det(\mathrm{Id} - (\mathrm{Ad}\, x \circ \theta)^{-1})_{\mathfrak{g}/\mathfrak{a}}|^{\frac{1}{2}} \int_{G(\mathbb{R}))/(G^{x\theta} \cap T_x)(\mathbb{R})} f(gx\theta(g^{-1}))\, d\dot{g}$$

where $\mathfrak{a} = \mathfrak{g}^{x\theta}$ and $d\dot{g}$ is the invariant measure on $G(\mathbb{R})/(G^{x\theta} \cap T_x)(\mathbb{R})$ normalized with our conventions. Note that if x is strongly θ-regular, then $(G^{x\theta} \cap T_x)(\mathbb{R}) = G^{x\theta}(\mathbb{R})$. These objects have been studied in [R]. We recall their properties. We need some notation. Recall that if x is a θ-regular element of G, with $\mathfrak{a} = \mathfrak{g}^{x\theta}$ and $\mathfrak{h}$ =Cent$(\mathfrak{a}, \mathfrak{g})$, then we have a decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\beta \in R(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^{\beta}$$

where $R(\mathfrak{g}, \mathfrak{a})$ is a (non-reduced) root system and $\mathfrak{g}^{\beta}$ is the root space for the root β. The root system $R(\mathfrak{g}, \mathfrak{a})$ is the set of restrictions to $\mathfrak{a}$ of the roots in $R(\mathfrak{g}, \mathfrak{h})$. Let A be the Cartan subspace $\exp \mathfrak{a}_{\mathbb{R}}\, x$. Let P be a system of positive imaginary roots in R. We introduce the 'Harish-Chandra' normalizing factor b_P on A_{reg}:

$$b_P(a) = \frac{\prod_{\alpha \in P} \det(\mathrm{Id} - (\mathrm{Ad}\, a \circ \theta)^{-1})_{|\mathfrak{g}^{\alpha}}}{|\prod_{\alpha \in P} \det(\mathrm{Id} - (\mathrm{Ad}\, a \circ \theta)^{-1})_{|\mathfrak{g}^{\alpha}}|}$$

Definition 5.2.1. *Let $A = \exp \mathfrak{a}_{\mathbb{R}} x$ be a Cartan subspace of $G(\mathbb{R})$. We will denote by $A_{\theta-I-reg}$ (resp. $A_{\theta-In-reg}$) the set of $a \in A$ such that the root system of $\mathfrak{a}$ in $\mathfrak{g}^{a\theta}$ has no imaginary (resp. non-compact imaginary) roots. This implies (in both cases) that $\mathfrak{a}_{\mathbb{R}}$ is a maximally split Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}^{a\theta}$.*

With the above notation, we denote by $S(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$, and we identify it with the algebra of differential operators on A which are invariant under left translations by elements of $\exp \mathfrak{a}_{\mathbb{R}}$. We denote by $\partial(u)$ the differential operator corresponding to $u \in S(\mathfrak{a})$.

We say that $x \in G(\mathbb{R})$ is semi-θ-regular when the derived algebra of $\mathfrak{g}_{\mathbb{R}}^{x\theta}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{su}(2)$. Suppose it is $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathfrak{a}_{\mathbb{R}}$ be a fundamental Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}^{x\theta}$, and $\pm\alpha$ the roots of $\mathfrak{a}$ in $\mathfrak{g}^{x\theta}$: they are non-compact imaginary and satisfy:

$$\det(\mathrm{Id} - (\mathrm{Ad}\, x \circ \theta)^{-1})_{|\mathfrak{g}^{\alpha}} = 0.$$

Pick a non-zero root vector X_α in the root space corresponding to α and fixed by $\mathrm{Ad}\, x \circ \theta$. Then $X_{-\alpha} := \overline{X_\alpha}$ is a root vector for $-\alpha$ (the conjugation in $\mathfrak{g}^{x\theta}$ is with respect to its real form $\mathfrak{g}_{\mathbb{R}}^{x\theta}$). Let $H_\alpha \in \mathfrak{a}$ be the coroot of α. We may chose X_α such that $(X_\alpha, X_{-\alpha}, H_\alpha)$ is a $\mathfrak{sl}_2$-triple, so we do. Then $\mathfrak{a}_{1,\mathbb{R}} := \mathrm{Ker}\,\alpha \oplus \mathbb{R}i(X_\alpha - X_{-\alpha})$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}^{x\theta}$. The adjoint automorphism c_α given by $\exp(-i\frac{\pi}{4}(X_\alpha + X_{-\alpha}))$ is such that $c_\alpha(\mathfrak{a}) = \mathfrak{a}_1$. Furthermore $A := \exp \mathfrak{a}_{\mathbb{R}}\, x$ and $A_1 := \exp \mathfrak{a}_{1,\mathbb{R}}\, x$ are Cartan subspaces of $G(\mathbb{R})$. We refer to this notation by saying that (x, A, A_1, c_α) is a jump datum for $G(\mathbb{R})$.

Let A be a Cartan subspace of $G(\mathbb{R})$, $y \in A$ and ϕ a function on $A_{\theta-reg}$. Let β be a imaginary root of $\mathfrak{a}$ in $\mathfrak{g}$, and $H_\beta \in i\mathfrak{a}_{\mathbb{R}}$ its coroot. Then, when the limits in the following formula exist we set:

$$[\phi]_{\beta}^{+}(y) = \lim_{t \to 0^+} \phi((\exp tiH_\beta)\, y) + \lim_{t \to 0^-} \phi((\exp tiH_\beta)\, y).$$

Let $\mathcal{I}^{\theta}(G(\mathbb{R}))$ be the subspace of $\mathcal{C}^{\infty}(G_{\theta-reg}(\mathbb{R}))$ of functions ψ which are constant on the θ-twisted conjugacy classes and have the following properties I_1^{θ}, I_2^{θ}, I_3^{θ}, I_4^{θ}:

I_1^{θ}: if $A = \exp \mathfrak{a}_{\mathbb{R}} x$ is a Cartan subspace of $G(\mathbb{R})$, for all compact subset K of A and for all $u \in S(\mathfrak{a})$ we have:

$$\sup_{a \in K_{\theta-reg}} |\partial(u) \cdot \psi_{|A}(a)| < \infty.$$

I_2^θ: if $A = \exp \mathfrak{a}_{\mathbb{R}} x$ is a Cartan subspace of $G(\mathbb{R})$, for all system P of positive imaginary roots in $R(\mathfrak{a}, \mathfrak{g})$, $b_P \psi_{|A}$ has a smooth extension on $A_{\theta-In-reg}$. This is equivalent to:

$I_2^{\theta\prime}$: $\psi_{|A}$ has a smooth extension on $A_{\theta-I-reg}$, and for all semi-θ-regular element $x \in A$ such that the roots $\pm\alpha$ of $\mathfrak{a}$ in $\mathfrak{g}^{x\theta}$ are compact imaginary, for all $u \in S(\mathfrak{a})$,

$$[\partial(u) \cdot \psi_{|A}]_\alpha^+(x) = 0$$

I_3^θ: for all jump data (x, A, A_1, c_α), and for all $u \in S(\mathfrak{a})$ we have:

$$[\partial(u) \cdot \psi_{|A}]_\alpha^+(x) = \mathrm{d}(x) \partial(c_\alpha \cdot u) \cdot \psi_{|A_1}(x),$$

where $\mathrm{d}(x)$ is equal to 2 if the reflection $s_\alpha \in W(G, A)$ is realized in $(G^{x\theta}(\mathbb{R}))$ and 1 otherwise.

I_4^θ: if $A = \exp \mathfrak{a}_{\mathbb{R}} x$ is a Cartan subspace of $G(\mathbb{R})$, $\mathrm{Supp}_A(\psi_{|A})$ is a compact subspace of A.

The space $\mathcal{I}^\theta(G(\mathbb{R}))$ is endowed with a topology of an inductive limit of Fréchet spaces, and we denote by $\mathcal{I}^\theta(G(\mathbb{R}))'$ its dual. We have the following result:

Theorem 5.2.2. *(*[R]*, théorème 9.4.) The map J_G^θ is linear, continuous and surjective from $\mathcal{C}_c^\infty(G(\mathbb{R}))$ onto $\mathcal{I}^\theta(G(\mathbb{R}))$, and its transpose ${}^tJ_G^\theta$ realizes a bijection from $\mathcal{I}^\theta(G(\mathbb{R}))'$ onto the space of θ-invariant distributions on $G(\mathbb{R})$.*

5.3. **Stable orbital integrals on $H_1(\mathbb{R})$.** Let $\mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ be the space of smooth functions f^{H_1} on $H_1(\mathbb{R})$ with compact support modulo Z_1 and such that:

$$f^{H_1}(zh) = \lambda_{H_1}(z)^{-1} f^{H_1}(h) \quad (h \in H_1(\mathbb{R})_{reg},\, z \in Z_1(\mathbb{R}))$$

The orbital integral of such a function is given by:

$$J_{H_1}(f^{H_1})(\gamma_1) = |\det(\mathrm{Id} - \mathrm{Ad}\, \gamma_1^{-1}))_{\mathfrak{h}_{\mathbb{R}}/\mathfrak{t}_{\mathbb{R}}}|^{\frac{1}{2}} \int_{H(\mathbb{R})/T(\mathbb{R})} f(h\gamma_1 h^{-1})\, d\dot{g}$$

where $\gamma_1 \in H_1(\mathbb{R})$ is regular, T is the projection on H of the Cartan subgroup T_1 containing γ_1 and $d\dot{h}$ is the invariant measure on $H(\mathbb{R})/T(\mathbb{R})$ normalized with our conventions. Notice that $H(\mathbb{R})/T(\mathbb{R}) \simeq H_1(\mathbb{R})/T_1(\mathbb{R})$, where T_1 is the inverse image of T in H_1. This is a well-defined converging integral since $\mathrm{Supp}\, f \cap \mathcal{O}_{\gamma_1}$ is compact.

The stable orbital integral of the function $f^{H_1} \in \mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ is defined by:

$$J_{H_1}^{st}(f^{H_1})(\gamma_1) = \sum_{\gamma_i} J_{H_1}(f^{H_1})(\gamma_i)$$

where the sum is taken over a system of representatives of conjugacy class in $H_1(\mathbb{R})$ in the stable conjugacy class of γ_1.

In [B1], A. Bouaziz gave a characterization of stable orbital integrals of compactly supported functions on a real algebraic reductive connected group. We rephrase his results for functions in $\mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})$, indicating briefly how the proof can be adapted. The notation is consistent with the one used in the preceding paragraph (take θ trivial).

The map $\gamma_1 \mapsto J_{H_1}^{st}(f^{H_1})(\gamma_1)$ is smooth on $H_1(\mathbb{R})_{reg}$, stably invariant (i.e. constant on stable conjugacy classes) and satisfies for all $h \in H_1(\mathbb{R})_{reg}$, $z \in Z_1(\mathbb{R})$

$$J_{H_1}^{st}(f^{H_1})(zh) = \lambda_{H_1}(z)^{-1} J_{H_1}^{st}(f^{H_1})(h). \tag{5.1}$$

Let $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$ be the subspace of $\mathcal{C}^\infty(H_1(\mathbb{R})_{reg})$ of functions ψ which are constant on the stable conjugacy classes and have the following properties: I_1^{st}, I_2^{st}, I_3^{st}, I_4^{st}, I_5^{st}:

I_1^{st}: if A is a maximal torus of H_1 defined over $\mathbb{R}$, for all compact subset of $A(\mathbb{R})$ and for all $u \in S(\mathfrak{a})$ we have

$$\sup_{a \in K_{reg}} |\partial(u) \cdot \psi_{|A}(a)| < \infty.$$

I_2^{st}: if A is a maximal torus of H_1 defined over $\mathbb{R}$, for all system P of positive imaginary roots in $R(\mathfrak{a}, \mathfrak{h}_1)$, $b_P \psi_{|A}$ has a smooth extension on $A(\mathbb{R})_{st-In-reg}$, where $A(\mathbb{R})_{st-In-reg}$ is the set of $a \in A$ such that all elements in A stably conjugate to a are in $A(\mathbb{R})_{In-reg}$. Note that this rather subtle definition is not really necessary here since for H_1 quasi-split, we have $A(\mathbb{R})_{st-In-reg} = A(\mathbb{R})_{I-reg}$ ([Sh1], prop. 4.11). So in fact I_2^{st} reduces to:

$I_2^{st\prime}$: $\psi_{|A}$ has a smooth extension on $A(\mathbb{R})_{I-reg}$.

