CHAPTER 12

THE CATEGORY OF TRIPLES OF $\widetilde{\mathcal{D}}_X$-MODULES

Summary. The category of triple of filtered $\widetilde{\mathcal{D}}_X$-modules has been considered in dimension one as a suitable abelian category containing the category of polarizable Hodge modules as a full subcategory (see Section 7.4.a). Our aim in this chapter is to extend the notion of triples in any dimension. For that purpose, we are led to extend the notion of $(\nabla, \overline{\nabla})$-flat sesquilinear pairing, used for the definition of a polarized variation of $\mathbb{C}$-Hodge structure (see Definition 4.1.4), to the case where the flat bundle is replaced with a $\mathcal{D}$-module. Such a sesquilinear pairing takes values in the sheaf of distributions or of currents of maximal degree. We also make precise its behaviour with respect to functors like pushforward, smooth pullback, nearby and vanishing cycles and localization.

In this chapter, we keep Notation 9.0.1. In the first sections, we will only consider $\mathcal{O}_X$-modules and $\mathcal{D}_X$-modules, as coherent filtrations will not play any role here. We will use the constructions and results of Chapter 9 in this framework. We come back to the filtered case in Section 12.7.

12.1. Introduction

One of the ingredients of a polarized variation of Hodge structure is a flat Hermitian pairing (that we have denoted by S), which is $(-1)^{w-p}$-definite on $\mathcal{H}^{p,w-p}$. In this chapter, we introduce the notion of sesquilinear pairing between holonomic $\mathcal{D}_X$-modules. It takes values in the sheaf of distributions (in fact a smaller sheaf, but we are not interested in characterizing the image). This notion will not be used directly as in classical Hodge theory to furnish the notion of polarization. Instead, we will take up the definition of a $\mathbb{C}$-Hodge structure as a triple (see Section 5.2) and mimic this definition in higher dimension. Our aim is therefore to define a category of $\mathcal{D}$-triples (an object consists of a pair of $\mathcal{D}_X$-modules and a sesquilinear pairing between the underlying $\mathcal{D}_X$-modules) and to extend to this abelian category the various functors considered in Chapter 9.

12.2. Distributions and currents on a complex manifold

12.2.a. Distributions and currents. Let $\overline{X}$ denote the complex manifold conjugate to X, i.e., with structure sheaf $\mathcal{O}_{\overline{X}}$ defined as the sheaf of anti-holomorphic functions $\overline{\mathcal{O}_X}$. Correspondingly is defined the sheaf of anti-holomorphic differential operators $\mathcal{D}_{\overline{X}}$. The sheaf of C^∞ functions on X is acted on by $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$ on the left and both actions commute, i.e., $\mathcal{C}_X^\infty$ is a left $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-module. Similarly, the sheaf of distributions $\mathfrak{Db}_X$ is a left $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-module: by definition, on any open set $U \subset X$, $\mathfrak{Db}_X(U)$ is dual to the space $\mathcal{E}_{\mathrm{c}}^{2n}(U)$ of C^∞ $2n$-forms with compact support, equipped with a suitable topology, and the presheaf defined in this way is a sheaf. On the other hand, the space of $\mathfrak{C}_X(U)$ of currents of degree 0 on X is dual to $C_{\mathrm{c}}^\infty(U)$ with suitable topology. Then $\mathfrak{C}_X$ is the right $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-module obtained from $\mathfrak{Db}_X$ by the left-to-right transformation for such objects, i.e.,

$$\mathfrak{C}_X = (\omega_X \otimes_{\mathbb{C}} \omega_{\overline{X}}) \otimes_{(\mathcal{O}_X \otimes \mathcal{O}_{\overline{X}})} \mathfrak{Db}_X\,.$$

12.2.1. Notation. From now on, the notation $\mathcal{A}_{X,\overline{X}}$ will mean $\mathcal{A}_X \otimes_{\mathbb{C}} \mathcal{A}_{\overline{X}}$ ($\mathcal{A} = \mathcal{O}$ or $\mathcal{D}$).

De Rham complex. One can easily adapt Exercise 8.22 to prove that the C^∞-de Rham complex $\mathcal{E}_X^{2n+\bullet} \otimes_{\mathcal{C}_X^\infty} \mathcal{D}_{X,\overline{X}} = \mathcal{E}_X^\bullet[2n] \otimes_{\mathcal{O}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}}$, where the differential is obtained from the standard differential on C^∞ k-forms and the universal connection $\nabla_X + \overline{\nabla}_X$ on $\mathcal{D}_{X,\overline{X}}$, is a resolution of $\mathcal{E}_X^{n,n} = \mathcal{E}_X^{2n}$ as a right $\mathcal{D}_{X,\overline{X}}$-module.

We denote by $\mathfrak{Db}_X^{n-p,n-q} = \mathcal{E}_X^{n-p,n-q} \otimes_{\mathcal{C}_X^\infty} \mathfrak{Db}_X$ or $\mathfrak{Db}_{X,p,q}$ the sheaf of currents of degree (p,q) (we also say of type $(n-p,n-q)$), that is, continuous linear forms on C_{c}^∞ differential forms of degree p,q.

The distributional de Rham complex yields then a resolution of $\mathfrak{C}_X$ as a right $\mathcal{D}_{X,\overline{X}}$-module:

$$\mathfrak{Db}_X^\bullet[2n] \otimes_{\mathcal{O}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}} \xrightarrow{\sim} \mathfrak{C}_X\,. \tag{12.2.2}$$

Let us make precise that the morphism is induced by

$$\mathfrak{Db}_X^{n,n} \otimes \mathcal{D}_{X,\overline{X}} = \mathfrak{C}_X \otimes \mathcal{D}_{X,\overline{X}} \longrightarrow \mathfrak{C}_X, \quad u \otimes P \longmapsto u \cdot P.$$

The Poincaré and Dolbeault lemmas for distributions and for L^2_{loc} forms. Here, the complex structure of X is not needed. It could be any differentiable manifold. The Poincaré lemma for distributions, due to de Rham [**dR73, dR84**], asserts that the complex $(\mathfrak{Db}_X^\bullet, \mathrm{d})$ is quasi-isomorphic to its subcomplex $(\mathcal{E}_X^\bullet, \mathrm{d})$, which is itself, by the standard Poincaré lemma, quasi-isomorphic to the constant sheaf $\mathbb{C}_X$. The notion of current is important in the proof since it involves integration of currents (see Definition 12.2.9 below). The basic regularization procedure is given by the following lemma.

***12.2.3. Lemma* (Regularization lemma, [dR73, dR84]).** *Let U be an open subset of $\mathbb{R}^m$. For each $\varepsilon \in (0,1)$ and each integer p, there exist $\mathbb{C}$-linear morphisms $R_\varepsilon : \mathfrak{Db}^p(U) \to \mathfrak{Db}^p(U)$ and $S_\varepsilon : \mathfrak{Db}^{p-1}(U) \to \mathfrak{Db}^p(U)$ such that*

1. *R_ε takes values in $\mathcal{E}^p(U)$,*
2. *$R_\varepsilon(u) - u = \mathrm{d}S_\varepsilon(u) - S_\varepsilon(\mathrm{d}u)$ for any $u \in \mathfrak{Db}^p(U)$,*
3. *$R_\varepsilon(\mathrm{d}u) = \mathrm{d}R_\varepsilon(u)$ and $\lim_{\varepsilon\to 0} R_\varepsilon(u) = u$ weakly.*

On the other hand, the L^2 de Rham complex $(\mathcal{L}^{\bullet}_{(2),X}, \mathrm{d})$ on X is the subcomplex of $(\mathfrak{Db}^{\bullet}_X, \mathrm{d})$ such that $\mathcal{L}^p_{(2),X}$ consists of p-forms with coefficients in L^2_{loc} in some (or any) basis $(\mathrm{d}x_I)_{\#I=p}$ *and* whose differentials are also L^2_{loc}. The construction of R_ε and S_ε also gives:

12.2.4. Lemma. *If u has L^2_{loc} coefficients, then so does $S_\varepsilon(u)$.*

***12.2.5. Corollary* (Poincaré lemmas).** *The inclusions of complexes*

$$(\mathbb{C}_X, 0) \hookrightarrow (\mathcal{E}^{\bullet}_X, \mathrm{d}) \hookrightarrow (\mathcal{L}^{\bullet}_{(2),X}, \mathrm{d}) \hookrightarrow (\mathfrak{Db}^{\bullet}_X, \mathrm{d})$$

are quasi-isomorphisms.

Proof. That the first inclusion is a quasi-isomorphism is the standard Poincaré lemma. Let us check the second inclusion for example. Let u be a local section of $\mathcal{L}^p_{(2),X}$ which is d closed. Then (for any $\varepsilon \in (0,1)$) $S_\varepsilon(u)$ has L^2_{loc} coefficients, and so does $\mathrm{d}S_\varepsilon(u) = u - R_\varepsilon(u)$, so that $S_\varepsilon(u)$ is a local section of $\mathcal{L}^{p-1}_{(2),X}$. It follows that u is cohomologous to $R_\varepsilon(u)$ in $\mathcal{L}^p_{(2),X}$, showing surjectivity $\mathcal{H}^p(\mathcal{E}^{\bullet}_X, \mathrm{d}) \to \mathcal{H}^p(\mathcal{L}^{\bullet}_{(2),X}, \mathrm{d})$, hence the latter is zero for $p > 0$, by the standard Poincaré lemma. For $p = 0$, we use that a distribution all of whose derivatives are zero is a locally constant function. □

One defines in a similar way the Dolbeault complexes $(\mathcal{E}^{p,\bullet}_X, \mathrm{d}'')$, $(\mathcal{L}^{p,\bullet}_{(2),X}, \mathrm{d}'')$ and $(\mathfrak{Db}^{p,\bullet}_X, \mathrm{d}'')$.

***12.2.6. Theorem* (Dolbeault lemmas).** *The inclusion of complexes*

$$(\Omega^p_X, 0) \hookrightarrow (\mathcal{E}^{p,\bullet}_X, \mathrm{d}'') \hookrightarrow (\mathcal{L}^{p,\bullet}_{(2),X}, \mathrm{d}'') \hookrightarrow (\mathfrak{Db}^{p,\bullet}_X, \mathrm{d}'')$$

are quasi-isomorphisms.

We refer e.g. to [**GH78**, p. 382–385] for the Dolbeault-Grothendieck theorem, i.e., $(\mathfrak{Db}^{p,\bullet}_X, \mathrm{d}'')$ is a resolution of Ω^p_X, and to [**Hör66**, Th. 4.2.2] for the L^2-Dolbeault lemma (the result proved in loc. cit. is stronger).

Distributions and currents depending continuously on a parameter. We wish to define the notion of distribution depending continuously on the parameter $s \in \boldsymbol{S}$. We will define such a sheaf on $X \times \boldsymbol{S}$ by $\mathfrak{Db}_{X\times\boldsymbol{S}/\boldsymbol{S}}$.

Let $\boldsymbol{S}$ be a C^∞ real manifold (we will mainly use $\boldsymbol{S} = \mathbb{C}_s$). Let $\mathcal{E}^{(n,n)}_{X\times\boldsymbol{S}/\boldsymbol{S},\mathrm{c}}$ the sheaf of C^∞ relative (with respect to the projection $X \times \boldsymbol{S} \to \boldsymbol{S}$) (n, n) forms with compact support. The sheaf on $X \times \boldsymbol{S}$ of distributions which are continuous with respect to $\boldsymbol{S}$ is defined as follows. Given any open set W of $X \times \boldsymbol{S}$, an element of $\mathfrak{Db}_{X\times\boldsymbol{S}/\boldsymbol{S}}(W)$ is a $C^\infty(\boldsymbol{S})$-linear map $\mathcal{E}^{(n,n)}_{X\times\boldsymbol{S}/\boldsymbol{S},\mathrm{c}}(W) \to C^0_{\mathrm{c}}(\boldsymbol{S})$ which is continuous with respect to the usual sup norm on $C^0_{\mathrm{c}}(\boldsymbol{S})$ and the family of semi-norms on $\mathcal{E}^{(n,n)}_{X\times\boldsymbol{S}/\boldsymbol{S},\mathrm{c}}(W)$ obtained by taking the sup on some compact set of W of the module of partial derivatives up to some order with respect to X. Given a compact set in W, the smallest order in ∂_x which is needed is called the *order* of u.

Currents of maximal degree which are continuous with respect to $\boldsymbol{S}$ are defined similarly. A section on W of $\mathfrak{C}_{X\times\boldsymbol{S}/\boldsymbol{S}}$ is a continuous $C^\infty(\boldsymbol{S})$-linear map $\mathcal{C}^\infty_{X\times\boldsymbol{S},\mathrm{c}}(W)\to C^0_{\mathrm{c}}(\boldsymbol{S})$.

12.2.7. Lemma. *Let $u\in\mathfrak{Db}_{X\times\boldsymbol{S}/\boldsymbol{S}}(U\times V)$. Then, for every $s_o\in V$, its restriction to $s=s_o$ is well-defined as a distribution on U, and similarly for currents.*

Proof. Let $\eta_o\in\mathcal{E}^{(n,n)}_{\mathrm{c}}(U)$ and let χ be a C^∞ function with compact support on V such that $\chi(s_o)=1$. Then $u(\eta_o\cdot\chi)$ is a continuous function on $\boldsymbol{S}$ that we can evaluate at $s=s_o$. The correspondence $\eta_o\mapsto u(\eta_o\cdot\chi)_{|s=s_o}$ obviously defines a distribution on U, because $|u(\eta_o\cdot\chi)_{|s=s_o}|\leqslant\sup_{\boldsymbol{S}}|u(\eta_o\cdot\chi)|$. If $\sigma(s)$ is another such function on $\boldsymbol{S}$ we have, by $C^\infty(\boldsymbol{S})$-linearity,

$$u(\eta_o\cdot\sigma\chi)=\sigma u(\eta_o\cdot\chi)=\chi u(\eta_o\cdot\sigma),$$

hence both take the same value at s_o. $\square$

12.2.b. Pushforward of currents.

12.2.b. Pushforward of currents. Let η be a C^∞ form of maximal degree on X. If $f:X\to Y$ is a proper holomorphic map which is *smooth*, then the integral of η in the fibers of f is a C^∞ form of maximal degree on Y, that one denotes by $\int_f\eta$.

If f is not smooth, then $\int_f\eta$ is only defined as a current of degree 0 on Y, and the definition extends to the case where η is itself a current of degree 0 on X (see Section 8.3.5 for the notion of current).

12.2.8. Remark. The definitions and properties below extend to the case when f is only assumed to be proper on the support of the currents involved, see Exercise 12.4.

***12.2.9. Definition* (Integration of currents of degree (p,q)).** Let $f:X\to Y$ be a proper holomorphic map and let u be a current of degree (p,q) on X. The current $\int_f u$ of degree (p,q) on Y is defined by

$$\Big\langle\int_f u,\eta\Big\rangle=\langle u,\eta\circ f\rangle,\quad\forall\,\eta\in\mathcal{E}^{p,q}_{\mathrm{c}}(Y).\tag{12.2.9 $*$}$$

This definition extends in a straightforward way if f is only assumed to be proper on the support of u.

We continue to assume that f is proper. We will now show how the integration of currents is used to defined a natural $\mathcal{D}_{Y,\overline{Y}}$-linear morphism ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_X\to\mathfrak{C}_Y$. The simpler case of a closed embedding is treated in Exercise 12.2.

The integration of currents is a morphism

$$\int_f\ :\ f_*\,\mathfrak{Db}_{X,p,q}\longrightarrow\mathfrak{Db}_{Y,p,q},$$

which is compatible with the d' and d'' differentials of currents on X and Y. In other words, taking the associated simple complex, it is a morphism of complexes

$$\int_f\ :\ f_*\,\mathfrak{Db}^\bullet_X[2n]\longrightarrow\mathfrak{Db}^\bullet_Y[2m].$$

Let us notice that the integration of currents is compatible with conjugation. Namely, given a current $u_{p,q} \in \Gamma(X, \mathfrak{Db}_X^{n-p,n-q})$, its conjugate $\overline{u_{p,q}} \in \Gamma(X, \mathfrak{Db}_X^{n-q,n-p})$ is defined by the relation

$$\langle \overline{u_{p,q}}, \eta^{q,p} \rangle := \overline{\langle u_{p,q}, \overline{\eta^{q,p}} \rangle}$$

for any test form $\eta^{q,p}$. Then we clearly have

$$\int_f \overline{u_{p,q}} = \overline{\int_f u_{p,q}}. \tag{12.2.10}$$

The notion of pushforward of a $\mathcal{D}_{X,\overline{X}}$-module is modeled on that of a $\mathcal{D}_X$-module in an obvious way (see Section 8.7.e). Since $\mathfrak{C}_X = (\mathfrak{Db}_X)^{\mathrm{right,right}}$ as a right $\mathcal{D}_{X,\overline{X}}$-module, we can apply the $\mathcal{D}_{X,\overline{X}}$-variant of Exercise 8.52(5) (more precisely, (8.52 $*$)) to get, since f is proper,

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_* \, \mathfrak{C}_X \simeq f_*(\mathfrak{Db}_X^{\bullet}[2n] \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}\mathcal{D}_{Y,\overline{Y}}) = f_* \, \mathfrak{Db}_X^{\bullet}[2n] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}. \tag{12.2.11}$$

The integration of currents $\int_f$ induces then a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism of complexes

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_* \, \mathfrak{C}_X \xrightarrow{\int_f^{\bullet}} \mathfrak{Db}_Y^{\bullet}[2m] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}} \simeq \mathfrak{C}_Y, \tag{12.2.12}$$

where we recall that the differential on the complex $\mathfrak{Db}_Y^{\bullet}[2m] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$ uses the universal connection $\nabla^Y + \overline{\nabla^Y}$ on $\mathcal{D}_{Y,\overline{Y}}$, and the isomorphism with $\mathfrak{C}_Y$ is given by (12.2.2).

On the left-hand side of (12.2.12), the term of degree zero reads $f_* \, \mathfrak{C}_X \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$ and, for a current u of degree zero on X and a differential operator P on Y, the morphism (12.2.12) sends $u \otimes P$ to the current $(\int_f u) \cdot P$ of degree zero on Y. It descends to a morphism $\mathcal{H}^0({}_{\mathrm{D},\overline{\mathrm{D}}}f_* \, \mathfrak{C}_X) \to \mathfrak{C}_Y$.

If we start from $\mathfrak{Db}_X$, considered as a left $\mathcal{D}_{X,\overline{X}}$-module, we have similarly by the $\mathcal{D}_{X,\overline{X}}$-variant of Exercise 8.52(5) (more precisely, (8.52 $**$)):

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_* \, \mathfrak{Db}_X \simeq \mathcal{H}om_{\mathcal{O}_{Y,\overline{Y}}}(\omega_{Y,\overline{Y}}, f_* \, \mathfrak{Db}_X^{\bullet}[2n] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}})$$

and the integration of currents induces a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_* \, \mathfrak{Db}_X \xrightarrow{\int_f} \mathcal{H}om_{\mathcal{O}_{Y,\overline{Y}}}(\omega_{Y,\overline{Y}}, \mathfrak{C}_Y) = \mathfrak{Db}_Y \, . \tag{12.2.13}$$

12.2.14. Remark **(Proper support).** One can relax the assumption that f is proper on X. If f is only proper on a closed analytic subset $Z \subset X$, one replaces $\mathfrak{C}_X$ resp. $\mathfrak{Db}_X$ in the previous arguments with the sheaves $\mathfrak{C}_{X,Z}$ resp. $\mathfrak{Db}_{X,Z}$ of currents resp. distributions supported on Z, i.e., vanishing when applied to any test function resp. form with compact support in $X \smallsetminus Z$ (see Exercise 12.4).

More generally, considering the functor ${}_{\mathrm{D},\overline{\mathrm{D}}}f_!$ instead of ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*$, by replacing f_* with $f_!$, enables one to only take into account currents with f-proper support, on which $\int_f$ is defined, so that (12.2.12) and (12.2.13) are well defined on ${}_{\mathrm{D},\overline{\mathrm{D}}}f_! \, \mathfrak{C}_X$ and ${}_{\mathrm{D},\overline{\mathrm{D}}}f_! \, \mathfrak{Db}_X$.

12.2.c. Moderate distributions. We refer to [**Mal66**, Chap. VII] for the results in this subsection.

Let D be a reduced divisor in X, let $\mathcal{O}_X(*D)$ be the sheaf of meromorphic functions on X with poles along D, and let $\mathfrak{Db}_{X,D}$ be the subsheaf of $\mathfrak{Db}_X$ consisting of distributions supported on D.

On the other hand, let $j : X \smallsetminus D \hookrightarrow X$ denote the open inclusion. By definition, there is an exact sequence of left $\mathcal{D}_{X,\overline{X}}$-modules

$$0 \longrightarrow \mathfrak{Db}_{X,D} \longrightarrow \mathfrak{Db}_X \longrightarrow j_* \mathfrak{Db}_{X\smallsetminus D}\,.$$

The image of the latter morphism is the sheaf on X of distributions on $X \smallsetminus D$ which are extendable as distributions on X. It can be characterized as the subsheaf of $j_* \mathfrak{Db}_{X\smallsetminus D}$ consisting of distributions which can be tested along C^∞ forms of maximal degree on $X \smallsetminus D$ having rapid decay along D. It is denoted by $\mathfrak{Db}_X^{\mathrm{mod}\,D}$ (sheaf on X of distributions having moderate growth along D). It can be characterized more algebraically. Indeed, we have

$$\mathfrak{Db}_X^{\mathrm{mod}\,D} = \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} \mathfrak{Db}_X = \mathcal{O}_{\overline{X}}(*\overline{D}) \otimes_{\mathcal{O}_{\overline{X}}} \mathfrak{Db}_X\,.$$

In other words, $\mathfrak{Db}_{X,D}$ is equal to the subsheaf of $\mathfrak{Db}_X$ consisting of local sections annihilated some power of g (or $\overline{f}$), and we have a short exact sequence

$$0 \longrightarrow \mathfrak{Db}_{X,D} \longrightarrow \mathfrak{Db}_X \longrightarrow \mathfrak{Db}_X^{\mathrm{mod}\,D} \longrightarrow 0.$$

The previous results apply to currents of degree 0 as well, and we keep similar notation.

12.2.15. Example **(The case where D is smooth).** If D is smooth, so that we denote it by H, the sheaf $\mathfrak{Db}_{X,H}$ is identified with the push-forward, in the sense of $\mathcal{D}_{X,\overline{X}}$-modules, of $\mathfrak{Db}_H$. If for example $X = H \times \mathbb{C}$, then, according to Exercise 12.2, we find exact sequences

$$\begin{gathered}0 \longrightarrow \iota_* \mathfrak{Db}_H[\partial_t, \partial_{\overline{t}}] \longrightarrow \mathfrak{Db}_X \longrightarrow \mathfrak{Db}_X[1/t] \longrightarrow 0,\\ 0 \longrightarrow \iota_* \mathfrak{C}_H[\partial_t, \partial_{\overline{t}}] \longrightarrow \mathfrak{C}_X \longrightarrow \mathfrak{C}_X[1/t] \longrightarrow 0.\end{gathered}$$

12.3. Sesquilinear pairings between $\mathcal{D}_X$-modules

The naive conjugation functor $\mathcal{M} \mapsto \overline{\mathcal{M}}$ transforms $\mathcal{O}_X$-modules (resp. $\mathcal{D}_X$-modules) into $\mathcal{O}_{\overline{X}}$-modules (resp. $\mathcal{D}_{\overline{X}}$-modules): let us regard $\mathcal{O}_{\overline{X}}$ as an $\mathcal{O}_X$-module by setting $f \cdot \overline{g} := \overline{fg}$, and similarly let us regard $\mathcal{D}_{\overline{X}}$ as a $\mathcal{D}_X$-module; for an $\mathcal{O}_X$-module (resp. a $\mathcal{D}_X$-module) $\mathcal{M}$ we then define $\overline{\mathcal{M}}$ as $\mathcal{O}_{\overline{X}} \otimes_{\mathcal{O}_X} \mathcal{M}$ (resp. $\mathcal{D}_{\overline{X}} \otimes_{\mathcal{D}_X} \mathcal{M}$). In other words, for a local section m of $\mathcal{M}$, we denote by $\overline{m}$ the same local section, that we act on by $\overline{f} \in \mathcal{O}_{\overline{X}}$ (resp. $\mathcal{D}_{\overline{X}}$) with the formula $\overline{f} \cdot \overline{m} := \overline{fm}$.

12.3.1. Definition **(Left sesquilinear pairing).**

(1) A *sesquilinear pairing* $\mathfrak{s}$ between left $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ is a $\mathcal{D}_{X,\overline{X}}$-linear morphism $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$. When $\mathcal{M}' = \mathcal{M}'' = \mathcal{M}$, we speak of a sesquilinear pairing on $\mathcal{M}$.

(2) The *Hermitian adjoint* of a *left* sesquilinear pairing $\mathfrak{s}$ is $\mathfrak{s}^* : \mathcal{M}'' \otimes_{\mathbb{C}} \overline{\mathcal{M}'} \to \mathfrak{Db}_X$ defined by $\mathfrak{s}^*(m'', \overline{m'}) = \overline{\mathfrak{s}(m', \overline{m''})}$, that is,

$$\langle \eta, \mathfrak{s}^*(m'', \overline{m'})\rangle := \overline{\langle \overline{\eta}, \mathfrak{s}(m', \overline{m''})\rangle}$$

for any test form of maximal degree η (see (8.3.0 $*$)).

12.3.2. *Definition* (Right sesquilinear pairing).

(1) A *sesquilinear pairing* $\mathfrak{s}$ between right $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ is a $\mathcal{D}_{X,\overline{X}}$-linear morphism $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$. When $\mathcal{M}' = \mathcal{M}'' = \mathcal{M}$, we speak of a sesquilinear pairing on $\mathcal{M}$.

(2) The *Hermitian adjoint* of a *right* sesquilinear pairing $\mathfrak{s}$ is $\mathfrak{s}^* : \mathcal{M}'' \otimes_{\mathbb{C}} \overline{\mathcal{M}'} \to \mathfrak{C}_X$ defined by $\mathfrak{s}^*(m'', \overline{m'}) = \overline{\mathfrak{s}(m', \overline{m''})}$, that is,

$$\langle \mathfrak{s}^*(m'', \overline{m'}), \eta\rangle := \overline{\langle \mathfrak{s}(m', \overline{m''}), \overline{\eta}\rangle}$$

for any test function η (see (8.3.0 $*$)).

12.3.3. Side-changing. If $\mathfrak{s} = \mathfrak{s}^{\mathrm{left}} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ is a sesquilinear pairing between left $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$, then it determines in a canonical way a sesquilinear pairing (recall $\mathrm{Sgn}(n) := \varepsilon(n+1)/(2\pi \mathrm{i})^n$, see Notation (0.2 $*$))

$$(12.3.3\,*)\qquad \begin{aligned} (\omega_X \otimes \mathcal{M}') \otimes_{\mathbb{C}} (\overline{\omega_X \otimes \mathcal{M}''}) &\xrightarrow{\mathfrak{s}^{\mathrm{right}}} \omega_X \otimes \overline{\omega_X} \otimes \mathfrak{Db}_X = \mathfrak{C}_X \\ (\omega' \otimes m', \overline{\omega'' \otimes m''}) &\longmapsto \mathrm{Sgn}(n)(\omega' \wedge \overline{\omega''}) \otimes \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}). \end{aligned}$$

Conversely, from a sesquilinear pairing between right $\mathcal{D}_X$-modules one recovers one for left $\mathcal{D}_X$-modules.