I_3^{st}: for all jump data (x, A, A_1, c_α), and for all $u \in S(\mathfrak{a})$ we have:

$$[\partial(u) \cdot \psi_{|A}]_\alpha^+(x) = 2\partial(c_\alpha \cdot u) \cdot \psi_{|A_1}(x)$$

Note that the right-hand side is well defined because $x \in A_{1,I-reg}$.

I_4^{st}: if A is a maximal torus of H_1 defined over $\mathbb{R}$, $\mathrm{Supp}_A(\psi_{|A(\mathbb{R})})$ is a compact subspace of $A(\mathbb{R})$ modulo $Z_1(\mathbb{R})$.

I_5^{st}: for all $h \in H_1(\mathbb{R})$, $z \in Z_1(\mathbb{R})$:

$$\psi(zh) = \lambda_{H_1}(z)^{-1}\psi(h) \tag{5.2}$$

The space $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$ is endowed with a topology of an inductive limit of Fréchet spaces and we denote by $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})'$ its dual. For all functions $f^{H_1} \in \mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})$, $J_{H_1}^{st}(f^{H_1}) \in \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$ (see [Sh4] and [B1] §6 for the case $H_1 = H$, i.e. orbital integrals of smooth functions with compact support, and see below for an argument of how this can be adapted to the general case). The last property is a easy consequence of (5.1).

Theorem 5.3.1. *(see* [B1]*, théorème 6.1) The map:*

$$J_{H_1}^{st}; \mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1}) \to \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$$

is linear, continuous, surjective and its transpose ${}^tJ_{H_1}^{st}$ *realizes a isomorphism between* $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})'$ *and the subspace of stable invariant 'distributions'* $\mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})'$.

Proof. . Suppose that the extension

$$1 \to Z_1(\mathbb{R}) \to H_1(\mathbb{R}) \to H(\mathbb{R}) \to 1 \tag{5.3}$$

is split, and let $c: H(\mathbb{R}) \to H_1(\mathbb{R})$ be a section. This section provides an isomorphism:

$$\text{Restr}: \mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}), \lambda_{H_1}) \to \mathcal{C}^\infty_c(H(\mathbb{R}))$$

by restricting a function to $c(H(\mathbb{R})) \simeq H(\mathbb{R})$.

Thus $\mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}), \lambda_{H_1})' \simeq \text{Distr}(H(\mathbb{R}))$ and we have the following commuting diagram:

$$\begin{array}{ccc} \mathcal{C}^\infty_{c,Z_1}(H_1(\mathbb{R}), \lambda_{H_1}) & \xrightarrow{\text{Restr}} & \mathcal{C}^\infty_c(H(\mathbb{R})) \\ \Big\downarrow J^{st}_{H_1} & & \Big\downarrow J^{st}_{H} \\ \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1}) & \xrightarrow{\text{Restr}} & \mathcal{I}^{st}(H(\mathbb{R})) \end{array}$$

When (5.3) is not split, we use the fact that $H_1(\mathbb{R}) \simeq H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R})/F$ where F is a finite subgroup and $Z(H_1)(\mathbb{R})$ is the center of $H_1(\mathbb{R})$. The theorem is established for functions in $\mathcal{C}^\infty_{c,Z_1}(H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R}), \lambda_{H_1})$. We deduce the statement for $\mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1})$ thanks to the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}^\infty_{c,Z_1}(H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R}), \lambda_{H_1}) & \xrightarrow{\text{M}} & \mathcal{C}^\infty_c(H_1(\mathbb{R}), \lambda_{H_1}) \\ \Big\downarrow J^{st}_{H_{1,der} \times Z(H_1)} & & \Big\downarrow J^{st}_{H_1} \\ \mathcal{I}^{st}(H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R}), \lambda_{H_1}) & \xrightarrow{\text{M}} & \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1}) \end{array}$$

where $M(\phi)(\gamma) = \sum_{z \in F} \phi(\gamma z)$ for any function ϕ on $H_{1,der}(\mathbb{R}) \times Z(H_1)(\mathbb{R})$. □

Let $T_{H_1}(\mathbb{R})$ be a Cartan subgroup of $H_1(\mathbb{R})$ with projection $T_H(\mathbb{R})$ on $H(\mathbb{R})$. We have then a exact sequence:

$$\{0\} \to (\mathfrak{z}_1)_\mathbb{R} \to (\mathfrak{h}_1)_\mathbb{R} \to \mathfrak{h}_\mathbb{R} \to \{0\} \tag{5.4}$$

of Lie algebras, with $(\mathfrak{z}_1)_\mathbb{R}$ central in $(\mathfrak{h}_1)_\mathbb{R}$. Since such a sequence always splits, we may, by fixing a section of 5.4 identify $\mathfrak{h}_\mathbb{R}$ with a subalgebra of $(\mathfrak{h}_1)_\mathbb{R}$, and:

$$(\mathfrak{h}_1)_\mathbb{R} = (\mathfrak{z}_1)_\mathbb{R} \oplus \mathfrak{h}_\mathbb{R} \tag{5.5}$$

This decomposition (5.5) induces:

$$(\mathfrak{t}_{H_1})_\mathbb{R} = (\mathfrak{z}_1)_\mathbb{R} \oplus (\mathfrak{t}_H)_\mathbb{R}$$

Furthermore, the decompositions:

$$\mathfrak{h}_1 = \mathfrak{t}_{H_1} \oplus \sum_{\alpha \in R(T_{H_1}, H_1)} \mathfrak{h}_1^\alpha$$

$$\mathfrak{h} = \mathfrak{t}_H \oplus \sum_{\alpha \in R(T_H, H)} \mathfrak{h}^\alpha$$

provide identification between $R(T_{H_1}, H_1) \simeq R(T_H, H)$ and $\mathfrak{h}_1^\alpha \simeq \mathfrak{h}^\alpha$.

Let $\gamma_1 \in T_{H_1}(\mathbb{R})$ and γ its projection on $T_H(\mathbb{R})$. Let $\alpha \in R(T_{H_1}, H_1) \simeq R(T_H, H)$; then we have $\alpha(\gamma_1) = \alpha(\gamma)$. Thus, if P is a system of positive imaginary roots in $R(T_{H_1}, H_1) \simeq R(T_H, H)$, then $b_P(\gamma_1) = b_P(\gamma)$.

We end this section by the following remark concerning differential operators from $S(T_{H_1})$. It is clear that for all smooth function ψ on $H_1(\mathbb{R})_{reg}$ satisfying (5.2) and for all $u, \in S(\mathfrak{z}_1)$ we have:

$$\partial(u) \cdot \psi = \mathrm{d}\lambda_{H_1}^{-1}(u)\psi$$

i.e. ψ is a eigenfunction for all $u \in S(\mathfrak{z}_1)$. Thus, to check properties $I_1^{st}, I_2^{st}, I_3^{st}$ we have only to consider differential operators coming from $S(\mathfrak{t}_H)$ (since $S(T_{H_1}) = S(\mathfrak{t}_H) \otimes S(\mathfrak{z}_1)$).

5.4. **Transfer of orbital integrals.** Let $(H, \mathcal{H}, s, \xi)$ be an endoscopic datum for (G, θ), and (H_1, ξ_{H_1}) be a z-pair for $\mathcal{H}$. We say that $\gamma_1 \in H_1(\mathbb{R})$ is G-regular (resp. strongly G-regular) if its projection γ on $H_\mathbb{R}$ is G-regular (resp. strongly G-regular). In this situation we say that γ_1 is a norm for $\delta \in G(\mathbb{R})$ if γ is a norm for δ.

Kottwitz and Shelstad have defined absolute transfer factors $\Delta(\gamma_1, \delta)$ where $\gamma_1 \in H_1(\mathbb{R})$ is G-regular and is a norm of $\delta \in G(\mathbb{R})$. This transfer factor is a product of three terms Δ_I, Δ_{II} and Δ_{III} (we omit their fourth term Δ_{IV} since it is already included in our definition of orbital integrals). We will recall the properties of these transfer factors when we will need it.

We say that the function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ and the function $f^{H_1} \in \mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ have matching orbital integrals if

$$J_{H_1}^{st}(f^{H_1})(\gamma_1) = \sum_{\delta \in \Sigma_{\gamma_1}} \Delta(\gamma_1, \delta) J_G^\theta(f)(\delta) \tag{5.6}$$

for every G-regular $\gamma_1 \in H_1(\mathbb{R})$. The sum (which might be empty, in which case the right-hand side is 0) is taken over a set of representative of θ-conjugacy classes under $G(\mathbb{R})$ of elements $\delta \in G(\mathbb{R})$ for which γ_1 is a norm.

Our principal result is that for every function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$, there is a function $f^{H_1} \in \mathcal{C}_{c,Z_1}^\infty(H_1(\mathbb{R}), \lambda_{H_1})$ having matching orbital integrals with f. Using the terminology of the previous paragraphs, we can rephrase it in the following form:

Theorem 5.4.1. *There is a transfer map:*

$$\begin{aligned} \mathit{Trans} : \mathcal{I}^\theta(G(\mathbb{R})) &\to \mathcal{I}^{st}(H_1(\mathbb{R}), \lambda_{H_1}) \\ \psi &\mapsto \mathit{Trans}(\psi) \end{aligned}$$

linear and continuous, such that

$$\mathit{Trans}(\psi)(\gamma_1) = \sum_{\delta \in \Sigma_{\gamma_1}} \Delta(\gamma_1, \delta)\psi(\delta) \tag{5.7}$$

when γ_1 is a G-regular element of $H_1(\mathbb{R})$. Furthermore, Trans(ψ) is defined on regular, non-G-regular elements of $H_1(\mathbb{R})$ by smooth extension.

The next section will be devoted to the proof of theorem 5.4.1.

6. Reduction of the proof of theorem 5.4.1

6.1. We sketch briefly the proof of the theorem before going into details. Let us first remark that the right-hand side of 5.7 is well defined, i.e. does not depend on the choices of representatives in Σ_γ. This is a consequence of the following lemma:

Lemma 6.1.1. *(*[KS]*, thm 5.1.D)*

$$\Delta(\gamma_1,\delta)=\Delta(\gamma_1,\delta')$$

when δ and δ' are θ-conjugate in $G(\mathbb{R})$.

Then, we will show how Trans(ψ) is defined on regular element of $H_1(\mathbb{R})$ (but not G-regular) by a smooth extension of 5.7. This is a generalization in our present context of lemma 4.3 of [Sh2].

The fact that Trans(ψ) is constant on stable conjugacy classes comes from the following lemma of [KS]:

Lemma 6.1.2. *(*[KS]*, lemma 5.1.B)*

$\Delta(\gamma_1,\delta)$ is unchanged when γ_1 is replaced with a stably conjugate element in $H_1(\mathbb{R})$.

To complete the theorem, we have to establish that properties $I_1^{st},\dots,I_5^{st}$ hold for Trans(ψ). As notation suggests, properties $I_1^{st},\dots,I_4^{st}$ for Trans(ψ) are consequences of properties $I_1^{\theta},\dots,I_4^{\theta}$ for ψ. Some are immediate $I_1^{st},I_2^{st},I_4^{st}$, the other one I_3^{st} requiring extra work. The last property I_5^{st}, is established by the following lemma:

Lemma 6.1.3. *(*[KS]*, lemma 5.1.C)*

$\Delta(z\gamma_1,\delta)=\lambda_{H_1}(z)^{-1}\Delta(\gamma_1,\delta)$ where $\gamma_1\in H_1(\mathbb{R})$ is a regular element and $z\in Z_1(\mathbb{R})$.

6.2. **More about geometric correspondences.** Fix a function $\psi\in\mathcal{I}^\theta(G(\mathbb{R}))$ and define $\varphi=\text{Trans}(\psi)$ by 5.7 on G-regular elements of $H_1(\mathbb{R})$. Let A_H be the Cartan subspace in $H(\mathbb{R})$ (i.e. a connected component of a Cartan subgroup $T_H(\mathbb{R})$ in $H(\mathbb{R})$), and let A_{H_1} be its inverse image in $H_1(\mathbb{R})$ (so that A_{H_1} is a Cartan subspace in $H_1(\mathbb{R})$).

If A_H does not originate in $G(\mathbb{R})$, then by definition the sum in the right-hand side of 5.7 is empty, so $\varphi_{|A_{H_1}}\equiv 0$.

If A_H originates in $G(\mathbb{R})$, then there exist a Cartan subspace A in $G(\mathbb{R})$, and a covering map:

$$\eta_{(A,A_H)}:A\to A_H$$

such that $\eta_{(A,A_H)}(\delta)$ is a norm of δ for all θ-regular $\delta\in A$. Recall that this map depends on the choices of a admissible embedding $T_H\xrightarrow{\eta}T_{\theta^*}$ and an element $x\in G^*_{sc}$. For any regular element $\delta\in A$, we have $\mathfrak{a}=\mathfrak{g}^{\delta\theta}$ and $A=(\exp\mathfrak{a}_{\mathbb{R}})\delta$. There is a chain of $\mathbb{R}$-isomorphisms:

$$\mathfrak{a}\xrightarrow{\text{Ad}_{x\circ\psi}}\mathfrak{t}^{\theta^*}\xrightarrow{N_\theta^*}\mathfrak{t}_{\theta^*}\xrightarrow{\eta^{-1}}\mathfrak{t}_H \tag{6.1}$$

Let us denote by $\eta_{(A,A_H)}$ again the isomorphism between the extreme terms of (6.1) and also for the induced isomorphism between $S(\mathfrak{a})$ and $S(\mathfrak{t}_H)$. This will enable us to transfer differential operators.

Let γ_1^0 be an element in A_{H_1}, We need to study $\varphi_{|A_{H_1}}$ in a neighborhood of γ_1^0 in A_{H_1}. Let γ_0 be the projection of γ_1^0 on A_H, and $\delta_0\in A$ be in the fiber above γ_0 of the covering map $\eta_{(A,A_H)}$. Then, there is a neighborhood $\mathcal{U}$ of δ^0 in A such that $\eta_{(A,A_H)}$ realizes a isomorphism between $\mathcal{U}$ and its image $\mathcal{V}$ in A_H.

In fact, thanks to the property I_5^{st} of φ, it will be sufficient to study its restriction in a neighborhood of γ_1^0 in

$$\exp(\mathfrak{t}_H)_{\mathbb{R}}\gamma_1^0 \subset A_{H_1}.$$

Let $\mathcal{V}_1$ be the inverse image of $\mathcal{V}$ in $\exp(\mathfrak{t}_H)_{\mathbb{R}}\gamma_1^0$. This yields an isomorphism:

$$\eta_{(\mathcal{U},\mathcal{V}_1)} : \mathcal{U} \to \mathcal{V}_1$$

Let ϕ be a smooth function on $\mathcal{V}_1$ and let $\tilde{\phi}$ be its pull-back on $\mathcal{U}$ by $\eta_{(\mathcal{U},\mathcal{V}_1)}$. Take $u \in S(\mathfrak{t}_H)$ and $u' = \eta^{-1}_{(A,A_H)}(u) \in S(\mathfrak{a})$. It is clear that we have for all $\delta \in \mathcal{U}$:

$$\partial(u') \cdot \tilde{\phi}(\delta) = \partial(u) \cdot \phi(\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta)). \tag{6.2}$$

Now, suppose that $\delta^0 \in \mathcal{U}$ is θ-semi-regular, such that the roots $\pm\alpha$ of $\mathfrak{a}$ in $\mathfrak{g}^{\delta^0\theta}$ are imaginary. Let $H_\alpha \in i\mathfrak{a}_{\mathbb{R}}$ be the coroot of α, and $\delta^\nu = \exp(i\nu H_\alpha)\delta^0 \in A$. Then for ν sufficiently small and non-zero, δ^ν is a θ-regular element in $\mathcal{U}$. Let $w_\alpha \in (G^{\delta^0\theta})_0$ be an element realizing the Weyl reflection s_α with respect to α. We have then :

$$w_\alpha \exp(i\nu H_\alpha)\delta^0\theta(w_\alpha)^{-1} = \exp(-i\nu H_\alpha)\delta^0.$$

Hence, δ^ν and $\delta^{-\nu}$ are stably θ-conjugate, and there are two possibilities: either they are in the same θ-conjugacy class in $G(\mathbb{R})$, or they are not. Let us consider the second case.

Our isomorphism $\eta_{(\mathcal{U},\mathcal{V}_1)}$ has been constructed from an admissible covering $\eta_{(A,A_H)}$. This admissible covering is itself obtained from an admissible embedding $T_H \xrightarrow{\eta} T_{\theta^*}$ and an element $x \in G^*_{sc}$ having some properties (see Section 4.3 above). Let ν be small enough and non-zero, so that $\delta^\nu \in \mathcal{U}$ is regular. Let $\gamma^\nu = \eta_{(\mathcal{U},\mathcal{V}_1)}(\delta^\nu)$, and Σ_{γ^ν} be a set of representatives of θ-conjugacy classes under $G(\mathbb{R})$ in the stable θ-conjugacy class of elements for which γ^ν is a norm. Then we may assume that δ^ν and $\delta^{-\nu}$ are in Σ_{γ^ν}.

We can construct another admissible covering:

$$\begin{aligned} \overline{\eta}_{(A,A_H)} :\ & A \to A_H \\ & \delta^{-\nu} \mapsto \gamma^\nu \end{aligned}$$

using the same admissible embedding $T_H \xrightarrow{\eta} T_{\theta^*}$ and the element $x\psi(w_\alpha^{-1})$, and from this, another isomorphism

$$\begin{aligned} \overline{\eta}_{(\mathcal{U},\mathcal{V}_1)} :\ & \mathcal{U} \to \mathcal{V}_1 \\ & \delta^{-\nu} \mapsto \gamma_1^\nu \end{aligned}$$

Then there is a complete system of coverings of A_H containing $\eta_{(A,A_H)}$ and $\overline{\eta}_{(A,A_H)}$. Let $u \in S(\mathfrak{t}_H)$. The following lemma will be useful in the next section.

Lemma 6.2.1. *Let* $u' = \eta^{-1}_{(A,A_H)}(u)$ *and* $\overline{u}' = \overline{\eta}^{-1}_{(A,A_H)}(u)$. *Then* $\overline{u}' = (u')^{s_\alpha}$

Proof. The proof is straightforward and left to the reader. □

6.3. **More about transfer factors.** Suppose that we are in the situation as above, in particular we have an isomorphism $\eta_{(\mathcal{U},\mathcal{V}_1)}$. Then, for any θ-regular element $\delta\in\mathcal{U}$, $\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta)$ is a norm of δ, so we have a well-defined map :

$$\widetilde{\Delta}_{\mathcal{U}} : \mathcal{U}\to\mathbb{C}$$
$$\delta\mapsto\Delta\left(\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta),\delta\right)$$

Note that this map depends on $\eta_{(\mathcal{U},\mathcal{V}_1)}$, although this is not apparent in the notation. This is of some importance, for example in the situation considered at the end of the preceding paragraph, where the construction obtained from $\eta_{(\mathcal{U},\mathcal{V}_1)}$ and $\overline{\eta}_{(\mathcal{U},\mathcal{V}_1)}$ are nor the same and will have to be distinguished. We will need the following properties of this function:

Proposition 6.3.1. *In the setting above:*

(i) $\widetilde{\Delta}_{\mathcal{U}}$ is smooth on $\mathcal{U}_{\theta-reg}$. Let $\delta\in A_{\theta-reg}$. Then there exists $\lambda\in\mathfrak{a}^*$ and $c\in\mathbb{C}$ such that

$$\widetilde{\Delta}_{\mathcal{U}}((\exp X)\,\delta)=c\,e^{\lambda(X)}$$

for all $X\in\mathfrak{a}_{\mathbb{R}}$ such that $(\exp X)\,\delta\in\mathcal{U}_{\theta-reg}$. Furthermore, $\widetilde{\Delta}_{\mathcal{U}}$ has a smooth continuation on $\mathcal{U}_{\theta-I-reg}$. Let τ_λ be the algebra automorphism of $S(\mathfrak{a})$ mapping $X\in\mathfrak{a}$ on $X+\lambda(X)$. We have the following identity of differential operators on $\mathcal{U}$: for all $u\in S(\mathfrak{a})$

$$\partial(u)\circ\widetilde{\Delta}_{\mathcal{U}}=\widetilde{\Delta}_{\mathcal{U}}\circ\partial(\tau_\lambda(u)).$$

Let $\delta^0\in\mathcal{U}$ be a θ-semi-regular element, such that the roots $\pm\alpha$ of $\mathfrak{a}$ in $\mathfrak{g}^{\delta^0\theta}$ are imaginary. Let $H_\alpha\in i\mathfrak{a}_{\mathbb{R}}$ be the coroot of α, and $\delta^\nu=\exp(i\nu H_\alpha)\delta^0\in A$. For ν small enough and non-zero, $\delta^\nu\in\mathcal{U}_{\theta-reg}$, we have:

(ii) if α is compact and $\gamma^0=\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta^0)\in(\mathcal{V}_1)_{reg}$, then

$$\lim_{\nu\to0^+}\Delta(\gamma_1^\nu,\delta^\nu)=-\lim_{\nu\to0^-}\Delta(\gamma_1^\nu,\delta^\nu),$$

(iii) if α is compact and $\gamma^0=\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta^0)$ is semi-regular, then

$$\lim_{\nu\to0^+}\Delta(\gamma_1^\nu,\delta^\nu)=\lim_{\nu\to0^-}\Delta(\gamma_1^\nu,\delta^\nu),$$

(iv) if α is non-compact and $\gamma^0=\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta^0)\in(\mathcal{V}_1)_{reg}$, then the reflection s_α with respect to the root α is not realized in $G^{\delta^0\theta}(\mathbb{R})$, i.e. δ^ν and $\delta^{-\nu}$ are not θ-conjugate in $G(\mathbb{R})$, and

$$\lim_{\nu\to0^+}\Delta(\gamma_1^\nu,\delta^\nu)=-\lim_{\nu\to0^-}\Delta(\gamma_1^\nu,\delta^\nu),$$

$$\Delta(\gamma_1^\nu,\delta^\nu)=-\Delta(\gamma_1^\nu,\delta^{-\nu}),$$

(v) if α is non-compact and $\gamma^0=\eta_{(\mathcal{U},\mathcal{V}_1)}(\delta^0)$ is semi-regular, then

$$\lim_{\nu\to0^+}\Delta(\gamma_1^\nu,\delta^\nu)=\lim_{\nu\to0^-}\Delta(\gamma_1^\nu,\delta^\nu),$$

$$\Delta(\gamma_1^\nu,\delta^\nu)=\Delta(\gamma_1^\nu,\delta^{-\nu}).$$

The proof of this proposition, which uses the fine properties of transfer factors, will be given in the next sections.