The compatibility with Hermitian adjunction is given by the following relation:

$$(12.3.3\,**)\qquad (\mathfrak{s}^{\mathrm{right}})^* = (\mathfrak{s}^{\mathrm{left}*})^{\mathrm{right}},$$

since $\overline{\mathrm{Sgn}(n)(\omega'' \wedge \overline{\omega'})} = \mathrm{Sgn}(n)(\omega' \wedge \overline{\omega''})$. In both left and right cases we have $\mathfrak{s}^{**} = \mathfrak{s}$.

12.3.4. Extension to C^∞ coefficients. Let us consider the right case for example. Let us define a right action of $\mathcal{D}_{X,\overline{X}}$ on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X$ by setting

$$(m \otimes \eta) \cdot \partial_{x_i} = m\partial_{x_i} \otimes \eta - m \otimes \partial\eta/\partial x_i \quad \text{and} \quad (m \otimes \eta) \cdot \partial_{\overline{x}_i} = -m \otimes \partial\eta/\partial\overline{x}_i.$$

Then $\mathfrak{s}$ extends in a unique way as a $\mathcal{C}^\infty_X$-linear morphism

$$(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X) \otimes_{\mathcal{C}^\infty_X} \overline{(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X)} \longrightarrow \mathfrak{C}_X$$

which satisfies, for any local section ξ of Θ_X or $\overline{\Theta_X}$,

$$\mathfrak{s}(\mu', \overline{\mu''})\xi = \mathfrak{s}(\mu'\xi, \overline{\mu''}) + \mathfrak{s}(\mu', \overline{\mu''\xi}),$$

by setting

$$\mathfrak{s}(m' \otimes \eta', \overline{m'' \otimes \eta''}) := \mathfrak{s}(m', \overline{m''})\eta'\overline{\eta''}.$$

Conversely, given such a pairing, one recovers the original $\mathfrak{s}$ by restricting to $\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''}$.

12.3.5. Example.

(1) Assume $\mathcal{M}'=\mathcal{M}''=\mathcal{O}_X$ with $\dim X=n$. We have a standard sesquilinear pairing $\mathfrak{s}_n=\mathfrak{s}_n^{\mathrm{left}}$ defined by

$$\boxed{\mathfrak{s}_n^{\mathrm{left}}(1,1)=1.} \tag{12.3.5$\,*$}$$

(2) If $\mathcal{M}'=\mathcal{M}''=\omega_X$, then $\mathfrak{s}_n=\mathfrak{s}_n^{\mathrm{right}}$ is defined by

$$\boxed{\mathfrak{s}_n^{\mathrm{right}}(\omega',\overline{\omega''})=\mathrm{Sgn}(n)(\omega'\wedge\overline{\omega''}),} \tag{12.3.5$\,**$}$$

in such a way that $\mathfrak{s}_n^{\mathrm{right}}=(\mathfrak{s}_n^{\mathrm{left}})^{\mathrm{right}}$.

Let us notice the following.

12.3.6. Lemma. *If $\mathcal{M}'$ and $\mathcal{M}''$ are $\mathcal{O}_X$-coherent (hence $\mathcal{O}_X$-locally free of finite rank), the pairing $\mathfrak{s}$ takes values in C^∞ functions (resp. forms of maximal degree).*

Proof. We know (see Example 8.3.2) that $\mathcal{M}',\mathcal{M}''$ are $\mathcal{O}_X$-generated by their flat local sections. For such local sections m',m'', the distribution (resp. current) $\mathfrak{s}(m',\overline{m''})$ is annihilated by d' and d'', hence is locally a constant. It follows that, for any local sections m',m'', $\mathfrak{s}(m',\overline{m''})$ is real-analytic, so in particular C^∞. More precisely, for local horizontal sections μ',μ'' and holomorphic functions h',h'', we can write

$$\mathfrak{s}(\mu'\otimes h',\overline{\mu''\otimes h''})=h'\overline{h''}\cdot\mathfrak{s}^\nabla(\mu',\overline{\mu''}), \tag{12.3.7}$$

where $\mathfrak{s}^\nabla:\mathcal{M}'^\nabla\otimes\overline{\mathcal{M}''^\nabla}\to\mathbb{C}$ is the sesquilinear pairing induced by $\mathfrak{s}$ on the underlying local systems. In other words, with respect to the above identification, we have $\mathfrak{s}=\mathfrak{s}^\nabla\cdot\mathfrak{s}_n$. ☐

12.3.8. Proposition **(Uniqueness across a non-characteristic divisor)**

Let $\mathcal{M}',\mathcal{M}''$ be coherent $\mathcal{D}_X$-modules and let H be a hypersurface which is non-characteristic for them. If two sesquilinear pairings $\mathfrak{s}_1,\mathfrak{s}_2:\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''}\to\mathfrak{C}_X$ (or $\mathfrak{Db}_X$) coincide when restricted to the open set $X\smallsetminus H$, then they coincide.

Proof. We will treat the case of right $\mathcal{D}_X$-modules. The question is local, so we can assume that $X=H\times\Delta_t$ and we can shrink Δ_t if needed. Set $\mathfrak{s}=\mathfrak{s}_1-\mathfrak{s}_2$ and let m',m'' be local sections of $\mathcal{M}',\mathcal{M}''$ defined on some neighbourhood $\mathrm{nb}(x_o)=\mathrm{nb}_H\times\Delta_t$ of $x_o\in H\times\{0\}$. Let $\eta\in C_{\mathrm{c}}^\infty(\mathrm{nb}(x_o))$, and let p be the order of $\mathfrak{s}(m',\overline{m''})$ on the compact set $\mathrm{Supp}\,\eta$. We aim at proving that $\langle\mathfrak{s}(m',\overline{m''}),\eta\rangle=0$.

We consider the current $\mathfrak{s}(m',\overline{m''})_\eta$ on Δ_t defined by

$$\chi\longmapsto\langle\mathfrak{s}(m',\overline{m''})_\eta,\chi\rangle:=\langle\mathfrak{s}(m',\overline{m''}),\chi\cdot\eta\rangle\quad\text{for }\chi\in C_{\mathrm{c}}^\infty(\Delta_t).$$

It is enough to prove that $\mathfrak{s}(m',\overline{m''})_\eta=0$ (by choosing $\chi\equiv1$ on the projection to Δ_t of $\mathrm{Supp}\,\eta$). This current has order $\leqslant p$ and is supported at the origin, hence can be written in a unique way, by using the Dirac current δ_0 at the origin, as

$$\mathfrak{s}(m',\overline{m''})_\eta=\sum_{0\leqslant a+b\leqslant p}c_{a,b}(\eta)\delta_0\partial_t^a\partial_{\overline{t}}^b,\quad c_{a,b}(\eta)\in\mathbb{C}.$$

We will prove that all the coefficients $c_{a,b}(\eta)$ vanish. This is obvious if $\eta = t^q\bar{t}^r\eta_{q,r}$ with $q + r > p$ for some C^∞ function $\eta_{q,r}$. We can thus assume that $\eta = \sum_{p+q\leqslant p} t^q\bar{t}^r\eta_{q,r}$, where $\eta_{q,r}$ is a test function H, and we are finally reduced to treating the case where η is equal to such an $\eta_{q,r}$, i.e., does not depend on $t, \bar{t}$.

We claim that there exists N large enough such that m' satisfies an equation of the form

$$m' \cdot b(t\partial_t) := m' \cdot \prod_{k=1}^{N}(t\partial_t + k) = m' \cdot t^{p+1}\sum_j P_j(t, x, \partial_x)(t\partial_t)^j,$$

where x is a local coordinate system on H. Indeed, H is also non-characteristic for the coherent sub-module $m' \cdot \mathcal{D}_X$, and the filtration $m' \cdot V_k\mathcal{D}_X$ is comparable with the V-filtration $V_\bullet(m'\cdot\mathcal{D}_X)$, so there exists N such that $V_{-N-1}(m'\cdot\mathcal{D}_X) \subset m'\cdot V_{-(p+1)}\mathcal{D}_X$. Since $m'\partial_t^N \in (m' \cdot \mathcal{D}_X) = V_{-1}(m' \cdot \mathcal{D}_X)$, we have $m'\partial_t^N t^N \in V_{-N-1}(m' \cdot \mathcal{D}_X)$, hence the assertion.

It follows that $\mathfrak{s}(m', \overline{m''})_\eta \cdot b(t\partial_t) = 0$. Since $\delta_0\partial_t^a\partial_{\bar{t}}^b \cdot (t\partial_t + k) = (a + k)\delta_0\partial_t^a\partial_{\bar{t}}^b$, we conclude that for every a, b, we have $c_{a,b}(\eta) \cdot \prod_{k=1}^N(a + k) = 0$, so $c_{a,b}(\eta) = 0$. □

We also have an analogue of Corollary 9.7.16 for sesquilinear pairings.

12.3.9. Proposition. *Let $\mathcal{M}', \mathcal{M}''$ be two holonomic $\mathcal{D}_X$-modules which are S-decomposable and let $(S_i)_{i\in I}$ be the family of their pure components. Then any sesquilinear pairing $\mathfrak{s} : \mathcal{M}'_{S_i} \otimes_{\mathbb{C}} \overline{\mathcal{M}''_{S_j}} \to \mathfrak{C}_X$ resp. $\mathfrak{Db}_X$ vanishes identically if $S_i \neq S_j$.*

This is reminiscent of Example 7.3.9(1). We will first prove a similar result related to the S-decomposition along a function.

12.3.10. Lemma. *Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\mathcal{M}', \mathcal{M}''$ be two coherent $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g). Assume that one of them, say $\mathcal{M}'$, is a middle extension along (g), and the other one, say $\mathcal{M}''$, is supported on $g^{-1}(0)$. Then any sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_X$ vanishes identically.*

Proof. By Kashiwara's equivalence (Proposition 12.4.7 below and Exercise 12.8), we can assume that g is the projection $X_0 \times \mathbb{C} \to \mathbb{C}$, and we choose a coordinate t on $\mathbb{C}$. We work locally near $x_o \in X_0$. Consider $\mathfrak{s}$ as a morphism $\mathcal{M}' \to \mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\overline{\mathcal{M}''}, \mathfrak{C}_X)$. Fix local $\mathcal{D}_X$-generators $m''_1, \dots, m''_\ell$ of $\mathcal{M}''_{x_o}$. By Kashiwara's equivalence 9.6.1, there exists $q \geqslant 0$ such that $m''_k t^q = 0$ for all $k = 1, \dots, \ell$. Let $m' \in \mathcal{M}'_{x_o}$ and let p be the maximum of the orders of the currents $\mathfrak{s}(m')(\overline{m''_k})$ on some neighbourhood of x_o. As $t^{p+1+q}/\bar{t}^q$ is C^p, we have, for every $k = 1, \dots, \ell$,

$$\mathfrak{s}(m')(\overline{m''_k})t^{p+1+q} = \mathfrak{s}(m')(\overline{m''_k})\bar{t}^q \cdot \frac{t^{p+1+q}}{\bar{t}^q} = \mathfrak{s}(m')(\overline{m''_k t^q}) \cdot \frac{t^{p+1+q}}{\bar{t}^q} = 0,$$

hence $\mathfrak{s}(m')t^{p+1+q} \equiv 0$. Applying this to generators of $\mathcal{M}'_{x_o}$ shows that all local sections of $\mathfrak{s}(\mathcal{M}'_{x_o})$ are killed by some power of t.

As $\mathcal{M}'$ is a middle extension along (t), we know from Proposition 9.7.2(2) that $V_{<0}\mathcal{M}'_{x_o}$ generates $\mathcal{M}'_{x_o}$ over $\mathcal{D}_X$. It is therefore enough to show that $\mathfrak{s}(V_{<0}\mathcal{M}'_{x_o}) = 0$. Let us fix a finite set of $V_0\mathcal{D}_X$-generators m'_i of $V_{<0}\mathcal{M}'_{x_o}$, and let N be such that

$\mathfrak{s}(m'_i)t^N = 0$ for all i. Since $t\partial_t \cdot t^N = t^N(t\partial_t + N)$, we conclude that t^N annihilates $\mathfrak{s}(V_<0\mathcal{M}'_{x_o})$. It follows that $\mathfrak{s}(V_{<-N}\mathcal{M}'_{x_o}) = 0$, since $V_{<-N}\mathcal{M}'_{x_o} = V_{<0}\mathcal{M}'_{x_o}t^N$.

Let now $\alpha < 0$ be such that $\mathfrak{s}(V_{<\alpha}\mathcal{M}'_{x_o}) = 0$, and let m' be a section of $V_\alpha\mathcal{M}'_{x_o}$; there exists $\nu_\alpha \geqslant 0$ such that, setting $b(s) = (s-\alpha)^{\nu_\alpha}$, we have $m'b(t\partial_t) \in V_{<\alpha}\mathcal{M}'_{x_o}$, hence $\mathfrak{s}(m')b(t\partial_t) = 0$; on the other hand, we have seen that there exists N such that $\mathfrak{s}(m')t^N = 0$, hence, putting $B(s) = \prod_{\ell=0}^{N-1}(s-\ell)$, it also satisfies $\mathfrak{s}(m')B(t\partial_t) = 0$; notice now that $b(s)$ and $B(s)$ have no common root, so $\mathfrak{s}(m') = 0$. □

***Proof of Proposition* 12.3.9.** The assertion is local on X, so we fix $x_o \in X$ and we work with germs at x_o. Assume for example that S_i is not contained in S_j and consider a germ g of analytic function, such that $g \equiv 0$ on S_j and $g \not\equiv 0$ on S_i. Then we can apply Lemma 12.3.10 to $\mathcal{M}'_{S_i}$ and $\mathcal{M}''_{S_j}$. □

12.4. Pushforward of sesquilinear pairings

We have already seen various occurrences of the pushforward of a sesquilinear pairing (Formulas (2.4.8), (4.2.17) and Definition 7.3.22). In this section, we take up the question in a general setting. The first observation is that considering sequilinear pairings between right $\mathcal{D}_X$-modules makes formulas simpler: this is explained with details in Section A.5. One reason is that the pushforward functor is simpler when expressed in the right setting. There is nevertheless a sign $\varepsilon(k)$ that has to be introduced in front of the natural formula for the kth pushforward of a right sesquilinear pairing. Various reasons lead to this sign. One is to obtain a definition which commutes with taking the Hermitian adjoint. Also, such a sign allows for compatibility with the Lefschetz morphism (12.4.13 ∗), in a way analogous to 2.4.13. Another one is of a cohomological nature and is explained in Section 12.7.d. Lastly, a detailed analysis for the sign is developed in Section A.5. In the left setting, sign changing as defined in 12.3.3 leads to formulas obtained in (2.4.8), (4.2.17) and Definition 7.3.22.

12.4.a. General definition.

Let $\mathcal{M}', \mathcal{M}''$ be coherent $\mathcal{D}_X$-modules and let $f\colon X \to Y$ be a holomorphic map which is *proper* when restricted to $S := \operatorname{Supp}\mathcal{M}' \cap \operatorname{Supp}\mathcal{M}''$. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$ or $\mathfrak{Db}_X$ be a sesquilinear pairing. Note that it takes values in currents or distributions supported on Z. Our aim is to define, for every $k \in \mathbb{Z}$, sesquilinear pairings:

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s} : ({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}') \otimes_{\mathbb{C}} \overline{({}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}'')} \longrightarrow \mathfrak{C}_Y \quad \text{resp. } \mathfrak{Db}_Y . $$

Of course, the sesquilinear pairing for left $\mathcal{D}_X$-modules is expected to be obtained from the one for right $\mathcal{D}_X$-modules by side-changing at the source and the target and conversely. We call ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}$ the *k-th pushforward* of the sesquilinear pairing $\mathfrak{s}$.

It is easier to start with right $\mathcal{D}_X$-modules. So, let $\mathcal{M}', \mathcal{M}''$ be coherent right $\mathcal{D}_X$-modules and let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_{X,S}$ be a sesquilinear pairing between them. We set (see Section 8.7.e)

$$ \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}} := \mathrm{Sp}_{X\to Y}(\mathcal{D}_X) \otimes_{\mathbb{C}} \overline{\mathrm{Sp}_{X\to Y}(\mathcal{D}_X)}, \tag{12.4.1} $$

which is a complex of left $\mathcal{D}_{X,\overline{X}}$-modules. Therefore, $\mathfrak{s}$ yields a morphism of complexes

$$\begin{aligned}\big(\mathcal{M}'\otimes_{\mathcal{D}_X}\mathrm{Sp}_{X\to Y}(\mathcal{D}_X)\big)\otimes_{\mathbb{C}}\overline{\big(\mathcal{M}''\otimes_{\mathcal{D}_X}\mathrm{Sp}_{X\to Y}(\mathcal{D}_X)\big)}\simeq(\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''})\otimes_{\mathcal{D}_{X,\overline{X}}}\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}\\ \longrightarrow\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}\,.\end{aligned}$$

By applying $\boldsymbol{R}f_*$, we thus obtained a morphism in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{Y,\overline{Y}})$:

$$\begin{aligned}\boldsymbol{R}f_*\big(\mathcal{M}'\otimes_{\mathcal{D}_X}\mathrm{Sp}_{X\to Y}(\mathcal{D}_X)\big)\otimes_{\mathbb{C}}\boldsymbol{R}f_*\overline{\big(\mathcal{M}''\otimes_{\mathcal{D}_X}\mathrm{Sp}_{X\to Y}(\mathcal{D}_X)\big)}\\ \longrightarrow\boldsymbol{R}f_*\Big((\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''})\otimes_{\mathcal{D}_{X,\overline{X}}}\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}\Big)\\ \longrightarrow\boldsymbol{R}f_*(\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}})={}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S}\xrightarrow{\int_f}\mathfrak{C}_Y\end{aligned}$$

(see Exercise 12.1 and (12.2.12)), and thus, for each $k\in\mathbb{Z}$, a morphism

$$(12.4.2)\qquad {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}:({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}')\otimes_{\mathbb{C}}\overline{{}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}''}\longrightarrow{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\,\mathfrak{C}_{X,S}\xrightarrow{\int_f}\mathfrak{C}_Y\,.$$

12.4.3. Definition.

(1) The sesquilinear pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}$ is the k-th pushforward of $\mathfrak{s}$. The k-th pushforward of $\mathfrak{s}$ in the left setting is obtained by side-changing at the source and target of f.

(2) The *signed right k-th pushforward* of $\mathfrak{s}$ is defined as (see Notation 0.2)

$$\boxed{{}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}:=\varepsilon(k)\,{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}}$$

and the signed left k-th pushforward of $\mathfrak{s}$ is obtained from the latter by side-changing at the source and target of f.

12.4.4. Pushforward and Hermitian adjunction. For sections ξ'_k,ξ''_ℓ of $\Theta_{X,k}$ and $\Theta_{X,\ell}$ respectively, we have the relation $\overline{\xi''_\ell\wedge\overline{\xi'_k}}=(-1)^{k\ell}\xi'_k\wedge\overline{\xi''_\ell}$. It follows that

$$({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(-k,k)}\mathfrak{s})^*=(-1)^k{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}(\mathfrak{s}^*).$$

The relation $\varepsilon(k)=(-1)^k\,\varepsilon(-k)$ enables us to absorb the sign, so it yields

$$(12.4.4\,*)\qquad ({}_{\mathrm{T}}f_*^{(-k,k)}\mathfrak{s})^*={}_{\mathrm{T}}f_*^{(k,-k)}(\mathfrak{s}^*).$$

12.4.5. C^∞ Computation of the pushforward of a sesquilinear pairing. Let us use the notation of Exercise 8.51, in particular $\mathrm{Sp}_X^{\infty,k}=\bigoplus_\ell\Theta_{X,\ell}\otimes\mathcal{E}_X^{(0,k+\ell)}$. We set

$$K'^{\bullet}=f_!(\mathcal{M}'\otimes_{\mathcal{O}_X}\mathrm{Sp}_X^{\infty,\bullet}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)\simeq f_!(\mathcal{M}'\otimes_{\mathcal{O}_X}\mathrm{Sp}_X^{\infty,\bullet})\otimes_{\mathcal{O}_Y}\mathcal{D}_Y$$

and similarly for $K''^{\bullet}$, both equipped with the differentials $(d',d'')=(f_!\delta^\infty_{\mathcal{M}',Y},f_!\delta^\infty_{\mathcal{M}'',Y})$ obtained by applying $f_!$ to $\delta^\infty_{\mathcal{M},Y}=\delta'^\infty_{\mathcal{M},Y}+\mathrm{d}''$ defined in Exercise 8.28. Furthermore, we can replace $f_!$ with f_* if f is assumed to be proper on $S=\operatorname{Supp}\mathcal{M}'\cup\operatorname{Supp}\mathcal{M}''$.

The sesquilinear pairing $\mathfrak{s}$ enables us to define termwise a pairing

$$\mathrm{Sp}^\infty(\mathfrak{s}):(\mathcal{M}'\otimes_{\mathcal{O}_X}\mathrm{Sp}_X^{\infty,k})\otimes_{\mathbb{C}}\overline{(\mathcal{M}''\otimes_{\mathcal{O}_X}\mathrm{Sp}_X^{\infty,-k})}\longrightarrow\mathfrak{C}_{X,S}$$

defined on local sections as

$$\begin{aligned}(m'\otimes\xi'_\ell\otimes\eta'_{k+\ell})\otimes\overline{(m''\otimes\xi''_{k+j}\otimes\eta''_j)}&\\ \longmapsto\begin{cases}0 & \text{if } \ell\neq j,\\ (-1)^\ell\mathfrak{s}(m',\overline{m''})\cdot\overline{\eta''_\ell}(\xi'_\ell)\cdot\eta'_{k+\ell}(\overline{\xi''_{k+\ell}}), & \text{if } \ell=j.\end{cases}\end{aligned}$$

We extend in a natural way this pairing as an $f^{-1}\mathcal{D}_{Y,\overline{Y}}$-sesquilinear pairing

$$\begin{aligned}\mathrm{Sp}_Y^\infty(\mathfrak{s}):(\mathcal{M}'\otimes_{\mathcal{O}_X}\mathrm{Sp}_X^{\infty,k}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)\otimes_{\mathbb{C}}\overline{(\mathcal{M}''\otimes_{\mathcal{O}_X}\mathrm{Sp}_X^{\infty,-k}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)}&\\ \longrightarrow\mathfrak{C}_{X,S}\otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}}f^{-1}\mathcal{D}_{Y,\overline{Y}}.&\end{aligned}$$

One checks that this defines a morphism of complexes. Applying $f_!$ (or f_*), yields a $\mathcal{D}_{Y,\overline{Y}}$-sesquilinear pairing

$$f_*\,\mathrm{Sp}_Y^\infty(\mathfrak{s}):K'^\bullet\otimes_{\mathbb{C}}\overline{K''^\bullet}\longrightarrow f_*\,\mathfrak{C}_{X,S}\otimes_{\mathcal{O}_{Y,\overline{Y}}}\mathcal{D}_{Y,\overline{Y}}.$$

Integrating along f we finally obtain

$$(12.4.5\,*)\qquad {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\mathfrak{s}=\textstyle\int f_*\,\mathrm{Sp}_Y^\infty(\mathfrak{s}):K'^\bullet\otimes_{\mathbb{C}}\overline{K''^\bullet}\longrightarrow\mathfrak{C}_Y\otimes_{\mathcal{O}_{Y,\overline{Y}}}\mathcal{D}_{Y,\overline{Y}}\longrightarrow\mathfrak{C}_Y,$$

where the second morphism is the right action of $\mathcal{D}_{Y,\overline{Y}}$ on $\mathfrak{C}_Y$.

12.4.b. Pushforward of a sesquilinear pairing by a closed embedding

Assume that $\iota:X\hookrightarrow Y$ is a closed immersion and let $\mathcal{M}=\mathcal{M}',\mathcal{M}''$ be *right* $\mathcal{D}_X$-modules. We then have ${}_{\mathrm{D}}\iota_*\mathcal{M}={}_{\mathrm{D}}\iota_*\mathcal{M}=\iota_*(\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})$. Let $\mathbf{1}$ denote the canonical section of $\mathcal{D}_{X\to Y}=\mathcal{O}_X\otimes_{\iota^{-1}\mathcal{O}_Y}\iota^{-1}\mathcal{D}_Y$. It is a generator of $\mathcal{D}_{X\to Y}$ as a right $\iota^{-1}\mathcal{D}_Y$-module. Any sesquilinear pairing $\mathfrak{s}_Y:{}_{\mathrm{D}}\iota_*\mathcal{M}'\otimes\overline{{}_{\mathrm{D}}\iota_*\mathcal{M}''}\to\mathfrak{C}_Y$ takes values in $\mathfrak{C}_{Y,X}$ (i.e., has support in X) and, by $\mathcal{D}_{Y,\overline{Y}}$-linearity, is determined by its restriction to $\iota_*(\mathcal{M}'\otimes\mathbf{1})\otimes\overline{\iota_*(\mathcal{M}''\otimes\mathbf{1})}$. Hence, for local sections m',m'' of $\mathcal{M}',\mathcal{M}''$, $\mathfrak{s}_Y(m'\otimes\mathbf{1},\overline{m''\otimes\mathbf{1}})$ must be the pushforward of some current on X. Conversely, given $\mathfrak{s}_X$, we *define* the sesquilinear pairing ${}_{\mathrm{T}}\iota_*\mathfrak{s}_X={}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_X^{(0,0)}$ in such a way that(1)

$$(12.4.6)\qquad {}_{\mathrm{T}}\iota_*\mathfrak{s}_X(m'\otimes\mathbf{1},\overline{m''\otimes\mathbf{1}})=\int_\iota\mathfrak{s}_X(m',\overline{m''}),$$

that is, for any test function η on Y,

$$\big\langle({}_{\mathrm{T}}\iota_*\mathfrak{s}_X)(m'\otimes\mathbf{1},\overline{m''\otimes\mathbf{1}}),\eta\big\rangle=\big\langle\mathfrak{s}_X(m',\overline{m''}),\eta_{|X}\big\rangle,$$

and we extend it by $\mathcal{D}_{Y,\overline{Y}}$-linearity.