Fix a function $\psi \in \mathcal{I}^\theta(G(\mathbb{R}))$ and define $\varphi = \mathrm{Trans}(\psi)$ by 5.7 on G-regular elements of $H_1(\mathbb{R})$.

Let $\gamma_1^0 \in H_1(\mathbb{R})$ be semi-simple, belonging to some Cartan subspace A_{H_1}, and let γ^0 its projection on $H(\mathbb{R})$. Let A_H be the projection of A_{H_1} on $H(\mathbb{R})$. Recall that in the non-twisted case, Cartan subspaces are just connected components of Cartan subgroups. We would like an expression of φ around γ_1^0.

Suppose that A_H originates in $G(\mathbb{R})$. This means that there exist $\gamma \in A_H$ G-regular, and a stable θ-conjugacy class of regular elements in $G(\mathbb{R})$ whose norm is γ. Fix a system Σ_γ of representatives of θ-conjugacy classes under $G(\mathbb{R})$ of elements in this stable θ-conjugacy class. For each $\delta_i \in \Sigma_\gamma$, let A_i be the Cartan subspace of $G(\mathbb{R})$ to which δ_i belongs. Recall the covering maps:

$$\eta_{(A_i,A_H)} : A_i = \exp(\mathfrak{g}_{\mathbb{R}}^{\delta_i\theta})\delta_i \to A_H.$$

They form a complete system of admissible coverings of A_H. Set $\delta_i^0 = \eta_{(A_i,A_H)}^{-1}(\gamma_0)$. Let the $\eta_{(\mathcal{U}_i,\mathcal{V}_1)}$'s be isomorphisms as constructed before from neighborhoods $\mathcal{U}_i$ of δ_i^0 onto a neighborhood $\mathcal{V}_1$ of γ_1^0. We can rewrite 5.7 as:

$$\varphi_{|\mathcal{V}_1}(\gamma_1) = \sum_{\eta_{(\mathcal{U}_i,\mathcal{V}_1)}} \tilde{\Delta}_{\mathcal{U}_i}\psi_{|\mathcal{U}_i}(\delta_i) \qquad (6.3)$$

6.4. **Smooth extension to $H_1(\mathbb{R})_{reg}$.** Let $\gamma_1^0 \in H_1(\mathbb{R})$ be regular but not G-regular, and let γ^0 its projection to $H(\mathbb{R})$. Let A_H (resp. A_{H_1}) be the Cartan subspace of $H(\mathbb{R})$ (resp. $H_1(\mathbb{R})$) to which γ^0 (resp. γ_1^0) belongs. If A_H does not originate in $G(\mathbb{R})$ (see 4.3.2) then by definition, the restriction of φ to A_{H_1} is zero, and there is certainly a smooth extension of φ around γ_1^0. Suppose that A_H originates in $G(\mathbb{R})$. We are in the situation considered at the end of the previous paragraph.

Suppose that one of the δ_i^0 is in $A_{i,\theta-I-reg}$. Then it will be the case for all of them. To see this, take two of them, say δ_i^0 and δ_j^0. Then, they are θ-conjugate by an element $g \in G$, such that $\mathrm{Int}\, g$ sends $G^{\delta_i^0\theta}$ isomorphically onto $G^{\delta_j^0\theta}$ and $\mathfrak{a}_i$ isomorphically onto $\mathfrak{a}_j$, this latter being defined over $\mathbb{R}$. It follows that $\mathrm{Int}\, g$ sends the roots of $\mathfrak{a}_i$ in $\mathfrak{g}^{\delta_i^0\theta}$ bijectively onto the roots of $\mathfrak{a}_j$ in $\mathfrak{g}^{\delta_j^0\theta}$, respecting their types (real, complex or imaginary).

It follows from proposition 6.3.1 (i) and property I_2^θ of ψ that there is a smooth extension of $\psi_{|A_{H_1}}$ around γ_1^0.

Now suppose that $\delta_i^0 \notin A_{i,\theta-I-reg}$ (if one is like that, all of them are). We want to study the behavior of $\delta_i \mapsto \tilde{\Delta}_{\mathcal{U}_i}\psi_{|A_i}$ around δ_i^0. We drop the indices 'i', since we are working with only one of them. Assume that δ^0 is θ-semi-regular. Then the roots $\pm\alpha$

of $\mathfrak{a}$ in $\mathfrak{g}^{\delta^0\theta}$ are imaginary. Suppose they are of compact type. Then for all $u \in S(\mathfrak{a})$:

$$\begin{aligned}
&[\partial(u)\cdot(\widetilde{\Delta}_{\mathcal{U}}\,\psi_{|A})]^-_\alpha(\delta^0)\\
&= \lim_{\nu\to 0^+}\partial(u)\cdot(\widetilde{\Delta}_{\mathcal{U}}\,\psi_{|A})(\delta^\nu) - \lim_{\nu\to 0^-}\partial(u)\cdot(\widetilde{\Delta}_{\mathcal{U}}\,\psi_{|A})(\delta^\nu)\\
&= \lim_{\nu\to 0^+}\widetilde{\Delta}_{\mathcal{U}}(\delta^\nu)\partial(\tau_\lambda(u))\cdot\psi_{|A}(\delta^\nu) - \lim_{\nu\to 0^-}\widetilde{\Delta}_{\mathcal{U}}(\delta^\nu)\partial(\tau_\lambda(u))\cdot\psi_{|A}(\delta^\nu)\\
&= \lim_{\nu\to 0^+}\widetilde{\Delta}_{\mathcal{U}}(\delta^\nu)[\partial(\tau_\lambda(u))\cdot\psi_{|A}]^+_\alpha = 0
\end{aligned}$$

We have used proposition 6.3.1, (i), (ii) and property $I_2^{\theta'}$ of ψ. Thus, the contribution of $\eta_{(\mathcal{U},\mathcal{V}_1)}$ to the right-hand side of (5.7) is smooth around δ^0.

Let us see what happens in the other case, i.e. when the roots $\pm\alpha$ are non-compact. Lemma 6.3.1 (iv) asserts that in that case δ^ν and $\delta^{-\nu}$ are not θ-conjugate in $G(\mathbb{R})$, i.e. the reflection s_α with respect to the root α is not realized in $G^{\delta^0\theta}(\mathbb{R})$.

Then, we are in the situation considered at the end of paragraph 6.2. To check that φ has a smooth extension around γ_1^0, we have to look at the contributions the right-hand side of (5.7) of $\eta_{(\mathcal{U},\mathcal{V}_1)}$ and $\overline{\eta}_{(\mathcal{U},\mathcal{V}_1)}$. We denote by $\widetilde{\overline{\Delta}}_{\mathcal{U}}$ the function obtained from $\overline{\eta}_{(\mathcal{U},\mathcal{V}_1)}$ (see 6.3). Let $u \in S(\mathfrak{t}_H)$ and $u' = \eta^{-1}_{(\mathcal{U},\mathcal{V}_1)}(u)$ and $\overline{u}' = \overline{\eta}^{-1}_{(\mathcal{U},\mathcal{V}_1)}(u)$. Recall that $\partial(u') = \partial(\overline{u}')^{s_\alpha}$ (lemma 6.2.1). This contribution is:

$$\begin{aligned}
&\lim_{\nu\to 0^+}\partial(u')\cdot(\widetilde{\Delta}_{\mathcal{U}}\,\psi_{|A})(\delta^\nu) - \lim_{\nu\to 0^-}\partial(u')\cdot(\widetilde{\Delta}_{\mathcal{U}}\,\psi_{|A})(\delta^\nu)\\
&+ \lim_{\nu\to 0^+}\partial(\overline{u}')\cdot(\widetilde{\overline{\Delta}}_{\mathcal{U}}\psi_{|A})(\delta^{-\nu}) - \lim_{\nu\to 0^-}\partial(\overline{u}')\cdot(\widetilde{\overline{\Delta}}_{\mathcal{U}}\psi_{|A})(\delta^{-\nu})\\
&= \lim_{\nu\to 0^+}\widetilde{\Delta}_{\mathcal{U}}(\delta^\nu)\;\partial(\tau_\lambda(u'))\cdot\psi_{|A}(\delta^\nu) - \lim_{\nu\to 0^-}\widetilde{\Delta}_{\mathcal{U}}(\delta^\nu)\;\partial(\tau_\lambda(u'))\cdot\psi_{|A}(\delta^\nu)\\
&+ \lim_{\nu\to 0^-}(\widetilde{\overline{\Delta}}_{\mathcal{U}})(\delta^\nu)\partial(\tau_\lambda(\overline{u}'))\cdot\psi_{|A}(\delta^\nu) - \lim_{\nu\to 0^+}(\widetilde{\overline{\Delta}}_{\mathcal{U}})(\delta^\nu)\partial(\tau_\lambda(\overline{u}'))\cdot\psi_{|A}(\delta^\nu),
\end{aligned}$$

We have $\lim_{\nu\to 0^+}\widetilde{\overline{\Delta}}_{\mathcal{U}}(\delta^\nu) = \lim_{\nu\to 0^+}(\widetilde{\Delta_{\mathcal{U}}})(\delta^\nu)$ (proposition 6.3.1. (iv)). Hence, if $\partial(\tau_\lambda(u')) = \partial(\tau_\lambda(\overline{u}')) = \partial(\tau_\lambda(u'))^{s_\alpha}$, the whole expression cancels. To complete the proof, it remains to check what happens when $\partial(\tau_\lambda(u')) = -\partial(\tau_\lambda(\overline{u}')) = -\partial(\tau_\lambda(u'))^{s_\alpha}$, the general case being deduced by linearity. Under this latter assumption we have using 6.3.1.(iv):

$$[\partial(u')\cdot(\widetilde{\Delta}_{\mathcal{U}}\,\psi_{|A})]^-_\alpha(\delta^0) = (\lim_{\nu\to 0^+}\widetilde{\Delta}_{\mathcal{U}}(\delta^\nu))[\partial(\tau_\lambda(u'))\cdot\psi_{|A})]^+_\alpha(\delta^0)$$

and a well-know principle of Harish-Chandra ([HC1]), valid also for the twisted case asserts that this is zero. Symmetrically,

$$[\partial(\overline{u}')\cdot(\widetilde{\overline{\Delta}}_{\mathcal{U}}\,\psi_{|A})]^-_\alpha(\delta^0) = 0.$$

To conclude, we have proved that if $\delta_i^0 \notin A_{i,\theta-I-reg}$ and is θ-semi-regular, then the same is true for all the δ_j^0's, and that the various contributions to the right-hand side of 5.7, when suitably grouped, extends to smooth functions around the δ_j^0's. Another principle of Harish-Chandra ([HC2]) asserts that these results still hold if the δ_j^0's are not semi-regular. Thus, $\varphi = \mathrm{Trans}(\psi)$ has a smooth extension to $H_1(\mathbb{R})_{reg}$.

6.5. **Properties $I_1^{st}, I_2^{st}, I_4^{st}$.** In this paragraph, we show that $\varphi = \mathrm{Trans}(\psi)$ satisfies the properties just listed.