12.4.7. Proposition **(Kashiwara's equivalence for sesquilinear pairings)**

Let $Z\xhookrightarrow{\iota}X$ be the inclusion of a closed submanifold and let $\mathcal{M}',\mathcal{M}''$ be coherent $\mathcal{D}_Z$-modules. There is a one-to-one correspondence between sesquilinear pairings $\mathfrak{s}_Z:\mathcal{M}'\otimes\overline{\mathcal{M}''}\to\mathfrak{C}_Z$ and sesquilinear pairings $\mathfrak{s}:{}_{\mathrm{D}}\iota_*\mathcal{M}'\otimes\overline{{}_{\mathrm{D}}\iota_*\mathcal{M}''}\to\mathfrak{C}_X$. In one

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(1) Since ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_X^{(k,-k)}=0$ for $k\neq0$ and $\varepsilon(0)=1$, we do not distinguish between ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_X^{(0,0)}$ and ${}_{\mathrm{T}}\iota_*\mathfrak{s}_X^{(0,0)}$.

direction, $\mathfrak{s} = {}_{\mathrm{T}}\iota_*\mathfrak{s}_Z$*. In the other direction,* $\mathfrak{s}_Z$ *is the pairing defined from* $\mathfrak{s}$ *by the formula*

$$\langle \mathfrak{s}_Z(m',\overline{m''}),\eta_Z\rangle = \langle \mathfrak{s}((m'\otimes\mathbf{1}),\overline{(m''\otimes\mathbf{1})}),\eta\rangle$$

for any test function η *on* X *such that* $\eta_{|Z}=\eta_Z$*.*

Proof. See Exercise 12.8. ☐

According to (12.2.10), the behaviour by Hermitian adjunction is expressed by

$$({}_{\mathrm{T}}\iota_*\mathfrak{s}_X)^* = {}_{\mathrm{T}}\iota_*(\mathfrak{s}_X^*).$$

The pushforward for a left sesquilinear pairing is defined by side-changing:

$${}_{\mathrm{T}}\iota_*(\mathfrak{s}_X^{\mathrm{left}}) := ({}_{\mathrm{T}}\iota_*\mathfrak{s}_X^{\mathrm{right}})^{\mathrm{left}}.$$

12.4.8. Example **(Pushforward of a left sesquilinear pairing by a closed embedding)**

Let us denote by $\iota_g : X \hookrightarrow X\times\mathbb{C}$ the graph embedding attached to a holomorphic function $g : X\to\mathbb{C}$. Let $\mathcal{M}',\mathcal{M}''$ be left $\mathcal{D}_X$-modules and let $\mathfrak{s}$ be a sesquilinear pairing between them. Let us identify $\mathcal{M}_g$ with $\iota_{g*}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\partial_t]\otimes \mathrm{d}t^\vee$ (see Example 8.7.7(2)). Let $m'\in\mathcal{M}'_{x_o}$ and $m''\in\mathcal{M}''_{x_o}$ be local sections and let η be a test form of maximal degree on a neighbourhood of $(x_o,0)$ in $X\times\mathbb{C}$ where m',m'' are defined. Set

$$\eta = \eta_1\wedge \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}(t-g)\wedge \mathrm{d}\overline{(t-g)}),$$

where η_1 is a relative form of degree (n,n), and set $\eta_o = \iota_g^*\eta_1$. Then we have

$$\boxed{\big\langle \eta, ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(m'\otimes 1,\overline{m''\otimes 1})\big\rangle = \big\langle \eta_o, \mathfrak{s}(m',\overline{m''})\big\rangle}$$

Indeed, let us write $\eta_1 = \varphi\mathrm{d}x\wedge\mathrm{d}\overline{x}$ in local coordinates, so that $\eta_o=\varphi_o\mathrm{d}x\wedge\mathrm{d}\overline{x}$, with $\varphi_o=\varphi_{|X\times\{0\}}$. Then, identifying m' with $m'\otimes 1$ in $\iota_{g*}\mathcal{M}'[\partial_t]$ and similarly with m'', it yields

$$\begin{aligned}
&\langle \eta, ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(m'\otimes\mathrm{d}t^\vee,\overline{m''\otimes\mathrm{d}t^\vee})\rangle\\
&\qquad= (-1)^n\,\tfrac{\mathrm{i}}{2\pi}\langle\varphi(\mathrm{d}x\otimes\mathrm{d}t)\wedge(\overline{\mathrm{d}x\otimes\mathrm{d}t}), ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(m'\otimes\mathrm{d}t^\vee,\overline{m''\otimes\mathrm{d}t^\vee})\rangle\\
&\qquad= (-1)^n\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{Sgn}(n+1)^{-1}\\
&\qquad\qquad\cdot\langle({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}^{\mathrm{right}})\big((\mathrm{d}x\otimes\mathrm{d}t)\otimes(m'\otimes\mathrm{d}t^\vee),\overline{(\mathrm{d}x\otimes\mathrm{d}t)\otimes(m''\otimes\mathrm{d}t^\vee)}\big),\varphi)\rangle\\
&\qquad= (-1)^n\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{Sgn}(n+1)^{-1}\langle\mathfrak{s}^{\mathrm{right}}(\mathrm{d}x\otimes m',\overline{\mathrm{d}x\otimes m''}),\varphi_o\rangle\\
&\qquad= (-1)^n\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{Sgn}(n+1)^{-1}\,\mathrm{Sgn}(n)\langle\eta_o,\mathfrak{s}(m',\overline{m''})\rangle.
\end{aligned}$$

The conclusion follows from the identity $(-1)^n\,\frac{\mathrm{i}}{2\pi}\,\mathrm{Sgn}(n+1)^{-1}\,\mathrm{Sgn}(n)=1$ (see Notation 0.2).

12.4.c. Pushforward of a sesquilinear pairing with differential forms

Let us now return to the general case of a map $f : X\to Y$ which is proper on $S=\operatorname{Supp}\mathcal{M}'\cup\operatorname{Supp}\mathcal{M}''$. We will use the formulas of Exercise 8.52 (i.e., (8.52 $*$) and (8.52 $**$)) for computing the direct image, as they happen to be more convenient at some place. Note that we already used them when expressing the integration morphism (12.2.12).

Starting from a *left* sesquilinear pairing, we aim at giving the formula for its direct images as *right* sesquilinear pairings, that is defined by side-changing at the source from ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{right}}$, and that we denote by $({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$. Similarly, $({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$ is defined by side-changing at the source from ${}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{right}}$, and in Proposition 12.4.12 we make precise the sign in the formula for $({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$.

Starting from a left $\mathcal{D}_X$-module $\mathcal{M}^{\mathrm{left}}$, let us consider the complex of right $\mathcal{D}_Y$-modules (we only indicate the shift of the complex, the sign change in the differential is understood)

$$K^\bullet := \boldsymbol{R}f_*\Omega_X^{n+\bullet}(\mathcal{M}^{\mathrm{left}}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y).$$

If $\mathcal{M}^{\mathrm{right}}$ is the right $\mathcal{D}_X$-module associated with $\mathcal{M}^{\mathrm{left}}$, we thus have

$${}_{\mathrm{D}}f_*(\mathcal{M}^{\mathrm{right}})\simeq K^\bullet \quad\text{and}\quad {}_{\mathrm{D}}f_*(\mathcal{M}^{\mathrm{left}})\simeq (K^\bullet)^{\mathrm{left}},$$

where the isomorphisms are induced termwise by the morphisms in Lemma 8.4.7. Moreover, it will be convenient to compute the direct image $\boldsymbol{R}f_*$ by using flabby sheaves more adapted to the computation than the Godement sheaves, so we will use the formula

$$\boldsymbol{R}f_*\Omega_X^{n+\bullet}(\mathcal{M}^{\mathrm{left}}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty\xrightarrow{\sim}f_*\mathcal{E}_X^{n+\bullet}(\mathcal{M}^{\mathrm{left}}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y),$$

obtained from the Dolbeault resolution $\Omega_X^i\xrightarrow{\sim}(\mathcal{E}^{(i,\bullet)},\mathrm{d}'')$ and by taking the associated simple complex. Lastly, we identify each term of this complex with

$$K_\infty^{n+\bullet} := f_*(\mathcal{E}_X^{n+\bullet}\otimes_{\mathcal{O}_X}\mathcal{M}^{\mathrm{left}})\otimes_{\mathcal{O}_Y}\mathcal{D}_Y \tag{12.4.9}$$

and, with this identification, the differential is given by the formula

$$(-1)^n\cdot\Big(\big[(\mathrm{d}\otimes\mathrm{Id}_{\mathcal{M}^{\mathrm{left}}})\otimes\mathrm{Id}\big]+\big[(\mathrm{Id}\otimes\nabla)\otimes\mathrm{Id}\big]+\big[(\mathrm{Id}\otimes\mathrm{Id})\otimes f_*f^*\nabla^Y\big]\Big),$$

where ∇^Y is the universal connection on $\mathcal{D}_Y$.

Let $\mathfrak{s}=\mathfrak{s}^{\mathrm{left}}:\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''}\to\mathfrak{Db}_X$ be a sesquilinear pairing between left $\mathcal{D}_X$-modules, and let $\mathfrak{s}_\infty$ denote its C^∞ extension (that suffices for our purpose, according to Remark 12.3.4). We first aim at defining a graded sesquilinear pairing whose degree k term ($k\in\mathbb{Z}$) is a pairing

$$\mathcal{H}^k(K'^{n+\bullet})\otimes\overline{\mathcal{H}^{-k}(K''^{n+\bullet})}\longrightarrow\mathfrak{C}_Y\,. \tag{12.4.10}$$

The C^∞ extension $\mathfrak{s}_\infty$ of $\mathfrak{s}$ induces a morphism

$$\mathfrak{s}_\infty^{k,\ell}:(\mathcal{E}_X^{n+k}\otimes_{\mathcal{O}_X}\mathcal{M}')\otimes_{\mathcal{C}_X^\infty}\overline{(\mathcal{E}_X^{n+\ell}\otimes_{\mathcal{O}_X}\mathcal{M}'')}\longrightarrow\mathfrak{Db}_{X,S}^{2n+k+\ell} \tag{12.4.11}$$

by the formula

$$(\eta'^{n+k}\otimes m')\otimes\overline{\eta''^{n+\ell}\otimes m''}\longmapsto\eta'^{n+k}\wedge\overline{\eta''^{n+\ell}}\,\mathfrak{s}(m',\overline{m''}),$$

and by applying f_*,

$$(f_*\mathfrak{s}_\infty)^{k,\ell}:f_*(\mathcal{E}_X^{n+k}\otimes_{\mathcal{O}_X}\mathcal{M}')\otimes_{f_*\mathcal{C}_X^\infty}f_*\overline{(\mathcal{E}_X^{n+\ell}\otimes_{\mathcal{O}_X}\mathcal{M}'')}\longrightarrow f_*\,\mathfrak{Db}_{X,S}^{2n+k+\ell},$$

so, by right $\mathcal{D}_{Y,\overline{Y}}$-linearity, a morphism

$$\begin{split}(f_*\mathfrak{s}_\infty)^{k,\ell}:\big[f_*(\mathcal{E}_X^{n+k}\otimes_{\mathcal{O}_X}\mathcal{M}')\otimes_{\mathcal{O}_Y}\mathcal{D}_Y\big]\otimes_{\mathcal{C}_Y^\infty}\overline{f_*(\mathcal{E}_X^{n+\ell}\otimes_{\mathcal{O}_X}\mathcal{M}'')\otimes_{\mathcal{O}_Y}\mathcal{D}_Y}\\ \longrightarrow f_*(\mathfrak{Db}_{X,S}^{2n+k+\ell})\otimes_{\mathcal{O}_{Y,\overline{Y}}}\mathcal{D}_{Y,\overline{Y}}.\end{split}$$

The compatibility of $\mathfrak{s}$ with the connections on $\mathcal{M}', \mathcal{M}''$ implies that this morphism is compatible with the differentials, so that, with respect to the identifications above and according to (12.2.11), we get a morphism of complexes of right $\mathcal{D}_{Y,\overline{Y}}$-modules

$$(f_*\mathfrak{s}_\infty)^{\bullet,\bullet}:(K'^{n+\bullet}_\infty\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty)\otimes_{\mathcal{C}_Y^\infty}\overline{(K''^{n+\bullet}_\infty\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty)}\longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S}\,.$$

Composing with the integration of currents (see Exercise 12.4)

$$\int_f : {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S}\longrightarrow\mathfrak{C}_Y$$

we finally get a morphism of complexes of right $\mathcal{D}_{Y,\overline{Y}}$-modules (where $\mathfrak{C}_Y$ is regarded as a complex having a single term in degree zero) that we denote by the same symbol:

$$(\textstyle\int f_*\mathfrak{s}_\infty)^{\bullet,\bullet}:(K'^{n+\bullet}_\infty\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty)\otimes_{\mathcal{C}_Y^\infty}\overline{(K''^{n+\bullet}_\infty\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty)}\longrightarrow\mathfrak{C}_Y\,.$$

At the cohomology level, we regard $\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty$ as a graded pairing, the degree k term being the induced pairing

$$(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)}:\big(\mathcal{H}^k(K'^{n+\bullet}_\infty\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty\big)\otimes\overline{\big(\mathcal{H}^{-k}(K''^{n+\bullet}_\infty\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty)\big)}\longrightarrow\mathfrak{C}_Y\,.$$

The natural morphism $\mathcal{H}^j(K^{n+\bullet})\to\mathcal{H}^j(K^{n+\bullet}_\infty)$ is an isomorphism and $\mathcal{H}^j(K^{n+\bullet})$ is thus an $\mathcal{O}_Y$-submodule of $\mathcal{H}^j(K^{n+\bullet}_\infty)\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty$. We can hence restrict $(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)}$ to obtain a graded sesquilinear pairing whose degree k term is

$$(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)}:\mathcal{H}^k(K'^{n+\bullet})\otimes\overline{\big(\mathcal{H}^{-k}(K''^{n+\bullet})\big)}\longrightarrow\mathfrak{C}_Y\,.$$

We finally adjust the sign in order to ensure compatibility with (12.4.2) (Recall $\mathrm{Sgn}(n,k)=(-1)^n\,\varepsilon(n+k)/(2\pi\mathsf{i})^n$, see Notation (0.2 $*$)).

12.4.12. Proposition. *The following equality holds between right sesquilinear pairings:*

$$\boxed{({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}=\mathrm{Sgn}(n,k)(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)}}$$

For example, if Y is a point, we recover Formula (2.4.8) for the case where $\mathcal{M}'=\mathcal{M}''=\mathcal{O}_X$, and Formula (4.2.17) for the more general case where $\mathcal{M}'=\mathcal{M}''$ underlies a polarized variation of Hodge structure. We also make explicit the case of a projection in Exercise 12.9.

Proof. We prove sign-commutativity, up to a precise constant in each bi-degree, of the following diagram:

$$\begin{tikzcd}
\left.\begin{array}{c}(\mathcal{M}'^{\mathrm{right}} \otimes \mathrm{Sp}_{X\to Y}) \\ \otimes \\ \overline{(\mathcal{M}''^{\mathrm{right}} \otimes \mathrm{Sp}_{X\to Y})}\end{array}\right\} \arrow[r, "\sim"] \arrow[d] & \left\{\begin{array}{c}(\Omega_X^{n+\bullet} \otimes \mathcal{M}'^{\mathrm{left}} \otimes f^{-1}\mathcal{D}_Y) \\ \otimes \\ \overline{(\Omega_X^{n+\bullet} \otimes \mathcal{M}''^{\mathrm{left}} \otimes f^{-1}\mathcal{D}_Y)}\end{array}\right. \arrow[d] \\
(\mathcal{M}'^{\mathrm{right}} \otimes \mathcal{M}''^{\mathrm{right}}) \otimes \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}} \arrow[d, "\mathfrak{s}^{\mathrm{right}}"'] & (\Omega_{X,\overline{X}}^{n+\bullet,n+\bullet} \otimes (\mathcal{M}'^{\mathrm{left}} \otimes \mathcal{M}''^{\mathrm{left}}) \otimes f^{-1}\mathcal{D}_{Y,\overline{Y}} \arrow[d, "\mathfrak{s}^{\mathrm{left}}"] \\
\mathfrak{C}_X \otimes \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}} \arrow[r, "\sim"] & \Omega_{X,\overline{X}}^{n+\bullet,n+\bullet} \otimes \mathfrak{Db}_X \otimes f^{-1}\mathcal{D}_{Y,\overline{Y}}
\end{tikzcd}$$

Let us consider local sections $\omega' \otimes m'$ of $\mathcal{M}'^{\mathrm{right}}$ and $\omega'' \otimes m''$ of $\mathcal{M}''^{\mathrm{right}}$, where m', m'' are local sections of $\mathcal{M}'^{\mathrm{left}}, \mathcal{M}''^{\mathrm{left}}$, and let ξ'_k, ξ''_ℓ be poly-vector fields of respective degree $k, \ell \geqslant 1$. Following the arrows downward for the local sections $\omega' \otimes m' \otimes \xi'_k \otimes \mathbf{1}_Y$ and $\overline{\omega'' \otimes m'' \otimes \xi''_\ell \otimes \mathbf{1}_Y}$ of the upper left terms, we obtain

$$\begin{aligned}
&(\omega' \otimes m' \otimes \xi'_k \otimes \mathbf{1}_Y) \otimes (\overline{\omega'' \otimes m'' \otimes \xi''_\ell \otimes \mathbf{1}_Y}) \\
&\qquad\qquad \longmapsto (\omega' \otimes \overline{\omega''}) \otimes (m' \otimes \overline{m''}) \otimes (\xi'_k \otimes \overline{\xi''_\ell}) \otimes \mathbf{1}_{Y,\overline{Y}} \\
&\qquad\qquad\qquad \longmapsto \mathfrak{s}^{\mathrm{right}}\big((\omega' \otimes \overline{\omega''}) \otimes (m' \otimes \overline{m''})\big) \otimes (\xi'_k \otimes \overline{\xi''_\ell}) \otimes \mathbf{1}_{Y,\overline{Y}}
\end{aligned}$$

and the image by the lower horizontal isomorphism of the last term above is, by mimicking Lemma 8.4.7,

$$\begin{aligned}
&\mathrm{Sgn}(n)(\omega' \otimes \overline{\omega''}) \otimes \mathfrak{s}^{\mathrm{left}}(m' \otimes \overline{m''}) \otimes (\xi'_k \otimes \overline{\xi''_\ell}) \otimes \mathbf{1}_{Y,\overline{Y}} \\
&\qquad\qquad = \mathrm{Sgn}(n)(-1)^{nk}((\xi'_k \lrcorner\, \omega') \otimes (\overline{\xi''_\ell \lrcorner\, \omega''})) \otimes \mathbf{1}_{Y,\overline{Y}},
\end{aligned}$$

where the sign $(-1)^{nk}$ comes from the commutation of ξ'_k with $\overline{\omega''}$.

On the other hand, the image of the first term above by the first horizontal isomorphism is, according to Lemma 8.4.7,

$$\begin{aligned}
&\big((\xi'_k \lrcorner\, \omega') \otimes m' \otimes \mathbf{1}_Y\big) \otimes \big(\overline{(\xi''_\ell \lrcorner\, \omega'') \otimes m'' \otimes \mathbf{1}_Y}\big) \\
&\qquad\qquad = \big((\xi'_k \lrcorner\, \omega') \otimes m' \otimes \mathbf{1}_Y\big) \otimes \big(\overline{(\xi''_\ell \lrcorner\, \omega'') \otimes m'' \otimes \mathbf{1}_Y}\big),
\end{aligned}$$

and the image of the latter term by the right vertical morphisms is

$$((\xi'_k \lrcorner\, \omega') \wedge (\overline{\xi''_\ell \lrcorner\, \omega''})) \otimes \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}) \otimes \mathbf{1}_{Y,\overline{Y}}.$$

Therefore, in bi-degree k, ℓ, the diagram commutes up to $(-1)^{nk}\,\mathrm{Sgn}(n)$.

The computation of $(\mathcal{H}^0 \textstyle\int f_* \mathfrak{s}_\infty)^{(k,-k)}$ makes use of the morphism of complexes deduced from the right vertical arrows, from the upper right term to the lower right one, while the computation of $({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$ uses the morphism obtained by composing the arrows between the same terms in the other path. It follows that

$$({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}} = (-1)^{nk}\,\mathrm{Sgn}(n)(\mathcal{H}^0 \textstyle\int f_* \mathfrak{s}_\infty)^{(k,-k)},$$

and the desired equality follows from the relation $(-1)^{nk}\,\varepsilon(k)\,\varepsilon(n)=\varepsilon(n+k)$. $\square$

12.4.13. The Lefschetz morphism. The left-to-right pushforward of a sesquilinear pairing is best suited to analyze the action of the Lefschetz morphism, i.e., the action of the external product by a closed $(1,1)$-form.

In the previous setting, let η be a closed $(1,1)$-form on X which is *real*, i.e., such that $\overline{\eta}=\eta$. This condition is satisfied if the cohomology class of η is equal to c_1 of some line bundle on X. The corresponding Lefschetz morphism $\mathrm{L}_\eta : {}_{\mathrm{D}}f_*\mathcal{M}\to{}_{\mathrm{D}}f_*\mathcal{M}[2]$ with $\mathcal{M}=\mathcal{M}',\mathcal{M}''$ (see Definition 8.7.20) satisfies

$$\int_f f_*\big(\mathfrak{s}_\infty^{k,-k}(\mathrm{L}_\eta m'^{k-2}_\infty,\overline{m''^{-k}_\infty})\big)=\int_f f_*\big(\mathfrak{s}_\infty^{k-2,-k+2}(m'^{k-2}_\infty,\overline{\mathrm{L}_\eta m''^{-k}_\infty})\big),$$

according to the definition of $\mathfrak{s}_\infty^{k,\ell}$ in (12.4.11), and therefore, by Proposition 12.4.12, since $\operatorname{Sgn}(n,k-2)=-\operatorname{Sgn}(n,k)$,

$${}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}(\mathrm{L}_\eta m',\overline{m''})=-{}_{\mathrm{T}}f_*^{(k-2,-k+2)}\mathfrak{s}(m',\overline{\mathrm{L}_\eta m''}),$$

if m' (resp. m'') is a local section of ${}_{\mathrm{D}}f_*^{k-2}\mathcal{M}'$ (resp. of ${}_{\mathrm{D}}f_*^{-k}\mathcal{M}''$). In order to eliminate the sign, we work as in (2.4.12) with

$$\mathrm{X}_\eta:=(2\pi\mathsf{i})\mathrm{L}_\eta,$$

so that

$$(12.4.13\,*)\qquad {}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}(\mathrm{X}_\eta m',\overline{m''})={}_{\mathrm{T}}f_*^{(k-2,-k+2)}\mathfrak{s}(m',\overline{\mathrm{X}_\eta m''}).$$

12.4.14. Composition with a closed embedding. Let us consider a composition

$$X\xrightarrow{\ f\ }Y\xrightarrow{\ f'\ }Z,$$

where X,Y,Z are complex manifolds. Let $\mathcal{M}',\mathcal{M}''$ be right $\mathcal{D}_X$-modules such that $f'\circ f$ is proper on $S=\operatorname{Supp}\mathcal{M}'\cup\operatorname{Supp}\mathcal{M}''$ (or $S=\operatorname{Supp}\mathcal{M}'\cap\operatorname{Supp}\mathcal{M}''$ if $\mathcal{M}',\mathcal{M}''$ are $\mathcal{D}_X$-coherent, see Remark 8.7.26), and let $\mathfrak{s}:\mathcal{M}'\otimes\overline{\mathcal{M}''}\to\mathfrak{C}_{X,S}$ be a sesquilinear pairing. Let us assume that f or f' is a closed embedding. We will prove that there is a natural identification for any $k\in\mathbb{Z}$:

$$(12.4.14\,*)\qquad {}_{\mathrm{T}}(f'\circ f)_*^{(k,-k)}\mathfrak{s}=\begin{cases}{}_{\mathrm{T}}f'^{(k,-k)}_*\big({}_{\mathrm{T}}f_*\mathfrak{s}\big) & \text{if } f \text{ is a closed embedding,}\\ {}_{\mathrm{T}}f'_*\big({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}\big) & \text{if } f' \text{ is a closed embedding.}\end{cases}$$

Using Notation (12.4.1) and arguing as in Theorem 8.7.23 with $\mathcal{D}_{X,\overline{X}}$-modules, we find, for any right $\mathcal{D}_{X,\overline{X}}$-module $\mathcal{N}$, a natural isomorphism

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_!(\mathcal{N})\simeq{}_{\mathrm{D},\overline{\mathrm{D}}}f'_!({}_{\mathrm{D},\overline{\mathrm{D}}}f_!\mathcal{N}).$$

We consider the case where and $f'\circ f$ is proper on $\operatorname{Supp}\mathcal{N}$. For example, let us assume that f is a closed embedding, the other case being treated in a similar way (a more general situation will be treated in Section 12.7.d). Then the isomorphism reads as follows:

$$\begin{aligned}
{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*(\mathcal{N}) &\simeq \boldsymbol{R}(f'\circ f)_*(\mathcal{N}\otimes_{\mathcal{D}_{X,\overline{X}}}\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}\otimes_{f^{-1}\mathcal{D}_{Y,\overline{Y}}} f^{-1}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}})\\
&\simeq \boldsymbol{R}f'_*(f_*((\mathcal{N}\otimes_{\mathcal{D}_{X,\overline{X}}}\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}})\otimes_{\mathcal{D}_{Y,\overline{Y}}}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}))\\
&\simeq {}_{\mathrm{D},\overline{\mathrm{D}}}f'_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\mathcal{N}).
\end{aligned}$$

Taking cohomology in degree zero yields a functorial isomorphism

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathcal{N})\simeq {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\mathcal{N}).$$

We apply this to $\mathcal{N}=\mathcal{M}'\otimes\overline{\mathcal{M}''}$ and to $\mathcal{N}=\mathfrak{C}_{X,S}$. By functoriality, we get a commutative diagram

$$\begin{tikzcd}
{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathcal{M}'\otimes\overline{\mathcal{M}''}) \arrow[r, "\sim"] \arrow[d, "{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathfrak{s})"'] & {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*(\mathcal{M}'\otimes\overline{\mathcal{M}''})) \arrow[d, "{}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*(\mathfrak{s}))"] \\
{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathfrak{C}_{X,S}) \arrow[r, "\sim"] & {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S})
\end{tikzcd}$$

On the one hand, we can complete this commutative diagram from above by adding the line

$${}_{\mathrm{D}}(f'\circ f)_*^{(k)}(\mathcal{M}')\otimes\overline{{}_{\mathrm{D}}(f'\circ f)_*^{(-k)}\mathcal{M}'')}\xrightarrow{\sim}{}_{\mathrm{D}}f'^{(k)}({}_{\mathrm{D}}f_*(\mathcal{M}'))\otimes\overline{{}_{\mathrm{D}}f'^{(-k)}({}_{\mathrm{D}}f_*(\mathcal{M}''))}$$

with the natural morphisms to the upper line of the diagram. On the other hand, we claim that the following diagram is commutative:

$$\begin{tikzcd}
{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathfrak{C}_{X,S}) \arrow[r, "\sim"] \arrow[ddd, "\int_{f'\circ f}"'] & {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S}) \arrow[d, "{}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*(\int_f)"] \\
 & {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(0)}_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{Y,f(S)}) \arrow[dd, "\int_{f'}"] \\
 & \\
\mathfrak{C}_Z \arrow[r, equal] & \mathfrak{C}_Z
\end{tikzcd}$$

This follows from the property that $\int_{f'}(\int_f u_{p,q})=\int_{f'\circ f}u_{p,q}$ for a current $u_{p,q}$ such that $f'\circ f$ is proper on $\operatorname{Supp}u_{p,q}$. All together, we obtain the commutativity corresponding to the first line of (12.4.14 $*$). The second line of (12.4.14 $*$) is obtained in a similar way. $\square$

12.4.d. An adjunction formula. We will verify the compatibility of the morphism of adjunction adj_f of Section 8.7.d with sesquilinear pairings. We consider a proper holomorphic map $f:X\to Y$ between complex manifolds of the same dimension $m=n$. In order to avoid any delicate question concerning the pullback of a sesquilinear pairing, we will assume that $\mathfrak{s}$ is a sesquilinear pairing between the left $\mathcal{D}_Y$-modules $\mathcal{M}',\mathcal{M}''$ that takes values in $\mathcal{C}^\infty_Y$. The main example is the case where $\mathcal{M}',\mathcal{M}''$ are holomorphic bundles with flat connection (Lemma 12.3.6). In such a case, since ${}_{\mathrm{D}}f^*\mathcal{M}$ is the $\mathcal{O}_X$-pullback module equipped with the pullback connection, we have a well-defined pullback sesquilinear pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}$ which satisfies