Let A_{H_1} be a Cartan subspace of $H_1(\mathbb{R})$ and A_H its projection on $H(\mathbb{R})$. If A_H does not originate in $G(\mathbb{R})$, then the support of $\varphi_{A_{H_1}}$ is empty. If A_H originates in $G(\mathbb{R})$, we choose a complete system of coverings of A_H. For each Cartan subspace A in $G(\mathbb{R})$ in this complete system, the restriction of ψ to A has compact support (I_4^θ). It follows easily from the definitions that the support of $\varphi_{|A_{H_1}}$ is compact modulo $Z_1(\mathbb{R})$.

The two other properties are local, so it is sufficient to check them in the setting of 6.3. Then, they are immediate consequences of I_1^θ, I_2^θ for ψ and 6.3.1 (i).

6.6. **Jumps relations.** In this section, we will prove the jump relations for $\varphi = \mathrm{Trans}(\psi)$. We postpone the proofs of the various lemmas until the next section. Let $(\gamma_1^0, T_{H_1}, T'_{H_1}, c_{\alpha_H})$ be a jump datum on $H_1(\mathbb{R})$, and γ^0, T_H, T'_H the corresponding projections on $H(\mathbb{R})$.

Lemma 6.6.1. *(see* [Sh1]*, §2) If $A_H = \gamma^0 \exp(\mathfrak{t}_H)_{\mathbb{R}}$ does not originate in $G(\mathbb{R})$, nor does $A'_H = \gamma^0 \exp(\mathfrak{t}'_H)_{\mathbb{R}}$.*

In that case, $\varphi_{|A_H} \equiv 0$ and $\varphi_{|A'_H} \equiv 0$, and I_3^{st} is satisfied at γ_1^0.

We suppose now that $A_H = \gamma^0 \exp(\mathfrak{t}_H)_{\mathbb{R}}$ does originate in $G(\mathbb{R})$. We fix a complete system of coverings $\{\eta_{(A_i, A_H)}\}$ as before, and a set of isomorphisms $\{\eta_{(\mathcal{U}_i, \mathcal{V}_1)}\}$, where $\mathcal{V}_1$ is a connected neighborhood of γ_1^0. We suppose first that $\delta_i^0 = \eta^{-1}_{(\mathcal{U}_i, \mathcal{V}_1)}(\gamma_1^0)$ is θ-semi-regular (recall that if one is, all of them are) and we denote by $\pm\alpha_i$ the roots of $\mathfrak{a}_i \mathfrak{g}^{\delta_i^0 \theta}$. It means that α_H and α_i have corresponding coroots (up to a change of signs for $\pm\alpha_i$) through the isomorphisms $T_{\theta^*} \simeq T_H$ and $A_i \simeq (T_{\mathbb{R}}^{\theta^*})_0$. Recall that $\delta_i^\nu = (\exp i\nu H_{\alpha_i})\, \delta_i^0$. For ν small enough, $\delta_i^\nu \in \mathcal{U}_i$, and let $\gamma_{1,i}^\nu = \eta_{(\mathcal{U}_i, \mathcal{V}_1)}(\delta_i^\nu)$. In fact, as the following lemma shows, this element does not depend on i.

Lemma 6.6.2. *$\gamma_{1,i}^\nu = \exp(i\nu m_\alpha H_{\alpha_H})\gamma_1^0$ where m_α is the dimension of the root space $\mathfrak{g}_{\mathbb{C}}^{\alpha_i}$ or twice this dimension. This integer does not depend on i, thus nor does $\gamma_{1,i}^\nu$, and we simply denote it by γ_1^ν.*

Notice that the notation is a little bit misleading but the lack of coherence due to the factor m_α is not really important since we eventually only consider limits when ν tends to 0.

Lemma 6.6.3. *(see* [Sh2]*, prop. 9.3) If for all i, $\delta_i^0 \in A_{i,\theta-In-reg}$, then $T'_H(\mathbb{R})$ does not originate in $G(\mathbb{R})$.*

It is then easily check that the two side of the jump relation are 0, and so that I_3^{st} is satisfied.

We are now dealing with the case where both A_H and A'_H originate in $G(\mathbb{R})$. Let $u \in S(\mathfrak{t}_H)$ and $v_i = \eta^{-1}_{(A_i, A_H)}(u)$. We have then:

$$[\partial(u) \cdot \varphi_{|A_{H_1}}]^+_{\alpha_H}(\gamma_1^0) = \sum_i [\partial(v_i) \cdot (\widetilde{\Delta}_{\mathcal{U}_i} \varphi_{|A_i})]^+_{\alpha_i}(\delta_i^0)$$

As in the previous paragraph, thanks to 6.3.1 (ii), if $\delta_i^0 \in A_{i,In-reg}$ the contribution of this term to the right-hand side is 0, so we are left only with the ones such that the root α_i is imaginary and non-compact in $\Delta(\mathfrak{a}_i, \mathfrak{g}^{\delta_i^0\theta})$. As we have already noticed, there are two exclusive possibilities:

- The reflection s_{α_i} is realized in $G^{\delta_i^0\theta}(\mathbb{R})$. Then $d(\delta_i^0) = 2$ and δ_i^ν, $\delta_i^{-\nu}$ are in the same θ-conjugacy class in $G(\mathbb{R})$. Let I_1 be the set of indices such this holds.

-The reflection s_{α_i} is not realized in $G^{\delta_i^0\theta}(\mathbb{R})$. Then $d(\delta_i^0) = 1$ and δ_i^ν, $\delta_i^{-\nu}$ are not θ-conjugate in $G(\mathbb{R})$. Let I_2 be the subset of indices i such this holds. Notice that we are then in the situation considered at the end of paragraph 6.2. We may suppose that our complete system of coverings is such that if $i \in I_2$ corresponds to a covering $\eta_{(A_i,A_H)}$, then the covering $\overline{\eta}_{(A_i,A_H)}$ constructed above is also in the system of coverings. Thus there is a set of indices I_2' such that $\{\eta_{(A_i,A_H)}\}_{i\in I_2} = \{\eta_{(A_i,A_H)}, \overline{\eta}_{(A_i,A_H)}\}_{i\in I_2'}$. Let $v_i = \eta^{-1}_{(A_i,A_H)}(u)$. A general principle of Harish-Chandra asserts that if $\partial(u)^{s_{\alpha_H}} = -\partial(u)$, then the jump relations are satisfied, the jump being 0. In the following computations we assume that $\partial(u)^{s_{\alpha_H}} = \partial(u)$, the general case being deduced by linearity. In particular, if $i \in I_2'$, we have $v_i = \overline{v}_i$. We compute:

$$[\partial(u)\cdot\varphi_{|A_{H_1}}]^+_{\alpha_H}(\gamma_1^0) = \sum_i [\partial(v_i)\cdot(\widetilde{\Delta}_{\mathcal{U}_i}\psi_{|A_i})]^+_{\alpha_i}(\delta_i^0) \tag{6.4}$$

$$= \sum_{i\in I_1}[\partial(v_i)\cdot(\widetilde{\Delta}_{\mathcal{U}_i}\psi_{|A_i})]^+_{\alpha_i}(\delta_i^0) + \sum_{i\in I_2}[\partial(v_i)\cdot(\widetilde{\Delta}_{\mathcal{U}_i}\psi_{|A_i})]^+_{\alpha_i}(\delta_i^0) \tag{6.5}$$

$$\begin{aligned} &= \sum_{i\in I_1} 2(\lim_{\nu\to 0}\widetilde{\Delta}_{\mathcal{U}_i}(\delta_i^\nu))\ \partial(c_{\alpha_i}\cdot\tau_\lambda(v_i))\cdot\psi_{|A_i'}(\delta_i^0) \\ &+ \sum_{i\in I_2'}(\lim_{\nu\to 0}\widetilde{\Delta}_{\mathcal{U}_i}(\delta_i^\nu))\ \partial(c_{\alpha_i}\cdot\tau_\lambda(v_i))\cdot\psi_{|A_i'}(\delta_i^0) \\ &+ (\lim_{\nu\to 0}\widetilde{\widetilde{\Delta}}_{\mathcal{U}_i}(\delta_i^\nu))\ \partial(c_{\alpha_i}\cdot\tau_\lambda(\overline{v}_i))\cdot\psi_{|A_i'}(\delta_i^0) \end{aligned} \tag{6.6}$$

$$= \sum_{i\in I_1\cup I_2'} 2(\lim_{\nu\to 0}\widetilde{\Delta}_{\mathcal{U}_i}(\delta_i^\nu))\ \partial(c_{\alpha_i}\cdot\tau_\lambda(v_i))\cdot\psi_{|A_i'}(\delta_i^0) \tag{6.7}$$

$$= \sum_{i\in I_1\cup I_2'} 2\partial(c_{\alpha_i}\cdot v_i)\cdot(\widetilde{\Delta}_{\mathcal{U}_i'}\psi_{|A_i'})(\delta_i^0) \tag{6.8}$$

$$= 2\ \partial(c_{\alpha_H}\cdot u)\cdot\varphi_{|A'_{H_1}}(\gamma_1^0) \tag{6.9}$$

Let us make some comments on these computations. The first line 6.4 is obtained by using the local expression of φ around γ_1^0. The contribution of the indices $i \in I$, not in $I_1 \cup I_2$ being 0, we have 6.5. In 6.6, we use 6.3.1 (i) and (v) and the jump relations for ψ, where c_{α_i} is the standard Cayley transform defined in section 5.2. For the last steps, we need one more lemma.

Lemma 6.6.4. *(See* [Sh1]*, Proposition 4.6 and* [Sh5]*, Theorem 6.1.1). Suppose that we are in the situation of Section 6.6. Then for each $i \in I_1 \cup I_2$, one can construct by Cayley transform an admissible covering $\eta_{(A_i',A_H')}$ of A_H'. Furthermore $\{\eta_{(A_i',A_H')}\}_{i\in I_1\cup I_2'}$*

is a complete set of admissible coverings of A'_H *and this induces a set of isomorphisms* $\{\eta_{(\mathcal{U}'_i,\mathcal{V}'_1)}\}_{i\in I_1\cup I_2}$ *where* $\mathcal{V}'_1$ *is a neighborood of* γ_1^0 *in* $\exp(\mathfrak{t}_H)_{\mathbb{R}}\gamma_1^0$ *and* $\mathcal{U}'_i$ *is a neighborood of* δ_i^0 *in* A'_i *having the following properties :*

1. $\lim_{\nu\to 0}\widetilde{\Delta}_{\mathcal{U}_i}(\delta_i^\nu) = \widetilde{\Delta}_{\mathcal{U}'_i}(\delta_i^0)$.

2. Fix $i \in I_1\cup I'_2$ *and let* $\lambda \in \mathfrak{a}_i^*$ *(resp.* $\lambda \in (\mathfrak{a}'_i)^*$*) be the linear functional appearing in the statement of Proposition 6.3.1(i) with respect to* $\mathcal{U}_i$ *(resp.* $\mathcal{U}'_i$*). Fix also* $u \in S(\mathfrak{t}_H)$ *and* $v_i = \eta^{-1}_{(A_i,A_H)}(u)$, $v'_i = \eta^{-1}_{(A'_i,A'_H)}(c_{\alpha_H}\cdot u)$*. Then*

$$c_{\alpha_i}\cdot\tau_\lambda(v_i) = \tau_{\lambda'}(v'_i).$$

The left-hand side in 1. is well defined because of property (v) in Proposition 6.3.1. So far, we have supposed the δ_i^0's θ-semi-regular. A well-known result of Harish-Chandra enable us to relax this assumption. Thus the φ satisfies the jump relations, and complete the proof of theorem 5.4.1.

The next sections are devoted to the proofs of the various lemma stated above.

7. Roots, coroots, Cayley transforms...

7.1. **Cayley transforms.** We need to be more precise about the standard Cayley transforms we use. Let $\mathfrak{g}_{\mathbb{R}}$ be a real reductive Lie algebra and $\mathfrak{g}$ its complexification. We denote by σ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_{\mathbb{R}}$. let $\mathfrak{b}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a Cartan subalgebra. Let $\alpha \in R(\mathfrak{g},\mathfrak{b})$ be an imaginary root, choose a root vector X_α for α and fix a root vector $X_{-\alpha}$ of $-\alpha$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. Then $\mathbb{C}\cdot X_\alpha + \mathbb{C}\cdot X_{-\alpha} + \mathbb{C}\cdot H_\alpha$ is a simple complex Lie algebra invariant under conjugation, $\sigma(H_\alpha) = -H_\alpha = H_{-\alpha}$ and $\sigma(X_\alpha) = cX_{-\alpha}$ for some $c \in \mathbb{C}$. Either there is a X_α for which $\sigma(X_\alpha) = -X_{-\alpha}$ or there is one for which $\sigma(X_\alpha) = X_{-\alpha}$. In the former case, α is compact, and non-compact in the latter. In section 5, we defined the standard Cayley transform c_α with respect to a non-compact imaginary root α to be the element $\exp(-i\pi/4(X_\alpha + X_{-\alpha}))$ of the adjoint group of $\mathfrak{g}^{\delta^0\theta}$ for some δ^0 in $G(\mathbb{R})$ θ-semi-regular. We will say that c_α is a standard Cayley transform with respect to α and δ^0. We have:

$$\mathfrak{b} = \operatorname{Ker}\alpha \oplus \mathbb{C}\cdot H_\alpha$$
$$\mathfrak{b}_{\mathbb{R}} = \operatorname{Ker}\alpha_{|\mathfrak{b}_{\mathbb{R}}} \oplus i\mathbb{R}\cdot H_\alpha.$$

Let $\mathfrak{a} := c_\alpha\cdot\mathfrak{b} = \operatorname{Ker}\alpha \oplus \mathbb{C}\cdot(X_\alpha - X_{-\alpha})$. This is a Cartan subalgebra defined over $\mathbb{R}$ and

$$\mathfrak{a}_{\mathbb{R}} = \operatorname{Ker}\alpha_{|\mathfrak{b}_{\mathbb{R}}} \oplus i\mathbb{R}\cdot(X_\alpha - X_{-\alpha}).$$

The root $\beta := c_\alpha\cdot\alpha$ of $R(\mathfrak{g},\mathfrak{a})$ is real and $c_\alpha\cdot H_\alpha = H_\beta = i(X_\alpha - X_{-\alpha})$. Furthermore:

$$\sigma(c_\alpha) = \exp(i\pi/4(X_\alpha + X_{-\alpha})) = c_\alpha^{-1}.$$

A standard Cayley transform is a particular case of a generalized Cayley transform defined by Shelstad (see [Sh1], §2), thus $\sigma(c_\alpha)^{-1}c_\alpha = c_\alpha^2$ realizes the Weyl reflection s_α with respect to the root α.

It will be useful to reverse the process, and define Cayley transform with respect to real root. If $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ defined over $\mathbb{R}$, and β a real root, we take root vectors X_β and $X_{-\beta}$ in $\mathfrak{g}_{\mathbb{R}}$ such that $[X_\beta, X_{-\beta}] = H_\beta$ and define $c_\beta :=$

$\exp(i\pi/4(X_\beta + X_{-\beta}))$. The root $c_\beta \cdot \beta$ of $\mathfrak{b} := c_\beta \cdot \mathfrak{a}$ is imaginary non-compact and we can make the choices such that $c_\beta = c_\alpha^{-1}$.

7.2. **Proof of lemma 6.6.2.** We need to introduce more material from [KS]. Suppose that θ^* preserves the pair (B,T) of G^*, with T defined over $\mathbb{R}$. Let us denote by $R(G^*,T)$ the system of roots of T in G^*, and by $R_{res}(G^*,T)$, the set of restricted roots, i.e. the set:

$$\{\alpha_{res} = \alpha_{|(T^{\theta^*})_0},\ \alpha \in R(G^*,T)\}.$$

There is a partition of $R_{res}(G^*,T)$ in three types of restricted roots, denoted R_1, R_2 and R_3 (see [KS], §1.3). Notice that unlike [KS], we do not suppose G^* to be simply connected, in particular T^{θ^*} doesn't have to be connected. Notice also that if $\delta \in T(\mathbb{R})$ is regular, then $A := \exp \mathfrak{t}_{\mathbb{R}}^{\theta^*} \delta$ is a Cartan subspace of G^* and that $R_{res}(G^*,T)$ is nothing but the root system of A in G^*.

Let $\alpha \in R(G^*,T) \subset X^*(T)$ and let $\check{\alpha} \in \check{R}(G^*,T) \subset X_*(T)$ be its coroot. Recall that we have constructed the L-group ${}^L G$ using a splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ of $\hat{G}$. This give an identification $\hat{T} = \mathcal{T}$. Thus we can consider the restriction of

$$\check{\alpha} \in \check{R}(G^*,T) = R(\hat{G},\mathcal{T}) \subset X^*(\mathcal{T})$$

to $(\mathcal{T}^{\hat{\theta}})_0$. Denoting also by $\hat{\theta}$ the transpose automorphism of $X^*(\mathcal{T})$, we have :

$$(\check{\alpha})_{res} \in X^*((\mathcal{T})^{\hat{\theta}})_0).$$

An explicit expression for $X^*((\mathcal{T}^{\hat{\theta}})_0)$ is $X^*((\mathcal{T}^{\hat{\theta}})_0) = X^*(\mathcal{T})/L$ where L is the sub-lattice of all $\tau \in X^*(\mathcal{T})$ with the property that there exists a non-zero integer n such that $n\tau \in (1-\hat{\theta})X^*(\mathcal{T})$ (see [ABV], lemma 9.5). Let us set also

$$X^*(\mathcal{T})_{\hat{\theta}} := X^*(\mathcal{T})/(1-\hat{\theta})X^*(\mathcal{T}).$$

We can also take the coroot of α_{res}:

$$\alpha_{res} \in R_{res}(G^*,T) \subset X^*((T^{\theta^*})_0),\quad (\alpha_{res})\check{} \in X_*((T^{\theta^*})_0) = X_*(T)^{\theta^*}.$$

At this point it must be clear to the reader that $(\alpha_{res})\check{} \neq (\check{\alpha})_{res}$. We have the following relations between their coroots (see [KS], §1.3 for notation and proofs of the various statements):

$$((\alpha_{res})\check{})\check{} = \alpha_{res} \tag{7.1}$$

$$((\check{\alpha})_{res})\check{} = \begin{cases} N\alpha & \text{if } \alpha_{res} \text{ of type } R_1 \text{ or } R_3, \\ 2N\alpha & \text{if } \alpha_{res} \text{ of type } R_2. \end{cases} \tag{7.2}$$

Suppose that we are in the situation of Section 4, i.e. we have a maximal torus T_H of H defined over $\mathbb{R}$, and an admissible embedding :

$$\eta :\ T_H \to T_{\theta^*}.$$

We say that $\alpha \in R_{res}(G^*,T)$ originates in H when it exists a root $\alpha_H \in R(H,T_H)$ such that $\check{\alpha_H} = (\check{\alpha})_{res}$.

Notice that $\check{\alpha_H} \in \check{R}(H,T_H) \subset X_*(T_H) \simeq X^*(\mathcal{T}_H) = X^*((\mathcal{T}^{\hat{\theta}})_0)$.

Let $\gamma \in T_H(\mathbb{R})$ be G^*-regular, and $\delta \in G^*$ such that $\eta(\gamma) = N_{\theta^*}(\delta)$. Notice that we changed our notation, because in what follows only G^* plays a role, and not G itself. Of course, all previous results apply to the case $G = G^*$ and this is what we use implicitely in this paragraph.

We have then:

$$\alpha_H(\gamma) = \begin{cases} N\alpha(\delta) & \text{if } \alpha_{res} \text{ of type } R_1 \text{ or } R_3 \\ (N\alpha(\delta))^2 & \text{if } \alpha_{res} \text{ of type } R_2 \end{cases} \tag{7.3}$$

Let $\eta_{(A,A_H)} : A := \exp \mathfrak{t}_{\mathbb{R}}^{\theta^*}\delta \to A_H = \exp(\mathfrak{t}_H)_{\mathbb{R}}\gamma$ be an admissible covering. Let $\delta_0 \in A$ be a θ^*-semi-regular element with respect to the roots $\pm\alpha_{res} \in R(G^*, A) = R_{res}(G^*, T)$ (i.e. $\det\left(\mathrm{Id} - (\mathrm{Ad}\,\delta_0 \circ \theta^*)^{-1}\right)_{|\mathfrak{g}_{\mathbb{C}}^{\pm\alpha}} = 0$ and $\pm\alpha_{res}$ are the only restricted roots with this property), and furthermore, suppose that α_{res} originates in H (i.e. $(\alpha_{res})\check{} = \alpha_H\check{}$ for some $\alpha_H \in R(H, T_H)$). Let $\gamma_0 = \eta_{(A,A_H)}(\delta^0)$ and suppose we are in the situation of lemma 6.6.2, i.e. γ_0 is semi-regular with respect to the roots $\pm\alpha_H$. The considerations of [KS], §1.3 (and 7.3 above) show that we are in one of the following two cases:

- α_{res} is of type R_1, $\alpha_H(\gamma_0) = N\alpha(\delta_0) = 1$,
- α_{res} is of type R_2, $\alpha_H(\gamma_0) = 1$ and $N\alpha(\delta_0) = 1$.

The restriction of N_{θ^*} to $(T^{\theta^*})_0$ defines a transpose map:

$$N^* : X(T_{\theta^*}) \to X((T^{\theta^*})_0.$$

For all $t \in (T^{\theta^*})_0$, for all $\tau \in X(T_{\theta^*})$,

$$N^*(\tau)(t) = \tau(N_{\theta^*}(t)).$$

In particular, for all $t \in (T^{\theta^*})_0$,

$$\begin{aligned} N^*(((\check{\alpha})_{res})\check{})(t) &= ((\check{\alpha})_{res})\check{}(N_{\theta^*}(t)) \\ &= \begin{cases} N\alpha(t) & \text{if } \alpha_{res} \text{ of type } R_1, \\ (N\alpha(t))^2 & \text{if } \alpha_{res} \text{ of type } R_2. \end{cases} \end{aligned}$$

Since $t \in (T^{\theta^*})_0$, $N\alpha(t) = \alpha(t)^{l_\alpha}$, where l_α is the multiplicity of the root α. The map N_{θ^*} defines also a map:

$$N_* : X_*((T^{\theta^*})_0) \to X_*(T_{\theta^*})$$

which is the transpose of N^*, and we have:

$$N_*((\alpha_{res})\check{}) = \begin{cases} l_\alpha(\check{\alpha})_{res} & \text{if } \alpha_{res} \text{ of type } R_1, \\ 2l_\alpha(\check{\alpha})_{res} & \text{if } \alpha_{res} \text{ of type } R_2. \end{cases}$$

Lemma 6.6.2 is a straightforward consequence of these computations, with $m_\alpha = l_\alpha$ if α_{res} of type R_1 and $m_\alpha = 2l_\alpha$ if α_{res} of type R_2.