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}(1\otimes m', \overline{1\otimes m''}) = f^*\mathfrak{s}(m', \overline{m''}) := \mathfrak{s}(m', \overline{m''})\circ f, $$

which is a C^∞ function on X. We can then consider the pushforward

$$ {}_{\mathrm{T}}f_*^{(0,0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}) : {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}') \otimes \overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \longrightarrow \mathfrak{Db}_Y\,. $$

12.4.15. Proposition. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{M}' \otimes \overline{\mathcal{M}''} \arrow[rr, "\mathfrak{s}"] \arrow[d, "\mathrm{adj}_f"'] \arrow[d, shift left=10, "\overline{\mathrm{adj}_f}"] & & \mathfrak{Db}_Y \arrow[d, equal] \\
{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}') \otimes \overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \arrow[rr, "{}_{\mathrm{T}}f_*^{(0,0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})"] & & \mathfrak{Db}_Y
\end{tikzcd}$$

Proof. It will be more convenient to work in the right setting. Extending the proof of Proposition 8.7.30 to the case of right $\mathcal{D}_{Y,\overline{Y}}$-modules, we obtain a functorial morphism

$$ \mathrm{adj} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{M}' \otimes \overline{\mathcal{M}''})), $$

which factorizes as

$$ \mathcal{M}' \otimes \overline{\mathcal{M}''} \xrightarrow{\mathrm{adj}_f \otimes \overline{\mathrm{adj}_f}} {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}') \otimes \overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{M}' \otimes \overline{\mathcal{M}''})\big). $$

We therefore obtain, by functoriality of adj, a commutative diagram

$$\begin{tikzcd}
\mathcal{M}' \otimes \overline{\mathcal{M}''} \arrow[rr, "\mathfrak{s}"] \arrow[d, "\mathrm{adj}"'] & & \mathcal{E}_Y^{n,n} \arrow[d, "\mathrm{adj}"] \\
{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{M}' \otimes \overline{\mathcal{M}''})\big) \arrow[rr, "{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})"] & & {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\big)
\end{tikzcd}$$

We have a natural morphism

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{C}_Y^\infty) = \mathcal{O}_{X,\overline{X}} \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}(\mathcal{C}_Y^\infty) \longrightarrow \mathcal{C}_X^\infty, $$

so that, by side-changing, a natural morphism

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n}) \longrightarrow \mathcal{E}_X^{n,n}, $$

and thus a composed morphism

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\big) \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}(\mathcal{E}_X^{n,n}) \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}(\mathfrak{C}_X) \xrightarrow{\int_f} \mathfrak{C}_Y\,. \tag{12.4.16} $$

In order to prove the proposition, it is enough to check that the composition on the left of (12.4.16) with adj is the natural inclusion $\mathcal{E}_Y^{n,n} \hookrightarrow \mathfrak{C}_Y$. Let us describe this morphism. Starting from a local section η of $\mathcal{E}_Y^{n,n}$, we lift it as the section $\eta\otimes 1$ of $\mathcal{E}_Y^{n,n} \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$, then we consider its image in $(f_*\mathcal{E}_X^{n,n}) \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$, that we integrate along f. We are thus left with checking that, for such an η, the integral of $u = f^*\eta$ as a current is equal to η. This follows from the property that, for any test function χ on Y and any (n,n)-form u on X, we have

$$\big\langle (\textstyle\int_f u), \chi \big\rangle = \langle u, \chi \circ f \rangle = \int_X (\chi \circ f) u,$$

so that, if $u = f^*\eta$,

$$\big\langle (\textstyle\int_f u), \chi \big\rangle = \int_X f^*(\chi\eta) = \int_Y \chi\eta = \langle \eta, \chi \rangle. \qquad \square$$

12.4.17. Example. In the setting of Example 8.7.31, the diagram of Proposition 12.4.15 can be completed as a commutative diagram

$$\begin{tikzcd}
\mathcal{M}' \arrow[d, "\mathrm{adj}_f"'] \arrow[dd, bend left, "\mathrm{Id}"] & \otimes & \overline{\mathcal{M}''} \arrow[d, "\overline{\mathrm{adj}_f}"] \arrow[dd, bend right] \arrow[r, "\mathfrak{s}"] & \mathfrak{Db}_Y \arrow[d, equal] \\
{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}') \arrow[d, "\mathrm{Tr}_f"'] & \otimes & \overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \arrow[d, "\overline{\mathrm{Tr}_f}"] \arrow[r, "{}_{\mathrm{T}}f_*^{(0,0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})"] & \mathfrak{Db}_Y \arrow[d, equal] \\
\mathcal{M}' & \otimes & \overline{\mathcal{M}''} \arrow[r, "\mathfrak{s}"] & \mathfrak{Db}_Y
\end{tikzcd}$$

The proof of commutativity of the lower diagram is very similar to that for the upper diagram. We define the trace Tr_f for $\mathcal{D}_{Y,\overline{Y}}$-modules and commutativity follows from identifying $\mathrm{Tr}_f : {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\big) \to \mathcal{E}_Y^{n,n}$ with the morphism (12.4.16). This identification follows from that of Exercise 12.7.

12.5. Pullback, specialization and localization of sesquilinear pairings

12.5.a. Pullback by a smooth morphism.

12.5.a. Pullback by a smooth morphism. The case of left $\mathcal{D}_X$-modules is easier to treat first. Let $f : X \to Y$ be a *smooth* holomorphic map (i.e., everywhere of maximal rank). Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_Y$ be a sesquilinear pairing between left $\mathcal{D}_Y$-modules. The pullback left $\mathcal{D}_X$-modules ${}_{\mathrm{D}}f^*\mathcal{M}'$, ${}_{\mathrm{D}}f^*\mathcal{M}''$ (we use the notation ${}_{\mathrm{D}}f^*$ instead of ${}_{\mathrm{D}}f^{*(0)}$ for the sake of simplicity) are defined in Section 8.6.a, and are equal to the derived pullback modules, since f is smooth (see Remark 8.6.7).

On the other hand, let η be a C^∞ form of maximal degree and compact support on X. It can be integrated along the fibers of f, to give rise, since f is smooth, to a C^∞ form $\int_f \eta$ of maximal degree and compact support on Y. This is a particular case of the pushforward of currents of maximal degree, as seen in Section 12.2.b: if φ is a C^∞ function on Y with compact support, we have

$$\Big\langle \int_f \eta, \varphi \Big\rangle = \int_X (\varphi \circ f) \cdot \eta.$$

Given a distribution u on Y, the pullback ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*u$ is the distribution on X defined by

$$\langle \eta, {}_{\mathrm{D},\overline{\mathrm{D}}}f^*u \rangle := \Big\langle \int_f \eta, u \Big\rangle, \qquad \eta \in \mathcal{E}_{\mathrm{c}}^{n,n}(X).$$

12.5.1. Definition **(Pullback of a sesquilinear pairing by a smooth morphism)**

The pullback ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s} : {}_{\mathrm{D}}f^*\mathcal{M}' \otimes_{\mathbb{C}} \overline{{}_{\mathrm{D}}f^*\mathcal{M}''} \to \mathfrak{Db}_X$ of a sesquilinear pairing between left $\mathcal{D}_X$-modules is defined as the morphism

$$\begin{aligned}\big(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}'\big) \otimes_{\mathbb{C}} \big(\overline{\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}''}\big) &\longrightarrow \mathfrak{Db}_X\\ (\varphi' \otimes m') \otimes (\overline{\varphi'' \otimes m''}) &\longmapsto \varphi' \cdot \overline{\varphi''} \cdot {}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}(m', \overline{m''})).\end{aligned}$$

The pullback of right sesquilinear pairings is obtained by the side-changing procedure, according to Remark 8.6.8:

$${}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}^{\mathrm{right}}) := ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}.$$

One checks that the above formulas define a sesquilinear pairing between ${}_{\mathrm{D}}f^*\mathcal{M}'$ and ${}_{\mathrm{D}}f^*\mathcal{M}''$.

12.5.2. Lemma **(Pullback and Hermitian adjunction).** *The pullback by a smooth morphism commutes with Hermitian adjunction:*

$${}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}^*) = ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})^*.$$

Proof. By definition and according to (12.3.3 $**$), it is enough to check the lemma for left sesquilinear pairings. We have

$$\begin{aligned}\big\langle \eta, {}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}^*)(1 \otimes m'', \overline{1 \otimes m'})\big\rangle &= \Big\langle \int_f \eta, \mathfrak{s}^*(m'', \overline{m'})\Big\rangle = \Big\langle \int_f \eta, \overline{\mathfrak{s}(m', \overline{m''})}\Big\rangle\\ &= \overline{\Big\langle \overline{\int_f} \eta, \mathfrak{s}(m', \overline{m''})\Big\rangle} = \overline{\Big\langle \int_f \overline{\eta}, \mathfrak{s}(m', \overline{m''})\Big\rangle} \quad \text{by (12.2.10)}\\ &= \overline{\big\langle \overline{\eta}, ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})(1 \otimes m', \overline{1 \otimes m''})\big\rangle}\\ &= \big\langle \eta, ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})^*(1 \otimes m'', \overline{1 \otimes m'})\big\rangle. \qquad \square\end{aligned}$$

12.5.3. Example **(Pullback of $\mathfrak{s}_m$ by a smooth morphism).**

(1) Assume $\mathcal{M}' = \mathcal{M}'' = \mathcal{O}_Y$ and $\mathfrak{s} = \mathfrak{s}_m^{\mathrm{left}}$ (see Example 12.3.5). Then ${}_{\mathrm{D}}f^*\mathcal{O}_Y = \mathcal{O}_X$ and

$$\boxed{{}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}_m^{\mathrm{left}} = \mathfrak{s}_n^{\mathrm{left}}.}$$

(2) If $\mathcal{M}' = \mathcal{M}'' = \omega_Y$ and $\mathfrak{s} = \mathfrak{s}_m^{\mathrm{right}}$, then ${}_{\mathrm{D}}f^*\omega_Y = \omega_X$ and

$$\boxed{{}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}_m^{\mathrm{right}} = \mathfrak{s}_n^{\mathrm{right}}.}$$

12.5.b. Specialization of a sesquilinear pairing. Let $g : X \to \mathbb{C}$ be a holomorphic function on X and let $\mathcal{M}', \mathcal{M}''$ be $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g). Assume that $\mathfrak{s}$ is a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$ with values in $\mathfrak{C}_X$ (right case) or $\mathfrak{Db}_X$ (left case). We wish to define sesquilinear pairings between the $\mathcal{D}_X$-modules $\psi_{g,\lambda}\mathcal{M}'$ and $\psi_{g,\lambda}\mathcal{M}'$ with values in $\mathfrak{C}_X$ resp. $\mathfrak{Db}_X$.

We start with the case where $g = t$ is the projection of a product $X = H \times \Delta_t$, and $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable $\mathcal{D}_X$-modules along H, equipped with a sesquilinear pairing $\mathfrak{s}$. In order to define a sesquilinear pairing on nearby cycles, we will use a

Mellin transform device by considering the residue of $\mathfrak{s}(m', \overline{m''})|t|^{2s}$ at various values of s.

It is important to notice that, while we need to restrict the category of coherent $\mathcal{D}_X$-modules in order to define nearby and vanishing cycles (i.e., to consider $\mathbb{R}$-*specializable* coherent $\mathcal{D}_X$-modules only), the specialization of a sesquilinear pairing between them does not need any new restriction: any sesquilinear pairing between such $\mathcal{D}_X$-modules can be specialized.

We assume that $\mathcal{M}', \mathcal{M}''$ are right $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along H, and we denote by $V_\bullet\mathcal{M}', V_\bullet\mathcal{M}''$ their V-filtration. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing. Fix $x_o \in H$. For local sections m', m'' of $\mathcal{M}', \mathcal{M}''$ defined in some open neighbourhood $\mathrm{nb}_X(x_o)$ of x_o in X, the current $\mathfrak{s}(m', \overline{m''})$ of degree 0 has finite order on any compact subset of $\mathrm{nb}_X(x_o)$. Let us shrink the neighbourhood $\mathrm{nb}(X, x_o)$ so that $\mathfrak{s}(m', \overline{m''})$ has finite order p on $\mathrm{nb}(X, x_o)$. Let η be a test function with compact support in $\mathrm{nb}_X(x_o)$. For $2\operatorname{Re} s > p$, the function $x \mapsto |t(x)|^{2s}$ is C^p as well as its s-derivative $2s|t|^{2s}\log|t|$, so for every such s, the function

$$s \longmapsto \big\langle \mathfrak{s}(m', \overline{m''})|t|^{2s}, \eta\big\rangle := \big\langle \mathfrak{s}(m', \overline{m''}), |t|^{2s}\eta\big\rangle$$

is well-defined and holomorphic on the half-plane $S_p := \{2\operatorname{Re} s > p\}$.

We also claim that, if we let η depend on s in such a way that $\eta \in \mathcal{C}^\infty_{\mathrm{c}}(W)$ for some $W \subset \mathrm{nb}(X, x_o) \times S_p$, then the correspondence $\eta \mapsto \big\langle \mathfrak{s}(m', \overline{m''}), |t|^{2s}\eta\big\rangle$ defines a current depending continuously on S_p (see Section 12.2.a). This is obvious since derivatives up to order p of $|t|^{2s}\eta$ introduce polynomials of degree at most p in s.

12.5.4. Proposition. *Let $\mathcal{M}', \mathcal{M}'', \mathfrak{s}$ be as above. Let $x_o \in H$ and let $\alpha', \alpha'' \in \mathbb{R}$. There exist $L \geqslant 0$ and a finite set of real numbers α satisfying*

$$(12.5.4\,*) \qquad \mathrm{gr}^V_\alpha \mathcal{M}'_{x_o}, \mathrm{gr}^V_\alpha \mathcal{M}''_{x_o} \neq 0, \quad \textit{and} \quad \alpha \leqslant \min(\alpha', \alpha''),$$

such that, for any sections $m' \in V_{\alpha'}\mathcal{M}'_{x_o}$ and $m'' \in V_{\alpha''}\mathcal{M}''_{x_o}$ defined on $\mathrm{nb}(X, x_o)$, *and any test function η on* $\mathrm{nb}(X, x_o)$, *the function*

$$(12.5.4\,**) \qquad s \longmapsto \big\langle \mathfrak{s}(m', \overline{m''})|t|^{2s}, \eta\big\rangle,$$

which is holomorphic on some half-plane $2\operatorname{Re} s > p$, *extends as a meromorphic function on $\mathbb{C}_\tau$ of the form $h(s)\prod_\alpha \Gamma(s-\alpha)^L$, with $h(s) \in \mathcal{O}(\mathbb{C}_\tau)$.*

Moreover, the correspondence $\eta \mapsto \prod_\alpha \Gamma(s-\alpha)^{-L}\big\langle \mathfrak{s}(m', \overline{m''}), |t|^{2s}\eta\big\rangle$ defines a current depending continuously on $S = \mathbb{C}_\tau$.

Proof. Let $b_{m'}(\mathrm{E})$ denote the Bernstein polynomial of m' (see Definition 9.3.7) and let $R(m')$ denote the set of its roots, so that we can write

$$b_{m'}(\mathrm{E}) = \prod_{\alpha \in R(m')} (\mathrm{E} - \alpha)^{\nu(\alpha)},$$

with $\nu(\alpha)$ bounded by the nilpotency index L of $\mathrm{E} - \alpha$. It is enough to prove that the product $\prod_{\alpha \in R(m')} \Gamma(s-\alpha)^{-\nu(\alpha)}$ of Γ factors can be used to make (12.5.4 $**$) an entire function (recall that the Γ function has no zeros and has simple poles at the non-positive integers, and no other poles). Indeed, arguing similarly for m'' and using that the set of roots $R(m'')$ of $b_{m''}(\mathrm{E})$ is real, one obtains that the product of Γ factors indexed by $R(m')\cap R(m'')$ can also be used to make (12.5.4 $**$) an entire function. It is then easy to check that Conditions (12.5.4 $*$) are satisfied by any $\alpha\in R(m')\cap R(m'')$.

We note first that, for every germ of operator $Q\in V_0\mathcal{D}_{X,x_o}$ and any test function η on $\mathrm{nb}_X(x_o)$, the function $Q\cdot(|t|^{2s}\eta)$ is C^p with compact support if $2\operatorname{Re}s>p$. Applying this to the Bernstein operator $Q=b_{m'}(\mathrm{E})-P$ for m' (see Definition 9.3.7), one gets

$$
\begin{aligned}
0&=\big\langle\mathfrak{s}(m',\overline{m''})\cdot[b_{m'}(\mathrm{E})-P],|t|^{2s}\eta\big\rangle\\
&=\big\langle\mathfrak{s}(m',\overline{m''}),[b_{m'}(\mathrm{E})-P]\cdot(|t|^{2s}\eta)\big\rangle\\
&=b_{m'}(s)\big\langle\mathfrak{s}(m',\overline{m''}),|t|^{2s}\eta\big\rangle+\big\langle\mathfrak{s}(m',\overline{m''}),|t|^{2s}t\eta_1\big\rangle
\end{aligned}
\tag{12.5.5}
$$

for some η_1, which is a polynomial in s with coefficients being C^∞ with compact support contained in that of η. As $|t|^{2s}t$ is C^p for $2\operatorname{Re}s+1>p$, we can argue by induction to show that, for every η and $k\in\mathbb{N}$,

$$
s\longmapsto b_{m'}(s+k-1)\cdots b_{m'}(s)\big\langle\mathfrak{s}(m',\overline{m''})|t|^{2s},\eta\big\rangle
\tag{12.5.6}
$$

extends as a holomorphic function on $\{s\mid 2\operatorname{Re}s>p-k\}$, and thus, letting $k\to\infty$,

$$
s\longmapsto\prod_{\alpha\in R(m')}\Gamma(s-\alpha)^{-\nu(\alpha)}\cdot\big\langle\mathfrak{s}(m',\overline{m''})|t|^{2s},\eta\big\rangle
$$

extends as an entire function.

Let q denote the order of Q. Then (12.5.5) also shows that $b_{m'}(s)\big\langle\mathfrak{s}(m',\overline{m''}),|t|^{2s}\eta\big\rangle$ defines a current of order $\leqslant p+q$ depending continuously on S_{p-1}, since the derivatives of order $\leqslant p$ of η_1 can be expressed in terms of derivatives of order $\leqslant p+q$ of η. By iterating this reasoning, we obtain the last part of the proposition. ☐

12.5.7. Remark. The previous proof also applies if we only assume that $\mathfrak{s}$ is $\mathcal{D}_{X,\overline{X}}$-linear *away from H*. Indeed, this implies that $\mathfrak{s}(m',\overline{m''})\cdot[b_{m'}(\mathrm{E})-P]$ is supported on H, and (12.5.5) only holds for $\operatorname{Re}s$ big enough, maybe $\gg p$. Then (12.5.6) coincides with a holomorphic current of degree 0 defined on $\{s\mid 2\operatorname{Re}s>p-k\}$ only for $\operatorname{Re}s\gg 0$. But, by uniqueness of analytic extension, it coincides with it on $\operatorname{Re}s>p$.

12.5.8. Corollary. *With the assumptions of Proposition* 12.5.4, *assume moreover that $\alpha'=\alpha''=:\alpha$. Let $[m']$ (resp. $[m'']$) be a germ of section of $\mathrm{gr}^V_\alpha\mathcal{M}'$ (resp. $\mathrm{gr}^V_\alpha\mathcal{M}''$) at x_o. Fix local liftings of m',m'' of $[m'],[m'']$ defined on $\mathrm{nb}(X,x_o)$. Then every polar coefficient at $s=\alpha$ of the meromorphic function $s\mapsto\big\langle\mathfrak{s}(m',\overline{m''}),|t|^{2s}\eta\big\rangle$ is the value of a well-defined current on $H\cap\mathrm{nb}(X,x_o)$ applied to the restriction $\eta_{|H}$. This current only depends on $[m'],[m'']$. It defines a sesquilinear pairing*

$$
\mathrm{gr}^V_\alpha\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathrm{gr}^V_\alpha\mathcal{M}''}\longrightarrow\mathfrak{C}_H\,.
$$

Proof. Any other local lifting of m' can be written as $m'+\mu'$, where μ' is a germ of section of $V_{<\alpha}\mathcal{M}'$. By the previous proposition, $\big\langle\mathfrak{s}(\mu',\overline{m''}),|t|^{2s}\eta\big\rangle$ is holomorphic at $s=\alpha$. If η vanishes on H, we have $\eta=t\eta_1+\bar t\eta_2$ for some test functions η_1,η_2, and we conclude similarly. The last part of Proposition 12.5.4, together with Lemma 12.2.7, shows each polar coefficient is a current on $\mathrm{nb}(X, x_o)$.

For the $\mathcal{D}_{H,\overline{H}}$-linearity, recall that $\mathrm{gr}_\alpha^V\mathcal{M}', \mathrm{gr}_\alpha^V\mathcal{M}''$ are $\mathcal{D}_H[\mathrm{E}]$-modules (see Remark 9.2.9). We can choose η of the form $\eta_o \cdot \chi(t)$, where η_o is a test function on H and χ is a cut-off function on Δ_t. Then the $\mathcal{D}_{H,\overline{H}}$-linearity of the pairing given by a polar coefficient is clear. ☐

***12.5.9. Remark* (The left case).** Assume that $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable left $\mathcal{D}_X$-modules equipped with a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ between them. Up to replacing X with a neighbourhood of H such that t has no critical point on $X \smallsetminus H$, any test form η on X, we can write $\eta = \eta_t \wedge \frac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t})$ for some relative differential form η_t on X. In particular, η_t restricts to a test form η_o on H. We consider the meromorphic function

$$s \longmapsto \big\langle |t|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\big\rangle,$$

and we link η to η_t by the previous relation. Then (left analogue of Corollary 12.5.8) the coefficients of the polar parts at α are the value of a distribution on H applied to η_o.

***12.5.10. Definition* (V-grading of a sesquilinear pairing).**

(1) (Left case) For every $\beta \in (-1, 0]$, the sesquilinear pairing

$$\mathrm{gr}_V^\beta\mathfrak{s} : \mathrm{gr}_V^\beta\mathcal{M}' \otimes \overline{\mathrm{gr}_V^\beta\mathcal{M}''} \longrightarrow \mathfrak{Db}_H$$

is well-defined by the formula

$$\big\langle \eta_o, \mathrm{gr}_V^\beta\mathfrak{s}([m'], \overline{[m'']})\big\rangle := \mathrm{Res}_{s=-\beta-1}\big\langle |t|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\big\rangle, \tag{12.5.10$\,*$}$$

where m', m'' are local liftings of $[m'], [m'']$ and η is any test form of maximal degree such that $\eta = \eta_1 \wedge \frac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t})$ with $\eta_{1|H} = \eta_o$.

(2) (Right case) For $\alpha \in [-1, 0)$, the sesquilinear pairing

$$\mathrm{gr}_\alpha^V(\mathfrak{s}) : \mathrm{gr}_\alpha^V\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{gr}_\alpha^V\mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

is well-defined by the formula

$$([m'], \overline{[m'']}) \longmapsto \Big[\eta_o \mapsto \mathrm{Res}_{s=\alpha}\big\langle \mathfrak{s}(m', \overline{m''}), |t|^{2s}\eta\big\rangle\Big], \tag{12.5.10$\,**$}$$

where m', m'' are local liftings of $[m'], [m'']$ and η is a test function such that $\eta|_H = \eta_o$.

***12.5.11. Lemma* (Side-changing).** *With the previous definition, we have*

$$\mathrm{gr}_\alpha^V\mathfrak{s}^{\mathrm{right}} = (\mathrm{gr}_V^\beta\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}} \qquad (\alpha = -\beta - 1).$$

Proof. Let η be a test function. We have

$$\big\langle \mathfrak{s}^{\mathrm{right}}(\omega' \otimes m', \overline{\omega'' \otimes m''}), |t|^{2s}\eta\big\rangle = \mathrm{Sgn}(n)\big\langle |t|^{2s}\eta\omega' \wedge \overline{\omega''}, \mathfrak{s}^{\mathrm{left}}(m', \overline{m''})\big\rangle,$$

hence, setting $\omega' = \omega'_o \wedge \mathrm{d}t$ and $\omega'' = \omega''_o \wedge \mathrm{d}t$, since $\mathrm{Sgn}(n-1) = (-1)^{n-1}\,\mathrm{Sgn}(n)\frac{2\pi}{\mathsf{i}}$,

$$\begin{aligned}\mathrm{Res}_{s=\alpha}&\big\langle \mathfrak{s}^{\mathrm{right}}(\omega'\otimes m', \overline{\omega''\otimes m''}), |t|^{2s}\eta\big\rangle\\ &= \mathrm{Sgn}(n)\,\mathrm{Res}_{s=-\beta-1}\big\langle |t|^{2s}\eta\,\omega'\wedge\overline{\omega''}, \mathfrak{s}^{\mathrm{left}}(m', \overline{m''})\big\rangle\\ &= (-1)^{n-1}\,\mathrm{Sgn}(n)\,\mathrm{Res}_{s=-\beta-1}\big\langle |t|^{2s}\eta\,\omega'_o\wedge\overline{\omega''_o}\wedge(\mathrm{d}t\wedge\overline{\mathrm{d}t}), \mathfrak{s}^{\mathrm{left}}(m', \overline{m''})\big\rangle\\ &= \mathrm{Sgn}(n-1)\big\langle \eta_o\,\omega'_o\wedge\overline{\omega''_o}, \mathrm{gr}_V^{\beta}\mathfrak{s}^{\mathrm{left}}([m'], \overline{[m'']})\big\rangle\\ &= \big\langle (\mathrm{gr}_V^{\beta}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}([\omega'\otimes m'], \overline{[\omega''\otimes m'']}), \eta_o\big\rangle. \qquad \square\end{aligned}$$

***12.5.12. Lemma* (N is self-adjoint).** *The nilpotent operator* $\mathrm{N} := -(\mathrm{E}-\beta)$ *is self-adjoint with respect to the pairing* (12.5.10 ∗)*, in the sense that*

$$\mathrm{gr}_V^{\beta}(\mathfrak{s})(\mathrm{N}[m'], \overline{[m'']}) = \mathrm{gr}_V^{\beta}(\mathfrak{s})([m'], \overline{\mathrm{N}[m'']}). \tag{12.5.12 ∗}$$

***Proof*.** The question is local, so we can assume $X = H\times\Delta_t$ and $\eta = \widetilde{\eta}_o\wedge\chi(t)\,\frac{\mathsf{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})$ with $\chi(t)\equiv 1$ near $t=0$. Then the statement is a consequence of the following properties (recall that $[|t|^{2s}\chi(t)\,\frac{\mathsf{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})]\cdot(-t\partial_t) = \partial_t t(|t|^{2s}\chi(t))\,\frac{\mathsf{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})$):

- β is real,
- $\bar t\partial_{\bar t}|t|^{2s} = t\partial_t|t|^{2s}$,
- $\bar t\partial_{\bar t}\chi(t)$ and $t\partial_t\chi(t)$ are zero in a neighbourhood of $t=0$. ☐

***12.5.13. Lemma* (Adjunction and V-grading).** *We have*

$$\mathrm{gr}_\alpha^V(\mathfrak{s}^*) = (\mathrm{gr}_\alpha^V\mathfrak{s})^*.$$

***Proof*.** Since α is real, we have $[\overline{m}] = \overline{[m]}$ in $\mathrm{gr}_\alpha^V\mathcal{M}$ and we can compute the residue by assuming that s varies in $\mathbb{R}$. We compute using Definition 12.3.2(2):

$$\begin{aligned}\langle\mathrm{gr}_\alpha^V(\mathfrak{s}^*)([m''], \overline{[m']}), \eta_o\rangle &= \mathrm{Res}_{s=\alpha}\big\langle\mathfrak{s}^*(m'', \overline{m'}), |t|^{2s}\eta\big\rangle\\ &= \mathrm{Res}_{s=\alpha}\,\overline{\big\langle\mathfrak{s}(m', \overline{m''}), |t|^{2s}\overline{\eta}\big\rangle}\\ &= \overline{\mathrm{Res}_{s=\alpha}\big\langle\mathfrak{s}(m', \overline{m''}), |t|^{2s}\overline{\eta}\big\rangle} = \overline{\big\langle\mathrm{gr}_\alpha^V\mathfrak{s}([m'], \overline{[m'']}), \overline{\eta}_o\big\rangle}\\ &= \big\langle(\mathrm{gr}_\alpha^V\mathfrak{s})^*([m''], \overline{[m']}), \eta_o\big\rangle. \qquad \square\end{aligned}$$

***12.5.14. Remark* (Properties for left sesquilinear pairings).** Due to Lemma 12.5.11, the previous properties also hold for right sesquilinear pairings.