7.3. **Proof of lemma 6.6.1.** Suppose we are in the setting of lemma 6.6.1 and that A'_H does originates in $G(\mathbb{R})$. We want to prove that A_H also originates in $G(\mathbb{R})$.

Our hypothesis is that there exist a θ^*-stable maximal torus T' of G^* defined over $\mathbb{R}$, with an admissible embedding $\eta' : T'_H \to T'_{\theta^*}$, elements $\gamma \in T'_H$, $\delta^* \in T'$, $\delta \in G(\mathbb{R})$, and $x \in G^*_{sc}$ such that δ is $=\theta$-regular, $\eta'(\gamma) = N_{\theta^*}(\delta^*)$, $\delta^* = xm(\delta)\theta^*(x)^{-1}$ and

$$\operatorname{Int} x \circ \psi : T_\delta^{\delta\theta} \to (T')^{\theta^*}$$

is defined over $\mathbb{R}$ (see section 4). From these data, we obtained an admissible covering:

$$\eta_{(A',A'_H)} : A' := \exp \mathfrak{g}_{\mathbb{R}}^{\delta\theta}\, \delta \to A'_H.$$

Let us denote by β_H the real root of $R(H, T'_H)$ obtained by taking the Cayley transform $c_{\alpha_H} \cdot \alpha_H$, and by $\mathfrak{B}_{res}$ the restricted root of $R_{res}(G^*, T')$ such that $(\mathfrak{B}^\vee)_{res} = (\beta_H)^\vee$. Since η' is defined over $\mathbb{R}$, $\mathfrak{B}_{res}$ takes real values on $(T')_0^{\theta^*}(\mathbb{R})$.

Let $\delta_0 \in A$ such that $\eta_{(A',A'_H)}(\delta_0) = \gamma_0$ and let $\delta_0^* = xm(\delta_0)\theta^*(x)^{-1}$. We have then:

$$\beta_H(\gamma_0) = 1 = \begin{cases} N\mathfrak{B}(\delta_0^*) & \text{if } \mathfrak{B}_{res} \text{ of type } R_1 \\ N\mathfrak{B}(\delta_0^*)^2 & \text{if } \mathfrak{B}_{res} \text{ of type } R_2 \end{cases}$$

Thus, if $\mathfrak{B}_{res}$ of type R_1, $N\mathfrak{B}(\delta_0^*) = 1$ and δ_0^* is singular with respect to $\mathfrak{B}_{res}$ (case 1). If $\mathfrak{B}_{res}$ of type R_2, $N\mathfrak{B}(\delta_0^*) = \pm 1$. If $N\mathfrak{B}(\delta_0^*) = 1$, then δ_0^* is singular with respect to $\mathfrak{B}_{res}$ (case 2). If $N\mathfrak{B}(\delta_0^*) = -1$, let $\mathfrak{C}_{res} = 2\mathfrak{B}_{res} \in R_{res}(G^*, T)$. This is a restricted root of type R_3 and $N\mathfrak{B} = N\mathfrak{C}$ (see [KS], §1.3). Then δ_0^* is singular with respect to $\mathfrak{C}_{res}$ (case 3). If $\mathfrak{B}_{res}$ of type R_3, let $\mathfrak{C}_{res} = 1/2\mathfrak{B}_{res} \in R_{res}(G^*, T)$. This is a restricted root of type R_2 and $N\mathfrak{B} = N\mathfrak{C}$. We have then $N\mathfrak{C}(\delta_0^*) = N\mathfrak{B}(\delta_0^*) = 1$ and δ_0^* is singular with respect to $\mathfrak{C}_{res}$ (case 4).

In cases 1 and 2, we set $\mathfrak{B}_{res} = \mathfrak{C}_{res}$. Let us denote by β the transport of $\mathfrak{C}_{res}$ to $R(G, A')$ by $\operatorname{Int} x \circ \psi$. This is a real root of $R(G, A')$. We take the standard Cayley transform c_β with respect to β and δ^0 and define $\mathfrak{a}_{\mathbb{C}} = c_\beta \cdot \mathfrak{a}'_{\mathbb{C}}$, $\mathfrak{a} = \mathfrak{a}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$.

We need also to use the standard Cayley transform $c_{\mathfrak{C}_{res}}$ with respect to the root $\mathfrak{C}_{res}$ of the Cartan subalgebra $(\mathfrak{t}')^{\theta^*} \subset \mathfrak{g}^{\delta_0^*\theta^*}$. Let $\mathfrak{u} := c_{\mathfrak{C}_{res}} \cdot (\mathfrak{t}')^{\theta^*}$. This is a abelian subalgebra of $\mathfrak{g}$, composed of semi-simple elements, and containing regular ones. Let $\mathfrak{t} := \operatorname{Centr}(\mathfrak{g}, \mathfrak{u})$. This is a θ^*-stable Cartan subalgebra. We denote by T its centralizer.

Notice that $\operatorname{Int} x \circ \psi_{|\mathfrak{a}_{\mathbb{C}}}$ is not necessary defined over $\mathbb{R}$, but reduction to the $SL(2)$ case shows that there exists $t \in (T^{\theta^*})_0$ such that

$$\operatorname{Int} tx \circ \psi : \mathfrak{a}_{\mathbb{C}} \to \mathfrak{t}^{\theta^*}$$

is defined over $\mathbb{R}$ and the following diagram commute:

$$\begin{array}{ccc} \mathfrak{a}'_{\mathbb{C}} & \xrightarrow{\operatorname{Ad} x\circ\psi} & (\mathfrak{t}')^{\theta^*} \\ \Big\downarrow c_\beta & & \Big\downarrow c_{\mathfrak{C}_{res}} \\ \mathfrak{a}_{\mathbb{C}} & \xrightarrow{\operatorname{Ad} tx\circ\psi} & \mathfrak{t}^{\theta^*} \end{array}$$

Since $c_{\mathfrak{C}_{res}} \in G^{\delta_0^*\theta^*}$, we have $\theta^*(c_{\mathfrak{C}_{res}}) = (\delta_0^*)^{-1} c_{\mathfrak{C}_{res}} \delta_0^*$, and for all $t \in T'$:

$$\theta^*(c_{\mathfrak{C}_{res}} t c_{\mathfrak{C}_{res}}^{-1}) = (\delta_0^*)^{-1} c_{\mathfrak{C}_{res}} \delta_0^* t (\delta_0^*)^{-1} c_{\mathfrak{C}_{res}}^{-1} \delta_0^* = c_{\mathfrak{C}_{res}} t c_{\mathfrak{C}_{res}}^{-1}$$

Thus $\operatorname{Ad} c_{\mathfrak{C}_{res}}$ maps $(1-\theta^*)T'$ onto $(1-\theta^*)T$ and induces a map $\overline{c}_{\mathfrak{C}_{res}} : T'_{\theta^*} \to T_{\theta^*}$. We define η by the following commutative diagram:

$$\begin{array}{ccccc} T'_H & \xrightarrow{\eta'} & T'_{\theta^*} & \xleftarrow{N_{\theta^*}} & (T')_0^{\theta^*} \\ \downarrow c_\beta & & \downarrow \overline{c}_{\mathfrak{C}_{res}} & & \downarrow c_{\mathfrak{C}_{res}} \\ T_H & \xrightarrow{\eta} & T_{\theta^*} & \xleftarrow{N_{\theta^*}} & (T^{\theta^*})_0 \end{array}$$

It remains now only to check that η is defined over $\mathbb{R}$ to complete the proof of lemma 6.6.1. For all $\gamma \in T_H$ we have:

$$\begin{aligned} \sigma_{G^*}(\eta(\gamma)) &= \sigma_{G^*}(\operatorname{Int} \overline{c}_{\mathfrak{C}_{res}} \circ \eta' \circ \operatorname{Int} c_{\beta_H}^{-1}(\gamma)) \\ &= \operatorname{Int}(\sigma_{G^*}(\overline{c}_{\mathfrak{C}_{res}}))(\sigma_{G^*}(\eta'(\operatorname{Int} c_{\mathfrak{C}_{res}}^{-1}(\gamma)))) \\ &= \operatorname{Int} \overline{c}_{\mathfrak{C}_{res}}^{-1}(\eta'(\sigma_H(\operatorname{Int} c_{\beta_H}^{-1}(\gamma)))) \\ &= \operatorname{Int} \overline{c}_{\mathfrak{C}_{res}}^{-1} \circ \eta' \circ \operatorname{Int} c_{\beta_H}(\sigma_H(\gamma)) \\ &= \operatorname{Int} \overline{c}_{\mathfrak{C}_{res}} \circ s_{\mathfrak{C}_{res}}(\eta'(s_{\beta_H}(\operatorname{Int} c_{\beta_H}(\sigma_H(\gamma))))) \\ &= \operatorname{Int} \overline{c}_{\mathfrak{C}_{res}} \circ \eta' \circ \operatorname{Int} c_{\beta_H}(\sigma_H(\gamma)) \\ &= \eta(\sigma_H(\gamma)) \end{aligned}$$

Thus we have constructed an admissible embedding $\eta : T_H \to T_{\theta^*}$ and this finishes the proof of lemma 6.6.1

7.4. **Proof of lemma 6.6.3.** Suppose that $T'_H(\mathbb{R})$ does originates in $G(\mathbb{R})$. Then it is clear from the constructions of section 7.3 that there exists an admissible covering:

$$\eta_{(A',A'_H)} : A' = \exp \mathfrak{a}'_{\mathbb{R}}\, \delta^0 \longrightarrow A'_H = \exp \mathfrak{t}'_{\mathbb{R}}\, \gamma^0$$

with $\eta_{(A',A'_H)}(\delta^0) = \gamma^0$ and we can make the choices such that δ^0 is one of the δ^0_i. Since $[\mathfrak{g}_{\mathbb{R}}^{\delta^0\theta}, \mathfrak{g}_{\mathbb{R}}^{\delta^0\theta}] \simeq \mathfrak{sl}(2,\mathbb{R})$, δ^0 is not in $A_{\theta-In-reg}$ which is what we wish to show.

8. Proof of proposition 6.3.1 and lemma 6.6.4

8.1. Let us fix $\delta \in A_{\theta-reg}$, $\gamma_1 = \eta_{(\mathcal{U},\mathcal{V}_1)}(\delta)$ and $X \in \mathfrak{a}_{\mathbb{R}}$ as in the proposition. We have :

$$\begin{aligned} \widetilde{\Delta}_{\mathcal{U}}((\exp X)\,\delta) &= \Delta(\eta_{(\mathcal{U},\mathcal{V}_1)}((\exp X)\,\delta), (\exp X)\,\delta) \\ &= \Delta(\eta_{(\mathcal{U},\mathcal{V}_1)}((\exp X)\,\delta), (\exp X)\,\delta; \gamma_1, \delta)\Delta(\gamma_1, \delta) \\ &= \Delta((\eta_{(\mathcal{U},\mathcal{V}_1)}(\exp X))\,\gamma_1, (\exp X)\,\delta; \gamma_1, \delta)\Delta(\gamma_1, \delta) \end{aligned}$$

where $\Delta(.,.;.,.)$ is the canonically defined relative transfer factor of [KS]. It is this relative transfer factor we will examine, and since δ, γ_1 are fixed, as well as $\eta_{(\mathcal{U},\mathcal{V}_1)}$ and $T_{\theta^*} \simeq T_H$ we will denote simply by:

$$\Delta(X) = \Delta((\eta_{(\mathcal{U},\mathcal{V}_1)}(\exp X))\,\gamma_1, (\exp X)\,\delta; \gamma_1, \delta).$$

8.2. The first assertion in (i) of 6.3.1 is obvious and the second is established in [Sh6], with an explicit determination of the λ appearing in 6.3.1 (i). Let us now prove the rest of (i) in 6.3.1. From the definition of transfer factors, we see that the obstruction to extend $\widetilde{\Delta}_{\mathcal{U}}$ to a smooth function on $\mathcal{U}$ comes from the Δ_2 term. More precisely, there might be an obstruction at the point $(\exp X)\,\delta$ if

- $N\alpha((\exp X^*)\delta^*) = 1$ with α orbit of the first kind.
- $N\alpha((\exp X^*)\delta^*) = \pm 1$ with α orbit of the second kind.
- $N\alpha((\exp X^*)\delta^*) = -1$ with α orbit of the third kind.