We now consider the general case of nearby cycles along any holomorphic function g, for which the functor ψ_g is needed.

***12.5.15. Definition* (Sesquilinear pairing on nearby cycles).** Let $g : X \to \mathbb{C}$ be any holomorphic function. Assume that $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable along (g). For a sesquilinear pairing $\mathfrak{s} : \mathcal{M}'\otimes\overline{\mathcal{M}''} \to \mathfrak{C}_X$ resp. $\mathfrak{Db}_X$ and for every $\lambda\in\mathbb{S}^1$ and $\alpha\in[-1,0)$ such that $\lambda = \exp(2\pi\mathsf{i}\,\alpha)$, we define

$$\boxed{\psi_{g,\lambda}\mathfrak{s} := \mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}) : \psi_{g,\lambda}\mathcal{M}'\otimes\overline{\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{C}_X} \tag{12.5.15 ∗}$$

resp. for every $\beta \in (-1,0]$ such that $\lambda = \exp(-2\pi \mathrm{i}\,\beta)$,

$$\tag{12.5.15 $**$} \boxed{\psi_{g,\lambda}\mathfrak{s} := \mathrm{gr}_V^{\beta}({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}) : \psi_{g,\lambda}\mathcal{M}' \otimes \overline{\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{Db}_X.}$$

12.5.16. Properties of $\psi_{g,\lambda}\mathfrak{s}$**.** The following properties are obviously obtained from similar properties for $\mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota_*\mathfrak{s})$.

(1) $\psi_{g,\lambda}\mathfrak{s}(\mathrm{N}m', \overline{m''}) = \psi_{g,\lambda}\mathfrak{s}(m', \overline{\mathrm{N}m''})$ $\quad(m' \in \psi_{g,\lambda}\mathcal{M}'_{x_o},\ m'' \in \psi_{g,\lambda}\mathcal{M}''_{x_o})$.

(2) We have induced pairings

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathfrak{s} : \mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathcal{M}' \otimes \overline{\mathrm{gr}_{-\ell}^{\mathrm{M}}\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{C}_X \quad \text{resp. } \mathfrak{Db}_X,$$

and, for every $\ell \geqslant 0$,

$$\mathrm{P}_\ell\psi_{g,\lambda}\mathfrak{s} : \mathrm{P}_\ell\psi_{g,\lambda}\mathcal{M}' \otimes \overline{\mathrm{P}_\ell\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{C}_X \quad \text{resp. } \mathfrak{Db}_X$$

is induced by $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathfrak{s}(\bullet, \overline{\mathrm{N}^\ell\bullet})$.

(3) $\psi_{g,\lambda}(\mathfrak{s}^*) = (\psi_{g,\lambda}\mathfrak{s})^*$, according to Section 12.4.b and Lemma 12.5.13.

(4) Recall that $\mathcal{M}_g \simeq \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$. Assume that $m' \otimes 1$ is a local section of $V_\alpha(\mathcal{M}'_g)$ and $m'' \otimes 1$ is a local section of $V_\alpha(\mathcal{M}''_g)$ $(1 \leqslant \alpha < 0)$. In such a case, $\psi_{g,\lambda}\mathfrak{s}([m' \otimes 1], \overline{[m'' \otimes 1]})$ is given by a formula similar to (12.5.10 $**$), for any test function η on an open set of X where m', m'' are defined:

$$\tag{12.5.16 $*$} \big\langle \psi_{g,\lambda}\mathfrak{s}([m' \otimes 1], \overline{[m'' \otimes 1]}), \eta\big\rangle = \mathrm{Res}_{s=\alpha}\big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\big\rangle.$$

Indeed, the left-hand term is equal to

$$\mathrm{Res}_{s=\alpha}\big\langle {}_{\mathrm{T}}\iota_{g*}\mathfrak{s}([m' \otimes 1], \overline{[m'' \otimes 1]}), |t|^{2s}\eta_1\big\rangle$$

if $\eta_1(x,t)$ satisfies $\eta_1(x,0) = \eta$. This is also written as

$$\mathrm{Res}_{s=\alpha}\big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta_1(x, g(x))\big\rangle$$

and by developing η_1 with respect to powers of $t, \bar{t}$, one checks by the same argument as that of Proposition 12.5.4 that only $\eta_1(x,0) = \eta(x)$ contributes to the residue.

12.5.17. Remark **(Independence of the embedding).** We now take up the setting of Proposition 9.6.6. Let $\iota : X \hookrightarrow X_1$ be a closed inclusion of complex manifolds, and let g_1 be a holomorphic function on X_1. Let us set $g = g_1 \circ \iota$.

Let $\mathcal{N}', \mathcal{N}''$ be coherent right $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g) and let $\mathfrak{s} : \mathcal{N}' \otimes \overline{\mathcal{N}''} \to \mathfrak{C}_X$ be a sesquilinear pairing between them. We deduce a sesquilinear pairing ${}_{\mathrm{T}}\iota_*\mathfrak{s}$ between $\mathcal{M}' = {}_{\mathrm{D}}\iota_*\mathcal{N}'$ and $\mathcal{M}'' = {}_{\mathrm{D}}\iota_*\mathcal{N}''$. We will prove the equality, for all $\lambda \in S^1$,

$$\tag{12.5.17 $*$} \psi_{g_1,\lambda}({}_{\mathrm{T}}\iota_*\mathfrak{s}) = {}_{\mathrm{T}}\iota_*(\psi_{g,\lambda}(\mathfrak{s})).$$

Recall that (Section 12.4.b), if η is any test function on X_1 and n', n'' are local sections of $\mathcal{N}', \mathcal{N}''$, so that $n' \otimes 1, n'' \otimes 1$ are local sections of $\mathcal{M}', \mathcal{M}''$, then

$$\langle {}_{\mathrm{T}}\iota_*\mathfrak{s}(n' \otimes 1, \overline{n'' \otimes 1}), \eta\rangle := \langle \mathfrak{s}(n', \overline{n''}), \eta_{|X}\rangle.$$

For n', n'' in $V_\alpha\mathcal{N}', V_\alpha\mathcal{N}''$, η a test function on X_1 as above, we thus have, due to Formula (12.5.16 $*$) (arguing for $\operatorname{Re} s \gg 0$ first and then using analytic continuation),

$$\begin{aligned}\langle \operatorname{gr}^V_\alpha({}_{\mathrm{T}}\iota_*\mathfrak{s})([n'\otimes 1],\overline{[n''\otimes 1]}), \eta_{|H_1}\rangle &= \operatorname{Res}_{s=\alpha}\langle {}_{\mathrm{T}}\iota_*\mathfrak{s}(n'\otimes 1, \overline{n''\otimes 1}), |g_1|^{2s}\eta\rangle\\ &= \operatorname{Res}_{s=\alpha}\langle \mathfrak{s}(n', \overline{n''}), |g|^{2s}\eta_{|X}\rangle \quad \text{by definition (12.4.6)}\\ &= \langle \operatorname{gr}^V_\alpha \mathfrak{s}([n'], \overline{[n'']}), \eta|_H\rangle\\ &= \langle ({}_{\mathrm{T}}\iota_*\operatorname{gr}^V_\alpha \mathfrak{s})([n'\otimes 1], \overline{[n''\otimes 1]}), \eta_{|H_1}\rangle.\end{aligned}$$

12.5.c. Non-characteristic restrictions

12.5.18. Pullback in the case of smooth $\mathcal{D}_X$-modules. Assume that the left $\mathcal{D}_Y$-modules $\mathcal{M}', \mathcal{M}''$ are $\mathcal{O}_Y$-locally free of finite rank, and let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_Y$ be a sesquilinear pairing. Recall (Lemma 12.3.6) that $\mathfrak{s}$ takes values in the sheaf of C^∞ functions. If $f : X \to Y$ is any holomorphic map between complex manifolds, the pullback $\mathcal{D}_X$-modules ${}_{\mathrm{D}}f^*\mathcal{M}', {}_{\mathrm{D}}f^*\mathcal{M}''$ are equal to the $\mathcal{O}_X$-pullbacks equipped with their natural pullback flat connection, hence are also $\mathcal{O}_X$-locally free of finite rank, and there is a natural pullback of $\mathfrak{s}$ as taking values in $\mathcal{C}^\infty_X$, that we denote by ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}$, defined such that

$${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}(1\otimes m', \overline{1\otimes m''}) = \mathfrak{s}(m', \overline{m''})\circ f.$$

For example, if $\mathcal{M}' = \mathcal{M}'' = \mathcal{O}_Y$, then

$${}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s}_m = \mathfrak{s}_n.$$

This definition is compatible with the general one when f is a smooth morphism (Section 12.5.1). We will check that it coincides with that of 12.5.19 when f is the inclusion $\iota : H \hookrightarrow X$ of a smooth hypersurface, and $\mathcal{M}', \mathcal{M}''$ are $\mathcal{O}_X$-locally free of finite rank.

Assume that $X = H \times \Delta_t$. Then $\mathcal{M} = V^0\mathcal{M}$, $\operatorname{gr}^\beta_V V\mathcal{M} = 0$ for $\beta \notin \mathbb{N}$, and $\operatorname{gr}^0_V\mathcal{M} = \mathcal{M}/t\mathcal{M}$. Denoting by h'_o etc. the restriction to H, we obtain from Exercise 6.13, by choosing $\chi(t) = \widetilde{\chi}(|t|^2)$, the expected formula analogous to (12.3.7):

$$(\operatorname{gr}^0_V\mathfrak{s})(\mu'_o\otimes h'_o, \overline{\mu''_o\otimes h''_o}) = (\mathfrak{s}^\nabla(\mu', \overline{\mu''})\cdot h'\overline{h''})|_H,$$

that is, $\operatorname{gr}^0_V(\mathfrak{s}^\nabla\cdot\mathfrak{s}_n) = \mathfrak{s}^\nabla|_H\cdot\mathfrak{s}_{n-1}$. Indeed, let η be a test form of maximal degree written as $\eta = \eta_t \wedge \chi(t)\,\frac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})$ (see Remark 12.5.9). Then

$$\begin{aligned}\operatorname{Res}_{s=-1}\big\langle \eta_t\wedge\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t}), \mathfrak{s}(h'\otimes\mu', \overline{\mu''\otimes h''})|t|^{2s}\big\rangle&\\ = \operatorname{Res}_{s=-1}\mathfrak{s}^\nabla(\mu', \overline{\mu''})|_H \int_X |t|^{2s}h'\overline{h''}\cdot\eta_t\wedge\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})&\\ = \mathfrak{s}^\nabla(\mu', \overline{\mu''})|_H \int_H h'_o\overline{h''_o}\,\eta_o \quad \text{(Exercise 6.13).}&\end{aligned}$$

12.5.19. Non-characteristic restriction of a sesquilinear pairing along a smooth divisor

Let H be a smooth hypersurface of X and let $\mathcal{M}', \mathcal{M}''$ be coherent left $\mathcal{D}_X$-modules such that H is non-characteristic for them (see Section 8.8.d). Then $\operatorname{gr}^\beta_V\mathcal{M}$ are zero except for $\beta \in \mathbb{N}$, and E acts by zero on $\operatorname{gr}^0_V\mathcal{M} = \mathcal{M}/t\mathcal{M}$, so that $\operatorname{gr}^0_V\mathcal{M}$ is naturally a $\mathcal{D}_H$-module. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing and let g be a local smooth function defining H.

We note that, if m', m'' are local sections of $\mathcal{M}', \mathcal{M}''$ and $\eta = \eta_g \wedge \frac{\mathrm{i}}{2\pi}\,\mathrm{d}g \wedge \mathrm{d}\overline{g}$ is a test form on an open set on which both sections are defined, the meromorphic function $s \mapsto \langle |g|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\rangle$ has poles at most at $-1, -2, \dots$, and the pole at -1 is simple. Indeed, this is a local question and we can assume that $g = t$ is part of a local coordinate system (t, x). Since m', m'' are sections of V^0, the poles of $\langle |t|^{2s}\eta, \mathfrak{s}(tm', \overline{m''})\rangle$ are at most integers $\leqslant -2$, and similarly for $\langle |t|^{2s}\eta, \mathfrak{s}(m', \overline{tm''})\eta\rangle$, so that, if η_x is a local C^∞ form in the variable x only whose restriction to H is equal to $\eta_o := \eta_g|_H$, it is enough to prove the assertion for $\langle |t|^{2s}\eta_x \wedge \frac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\overline{t}, \mathfrak{s}(m', \overline{m''})\rangle$. Let us check that $(s+1)\langle |t|^{2s}\eta_x \wedge \frac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\overline{t}, \mathfrak{s}(m', \overline{m''})\rangle$ is holomorphic at $s = -1$. We have, since η_o is independent of t,

$$\begin{aligned}(s+1)\langle |t|^{2s}\eta_x \wedge \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\overline{t}, \mathfrak{s}(m', \overline{m''})\rangle &= -\langle (|t|^{2s}\eta_x \wedge \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\overline{t})\partial_t, \mathfrak{s}(m', \overline{m''})\rangle\\ &= -\langle |t|^{2s}\eta_x \wedge \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\overline{t}, \mathfrak{s}(\partial_t m', \overline{m''})\rangle,\end{aligned}$$

and since $\partial_t m'$ also belongs to $V_{-1}\mathcal{M}'$, the poles of this function are at most for $s + 1 = -1, -2, \dots$, hence the assertion.

We claim that the formula

$$\eta_o \longmapsto \operatorname{Res}_{s=-1}\langle |g|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\rangle, \quad \eta_o = \eta_g|_H,$$

similar to (12.5.10 ∗) with $\beta = 0$, is independent of the function g. Indeed, assume that $g' = hg$, where h is an invertible holomorphic function. Then modulo $g, \overline{g}$ we have $\mathrm{d}g' \wedge \mathrm{d}\overline{g'} = |h|^2 \mathrm{d}g \wedge \mathrm{d}\overline{g}$ and thus $\eta'_o := \eta_{g'|H} = |h|_H|^{-2}\eta_o$. The distributions

$$\eta_o \longmapsto \operatorname{Res}_{s=-1}\langle |g|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\rangle \quad \text{and} \quad \eta_o \longmapsto \operatorname{Res}_{s=-1}\langle |g'|^{2s}|h|^2\eta, \mathfrak{s}(m', \overline{m''})\rangle$$

coincide, as claimed. The formula above thus defines globally a sesquilinear pairing between $\mathrm{gr}_V^0\mathcal{M}' = {}_{\mathrm{D}}\iota^*\mathcal{M}'$ and $\mathrm{gr}_V^0\mathcal{M}'' = {}_{\mathrm{D}}\iota^*\mathcal{M}''$. We set

$$(12.5.19\,*) \qquad {}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s} := \mathrm{gr}_V^0\mathfrak{s} : {}_{\mathrm{D}}\iota^*\mathcal{M}' \otimes_{\mathbb{C}} \overline{{}_{\mathrm{D}}\iota^*\mathcal{M}''} \longrightarrow \mathfrak{Db}_H.$$

12.5.20. Non-characteristic restriction of a sesquilinear pairing along a submanifold

Let Y be a closed submanifold of X of any codimension and let U be an open chart in X. Let $u \in \mathfrak{Db}(U)$ be a distribution on U. We say that u *can be restricted to* $U \cap Y$ if there exists $v \in \mathfrak{Db}(U \cap Y)$ such that, for any sequence $\varphi_j \in \mathcal{C}_{\mathrm{c}}^\infty(U)$ which tends to u in $\mathfrak{Db}(U)$, the sequence $\varphi_j|_Y \in \mathcal{C}_{\mathrm{c}}^\infty(U \cap Y)$ tends to v in $\mathfrak{Db}(U \cap Y)$. We then set $v = {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_Y^* u$.

Let $\mathcal{M}', \mathcal{M}''$ be coherent left $\mathcal{D}_X$-modules which are non-characteristic along Y (see Section 8.8.d) and let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing.

Claim. *For any sections $m' \in \mathcal{M}'(U)$ and $m'' \in \mathcal{M}''(U)$, the distribution $\mathfrak{s}(m', m'') \in \mathfrak{Db}(U)$ can be restricted to $U \cap Y$. If* $\operatorname{codim} Y = 1$, *its restriction ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s}(m', m'')$ coincides with that defined by* (12.5.19 ∗).

Proof. The question is local as the restriction is unique if it exists. We can thus assume that $U = Y \times \Delta^r$ with coordinates $t_1, \dots, t_r$ on Δ^r. We argue by induction on r. Let us assume that the claim holds if $r = 1$. Set $H = \{t_1 = 0\}$. According to Remark 8.8.19, H is non-characteristic for $\mathcal{M}', \mathcal{M}''$, and Y is non-characteristic for ${}_{\mathrm{D}}\iota_H^*\mathcal{M}', {}_{\mathrm{D}}\iota_H^*\mathcal{M}''$.

By the case $r = 1$, $\mathfrak{s}(m', m'')$ can be restricted to H in some neighborhood of Y, and by induction ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_H^*\mathfrak{s}(m', m'')$ can be restricted to Y. It follows that $\mathfrak{s}(m', m'')$ can be restricted to Y and ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_Y^*\mathfrak{s}(m', m'') = {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_Y^*\big({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_H^*\mathfrak{s}(m', m'')\big)$.

It is thus enough to prove the last assertion of the claim. Let $\varphi_j \in \mathcal{C}_{\mathrm{c}}^\infty(U)$ be a sequence that converges to $\mathfrak{s}(m', m'')$ in $\mathfrak{Db}(U)$. By the computation of Section 12.5.18, we have for each test form η of maximal degree on U the equality

$$\operatorname{Res}_{s=-1}\langle |t|^{2s}\eta, \varphi_j\rangle = \langle \eta_o, \varphi_j|_H\rangle = \int_H (\varphi_j|_H)\eta_o.$$

Let Δ_s be a disc of radius $< 1/2$ centered at -1. For any j, the correspondences $\eta \mapsto (s+1)\langle |t|^2 s\eta, \varphi_j\rangle$ and $\eta \mapsto (s+1)\langle |t|^2 s\eta, \mathfrak{s}(m', m'')\rangle$ define distributions on $U \times \Delta_s$ depending continuously on $s \in \Delta_s$ and the latter is the limit of the family defined by the formers. Arguing as in Lemma 12.2.7, one checks that the same result holds after restricting to $s = -1$. In other words, $\lim_j \operatorname{Res}_{s=-1}\langle |t|^{2s}\eta, \varphi_j\rangle$ exists and is equal to $\operatorname{Res}_{s=-1}\langle |t|^{2s}\eta, \mathfrak{s}(m', m'')\rangle$. We conclude that the limit $\varphi_j|_H$ exists in $\mathfrak{Db}(U \cap Y)$ and is equal to ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_H^*\mathfrak{s}(m', m'')$ as defined by (12.5.19 $*$). In particular, this limit is independent of the choice of the sequence (φ_j) converging to $\mathfrak{s}(m', m'')$. $\square$

12.5.d. Examples

12.5.21. Specialization along a non-characteristic divisor with normal crossings

We take up the setting of Section 9.9.b and analyze how the isomorphisms (9.9.3 $*$) are compatible with sesquilinear pairings. Assume thus that the right $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ are non-characteristic along D_1, D_1, D_{12} and let $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing between them. We work in the local setting of Proposition 9.9.3.

Complementing Lemma 9.9.2, we note that, if η_o is a C^∞ function on X,

(a) the two-variable Mellin transform $(s_1, s_2) \mapsto \langle \mathfrak{s}(m', \overline{m''}), |x_1|^{2s_1}|x_2|^{2s_2}\eta_o\rangle$ has only simple poles along the lines $s_1 = -1 - k_1$, $s_2 = -1 - k_2$, with $k_1, k_2 \in \mathbb{N}$, and no other poles,

(b) $\mathrm{gr}_{-1}^{V^{(1)}}\mathrm{gr}_{-1}^{V^{(2)}}\mathfrak{s} = \mathrm{gr}_{-1}^{V^{(2)}}\mathrm{gr}_{-1}^{V^{(1)}}\mathfrak{s}$.

We will prove that, under the isomorphisms (9.9.3 $*$),

$$\mathrm{P}_0\psi_{g,1}\mathfrak{s} = \mathrm{gr}_0^{\mathrm{M}}\psi_{g,1}\mathfrak{s} = \psi_{x_1,1}\mathfrak{s} \oplus \psi_{x_2,1}\mathfrak{s}, \tag{12.5.21 $*$}$$

$$\mathrm{P}_1\psi_{g,1}\mathfrak{s} = \psi_{x_1,1}\psi_{x_2,1}\mathfrak{s} = \psi_{x_2,1}\psi_{x_1,1}\mathfrak{s}. \tag{12.5.21 $**$}$$

Let m' be a local section of $\mathcal{M}'$, let $m' \otimes 1$ its image in $\iota_{g*}\mathcal{M}'[\partial_t] = V_{-1}(\mathcal{M}'_g)$ and let $[m' \otimes 1]$ its class in $\mathrm{gr}_{-1}^V(\mathcal{M}'_g)$. Similar notation for m''. We have

$$\begin{aligned}\big\langle (\mathrm{gr}_{-1}^V{}_{\mathrm{T}}\iota_{g*}\mathfrak{s})([m' \otimes 1], \overline{[m'' \otimes 1]}), \eta_o\big\rangle \\ &= \operatorname{Res}_{s=-1}\big\langle {}_{\mathrm{T}}\iota_{g*}\mathfrak{s}(m' \otimes 1, \overline{m'' \otimes 1}), |t|^{2s}\eta_o\chi(t)\big\rangle \\ &= \operatorname{Res}_{s=-1}\big\langle \mathfrak{s}(m', \overline{m''}), |x_1x_2|^{2s}\eta_o\chi(g)\big\rangle \quad \text{(by (12.5.16 $*$)).}\end{aligned}$$

Let us consider the morphism $\varphi_1 : \mathcal{N}_1 = {}_{\mathrm{D}}\iota_{1*}\mathrm{gr}_{-1}^{V^{(1)}}\mathcal{M} \to \mathrm{gr}_0^{\mathrm{M}}\mathrm{gr}_{-1}^V(\mathcal{M}_g)$ as described in (9.9.11). It sends $m \otimes 1$ to $mx_2 \otimes 1$ $(m = m', m'')$, so that

$$\big\langle (\mathrm{gr}_0^{\mathrm{M}}\mathrm{gr}_{-1}^V{}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(\varphi_1[m' \otimes 1], \overline{\varphi_1[m'' \otimes 1]}), \eta_o\big\rangle = \operatorname{Res}_{s=-1}\big\langle \mathfrak{s}(m'x_2, \overline{m''x_2}), |x_1x_2|^{2s}\eta_o\big\rangle.$$

On the other hand, we have

$$\begin{aligned}\langle({}_{\mathrm{D}}\iota_{1*}\mathrm{gr}_{-1}^{V^{(1)}}\mathfrak{s})([m'],\overline{[m'']}),\eta_o\rangle &= \mathrm{Res}_{s=-1}\langle\mathfrak{s}(m',\overline{m''}),|x_1|^{2s}\eta_o\rangle\\ &\overset{(*)}{=} \mathrm{Res}_{s=-1}\langle\mathfrak{s}(m',\overline{m''}),|x_1|^{2s}|x_2|^{2s+2}\eta_o\rangle\\ &= \mathrm{Res}_{s=-1}\langle\mathfrak{s}(m'x_2,\overline{m''x_2}),|x_1x_2|^{2s}\eta_o\rangle.\end{aligned}$$

The equality $(*)$ is obtained by using that the two-variable Mellin transform $\langle\mathfrak{s}(m',\overline{m''}),|x_1|^{2s_1}|x_2|^{2s_2+2}\eta_o\rangle$ has no pole along $s_2=-1$. We argue similarly by changing the roles of x_1 and x_2, and we obtain (12.5.21 $*$) in this way, according to (12.5.15 $*$).

For (12.5.21 $**$), we wish to compute $\psi_{g,1}\mathfrak{s}([m'\otimes 1],\overline{\mathrm{N}[m''\otimes 1]})$ for

$$[m'\otimes 1]\in\mathrm{gr}_1^{\mathrm{M}}\mathrm{gr}_{-1}^V(\mathcal{M}'_g),\quad [m''\otimes 1]\in\mathrm{gr}_1^{\mathrm{M}}\mathrm{gr}_{-1}^V(\mathcal{M}''_g).$$

By using the isomorphisms (9.9.9) and (9.9.8), we find

$$\big\langle(\mathrm{gr}_{-1}^V{}_{\mathrm{T}}\iota_{g*}\mathfrak{s})([m'\otimes 1],\overline{\mathrm{N}[m''\otimes 1]}),\eta_o\big\rangle = \mathrm{Res}_{s=-1}(s+1)\big\langle\mathfrak{s}(m',\overline{m''}),|x_1x_2|^{2s}\eta_o\big\rangle,$$

since $\mathrm{N}=\partial_t t$. One notices that, for a meromorphic function $\varphi(s_1,s_2)$ on $\mathbb{C}^2$ as in (a) above, we have

$$\begin{aligned}\mathrm{Res}_{s=-1}(s+1)\varphi(s,s) &= \mathrm{Res}_{s_1=-1}\mathrm{Res}_{s_2=-1}\varphi(s_1,s_2)\\ &= \mathrm{Res}_{s_2=-1}\mathrm{Res}_{s_1=-1}\varphi(s_1,s_2).\end{aligned}$$

Then (12.5.21 $**$) follows.

12.5.22. Nearby cycles along a monomial function of a smooth $\mathcal{D}$-module

We take up the setting of Section 9.9.c. Since the question is local, we set $X=\mathbb{C}^n$. Let us assume that $\mathcal{M}',\mathcal{M}''$ are smooth left $\mathcal{D}_X$-modules and let $\mathfrak{s}:\mathcal{M}'\otimes\overline{\mathcal{M}''}\to\mathfrak{Db}_X$ be a sesquilinear pairing. We know that $\mathfrak{s}$ takes values in $\mathcal{C}_X^\infty$ (Lemma 12.3.6). In particular, the restriction ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_I^*\mathfrak{s}$ is a well-defined sesquilinear pairing between ${}_{\mathrm{D}}\iota_I^*\mathcal{M}'$ and ${}_{\mathrm{D}}\iota_I^*\mathcal{M}''$. We will show that, under the isomorphism of Proposition 9.9.12(3), we have

$$\text{(12.5.22 $*$)}\qquad \mathrm{P}_\ell\psi_{g,1}\mathfrak{s} = \bigoplus_{J\in\mathfrak{J}_{\ell+1}}{}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_I^*\mathfrak{s})\qquad (I=J^{\mathrm{c}}).$$

We will restrict to the case where $\mathcal{M}'=\mathcal{M}''=\mathcal{O}_X$ and $\mathfrak{s}=\mathfrak{s}_n$, the general case being similar (see Example 12.5.18). By the $\mathcal{D}$-linearity of the isomorphism in Lemma 9.9.21, it is enough to compute, for $J',J''\in\mathfrak{J}_{\ell+1}$, $\mathrm{P}_\ell\psi_{g,1}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})$ and to show that

- it is zero if $J'\neq J''$,
- if $J'=J''=J$, it is equal to ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{I*}\mathfrak{s}_I(\delta_J,\overline{\delta_J})$, where $\mathfrak{s}_I={}_{\mathrm{D},\overline{\mathrm{D}}}\iota_I^*\mathfrak{s}_n$ is the standard pairing on $\mathcal{O}_I$.

By definition, for a test form of maximal degree $\eta(x)$ and a cutoff function $\chi(t)$,

$$\begin{aligned}\big\langle\eta,\mathrm{P}_\ell\psi_{g,1}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle &= \big\langle\eta,\psi_{g,1}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{\mathrm{N}^\ell x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle\\ &= \mathrm{Res}_{s=-1}\big\langle\eta\chi(t)|t|^{2s}(\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}),\overline{(-t\partial_t)^\ell}{}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{g*}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle\\ &= \mathrm{Res}_{s=-1}(s+1)^\ell\big\langle\eta\chi(t)|t|^{2s}(\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}),{}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{g*}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle,\end{aligned}$$

where we have used, for the third equality, the property that the expression $\langle\eta\partial_{\overline{t}}\chi|t|^{2s}\cdots,\cdots\rangle$ is an entire function of s. By a computation already done, the last term can be expressed as

$$\mathrm{Res}_{s=-1}(s+1)^\ell\big\langle\eta\chi(g)|g|^{2s},\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}},\overline{x_{I''}^{\mathbf{1}_{I''}}})\big\rangle = \mathrm{Res}_{s=-1}(s+1)^\ell\int_{\Delta_r^n} x_{I'}^{\mathbf{1}_{I'}}\overline{x_{I''}^{\mathbf{1}_{I''}}}\chi(g)|g|^{2s}\eta,$$

where $\Delta_r\subset\mathbb{C}$ is a disc of radius $r>0$ large enough so that Δ_r^n contains the compact support of η. Let us forget $\chi(g)$, which plays no role, and let us assume that $p=n$, since the variables $x_{p+1},\dots,x_n$ do not play any important role here. Due to the formula (obtained by computing with polar coordinates)

$$\int_{\Delta_r}|t|^{2s}t^p\overline{t}^q\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t} = \begin{cases}0 & \text{if } p\neq q,\\ \dfrac{r^{2(s+p+1)}}{s+p+1} & \text{if } p=q,\end{cases}$$

we find that, for a monomial $\eta_{\boldsymbol{p},\boldsymbol{q}} = x^{\boldsymbol{p}}\overline{x}^{\boldsymbol{q}}\prod_i(\frac{\mathrm{i}}{2\pi}\,\mathrm{d}x_i\wedge\mathrm{d}\overline{x}_i)$ $(p_i,q_i\geqslant 0)$,

$$\int_{\Delta_r^n} x_{I'}^{\mathbf{1}_{I'}}\overline{x_{I''}^{\mathbf{1}_{I''}}}|g|^{2s}\eta_{\boldsymbol{p},\boldsymbol{q}} = \begin{cases}0 & \text{if } \exists\, i,\ (\boldsymbol{p}+\mathbf{1}_{I'})_i\neq(\boldsymbol{q}+\mathbf{1}_{I''})_i,\\ \prod_i\dfrac{r^{2(s+1+(\boldsymbol{p}+\mathbf{1}_{I'})_i)}}{s+1+(\boldsymbol{p}+\mathbf{1}_{I'})_i} & \text{if } \forall\, i,\ (\boldsymbol{p}+\mathbf{1}_{I'})_i=(\boldsymbol{q}+\mathbf{1}_{I''})_i.\end{cases}$$

A pole at $s=-1$ can occur only if $J'=J''$, that we denote by J (hence $I'=I''$) and $p_J=q_J=0_J$. It is then of order $\ell+1$. We conclude that the residue above is nonzero only if $I'=I''$, that we denote by I, and if we set $\eta=\eta_I\prod_{j\in J}(\frac{\mathrm{i}}{2\pi}\,\mathrm{d}x_j\wedge\mathrm{d}\overline{x}_j)$, we obtain

$$\mathrm{Res}_{s=-1}(s+1)^\ell\int_{\Delta_r^n} x_{I'}^{\mathbf{1}_{I'}}\overline{x_{I''}^{\mathbf{1}_{I''}}}|g|^{2s}\eta = \int_{\Delta_r^I}\eta_{I|X_I},$$

where $X_I=\{x_j=0\mid\forall\, j\in J\}$. According to Example 12.4.8, this is nothing but $\langle\eta,({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{I*}\mathfrak{s}_I)(\delta_J,\overline{\delta_J})\rangle$.

12.5.e. Sesquilinear pairing on vanishing cycles. If $\mathcal{M}',\mathcal{M}''$ are supported on $g^{-1}(0)$, the residue formulas (12.5.10 $*$) or (12.5.10 $**$) with kernel $|g|^{2s}$ return the value zero since any local section m of $\mathcal{M}$ is annihilated by some power of g and, for $\mathrm{Re}(s)\gg 0$, the function (12.5.6) is identically zero (see Exercise 12.10). Therefore, these formulas do not lead to a definition of an interesting sesquilinear pairing

$$\phi_{t,1}\mathfrak{s}:\phi_{t,1}\mathcal{M}'\otimes\overline{\phi_{t,1}\mathcal{M}''}\longrightarrow\mathfrak{C}_X\quad\text{resp. }\mathfrak{Db}_X$$

for every $\mathcal{M}',\mathcal{M}''$ which are coherent and $\mathbb{R}$-specializable along $t=0$.

On the other hand, the Thom-Sebastiani formula for vanishing cycles (Section 11.8) led us to interpret vanishing cycles in terms of the algebraic microlocalized modules, by introducing the operator ∂_t^{-1} as a new variable θ. Since, by definition, θ acts in an invertible way on the microlocalized module, we could try to apply a residue formula with kernel $|\theta|^{2s}$ in order to define $\phi_{t,1}\mathfrak{s}$. Interpreting the variable $\tau = \theta^{-1}$ as the Fourier dual of the variable t, we will apply a residue formula with the kernel obtained by inverse Fourier transform from $|\theta|^{2(s-1)}$ (a similar shift by -1 having been already observed in Remark 11.8.3). We will use the properties of the functions $I_{\widehat{\chi}}(t,s)$ and $\widehat{I}_{\chi}(\tau,s)$ introduced in Exercises 7.21 and 7.22, in order to extend Definition 7.3.15 and the properties of $\phi_{t,1}\mathfrak{s}$ in dimension one.

12.5.23. The function $s\mapsto \langle\mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\rangle$. We take up the setting and notation of Proposition 12.5.4 with right $\mathcal{D}_X$-modules $\mathcal{M}',\mathcal{M}''$. The properties of the functions $I_{\widehat{\chi},k,k}$ obtained in Exercise 7.21 enable us to apply arguments similar to those of Proposition 12.5.4 and Corollary 12.5.8 to obtain that, for $x_o\in X$, for local sections $m'\in V_0\mathcal{M}'_{x_o}$ and $m''\in V_0\mathcal{M}''_{x_o}$, and for any test function on $\mathrm{nb}(X,x_o)$, the function

$$s\longmapsto \big\langle\mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle$$

extends as a meromorphic function on the plane $\mathbb{C}$ with possible poles contained in $\mathbb{R}_{\leqslant 0}$. The correspondence

$$\eta_o\longmapsto \operatorname{Res}_{s=0}\big\langle\mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle,$$

for any test function η with $\eta|_H=\eta_o$, well-defines a current on H, which only depends on the classes $[m'],[m'']$ in $\mathrm{gr}_0^V\mathcal{M}',\mathrm{gr}_0^V\mathcal{M}''$. There is a left analogue, as in Remark 12.5.9.

We can now mimic Definition 12.5.10.

12.5.24. Definition **(V-grading of a sesquilinear pairing, continued)**

(1) (Left case) The sesquilinear pairing

$$\mathrm{gr}_V^{-1}\mathfrak{s}:\mathrm{gr}_V^{-1}\mathcal{M}'\otimes\overline{\mathrm{gr}_V^{-1}\mathcal{M}''}\longrightarrow\mathfrak{Db}_H$$

is well-defined by the formula

$$\big\langle\eta_o,\mathrm{gr}_V^{-1}\mathfrak{s}([m'],\overline{[m'']})\big\rangle := \operatorname{Res}_{s=0}\big\langle I_{\widehat{\chi}}(g,s)\eta,\mathfrak{s}(m',\overline{m''})\big\rangle, \tag{12.5.24 $*$}$$

where m',m'' are local liftings of $[m'],[m'']$ and η is any test form of maximal degree such that $\eta=\widetilde{\eta}_o\wedge\frac{\mathrm{i}}{2\pi}(\mathrm{d}g\wedge\mathrm{d}\overline{g})$ with $\widetilde{\eta}_{o|H}=\eta_o$.

(2) (Right case) The sesquilinear pairing

$$\mathrm{gr}_0^V(\mathfrak{s}):\mathrm{gr}_0^V\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathrm{gr}_0^V\mathcal{M}''}\longrightarrow\mathfrak{C}_H$$

is well-defined by the formula

$$([m'],\overline{[m'']})\longmapsto\Big[\eta_o\mapsto\operatorname{Res}_{s=0}\big\langle\mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle\Big], \tag{12.5.24 $**$}$$

where m',m'' are local liftings of $[m'],[m'']$ and η is a test function such that $\eta|_H=\eta_o$.

12.5.25. Some properties of $\mathrm{gr}_V^{-1}\mathfrak{s}$ ***and*** $\mathrm{gr}_0^V\mathfrak{s}$**.** The proof of Lemma 12.5.11 extends to the present case and shows that

$$\mathrm{gr}_0^V\mathfrak{s}^{\mathrm{right}} = (\mathrm{gr}_V^{-1}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}.$$

The proofs of Lemmas 12.5.12, 12.5.13 and of $(12.5.17\,*)$ also extend to the present case, due to the properties of $I_{\widehat{\chi}}$ given in Exercise 7.22. We conclude that N is self-adjoint for $\mathrm{gr}_V^{-1}\mathfrak{s}$ and $\mathrm{gr}_0^V\mathfrak{s}$, that the functors gr_V^{-1} and gr_0^V on sesquilinear forms commute with Hermitian adjunction and they do not depend on the embedding.

12.5.26. Definition **(Vanishing cycles of a sesquilinear pairing)**

Let $g : X \to \mathbb{C}$ be any holomorphic function and let $\mathcal{M}', \mathcal{M}''$ be $\mathbb{R}$-specializable. If $\mathfrak{s}$ is a sesquilinear pairing between them, we set

$$\phi_{g,1}\mathfrak{s} := \mathrm{gr}_V^{-1}({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}) \quad \text{resp. } \mathrm{gr}_0^V({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}),$$

which is a sesquilinear pairing between $\phi_{g,1}\mathcal{M}'$ and $\phi_{g,1}\mathcal{M}''$.

12.5.27. Remark **(Properties of** $\phi_{g,1}\mathfrak{s}$**).** The properties of Remark 12.5.16 extend to similar properties for $\phi_{g,1}\mathfrak{s}$. In particular, 12.5.16(4) reads, for $m'\otimes 1 \in V_0({}_{\mathrm{D}}\iota_*\mathcal{M}')$ and $m''\otimes 1 \in V_0({}_{\mathrm{D}}\iota_*\mathcal{M}'')$,

$$(12.5.27\,*) \qquad \big\langle \phi_{g,1}\mathfrak{s}([m'\otimes 1], \overline{[m''\otimes 1]}), \eta\big\rangle = \mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle.$$

12.5.28. Proposition **(Behaviour with respect to** can **and** var**)**

The following equalities holds for $[m']$ *in* $\psi_{g,1}\mathcal{M}'$ *and* $[m''] \in \phi_{g,1}\mathcal{M}''$*, resp.* $[m']$ *in* $\phi_{g,1}\mathcal{M}'$ *and* $[m''] \in \psi_{g,1}\mathcal{M}''$*:*

$$(\phi_{g,1}\mathfrak{s})(\mathrm{can}[m'], \overline{[m'']}) = -(\psi_{g,1}\mathfrak{s})([m'], \overline{\mathrm{var}[m'']})$$

$$\text{resp.} \qquad (\phi_{g,1}\mathfrak{s})([m'], \overline{\mathrm{can}[m'']}) = -(\psi_{g,1}\mathfrak{s})(\mathrm{var}[m'], \overline{[m'']}).$$

Proof. Let us show the first equality in the right setting for example. We choose a test function η on $X \times \mathbb{C}$ of the form $\eta = \eta_o\chi(t)$, where η_o is C^∞ with compact support on X, χ is a cut-off function near $t = 0$. We need to show

$$(12.5.29) \quad -\mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), \partial_t\big(I_{\widehat{\chi}}(t,s)\chi(t)\big)\eta_o(x)\big\rangle = \mathrm{Res}_{s=-1}\big\langle \mathfrak{s}(m', \overline{m''}), \bar{t}|t|^{2s}\chi(t)\eta_o(x)\big\rangle,$$

where $m' \in V_{-1}(\mathcal{M}'_g)_{(x_o,0)}$ and $m'' \in V_0(\mathcal{M}''_g)_{(x_o,0)}$ are respective liftings of $[m']$ and $[m'']$, and we have written $\mathfrak{s}$ instead of ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{g*}\mathfrak{s}$. Let us consider the left-hand side. By Exercise 7.21(6), the function $s \mapsto \big\langle \mathfrak{s}(m', \overline{m''}), I_{\widehat{\chi}}(t,s)\partial_t(\chi(t))\eta_o(x)\big\rangle$ is holomorphic. The left-hand side is thus equal to

$$-\mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), \partial_t(I_{\widehat{\chi}}(t,s))\chi(t)\eta_o(x)\big\rangle$$

and, using Exercise 7.21(7) and arguing for $\mathrm{Re}\, s > 1$ first, it is equal to

$$(12.5.30) \qquad \mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), I_{\widehat{\chi},-1,0}(t,s)\chi(t)\eta_o(x)\big\rangle.$$

Let us denote by T the one-variable current with compact support (by definition of χ)

$$\mathcal{C}^\infty(\mathbb{C}) \ni \varphi(t) \longmapsto \langle \mathfrak{s}(m', \overline{m''}), \varphi(t)\chi(t)\eta_o\rangle.$$

Its Fourier transform $\mathcal{F}T := \langle T, e^{\overline{t\tau}-t\tau}\rangle$ is a C^∞ function of τ, which has moderate growth, as well as all its derivatives, when $\tau \to \infty$. The function whose residue is taken in (12.5.30) is then written as

$$\int \tau^{-1} |\tau|^{-2(s+1)} \widehat{\chi}(\tau)\mathcal{F}T(\tau) \tfrac{\mathrm{i}}{2\pi}(d\tau \wedge \mathrm{d}\overline{\tau}). \tag{12.5.31}$$

On the other hand, up to replacing χ with χ^2 in (12.5.29), which does not change the residue, as previously remarked, the function in the RHS of (12.5.29) is

$$\begin{aligned}\langle T, t|t|^{2s}\chi(t)\rangle &= \big\langle \mathcal{F}T \tfrac{\mathrm{i}}{2\pi}(d\tau \wedge \mathrm{d}\overline{\tau}), \mathcal{F}^{-1}(t|t|^{2s}\chi(t))\big\rangle\\ &= \int \widehat{I}_{\chi,1,0}(\tau,s)\cdot \mathcal{F}T(\tau) \tfrac{\mathrm{i}}{2\pi}(d\tau \wedge \mathrm{d}\overline{\tau}),\end{aligned} \tag{12.5.32}$$

where we have set

$$\widehat{I}_{\chi,k,\ell}(\tau,s) := \mathcal{F}^{-1}(|t|^{2s}t^k\overline{t}^\ell\chi(t)),$$

and the properties we need for the function $\widehat{I}_{\chi,1,0}(\tau,s)$ are made precise in Exercise 7.22. Using the function $\widehat{\chi}(\tau)$ as above, we conclude from Exercise 7.22(7) that the integral

$$\int \mathcal{F}T(\tau)\cdot \widehat{I}_{\chi,1,0}(\tau,s)\cdot (1-\widehat{\chi}(\tau)) \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} \tag{12.5.33}$$

is holomorphic with respect to s for $\operatorname{Re} s > -3/2$. It can thus be neglected when computing the residue at $s=-1$. The question reduces therefore to the comparison between $\widehat{I}_{\chi,1,0}(\tau,s)$ and $\tau^{-1}|\tau|^{-2(s+1)}$ when $\tau \to \infty$.

Let us set $\widehat{J}_{\chi,1,0}(\tau,s) = \tau|\tau|^{2(s+1)}\widehat{I}_{\chi,1,0}(\tau,s)$. Then, by (7.7.0 $**$), we have

$$\tau\frac{\partial \widehat{J}_{\chi,1,0}}{\partial\tau} = -\widehat{J}_{\partial\chi/\partial t,2,0}, \quad \overline{\tau}\frac{\partial \widehat{J}_{\chi,1,0}}{\partial\overline{\tau}} = -\widehat{J}_{\partial\chi/\partial\overline{t},1,1},$$

and both functions $\widehat{J}_{\partial\chi/\partial t,2,0}$ and $\widehat{J}_{\partial\chi/\partial\overline{t},1,1}$ can be extended as C^∞ functions, infinitely flat at $\tau=\infty$ and holomorphic with respect to $s\in\mathbb{C}$.

12.5.34. Lemma. *For s in the strip $\operatorname{Re}(s+1)\in(-1,-1/4)$, the function $\tau\mapsto\widehat{J}_{\chi,1,0}(\tau,s)$ satisfies*

$$\lim_{\tau\to\infty}\widehat{J}_{\chi,1,0}(\tau,s) = -\frac{\Gamma(s+2)}{\Gamma(-s)}.$$

Proof. We can assume that χ is a C^∞ function of $|t|^2$. For simplicity, we assume that $\chi\equiv 1$ for $|t|\leqslant 1$. Then the limit of $\widehat{J}_{\chi,1,0}$ is also equal to the limit of the integral

$$J(\tau,s) = \int_{|t|\leqslant 1} e^{-\overline{t\tau}+t\tau}\, t\tau\, |t\tau|^{2(s+1)} \tfrac{\mathrm{i}}{2\pi}\frac{\mathrm{d}t}{t}\wedge\frac{\mathrm{d}\overline{t}}{\overline{t}}.$$

By a simple change of variables, we have

$$J(\tau,s) = \int_{|u|\leqslant|\tau|} e^{2i\operatorname{Im} u} u\, |u|^{2s} \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}u\wedge\mathrm{d}\overline{u}.$$

Using the Bessel function $J_{\pm 1}(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ix\sin y}e^{\pm iy}\mathrm{d}y$, we can write

$$\begin{aligned}J(\tau,s) &= 2\int_{\rho\leqslant|\tau|} J_{-1}(2\rho)\rho^{2(s+1)}\mathrm{d}\rho\\ &= -2^{-2(s+1)}\int_{\rho\leqslant 2|\tau|} J_1(\rho)\rho^{2(s+1)}\mathrm{d}\rho, \quad \text{as } J_1 = -J_{-1}.\end{aligned}$$

For $\operatorname{Re}(s+1)\in(-1,-1/4)$, the limit when $|\tau|$ of the previous integral is equal to $2^{2(s+1)}\Gamma(s+2)/\Gamma(-s)$ (see [**Wat22**, §13.24, p. 391]). ☐

Let us set

$$\widehat{K}_\chi(\tau,s) = -\int_0^1 \big[\widehat{J}_{\partial\chi/\partial t,1,0}(\lambda\tau,s) + \widehat{J}_{\partial\chi/\partial\bar{t},0,1}(\lambda\tau,s)\big]\mathrm{d}\lambda.$$

Then $\widehat{K}_\chi$ is also C^∞, infinitely flat at $\tau=\infty$ and holomorphic with respect to $s\in\mathbb{C}$. According to Lemma 12.5.34, we can write, on the strip $\operatorname{Re}(s+1)\in(-1/2,-1/4)$,

$$\widehat{I}_{\chi,1,0}(\tau,s) = -\tau^{-1}\,|\tau|^{-2(s+1)}\,\frac{\Gamma(s+2)}{\Gamma(-s)} + K_\chi(\tau,s) \tag{12.5.35}$$

where $K_\chi(\tau,s) = -\tau^{-1}\,|\tau|^{-2(s+1)}\,\widehat{K}_\chi(\tau,s)$ is C^∞ on $\mathbb{C}\times\mathbb{C}$, infinitely flat at $\tau=\infty$ and holomorphic with respect to s. For any $p\geqslant 0$, let us apply $(\partial_\tau\partial_{\overline{\tau}})^p$ to the previous equality restricted to $\tau\neq 0$ (where both sides are C^∞ in τ and holomorphic with respect to s; preferably, multiply both sides by $\widehat{\chi}(\tau)$), to get, for s in the same strip,

$$\widehat{I}_{\chi,1,0}(\tau,s+p) = -\tau^{-1}\,|\tau|^{-2(s+p+1)}\,\frac{\Gamma(s+p+2)}{\Gamma(-s-p)} + (\partial_\tau\partial_{\overline{\tau}})^p K_\chi(\tau,s)$$

where the last term remains infinitely flat at $\tau=\infty$. It follows that (12.5.35) remains true on any strip $\operatorname{Re}(s+1)\in(p-1/2,p-1/4)$ with $p\geqslant 0$ and a function $K_\chi^{(p)}$ instead of K_χ.

Choose p such that the two the meromorphic functions considered in (12.5.29) are holomorphic on the strip $\operatorname{Re}(s+1)\in(p-1/2,p-1/4)$. The difference between $\Gamma(s+2)/\Gamma(-s)$ times the function in the LHS and the function in the RHS coincides, on this strip, with the restriction of a holomorphic function defined on the half-plane $\{s\mid \operatorname{Re}s>-3/2\}$ (taking into account (12.5.33) and $K_\chi^{(p)}$). It is then equal to it on this whole half-plane, hence has residue 0 at $s=-1$. ☐

12.5.36. Corollary.

(1) *Assume that* $\mathcal{M}',\mathcal{M}''$ *are middle extensions along* $g^{-1}(0)$ *(see Definition* 9.7.3*), so that* $\phi_{g,1}\mathcal{M}\simeq\operatorname{Im}[\mathrm{N}:\psi_{g,1}\mathcal{M}\to\psi_{g,1}\mathcal{M}]$ *for* $\mathcal{M}=\mathcal{M}',\mathcal{M}''$*. Then, for* $[\mu']\in\phi_{g,1}\mathcal{M}'$ *and* $[\mu'']\in\phi_{g,1}\mathcal{M}''$*, and* $[\mu']=\mathrm{N}[m']$, $[\mu'']=\mathrm{N}[m'']$*, we have*

$$\phi_{g,1}\mathfrak{s}([\mu'],\overline{[\mu'']}) = -\psi_{g,1}\mathfrak{s}([m'],\overline{\mathrm{N}[m'']}) = -\psi_{g,1}\mathfrak{s}(\mathrm{N}[m'],\overline{[m'']}).$$

(2) *Assume that* $\mathcal{M}',\mathcal{M}''$ *are supported on* $g^{-1}(0)$*, so that* $\mathcal{M}=\phi_{g,1}\mathcal{M}$ *for* $\mathcal{M}=\mathcal{M}',\mathcal{M}''$*. Then* $\mathfrak{s}=\phi_{g,1}\mathfrak{s}$.

Proof.

(1) The second equality is due to Remark 12.5.16(1). The identification of $\phi_{g,1}\mathcal{M}$ with $\operatorname{Im}\mathrm{N}$ implies precisely that $[\mu'] = \mathrm{can}[m']$ and $[\mu''] = \mathrm{can}[m'']$. Then, according to Proposition 12.5.28,

$$\begin{aligned}\phi_{g,1}\mathfrak{s}([\mu'], \overline{[\mu'']}) &= \phi_{g,1}\mathfrak{s}(\mathrm{can}[m'], \overline{\mathrm{can}[m'']})\\ &= -\psi_{g,1}\mathfrak{s}([m'], \overline{\mathrm{var}\,\mathrm{can}[m'']}) = -\psi_{g,1}\mathfrak{s}([m'], \overline{\mathrm{N}[m'']}).\end{aligned}$$

(2) We can assume that $X = H\times\mathbb{C}$ and g is the projection $(x,t)\mapsto t$ (x is the variable in H) and we set $\iota : H\times\{0\}\hookrightarrow X$. We can assume that we are given $\mathcal{D}_H$-modules $\mathcal{M}'_o, \mathcal{M}''_o$ and a sesquilinear pairing between them. For local sections m'_o, m''_o, we have to identify $\mathrm{gr}^V_{-1}({}_{\mathrm{T}}\iota_*\mathfrak{s}_o)([m'_o\otimes 1], \overline{([m''_o\otimes 1]})$ with $\mathfrak{s}_o([m'_o], \overline{([m''_o]})$. Setting $\eta = \eta_o(x)\chi(t)$, we have, since $\chi(0) = 1$,

$$\big\langle({}_{\mathrm{T}}\iota_*\mathfrak{s}_o)((m'_o\otimes 1), \overline{((m''_o\otimes 1))}), \eta_o I_{\widehat{\chi}}(t,s)\chi(t)\big\rangle = \big\langle\mathfrak{s}_o(m'_o, \overline{(m''_o}), \eta_o\big\rangle\cdot I_{\widehat{\chi}}(0,s).$$

The assertion follows since $\mathrm{Res}_{s=0}\, I_{\widehat{\chi}}(0,s) = 1$ (see Exercise 7.21(2)). ☐

12.5.37. Remark **(Kashiwara's equivalence and $\mathrm{gr}^V_0\,\mathfrak{s}$).** Let $H\xhookrightarrow{\iota} X$ be the inclusion of a smooth hypersurface (not necessarily defined by a global equation). If $\mathcal{M} = \mathcal{M}', \mathcal{M}''$ are coherent $\mathcal{D}_H$-modules, we have $V_0({}_{\mathrm{D}}\iota_*\mathcal{M}) = \mathcal{M}\otimes\mathbf{1}$ (notation of Section 12.4.b) and, for a sesquilinear pairing $\mathfrak{s}_H$ between $\mathcal{M}'$ and $\mathcal{M}''$, we recover $\mathfrak{s}_H$ from ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_H$ by Formula (12.5.24 $**$) for any local equation of H, which therefore does not depend on the choice of such a local equation. We then denote $\mathfrak{s}_H = \mathrm{gr}^V_0({}_{\mathrm{T}}\iota_*\mathfrak{s}_H)$.

12.5.f. (Dual) localization of a sesquilinear pairing.