So $\widetilde{\Delta}$ extends to a smooth function on $\mathcal{U}_{\theta-I-reg}$.

8.3. Let us recall some notation from proposition 6.3.1. Let $\delta^0 \in \mathcal{U}$ be a θ-semi-regular element such that the roots $\pm\alpha$ of $\mathfrak{a}$ in $\mathfrak{g}^{\delta^0\theta}$ are imaginary. If $\gamma_1^0 = \eta_{(\mathcal{U},\mathcal{V}_1)}(\delta^0)$ is semi-regular, that means that the transport of α to $\mathfrak{t}^{\theta^*}$ by $\operatorname{Int} x \circ \psi$ is not of one of the type above, so $\widetilde{\Delta}_{\mathcal{U}}$ has a smooth continuation at δ^0. This can be rewritten

$$\lim_{\nu\to 0^+} \Delta(\gamma_1^\nu, \delta^\nu) = \lim_{\nu\to 0^-} \Delta(\gamma_1^\nu, \delta^\nu)$$

proving the corresponding assertions in (iii) and (v). If γ_1^0 is regular, we are in the opposite situation, and the transport of α to $\mathfrak{t}^{\theta^*}$ by $\operatorname{Int} x \circ \psi$ (denoted by β_{res}) is of one of the type above. Suppose that it is of the first kind, i.e. $N\alpha(\delta^{0,*}) = 1$ with α orbit of type R_1 not from H. The obstruction to the smoothness of $\widetilde{\Delta}_{\mathcal{U}}$ at δ_0 comes from the factor

$$\chi_{\beta_{res}}(N\beta(\delta^{\nu,*}) - 1)$$

and it is easy to see that

$$\lim_{\nu\to 0^+} \chi_{\beta_{res}}(N\beta(\delta^{\nu,*} - 1) = -\lim_{\nu\to 0^-} \chi_{\beta_{res}}(N\beta(\delta^{\nu,*} - 1).$$

We can make similar computation in the two remaining case and conclude that

$$\lim_{\nu\to 0^+} \Delta(\gamma_1^\nu, \delta^\nu) = -\lim_{\nu\to 0^-} \Delta(\gamma_1^\nu, \delta^\nu),$$

proving the assertions in (ii) and (iv).

8.4. For the remaining assertions of Proposition 6.3.1, we need to compare

$$\Delta(\gamma_1^\nu, \delta^\nu) \quad \text{and} \quad \Delta(\gamma_1^\nu, \delta^{-\nu}).$$

By [KS], Theorem 5.1.D, the quotient of these two terms is given by the quantity

$$\langle \operatorname{inv}(\delta^\nu, \delta^{-\nu}), \kappa_{\delta^\nu} \rangle.$$

Let w_α be an element in $G_{sc}^{\delta^0\theta}$ realizing the Weyl group reflection s_α. We have

$$w_\alpha \delta^\nu \theta(w_\alpha)^{-1} = \delta^{-\nu}.$$

Thus the hypercocycle

$$\operatorname{inv}(\delta^\nu, \delta^{-\nu}) \in H^1(\Gamma, T_{\delta^\nu}^{sc} \stackrel{(1-\theta^{\delta^\nu})\pi}{\longrightarrow} V_{\delta^\nu})$$

is given by the pair $(\sigma_G(w_\alpha)w_\alpha^{-1}, 1)$. We transport this by $\mathrm{Int}\, x \circ \psi$ to an element of $(y_\sigma, 1)$ of $H^1(\Gamma, T^{sc} \stackrel{(1-\theta^*)\pi}{\longrightarrow} V)$ (recall that $\mathrm{Int}\, x \circ \psi$ transports T_{δ^ν} to T over $\mathbb{R}$ and transports θ^{δ^ν} to θ^*).

Let us recall some properties of the pairing in hypercohomology. From [KS], end of Section A.1, we extract two relevant portions of exact sequences :

$$\ldots \to H^1(\Gamma, \mathrm{Ker}(1-\theta^*)) \xrightarrow{i'} H^1(\Gamma, T^{sc} \stackrel{(1-\theta^*)\pi}{\longrightarrow} V) \xrightarrow{j'} H^0(\Gamma, \mathrm{coker}(1-\theta^*)) \to \ldots$$

$$\ldots \to H^1(W_\mathbb{R}, \mathrm{Ker}(1-\hat{\theta})) \xrightarrow{\hat{i}'} H^1(W_\mathbb{R}, \widehat{V} \stackrel{1-\hat{\theta}}{\longrightarrow} \widehat{T}^{ad}) \xrightarrow{\hat{j}'} H^0(W_\mathbb{R}, \mathrm{coker}(1-\hat{\theta})) \to \ldots$$

Now $\mathrm{Ker}(1-\theta^*) = (T^{sc})^{\theta^*}$, $\mathrm{coker}(1-\hat{\theta})) = (T^{ad})^{\hat{\theta}}$ and the transport y_σ of $\sigma_G(w_\alpha)w_\alpha^{-1}$ by $\mathrm{Int}\, x \circ \psi$ defines a cocycle in $H^1(\Gamma, (T^{sc})^{\theta^*})$. Furthermore $i'(y_\sigma) = (y_\sigma, 1)$, and by the compatibility of the pairing with the above exact sequences (see [KS], A.3.13), we get :

$$\langle \mathrm{inv}(\delta^\nu, \delta^{-\nu}), \kappa_{\delta^\nu} \rangle = \langle y_\sigma, \hat{j}'(b_T^{-1}, s) \rangle = \langle y_\sigma, s \rangle,$$

where the pairing on the right hand side is the Tate-Nakayama pairing between $H^1(\Gamma, (T^{sc})^{\theta^*})$ and $H^0(W_\mathbb{R}, (T^{ad})^{\hat{\theta}})$. Notice that since we take invariants in G^{sc}, $(T^{sc})^{\theta^*}$ is connected. The properties of this pairing are well known (see [Sh3] for instance), and $\langle y_\sigma, s \rangle = 1$ if γ^0 is singular in $H_\mathbb{R}$, and $\langle y_\sigma, s \rangle = -1$ if γ^0 is regular in $H_\mathbb{R}$. Q.E.D.

8.5. **About Lemma 6.6.4.** Suppose that we are in the situation of the lemma. Recall the complete set of admissible coverings $\{\eta_{(A_i, A_H)}\}_{i \in I}$. Let $i \in I_1 \cup I_2$. Then δ_i^0 is semi-regular with respect to a non compact imaginary root α_i of $\mathfrak{a}_i$ in $\mathfrak{g}$. This root originates from a root α_H of $\mathfrak{t}_H$ in $\mathfrak{h}$. Consider the Cayley transforms $c_{\alpha_i} : \mathfrak{a}_i \to \mathfrak{a}'_i$ and $c_{\alpha_H} : \mathfrak{t}_H \to \mathfrak{t}'_H$. From the constructions of Section 7.3, it is clear that we have an admissible covering

$$\eta_{(A'_i, A'_H)} : A'_i = \exp \mathfrak{a}'_{i,\mathbb{R}}\, \delta_i^0 \longrightarrow A'_H = \exp \mathfrak{t}'_{H,\mathbb{R}}\, \gamma^0.$$

If $i \in I_2$ let $\overline{\eta}_{(A'_i, A'_H)}$ be the admissible covering covering obtained as above from $\overline{\eta}_{(A_i, A_H)}$ instead of $\eta_{(A_i, A_H)}$. It is easy to check that $\overline{\eta}_{(A'_i, A'_H)}$ and $\eta_{(A'_i, A'_H)}$ are equivalent, i.e. they are θ-conjugate by an element of $G_\mathbb{R}$ (the computations essentially take place, modulo center, in a copy of SL_2). It is also straitforward to check that if $i, j \in I_1 \cup I'_2$, then $\overline{\eta}_{(A'_i, A'_H)}$ is equivalent to $\overline{\eta}_{(A'_j, A'_H)}$ if and only if $i = j$.

Now suppose that $\eta_{(A', A'_H)} : \exp \mathfrak{a}'_\mathbb{R}\, \delta^0 \to A'_H = \exp \mathfrak{t}_H\, \gamma^0$ is an admissible covering of A'_H, where γ^0 is semi-regular with respect to a real root α_H. Then the constructions can be inverted, using inverse Cayley transforms. Thus we obtain an admissible covering

$$\eta_{(A, A'_H)} : A = \exp \mathfrak{a}_\mathbb{R}\, \delta^0 \longrightarrow A_H = \exp \mathfrak{t}_{H,\mathbb{R}}\, \gamma^0$$

which has to be equivalent to one of the $\eta_{(A_i, A_H)}$. Then $\eta_{(A', A'_H)}$ is equivalent to $\eta_{(A'_i, A'_H)}$. This proves the first assertion of the lemma.

8.6. The second assertion of the lemma will be a consequence of the results in [Sh6]. Let us write Δ and Δ' for $\widetilde{\Delta}_{\mathcal{U}_i}$ and $\widetilde{\Delta}_{\mathcal{U}'_i}$, with $\Delta = \Delta_1\Delta_2\Delta_3$ and $\Delta' = \Delta'_1\Delta'_2\Delta'_3$. Adapting the proof of Theorem 6.1.1 in [Sh5] to the twisted case, we obtain that

$$\Delta_3(\delta_i^0) = \Delta'_3(\delta_i^0),$$

and thus

$$\lim_{\nu\to 0} \Delta_2\Delta_3(\delta_i^\nu) = \pm\Delta'_2\Delta'_3(\delta_i^0).$$

The factor Δ_1 is taylored exactly to get the sign right. To show this we need to extend the results about descent for transfer factors of [LS2] to the present setting of twisted endoscopy.

8.7. For the last assertion of the lemma, we start by noticing that it is enough to prove that

$$c_{\alpha_i}(v_i) = v'_i.$$

This is because $\lambda \in \mathfrak{a}^*$ and $\lambda' \in (\mathfrak{a}')^*$ correspond to elements $\mu^* \in \mathfrak{t}_H^*$ and $(\mu')^* \in (\mathfrak{t}'_H)^*$ which are both the restriction of the same element in $\mathfrak{h}^*$. What remains is straitforward.

8.8. For the convenience of the reader and for references, we recall once more what is left to get a complete proof of the transfer :

- From Proposition 6.3.1 (see notation there) : Let $\delta \in A_{\theta-reg}$. Then there exists $\lambda \in \mathfrak{a}^*$ and $c \in \mathbb{C}$ such that

$$\widetilde{\Delta}_{\mathcal{U}}((\exp X)\,\delta) = c\,e^{\lambda(X)}$$

for all $X \in \mathfrak{a}_{\mathbb{R}}$ such that $(\exp X)\,\delta \in \mathcal{U}_{\theta-reg}$.

we need also an explicit determination of this λ in terms of endoscopic data ...

- From Lemma 6.6.4 : $\lim_{\nu\to 0} \widetilde{\Delta}_{\mathcal{U}_i}(\delta_i^\nu) = \widetilde{\Delta}_{\mathcal{U}'_i}(\delta_i^0)$.

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