Let $\mathfrak{s} : \mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''}\to\mathfrak{C}_X$ be a sesquilinear pairing between right $\mathcal{D}_X$-modules, and let D be an effective divisor in X. Recall that localization and dual localization are defined for $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along D and that we have natural morphisms (see Corollaries 11.3.10(2) and 11.4.9(2))

$$\mathcal{M}(!D)\xrightarrow{\mathrm{dloc}}\mathcal{M}\xrightarrow{\mathrm{loc}}\mathcal{M}(*D).$$

According to the results recalled above, $\mathfrak{s}$ defines a moderate sesquilinear pairing by localization:

$$\mathfrak{s}^{\mathrm{mod}\,D} : \mathcal{M}'(*D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(*D)}\longrightarrow\mathfrak{C}_X^{\mathrm{mod}\,D}.$$

Our aim is to refine it as a pairing taking values in $\mathfrak{C}_X$. We note that $\mathfrak{s}^{\mathrm{mod}\,D}$ naturally induces sesquilinear pairings

$$\begin{aligned}\mathfrak{s}^{(*D)} &: \mathcal{M}'(*D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(!D)}\longrightarrow\mathfrak{C}_X^{\mathrm{mod}\,D},\\ \mathfrak{s}^{(!D)} &: \mathcal{M}'(!D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(*D)}\longrightarrow\mathfrak{C}_X^{\mathrm{mod}\,D},\end{aligned}\tag{12.5.38}$$

by setting

$$\mathfrak{s}^{(*D)}(\bullet,\overline{\bullet}) = \mathfrak{s}^{\mathrm{mod}\,D}(\bullet,\overline{\mathrm{loc}''\circ\mathrm{dloc}''(\bullet)}),\quad \mathfrak{s}^{(!D)}(\bullet,\overline{\bullet}) = \mathfrak{s}^{\mathrm{mod}\,D}(\mathrm{loc}'\circ\mathrm{dloc}'(\bullet),\overline{\bullet}).$$

Moreover, the second one is obtained by adjunction of the first one, that is,

$$\mathfrak{s}^{(!D)} = \big[\mathfrak{s}^{*(*D)}\big]^*,$$

since $(\mathfrak{s}^*)^{\mathrm{mod}\,D} = (\mathfrak{s}^{\mathrm{mod}\,D})^*$.

12.5.39. Proposition. *Assume that $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable along D. Then $\mathfrak{s}^{(*D)}$ and $\mathfrak{s}^{(!D)}$ take values in $\mathfrak{C}_X$ and are compatible with $\mathfrak{s}$, in the sense that the following diagram commutes:*

(12.5.39 $*$)

$$\begin{tikzcd}
\mathcal{M}'(*D) & \otimes_{\mathbb{C}} & \overline{\mathcal{M}''(!D)} \arrow[r, "\mathfrak{s}^{(*D)}"] \arrow[d, "\overline{\mathrm{dloc}''}"] & \mathfrak{C}_X \arrow[d, equal] \\
\mathcal{M}' \arrow[u, "\mathrm{loc}'"] & \otimes_{\mathbb{C}} & \overline{\mathcal{M}''} \arrow[r, "\mathfrak{s}"] \arrow[d, "\overline{\mathrm{loc}''}"] & \mathfrak{C}_X \arrow[d, equal] \\
\mathcal{M}'(!D) \arrow[u, "\mathrm{dloc}'"] & \otimes_{\mathbb{C}} & \overline{\mathcal{M}''(*D)} \arrow[r, "\mathfrak{s}^{(!D)}"] & \mathfrak{C}_X
\end{tikzcd}$$

that is, for any local sections m', m'' of $\mathcal{M}', \mathcal{M}''$ and $m'_!, m''_!$ of $\mathcal{M}'(!D), \mathcal{M}''(!D)$, the following equalities hold:

$$\mathfrak{s}^{(*D)}(\mathrm{loc}'(m'), \overline{m''_!}) = \mathfrak{s}(m', \overline{\mathrm{dloc}''(m''_!)}), \quad \mathfrak{s}^{(!D)}(m'_!, \overline{\mathrm{loc}''(m'')}) = \mathfrak{s}(\mathrm{dloc}'(m'_!), \overline{m''}).$$

Furthermore, if $\mathfrak{s}_1, \mathfrak{s}_2$ are two such sesquilinear pairings which coincide when restricted to $X \smallsetminus D$, then $\mathfrak{s}_1^{(*D)}, \mathfrak{s}_2^{(*D)}$, resp. $\mathfrak{s}_1^{(!D)}, \mathfrak{s}_2^{(!D)}$, coincide.

***12.5.40. Remark* ((Dual) localization for left pairings).** The pairings $\mathfrak{s}^{(*D)}$ and $\mathfrak{s}^{(!D)}$ are defined by side-changing (12.3.3 $*$) from their right analogues.

***Proof of Proposition* 12.5.39.** The question is local, and we can reduce to the case where $D = H$ is smooth and $X = H \times \mathbb{C}$. The V-filtration is then well-defined for an $\mathbb{R}$-specializable $\mathcal{D}_X$-module. Furthermore, it is enough to construct $\mathfrak{s}^{(*H)}$ and $\mathfrak{s}^{(!H)}$ such that the diagram (12.5.39 $*$) commutes, as these pairings will then coincide with those defined by (12.5.38). Since the morphisms $\mathcal{M}(!H) \to \mathcal{M}$ and $\mathcal{M} \to \mathcal{M}(*H)$ have kernels and cokernels supported in H, they induce isomorphisms between the $V_{<0}$ of these modules. In particular, since $\mathfrak{s}$ takes values in $\mathfrak{C}_X$, so does the restriction of $\mathfrak{s}^{\mathrm{mod}\, H}$ to $V_{<0}\mathcal{M}' \otimes_{\mathbb{C}} \overline{V_{<0}\mathcal{M}''}$. We will construct $\mathfrak{s}^{(*H)}$, the case of $\mathfrak{s}^{(!H)}$ being similar.

An unsuccessful tentative. Since $\mathrm{loc} : V_{<0}\mathcal{M}' \to V_{<0}(\mathcal{M}'(*H))$ is an isomorphism, we expect that the restriction of $\mathfrak{s}^{(*H)}$ to $V_{<0}(\mathcal{M}'(*H)) \otimes_{\mathbb{C}} \overline{\mathcal{M}''(!H)}$ is given by

$$\mathfrak{s}^{(*H)}(m'_{*<0}, \overline{m''_!}) = \mathfrak{s}^{(*H)}(\mathrm{loc}(m'_{<0}), \overline{m''_!}) = \mathfrak{s}(m'_{<0}, \overline{\mathrm{dloc}(m''_!)}).$$

Since any m'_* can be written as $m'_{*<0}t^{-k}$ for some $k \geqslant 0$ and $m'_{*<0} \in V_{<0}\mathcal{M}'$, it would be natural to define $\mathfrak{s}^{(*H)}(m'_*, \overline{m''_!})$ as the result of the division of the current $\mathfrak{s}(m'_{<0}, \overline{\mathrm{dloc}(m''_!)})$ by t^k. However, this result is not unique, and is determined only up to a current supported on H, so this method does not lead to a well-defined result.

An successful tentative. We will proceed the other way round, starting by extending $\mathfrak{s}$ to $\mathcal{M}(*H) \otimes_{\mathbb{C}} \overline{V_{<0}\mathcal{M}''}$. For every $\ell \geqslant 1$, we extend $\mathfrak{s}$ as a sesquilinear pairing

$$\mathfrak{s}_\ell : (V_{<0}\mathcal{M}' \cdot t^{-\ell}) \otimes_{\mathbb{C}} \overline{V_{<0}\mathcal{M}''} \longrightarrow \mathfrak{C}_X\,.$$

We argue exactly as in the proof of Proposition 12.5.4 by extending, for every test function η on $\mathrm{nb}(x_o)$ and each local section m' of $V_{<0}\mathcal{M}'$ and m'' of $V_{<0}\mathcal{M}''$, the holomorphic function (for $\operatorname{Re} s \gg 0$)

$$s \longmapsto \big\langle \mathfrak{s}(m', \overline{m''})|t|^{2(s-\ell)}\bar{t}^{\ell}, \eta\big\rangle = \mathfrak{s}(m', \overline{m'' t^{\ell}})|t|^{2(s-\ell)}, \eta\big\rangle$$

as a meromorphic function on $\mathbb{C}$. Since $m'' t^{\ell}$ is a local section of $V_{<\ell}\mathcal{M}''$, it has no pole at $s-\ell=-\ell$, that is, at $s=0$. Taking the value of this function at $s=0$ gives the desired extension of $\mathfrak{s}$, since $|t|^{2(s-\ell)}\bar{t}^{\ell} = |t|^{2s}t^{-\ell}$. Moreover, one checks that $\mathfrak{s}_{\ell}$ restricts to $\mathfrak{s}_{\ell-1}$ on $(V_{<0}\mathcal{M}' \cdot t^{-\ell+1}) \otimes_{\mathbb{C}} \overline{V_{<0}\mathcal{M}''}$, and thus defines a sesquilinear pairing

$$\mathfrak{s}^{(*H)} : \mathcal{M}'(*H) \otimes_{\mathbb{C}} \overline{V_{<0}\mathcal{M}''} \longrightarrow \mathfrak{C}_X .$$

This pairing can be extended in at most one way as a pairing

$$\mathfrak{s}^{(*H)} : \mathcal{M}'(*H) \otimes_{\mathbb{C}} \overline{\mathcal{M}''(!H)} \longrightarrow \mathfrak{C}_X,$$

due to the $\mathcal{D}_{\overline{X}}$-linearity and the equality $\mathcal{M}''(!H) = V_{<0}\mathcal{M}'' \otimes_{V_0\mathcal{D}_X} \mathcal{D}_X$. However, since $\mathcal{D}_X$ is not locally free as a $V_0\mathcal{D}_X$-module, the existence of such an extension is not a priori obvious. Such an extension will exist near x_o if, for any finite family (m''_j) of elements of $V_{<0}\mathcal{M}''_{x_o}$, any finite family $(P_j)_j$ of germs of differential operators at x_o, and any $m' \in \mathcal{M}(*H)_{x_o}$, the condition $\sum_j m''_j \otimes P_j = 0$ implies $\sum_j \mathfrak{s}^{(*H)}(m', \overline{m''_j}) \cdot \overline{P}_j = 0$. This holds by definition if all P_j belong to $V_0\mathcal{D}_{X,x_o}$. Therefore, one can reduce to the case where $j=0,\dots,N$ and $P_j = \partial_t^j$.

We argue by induction on N, the case where $N=0$ being clear. We thus assume that $\sum_j m''_j \otimes \partial_t^j = 0$ and we first claim that $m''_N \otimes \partial_t \in V_{<0}\mathcal{M}''(!H)$. Indeed, $\mathcal{M}''(!H)$ has the property that $\partial_t : \operatorname{gr}^V_{\alpha}\mathcal{M}''(!H) \to \operatorname{gr}^V_{\alpha+1}\mathcal{M}''(!H)$ is an isomorphism if $\alpha=-1$, and on the other hand it is an isomorphism for any other α (this holds for any $\mathbb{R}$-specializable coherent $\mathcal{D}_X$-module). This implies that

$$\partial_t^N : V_{<0}\mathcal{M}''(!H)/V_{<-1}\mathcal{M}''(!H) \longrightarrow V_{<N}\mathcal{M}''(!H)/V_{<N-1}\mathcal{M}''(!H)$$

is an isomorphism. Since

$$m''_N \otimes \partial_t^N = -\sum_{j=0}^{N-1} m''_j \otimes \partial_t^j \in V_{<N-1}\mathcal{M}''(!H)_{x_o},$$

we conclude that $m''_N \otimes 1 \in V_{<-1}\mathcal{M}''(!H)_{x_o}$, hence the assertion.

By induction, we thus have

$$\sum_{j=0}^{N-1} \mathfrak{s}^{(*H)}(m', \overline{m''_j}) \cdot \partial_{\bar{t}}^j + \mathfrak{s}^{(*H)}(m', \overline{m''_N \otimes \partial_t}) \cdot \partial_{\bar{t}}^{N-1} = 0 \in \mathfrak{C}_X .$$

It is therefore enough to check that, for $m' \in \mathcal{M}'(*H)_{x_o}$ and $m'' \in V_{<-1}\mathcal{M}''_{x_o}$, we have

$$\mathfrak{s}^{(*H)}(m', \overline{m'' \otimes \partial_t}) = \mathfrak{s}^{(*H)}(m', \overline{m''}) \cdot \partial_{\bar{t}}.$$

Notice now that $t : V_{<0}\mathcal{M}''_{x_o} \to V_{<-1}\mathcal{M}''_{x_o}$ is an isomorphism, hence $m'' = n''t$ for some $n'' \in V_{<0}\mathcal{M}''_{x_o}$. We thus have

$$\begin{aligned}\mathfrak{s}^{(*H)}(m', \overline{m'' \otimes \partial_t}) &= \mathfrak{s}^{(*H)}(m', \overline{n''t \otimes \partial_t}) = \mathfrak{s}^{(*H)}(m', \overline{n'' \otimes t\partial_t})\\ &= \mathfrak{s}^{(*H)}(m', \overline{n''t\partial_t \otimes 1}) = \mathfrak{s}^{(*H)}(m', \overline{n'' \otimes 1}) \cdot \bar{t}\partial_{\bar{t}}\\ &= \mathfrak{s}^{(*H)}(m', \overline{n''t \otimes 1}) \cdot \partial_{\bar{t}} = \mathfrak{s}^{(*H)}(m', \overline{m'' \otimes 1}) \cdot \partial_{\bar{t}}.\end{aligned}$$

The relation

$$\mathfrak{s}^{(*H)}(\mathrm{loc}'(m'), \overline{m''_!}) = \mathfrak{s}(m', \overline{\mathrm{dloc}''(m''_!)})$$

is not obvious with this method and we will check it. For the sake of simplicity, we assume that $m' = m'_0 \in V_0\mathcal{M}'$ and we write $m''_! = \sum_j m''_j \otimes \partial_t^j$ with $m''_j \in V_{<0}\mathcal{M}''$ (any such writing leads to the same result). We have $\mathrm{loc}'(m'_0) = (m'_0 t) \otimes t^{-1}$ and $\mathrm{dloc}''(m''_!) = \sum_j m''_j \partial_t^j$. Then (the restriction at $s = 0$ is justified by the fact that $m''_j \in V_{<0}(\mathcal{M}''))$

$$\begin{aligned}\big\langle \mathfrak{s}^{(*H)}(\mathrm{loc}'(m'_0), \overline{m''_!}), \eta \big\rangle &= \sum_j \big\langle \mathfrak{s}^{(*H)}(\mathrm{loc}'(m'_0), \overline{m''_j}), \partial_{\bar t}^j \eta \big\rangle \\ &= \sum_j \big\langle \mathfrak{s}(m'_0 t, \overline{m''_j}), |t|^{2(s-1)} \bar t \partial_{\bar t}^j \eta \big\rangle \big|_{s=0} \\ &= \sum_j \big\langle \mathfrak{s}(m'_0, \overline{m''_j}), |t|^{2s} \partial_{\bar t}^j \eta \big\rangle \big|_{s=0} \\ &= \sum_j \big\langle \mathfrak{s}(m'_0, \overline{m''_j}), \partial_{\bar t}^j \eta \big\rangle = \mathfrak{s}(m'_0, \overline{\mathrm{dloc}''(m''_!)}).\end{aligned}$$

Let us end with the question of uniqueness. Clearly, it is enough to prove that if $\mathfrak{s}_1, \mathfrak{s}_2$ coincide on $X \smallsetminus D$, then $\mathfrak{s}_1^{\mathrm{mod}\,D}, \mathfrak{s}_2^{\mathrm{mod}\,D}$ coincide. This is tautological, since for an open subset U of X, $\mathfrak{Db}^{\mathrm{mod}\,D}(U)$ is by definition equal to the image of the restriction morphism $\mathfrak{Db}(U) \to \mathfrak{Db}(U \smallsetminus D)$. ☐

12.5.41. Corollary. *To any sesquilinear pairing $\mathfrak{s}$ between $\mathcal{D}_X$-modules $\mathcal{M}'$, $\mathcal{M}''$ which are $\mathbb{R}$-specializable along D is associated a sesquilinear pairing $\mathfrak{s}^{(!*)}$ between $\mathcal{M}'(!{*}D)$ and $\mathcal{M}''(!{*}D)$, which is uniquely determined from $\mathfrak{s}|_{X \smallsetminus D}$. Furthermore, any sesquilinear pairing between $\mathcal{M}'(!{*}D)$ and $\mathcal{M}''(!{*}D)$ is uniquely determined from its restriction to $X \smallsetminus D$.*

Proof. Recall that $\mathcal{M}(!{*}D)$ is the image of the natural morphism $\mathrm{loc} \circ \mathrm{dloc} : \mathcal{M}(!D) \to \mathcal{M}(*D)$. For the first point, it is enough to check that, for $m''_! \in \mathrm{Ker}(\mathrm{loc}'' \circ \mathrm{dloc}'') \subset \mathcal{M}''(!D)$ and $m'_! \in \mathcal{M}'(!D)$, we have $\mathfrak{s}^{(*D)}(\mathrm{loc}' \circ \mathrm{dloc}'(m'_!), m''_!) = 0$. But (12.5.39 $*$) shows that the left-hand side is equal to $\mathfrak{s}^{(!D)}(m'_!, \mathrm{loc}'' \circ \mathrm{dloc}''(m''_!)$, which is equal to zero by assumption on $m''_!$.

For the last assertion, we can assume, due to Kashiwara's equivalence, that $X = H \times \Delta_t$. Then $\mathcal{M}(!{*}H)$ is $\mathcal{D}_X$-generated by $V_{<0}\mathcal{M}(!{*}D)$. Let $(m'_i), (m''_j)$ be a finite family of local $\mathcal{D}_X$-generators in $V_{<0}\mathcal{M}'(!{*}D), V_{<0}\mathcal{M}'(!{*}D)$. Assume that a sesquilinear pairing $\mathfrak{s}$ vanishes on $X \smallsetminus H$. Then there exist $k \geqslant 1$ such that $\mathfrak{s}(m'_i, \overline{m''_j}) t^k = 0$ for any i, j. It follows that $\mathfrak{s}$ vanishes on $V_{-k}\mathcal{M}'(!{*}D) \otimes \overline{V_{<0}\mathcal{M}'(!{*}D)}$. Since $V_{-k}\mathcal{M}'(!{*}D)$ also generates $\mathcal{M}'(!{*}D)$ as a $\mathcal{D}_X$-module, this implies that $\mathfrak{s} = 0$. ☐

12.5.g. (Dual) localization in the non-characteristic case. We keep the setting of Section 12.5.f and we moreover assume that $D = H$ is a smooth hypersurface which is non-characteristic with respect to $\mathcal{M}', \mathcal{M}''$. In such a case, loc is injective and dloc is surjective. Due to the commutative diagram of Proposition 12.5.39, we have natural sesquilinear pairings

$$\mathfrak{s}' : \operatorname{Coker}\operatorname{loc}' \otimes_{\mathbb{C}} \overline{\operatorname{Ker}\operatorname{dloc}''} \longrightarrow \mathfrak{C}_X \quad \text{and} \quad \mathfrak{s}'' : \operatorname{Ker}\operatorname{dloc}' \otimes_{\mathbb{C}} \overline{\operatorname{Coker}\operatorname{loc}''} \longrightarrow \mathfrak{C}_X\,.$$

Since these $\mathcal{D}_X$-modules are supported on H, Kashiwara's equivalence (Proposition 12.4.7), together with the identification of Propositions 11.2.9 and 11.2.13, yields that $\mathfrak{s}', \mathfrak{s}''$ are the pushforwards of sesquilinear pairings

$$\mathfrak{s}'_H : \mathcal{M}'_H \otimes_{\mathbb{C}} \overline{\mathcal{M}''_H} \longrightarrow \mathfrak{C}_H, \quad \mathfrak{s}''_H : \mathcal{M}'_H \otimes_{\mathbb{C}} \overline{\mathcal{M}''_H} \longrightarrow \mathfrak{C}_H\,.$$

12.5.42. Proposition. *The pairings $\mathfrak{s}'_H, \mathfrak{s}''_H$ are equal and correspond, by side-changing, to the pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s}$ defined by* (12.5.19 $*$).

Proof. We will treat the case of $\mathfrak{s}'$, that of $\mathfrak{s}''$ being similar. The question is local, and we can assume that $X = H \times \Delta_t$. After the identifications of Remarks 11.2.11 and 11.2.14, the right version of ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s}$ defined by (12.5.19 $*$) is nothing but $\operatorname{gr}^V_{-1}(\mathfrak{s}) : \operatorname{gr}^V_{-1}(\mathcal{M}') \otimes \overline{\operatorname{gr}^V_{-1}(\mathcal{M}'')} \to \mathfrak{C}_H$, and we use the isomorphisms $\operatorname{gr}^V_0(\mathcal{M}'(*H)) \xrightarrow{\sim} \operatorname{gr}^V_{-1}\mathcal{M}'$ and $\operatorname{gr}^V_0(\mathcal{M}''(!H)) \xrightarrow{\sim} \operatorname{gr}^V_{-1}\mathcal{M}''$ defined resepctively as $\operatorname{gr}^V_{-1}\operatorname{loc}^{-1}\circ t$ and $\operatorname{gr}^V_{-1}\operatorname{dloc}\circ\partial_t^{-1}$. On the other hand, $\mathfrak{s}'_H$ is nothing but $\operatorname{gr}^V_0(\mathfrak{s}')$. Since $\operatorname{gr}^V_0\mathcal{M} = 0$ for $\mathcal{M} = \mathcal{M}', \mathcal{M}''$ (non-characteristicity of H), $\operatorname{gr}^V_0(\mathfrak{s}')$ is equal to

$$\operatorname{gr}^V_0(\mathfrak{s}^{(*H)}) : \operatorname{gr}^V_0(\mathcal{M}'(*H)) \otimes \overline{\operatorname{gr}^V_0(\mathcal{M}''(!H))} \longrightarrow \mathfrak{C}_H\,.$$

Let m'_{*0} be a local section of $V_0\mathcal{M}'(*H)$ and $m''_{!0}$ a local section of $V_0\mathcal{M}''(!H)$. We can write $m''_{!0} = m''_{-1} \otimes \partial_t + m''_{<0} \otimes 1$ and we will neglect $m''_{<0}$ which does not contribute to $\operatorname{gr}^V_0(\mathfrak{s}^{(*H)})([m'_{*0}], \overline{[m''_{!0}]})$. We can also write $m'_{*0} = \operatorname{loc}(m'_{-1})t^{-1}$. We have

$$\begin{aligned}\big\langle \operatorname{gr}^V_0(\mathfrak{s}^{(*H)})([m'_{*0}], \overline{[m''_{!0}]}), \eta_o\big\rangle &= \operatorname{Res}_{s=0}\big\langle \mathfrak{s}^{(*H)}(m'_{*0}, \overline{m''_{!0}}), |t|^{2s}\eta\big\rangle \quad \text{with } \eta|_H = \eta_o\\ &= \operatorname{Res}_{s=0}\big\langle \mathfrak{s}^{(*H)}(\operatorname{loc}(m'_{-1})t^{-1}, \overline{m''_{-1} \otimes \partial_t}), |t|^{2s}\eta\big\rangle\\ &= \operatorname{Res}_{s=0}\big\langle \mathfrak{s}^{(*H)}(\operatorname{loc}(m'_{-1}), \overline{m''_{-1} \otimes \partial_t}), |t|^{2s}t^{-1}\eta\big\rangle\\ &= \operatorname{Res}_{s=0}\big\langle \mathfrak{s}(m'_{-1}, \overline{m''_{-1}\partial_t}), |t|^{2s}t^{-1}\eta\big\rangle\\ &= \operatorname{Res}_{s=0}\big\langle \mathfrak{s}(m'_{-1}, \overline{m''_{-1}\partial_t t}), |t|^{2(s-1)}\eta\big\rangle\\ &= \big\langle \operatorname{gr}^V_{-1}(\mathfrak{s})([m'_{-1}], \overline{[m''_{-1}]}), \eta_o\big\rangle,\end{aligned}$$

where the last equality follows from the fact that $\partial_t t = \mathrm{Id} + \mathrm{E}$ acts as the identity on $\operatorname{gr}^V_{-1}\mathcal{M}''$, due to non-characteristicity of H. ☐

12.5.h. Maximal extension of a sesquilinear pairing. We take up the notation of Sections 11.6.b–11.6.d, omitting the F-filtration however. The conjugation that we consider on $\mathbb{C}[s, s^{-1}], A, B, A_{(k)}, B^{(k)}$ is the conjugation of the coefficients. In other words, we consider s as a real variable. Let $g : X \to \mathbb{C}$ be a holomorphic function and let D be the reduced divisor underlying (g). Let $\mathcal{M}_*$ be a left $\mathcal{D}_X(*g)$-module. In the absence of a filtration, we can directly define $\mathcal{M}_{*(k)} = g^s(\mathcal{M}_* \otimes_{\mathbb{C}} A_{(k)})$ (with connection $\nabla \otimes \mathrm{Id} + (\mathrm{d}g/g) \otimes s$) and $\mathcal{M}^{*(k)} = g^s(\mathcal{M}_* \otimes_{\mathbb{C}} B^{(k)})$.

Let $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing between left $\mathcal{D}_X$-modules. It induces a sesquilinear pairing $\mathfrak{s}_* : \mathcal{M}'_* \otimes \overline{\mathcal{M}''_*} \to \mathfrak{Db}_X^{\mathrm{mod}\,D}$, where we have set $\mathcal{M}_* = \mathcal{M}(*g)$. We deduce sesquilinear pairings

$$\begin{aligned}\mathcal{M}'_{*(k)} \otimes_{\mathbb{C}} \overline{\mathcal{M}''^{(k)}_*} &\xrightarrow{\ \mathfrak{s}_{*(k)}\ } \mathfrak{Db}_X^{\mathrm{mod}\,D},\\ g^s(m' \otimes a(s)) \otimes \overline{g^s(m'' \otimes b(s^{-1}))} &\longmapsto \operatorname{Res}_{s=0}\Big(|g|^{2s} a(s) b(s^{-1}) \frac{\mathrm{d}s}{s}\Big) \mathfrak{s}_*(m', \overline{m''})\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}'^{(k)}_* \otimes_{\mathbb{C}} \overline{\mathcal{M}''_{*(k)}} &\xrightarrow{\ \mathfrak{s}^{(k)}_*\ } \mathfrak{Db}_X^{\mathrm{mod}\,D},\\ g^s(m' \otimes b(s^{-1})) \otimes \overline{g^s(m'' \otimes a(s))} &\longmapsto \operatorname{Res}_{s=0}\Big(|g|^{2s} b(s^{-1}) a(s) \frac{\mathrm{d}s}{s}\Big) \mathfrak{s}_*(m', \overline{m''}).\end{aligned}$$

Expressing $|g|^{2s}$ as $\exp(s \log |g|^2)$, we note that the residue in the right-hand sides is equal, up to a constant, to some power of $\log |g|^2$, hence is a multiplier on $\mathfrak{Db}_X^{\mathrm{mod}\,D}$.

By means of Proposition 12.5.39, we obtain sesquilinear pairings

$$\begin{aligned}\mathfrak{s}^{(!g)}_{*(k)} &: \mathcal{M}'_{*(k)}(!g) \otimes_{\mathbb{C}} \overline{\mathcal{M}''^{(k)}_*(*g)} \longrightarrow \mathfrak{Db}_X,\\ \mathfrak{s}^{(*g)}_{*(k)} &: \mathcal{M}'_{*(k)}(*g) \otimes_{\mathbb{C}} \overline{\mathcal{M}''^{(k)}_*(!g)} \longrightarrow \mathfrak{Db}_X\,.\end{aligned}$$

12.5.43. Proposition. *The pairings* $\mathfrak{s}^{(!g)}_{*(k)}$ *and* $\mathfrak{s}^{(*g)}_{*(k)}$ *are compatible, in a natural way, with the natural morphisms* $\mathrm{dloc} \circ \mathrm{loc} : (!g) \to (*g)$.

- *They define on* $\mathrm{Ker}(\mathrm{dloc} \circ \mathrm{loc})$, *resp.* $\mathrm{Coker}(\mathrm{dloc} \circ \mathrm{loc})$, *for k large enough locally on D, the same pairing, which also coincides with $\psi_{g,1}\mathfrak{s}$ via the identification of Proposition* 11.6.10(2).
- *They define on* $\mathrm{Ker}(s \cdot \mathrm{dloc} \circ \mathrm{loc})$, *resp.* $\mathrm{Coker}(\mathrm{dloc} \circ \mathrm{loc} \cdot s)$, *for k large enough locally on D, the same pairing $\Xi_g \mathfrak{s} : \Xi_g \mathcal{M}' \otimes \overline{\Xi_g \mathcal{M}''} \to \mathfrak{Db}_X$ (see Proposition* 11.6.12*).*

Proof. It is left as an exercise. ☐

12.6. Compatibility between functors on sesquilinear pairings

In this section, we consider the following setting. Let $f : X \to X'$ be a holomorphic map between complex manifolds and let $h : X' \to \mathbb{C}$ be a holomorphic function. Set $g = h \circ f$. Let $\mathcal{M}', \mathcal{M}''$ be right $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g). Let $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing. Assume that f is proper on the support of $\mathcal{M}', \mathcal{M}''$. Recall that Theorem 9.8.8 implies:

- for every $k \in \mathbb{Z}$, ${}_{\mathrm{D}}f^{(k)}_*\mathcal{M}$ is $\mathbb{R}$-specializable along (h),
- for every $\alpha \in \mathbb{R}$, the natural morphism ${}_{\mathrm{D}}f^{(k)}_* V_\alpha \mathcal{M} \to {}_{\mathrm{D}}f^{(k)}_*\mathcal{M}$ is injective and its image is equal to $V_\alpha({}_{\mathrm{D}}f^{(k)}_*\mathcal{M})$.

12.6.a. Pushforward and specialization of sesquilinear pairings

12.6.1. Theorem. *With respect to the previous natural morphism, we have*

$${}_{\mathrm{T}}f^{(k,-k)}_* \psi_{g,\lambda}\mathfrak{s} = \psi_{h,\lambda}({}_{\mathrm{T}}f^{(k,-k)}_*\mathfrak{s}), \quad {}_{\mathrm{T}}f^{(k,-k)}_* \phi_{g,1}\mathfrak{s} = \phi_{h,1}({}_{\mathrm{T}}f^{(k,-k)}_*\mathfrak{s}).$$

Proof. We start with the case of a map $f \times \mathrm{Id} : X \times \mathbb{C} \to X' \times \mathbb{C}$ and we take for the function $h : X' \times \mathbb{C} \to \mathbb{C}$ the second projection. We assume that $\mathcal{M}', \mathcal{M}''$ are right $\mathcal{D}_{X\times\mathbb{C}}$-modules.

12.6.2. Lemma. *With these assumptions, for every $\alpha \in [-1,0]$ and $k \in \mathbb{Z}$,*

$$ {}_{\mathrm{T}}f_*^{(k,-k)}(\mathrm{gr}_\alpha^V \mathfrak{s}) = \mathrm{gr}_\alpha^V\big({}_{\mathrm{T}}(f\times\mathrm{Id})_*^{(k,-k)}\mathfrak{s}\big).$$

Proof. Set $\beta = -\alpha - 1$ and let

$$\begin{aligned} m'^{k}_{\infty} &\in \Gamma(U, f_*(\mathcal{E}^{n+1+k}_{X\times\mathbb{C}} \otimes_{\mathcal{O}_{X\times\mathbb{C}}} V^\beta\mathcal{M}'^{\mathrm{left}})),\\ m''^{-k}_{\infty} &\in \Gamma(U, f_*(\mathcal{E}^{n+1-k}_{X\times\mathbb{C}} \otimes_{\mathcal{O}_{X\times\mathbb{C}}} V^\beta\mathcal{M}''^{\mathrm{left}})).\end{aligned}$$

The cohomology classes $[m'^{k}_{\infty}]$ and $[m''^{-k}_{\infty}]$ can be regarded as sections of the modules $V_\alpha({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}') \otimes_{\mathcal{O}_Y} \mathcal{C}^\infty_Y$ and $V_\alpha({}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}'') \otimes_{\mathcal{O}_Y} \mathcal{C}^\infty_Y$ respectively, according to the result recalled above. We can then compute with these classes. Let us also denote by $[m]_\alpha$ the class of $m \in V_\alpha$ modulo $V_{<\alpha}$. Let us assume $\alpha \in [-1,0)$ (the case of $\alpha = 0$ is similar by using the function $I_{\widehat{\chi}}$). We have, for $\eta \in \mathcal{C}^\infty_Y(U)$,

$$\begin{aligned} &\Big\langle \mathrm{gr}_\alpha^V\big({}_{\mathrm{T}}(f\times\mathrm{Id})_*^{(k,-k)}\mathfrak{s}\big)([m'^{k}_{\infty}]_\alpha, \overline{[m''^{-k}_{\infty}]_\alpha}), \eta(y)\Big\rangle\\ &\qquad = \mathrm{Res}_{s=\alpha}\Big\langle \big({}_{\mathrm{T}}(f\times\mathrm{Id})_*^{(k,-k)}\mathfrak{s}\big)([m'^{k}_{\infty}], \overline{[m''^{-k}_{\infty}]}), \eta(y)|t|^{2s}\chi(t)\Big\rangle\\ &\qquad = \mathrm{Res}_{s=\alpha}\Big\langle \mathfrak{s}(m'^{k}_{\infty}, \overline{m''^{-k}_{\infty}}), \eta\circ f(x)|t|^{2s}\chi(t)\Big\rangle\\ &\qquad = \Big\langle \mathrm{gr}_\alpha^V\mathfrak{s}((m'^{k}_{\infty})_\alpha, \overline{(m''^{-k}_{\infty})_\alpha}), \eta\circ f(x)\Big\rangle\\ &\qquad = \Big\langle \big({}_{\mathrm{T}}f_*^{(k,-k)}\mathrm{gr}_\alpha^V\mathfrak{s}\big)\big([(m'^{k}_{\infty})_\alpha], \overline{[(m''^{-k}_{\infty})_\alpha]}\big), \eta(y)\Big\rangle, \end{aligned}$$

and we obtain the desired equality since, as recalled, $[m^{\pm k}_{\infty}]_\alpha = [(m^{\pm k}_{\infty})_\alpha]$ in $\Gamma(U, \mathrm{gr}_\alpha^V[{}_{\mathrm{D}}(f\times\mathrm{Id})_*^{(\pm k)}\mathcal{M}]) = \Gamma(U, {}_{\mathrm{D}}f_*^{(\pm k)}\mathrm{gr}_\alpha^V\mathcal{M})$. □

We can now end the proof of Theorem 12.6.1. We have

$$\begin{aligned} {}_{\mathrm{T}}f_*^{(k,-k)}\psi_{g,\lambda}\mathfrak{s} &= {}_{\mathrm{T}}f_*^{(k,-k)}\mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota^0_{g*}\mathfrak{s}) \quad \text{(see (12.5.15 $*$))}\\ &= \mathrm{gr}_\alpha^V({}_{\mathrm{T}}(f\times\mathrm{Id})_*^{(k,-k)}({}_{\mathrm{T}}\iota^0_{g*}\mathfrak{s})) \quad \text{(Lemma 12.6.2)}\\ &= \mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota^0_{h*}({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s})) \quad \text{(after (12.4.14 $*$))}\\ &= \psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}). \end{aligned}$$
□

12.6.b. Pushforward and localization of sesquilinear pairings

Let D' be an effective divisor in X', locally equal to (h) for some holomorphic function h, and set $D = f^*(D')$. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along D. Then we have natural morphisms (see Corollary 11.7.1(1))

$$({}_{\mathrm{D}}f_*^{(k)}\mathcal{M})(!D') \longrightarrow {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}(!D)) \quad\text{and}\quad {}_{\mathrm{D}}f_*^{(k)}(\mathcal{M}(*D)) \longrightarrow ({}_{\mathrm{D}}f_*^{(k)}\mathcal{M})(*D').$$

12.6.3. Proposition. *With respect to the previous natural morphism, the sesquilinear pairings ${}_{\mathrm{T}}f_*^{(k,-k)}(\mathfrak{s}^{(\star D)})$ and $({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s})^{(\star D')}$ coincide ($\star = *, !$).*

Proof. One first considers the naive localization, and recall that $\mathfrak{C}_X^{\mathrm{mod}\,D} = \mathfrak{C}_X(*D)$. One then easily checks that ${}_{\mathrm{T}}f_*^{(k,-k)}(\mathfrak{s}(*D)) = ({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s})(*D')$ with values in $\mathfrak{C}_{X'}(*D')$. By definition and from the commutativity above, $({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s})^{(*D')}$ is the restriction of the latter to ${}_{\mathrm{D}}f_*^{(k)}(\mathcal{M}(*D)) \otimes \overline{{}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}(!D)}$, and the assertion follows for $\star = *$. The case $\star = !$ is similar. □

12.6.c. Pushforward and Beilinson extension of sesquilinear pairings

12.6.4. Corollary. *Via the isomorphisms of Corollary* 11.7.1*, the sesquilinear pairings* ${}_{\mathrm{T}}f_*^{(k,-k)}(\Xi_g\mathfrak{s})$ *and* $\Xi_h({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s})$ *coincide.* □

12.7. The category $\widetilde{\mathcal{D}}$-Triples and its functors

We now come back to the filtered setting, and consider $\widetilde{\mathcal{D}}_X$-modules, with $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$. Given a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we denote by $\mathcal{M} := \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ the associated $\mathcal{D}_X$-module.

12.7.a. The category of $\widetilde{\mathcal{D}}$-triples.

We extend to arbitrary dimensions the constructions of Section 7.4.a. The category $\widetilde{\mathcal{D}}$-Triples(X) is an abelian category, and possesses the basic functors we need for studying pure Hodge modules. For example, the pushforward functor will be denoted by ${}_{\mathrm{T}}f_*$, etc.

12.7.1. Definition. The category $\widetilde{\mathcal{D}}$-Triples(X) has

- objects consisting of triples $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$, where $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{D}}_X$-modules and $\mathfrak{s}$ is a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$ (with values in $\mathfrak{Db}_X$ in the left case, and in $\mathfrak{C}_X$ in the right case),
- morphisms $\varphi : \widetilde{\mathcal{T}}_1 \to \widetilde{\mathcal{T}}_2$ consisting of pairs $\varphi = (\varphi', \varphi'')$, where $\varphi' : \widetilde{\mathcal{M}}'_1 \to \widetilde{\mathcal{M}}'_2$ and $\varphi'' : \widetilde{\mathcal{M}}''_2 \to \widetilde{\mathcal{M}}''_1$ are $\widetilde{\mathcal{D}}_X$-linear, such that for all local sections m'_1 of $\mathcal{M}'_1$ and m''_1 of $\mathcal{M}''_2$,

$$\mathfrak{s}_1(m'_1, \overline{\varphi''(m''_2)}) = \mathfrak{s}_2(\varphi'(m'_1), \overline{m''_2}). \tag{12.7.1 $*$}$$

In particular, $\widetilde{\mathcal{D}}$-Triples(X) is an abelian subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X) \times \mathsf{Mod}(\widetilde{\mathcal{D}}_X)^{\mathrm{op}}$.

We say that an object $\widetilde{\mathcal{T}}$ of $\widetilde{\mathcal{D}}$-Triples(X) is *coherent*, resp. *strictly* $\mathbb{R}$*-specializable*, resp. *smooth*, if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{D}}_X$-coherent, resp. strictly $\mathbb{R}$-specializable, resp. $\widetilde{\mathcal{O}}_X$-locally free of finite rank. The *support* of $\widetilde{\mathcal{T}}$ is $\operatorname{Supp}\widetilde{\mathcal{T}} = \operatorname{Supp}\widetilde{\mathcal{M}}' \cup \operatorname{Supp}\widetilde{\mathcal{M}}''$.

12.7.2. Complexes in $\widetilde{\mathcal{D}}$***-Triples(X).*** A complex $(\widetilde{\mathcal{T}}^\bullet, d)$ consists of a graded object $\widetilde{\mathcal{T}}^\bullet = \bigoplus_k \widetilde{\mathcal{T}}^k$ of $\widetilde{\mathcal{D}}$-Triples(X) together with a differential $d : \widetilde{\mathcal{T}}^k \to \widetilde{\mathcal{T}}^{k+1}$ such that $d^2 = 0$. We write $\widetilde{\mathcal{T}}^k = (\widetilde{\mathcal{M}}'^k, \widetilde{\mathcal{M}}''^{-k}, \mathfrak{s}_k)$ and $d = (d', d'')$, so that $(\widetilde{\mathcal{M}}'^\bullet, d')$ and $(\widetilde{\mathcal{M}}''^\bullet, d'')$ are complexes and $\mathfrak{s}_k : \mathcal{M}'^k \otimes \overline{\mathcal{M}''^{-k}} \to \mathfrak{Db}_X$ (left case) satisfies $\mathfrak{s}_k(d'm'_{k-1}, \overline{m''_{-k}}) = \mathfrak{s}(m'_{k-1}, \overline{d''m''_{-k}})$.

12.7.3. Side-changing in **$\widetilde{\mathcal{D}}$-Triples(X).** Let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be a *left* $\widetilde{\mathcal{D}}_X$-triple. We set

$$\widetilde{\mathcal{T}}^{\mathrm{right}} := (\widetilde{\mathcal{M}}'^{\mathrm{right}}, \widetilde{\mathcal{M}}''^{\mathrm{right}}, \mathfrak{s}^{\mathrm{right}}),$$

where $\mathfrak{s}^{\mathrm{right}}$ is defined by (12.3.3 $*$). The right-to-left side changing is defined correspondingly, so that the composition of both is the identity.

12.7.4. Hermitian duality. The Hermitian dual of an object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-Triples(X) is the object $\widetilde{\mathcal{T}}^* := (\widetilde{\mathcal{M}}'', \widetilde{\mathcal{M}}', \mathfrak{s}^*)$, where $\mathfrak{s}^*$ is the Hermitian adjoint sesquilinear pairing (see Definitions 12.3.1(2) and 12.3.2(2)). The Hermitian adjoint of a morphism $\varphi = (\varphi', \varphi'')$ is the morphism $\varphi^* := (\varphi'', \varphi')$. We clearly have $\widetilde{\mathcal{T}}^{**} = \widetilde{\mathcal{T}}$ and $\varphi^{**} = \varphi$.

12.7.5. Side-changing and Hermitian duality in **$\widetilde{\mathcal{D}}$-Triples(X).** With the previous definitions, Hermitian duality commutes with side-changing, because of (12.3.3 $**$).

12.7.6. Hermitian dual of a graded triple. Let $\widetilde{\mathcal{T}}^\bullet = \bigoplus_k \widetilde{\mathcal{T}}^k$ be a graded object in $\widetilde{\mathcal{D}}$-Triples(X). We write $\widetilde{\mathcal{T}}^k$ as $(\widetilde{\mathcal{M}}'^k, \widetilde{\mathcal{M}}''^{-k}, \mathfrak{s}_k)$. The Hermitian dual object is then

$$(\widetilde{\mathcal{T}}^*)^\bullet = \bigoplus_k (\widetilde{\mathcal{T}}^*)^k := \bigoplus_k (\widetilde{\mathcal{T}}^{-k})^*.$$

12.7.7. Tate twist. The definition of the twist follows the general definition 5.2.2(7), and the Tate twist is as in Notation 5.2.3:

$$\widetilde{\mathcal{T}}(\ell) = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})(\ell) := (\widetilde{\mathcal{M}}'(\ell), \widetilde{\mathcal{M}}''(-\ell), \mathfrak{s}).$$

12.7.8. Pre-polarization and Hermitian pairs. A pre-polarization of weight w of $\widetilde{\mathcal{T}}$ is an isomorphism $\mathrm{S} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}^*(-w)$ which is Hermitian. Tate twist acts as $(\widetilde{\mathcal{T}}, \mathrm{S})(\ell) = (\widetilde{\mathcal{T}}(\ell), (-1)^\ell \mathrm{S})$. Any pre-polarized triple of weight w is isomorphic to a triple $(\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'(w), \mathcal{S})$ with pre-polarization $(\mathrm{Id}, \mathrm{Id})$. Hence, giving a pre-polarized triple $(\widetilde{\mathcal{T}}, \mathrm{S})$ of weight w is equivalent to giving the Hermitian pair $(\widetilde{\mathcal{M}}', \mathcal{S})$ and the weight w. Tate twist acts as

$$(\widetilde{\mathcal{M}}', \mathcal{S}, w)(\ell) = (\widetilde{\mathcal{M}}'(\ell), (-1)^\ell \mathcal{S}, w - 2\ell).$$

12.7.9. Two basic examples. Let us keep the notation of Examples 5.4.4 and 12.3.5.

(1) (Left case) The triple ${}_{\mathrm{H}}\widetilde{\mathcal{O}}_X = (\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X(n), \mathfrak{s}_n)$ is the smooth left triple with $\mathfrak{s}_n(1, 1) = 1$. It satisfies $({}_{\mathrm{H}}\widetilde{\mathcal{O}}_X)^*(-n) = {}_{\mathrm{H}}\widetilde{\mathcal{O}}_X$.

(2) (Right case) The triple ${}_{\mathrm{H}}\widetilde{\omega}_X = (\widetilde{\omega}_X, \widetilde{\omega}_X(n), \mathfrak{s}_n)$ is the smooth right triple with $\mathfrak{s}_n(\omega', \overline{\omega''}) = \mathrm{Sgn}(n)(\omega' \wedge \overline{\omega''})$.

In both cases we have $\mathfrak{s}_n^* = \mathfrak{s}_n$.

12.7.10. Smooth triples. We say that $\widetilde{\mathcal{T}}$ is smooth if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{O}}_X$-locally free of finite rank. Then the corresponding sesquilinear pairing reads $\mathfrak{s}^{\widetilde{\nabla}} \cdot \mathfrak{s}_n$ (see Lemma 12.3.6).

12.7.11. Lefschetz triples. The notion of Lefschetz structure $(\widetilde{\mathcal{T}}, \mathrm{N})$ in the abelian category $\widetilde{\mathcal{D}}$-Triples(X), or that of $\mathfrak{sl}_2$-structure $(\widetilde{\mathcal{T}}_\bullet, \mathrm{X}, \mathrm{Y})$, is obtained, as in Section 5.3. Using Hermitian duality in $\widetilde{\mathcal{D}}$-Triples(X), we obtain as in Definition 5.3.2 the notion of Hermitian duality for a Lefschetz $\widetilde{\mathcal{D}}$-triple $(\widetilde{\mathcal{T}}, \mathrm{N})$. Therefore, if $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$, the nilpotent endomorphism $\mathrm{N} = (\mathrm{N}', \mathrm{N}'')$ consists of nilpotent endomorphisms

$$\mathrm{N}' : \widetilde{\mathcal{M}}' \longrightarrow \widetilde{\mathcal{M}}' \quad \text{and} \quad \mathrm{N}'' : \widetilde{\mathcal{M}}'' \longrightarrow \widetilde{\mathcal{M}}''$$

such that $\mathfrak{s}(\mathrm{N}'\bullet, \overline{\bullet}) = \mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet})$ (see also Section 5.3.a). The notion of pre-polarization of weight w is defined as in Section 5.3.1.

12.7.b. Pullback, specialization and localization in $\widetilde{\mathcal{D}}$-Triple

12.7.12. Pullback by a smooth morphism. Let $f : X \to Y$ be a *smooth* holomorphic map of relative dimension p, as in Definition 12.5.1, and let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be a *left* $\widetilde{\mathcal{D}}_X$-triple. We set

$${}_{\mathrm{T}}f^*\widetilde{\mathcal{T}} = ({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}', {}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}''(p), {}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}).$$

For a right $\widetilde{\mathcal{D}}_X$-triple, we use side-changing at the source and target to define ${}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}$, i.e.,

$${}_{\mathrm{T}}f^*(\widetilde{\mathcal{T}}^{\mathrm{right}}) := ({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}^{\mathrm{left}})^{\mathrm{right}}.$$

If S is a pre-polarization of weight w of $\widetilde{\mathcal{T}}$, we regard $f^*\mathrm{S}$ as a pre-polarization of weight $w+p$ of ${}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}$ and we set

$${}_{\mathrm{T}}f^*(\widetilde{\mathcal{T}}, \mathrm{S}) = ({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}, (-1)^p f^*\mathrm{S}).$$

12.7.13. Pullback of a smooth triple. Formulas similar to those in 12.7.12 hold if, instead of assuming f smooth and $\widetilde{\mathcal{T}}$ arbitrary, we assume f arbitrary but $\widetilde{\mathcal{T}}$ smooth.

12.7.14. Specialization in $\widetilde{\mathcal{D}}$-Triples. An object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-Triples(X) is said to be strictly $\mathbb{R}$-specializable along (g) if $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so. In a way similar to 7.4.2, we then define, for $\lambda \in \mathbb{S}^1$,

$$\begin{aligned} \psi_{g,\lambda}\widetilde{\mathcal{T}} &:= (\psi_{g,\lambda}\widetilde{\mathcal{M}}', \psi_{g,\lambda}\widetilde{\mathcal{M}}''(-1), \psi_{g,\lambda}\mathfrak{s}), \\ \phi_{g,1}\widetilde{\mathcal{T}} &:= (\phi_{g,1}\widetilde{\mathcal{M}}', \phi_{g,1}\widetilde{\mathcal{M}}'', \phi_{g,1}\mathfrak{s}). \end{aligned} \tag{12.7.14 $*$}$$

Then $\psi_{g,\lambda}, \phi_{g,1}$ are functors from the full subcategory of strictly $\mathbb{R}$-specializable objects of $\widetilde{\mathcal{D}}$-Triples(X) to the category of objects supported on $g^{-1}(0)$. From Proposition 9.7.1 and Corollary 12.5.36(2) we deduce:

12.7.15. Proposition. *Assume $\widetilde{\mathcal{T}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along (g).*

(1) *For every $\lambda \in \mathbb{S}^1$, $\dim \operatorname{Supp} \psi_{g,\lambda}\widetilde{\mathcal{T}} < \dim \operatorname{Supp} \widetilde{\mathcal{T}}$.*
(2) *If $\operatorname{Supp} \widetilde{\mathcal{T}} \subset g^{-1}(0)$, then $\psi_{g,\lambda}\widetilde{\mathcal{T}} = 0$ for any $\lambda \in \mathbb{S}^1$ and $\widetilde{\mathcal{T}} \simeq \phi_{g,1}\widetilde{\mathcal{T}}$.* □

According to Remark 12.5.16(3) and Remark 12.5.27, these functors commute with Hermitian duality 12.7.4 as follows:

$$\begin{aligned} \psi_{g,\lambda}(\widetilde{\mathcal{T}}^*) &= (\psi_{g,\lambda}\widetilde{\mathcal{T}})^*(-1), \\ \phi_{g,1}(\widetilde{\mathcal{T}}^*) &= (\phi_{g,1}\widetilde{\mathcal{T}})^*. \end{aligned} \tag{12.7.15 $*$}$$

If S is a pre-polarization of $\widetilde{\mathcal{T}}$ of weight w, then

- $\psi_{g,\lambda}\mathrm{S}$ is a pre-polarization of $\psi_{g,\lambda}\widetilde{\mathcal{T}}$ of weight $w-1$,
- $\phi_{g,1}\mathrm{S}$ is a pre-polarization of $\phi_{g,1}\widetilde{\mathcal{T}}$ of weight w,

and we set

$$(12.7.15\,{**})\qquad \begin{aligned}\psi_{g,\lambda}(\widetilde{\mathcal{T}},\mathrm{S}) &= (\psi_{g,\lambda}\widetilde{\mathcal{T}},\psi_{g,\lambda}\mathrm{S}),\\ \phi_{g,1}(\widetilde{\mathcal{T}},\mathrm{S}) &= (\phi_{g,1}\widetilde{\mathcal{T}},\phi_{g,1}\mathrm{S}).\end{aligned}$$

12.7.16. Properties of* N*,* can *and* var*. The properties analogous to those of a Hodge-Lefschetz quiver explained in Section 5.3.6 also hold in the present setting, as follows from Remark 12.5.16(1), Remark 12.5.27 and Proposition 12.5.28. Let us denote by $\mathrm{N}', \mathrm{can}'\,\mathrm{var}'$ resp. $\mathrm{N}'', \mathrm{can}''\,\mathrm{var}''$ the morphisms relative to $\widetilde{\mathcal{M}}'$ resp. $\widetilde{\mathcal{M}}''$. Then

(1) $\mathrm{N} := (\mathrm{N}', \mathrm{N}'')$ is a nilpotent morphism

$$\psi_{g,\lambda}\widetilde{\mathcal{T}} \longrightarrow \psi_{g,\lambda}\widetilde{\mathcal{T}}(-1) \quad\text{and}\quad \phi_{g,1}\widetilde{\mathcal{T}} \longrightarrow \phi_{g,1}\widetilde{\mathcal{T}}(-1),$$

(2) $\mathrm{can} = (\mathrm{can}', -\mathrm{var}'')$ is a morphism $\psi_{g,1}\widetilde{\mathcal{T}} \to \phi_{g,1}\widetilde{\mathcal{T}}$ commuting with the morphisms $\psi_{g,1}\varphi, \phi_{g,1}\varphi$ associated with any morphism φ between strictly $\mathbb{R}$-specializable objects of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$,

(3) $\mathrm{var} = (\mathrm{var}', -\mathrm{can}'')$ is a morphism $\phi_{g,1}\widetilde{\mathcal{T}} \to \psi_{g,1}\widetilde{\mathcal{T}}(-1)$ with the same commutation property as above,

(4) $\mathrm{N} = \mathrm{var}\circ\mathrm{can}$ on $\psi_{g,1}\widetilde{\mathcal{T}}$ and $\mathrm{can}\circ\mathrm{var}$ on $\phi_{g,1}\widetilde{\mathcal{T}}$, that is, on $\psi_{g,1}\widetilde{\mathcal{T}}$ for example, $\mathrm{N}' = \mathrm{var}'\circ\mathrm{can}'$ and $\mathrm{N}'' = \mathrm{var}''\circ\mathrm{can}''$,

(5) $\mathrm{can}(\widetilde{\mathcal{T}})^* = -\mathrm{var}(\widetilde{\mathcal{T}}^*)$ and $\mathrm{var}(\widetilde{\mathcal{T}})^* = -\mathrm{can}(\widetilde{\mathcal{T}}^*)$, so that $\mathrm{N}(\widetilde{\mathcal{T}}^*) = \mathrm{N}(\widetilde{\mathcal{T}})^*$ (where $\mathrm{var}(\widetilde{\mathcal{T}})$, etc. means var relative to $\widetilde{\mathcal{T}}$, etc.),

(6) If S is a morphism $\widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}^*(-w)$ (e.g. a pre-polarization), then the following diagram commutes:

$$\begin{tikzcd}
\phi_{g,1}\widetilde{\mathcal{T}} \arrow[r, "\phi_{g,1}\mathrm{S}"] \arrow[d, "\mathrm{var}"'] & (\phi_{g,1}\widetilde{\mathcal{T}})^*(-w) \arrow[d, "-\mathrm{can}^*"] \\
\psi_{g,1}\widetilde{\mathcal{T}}(-1) \arrow[r, "\psi_{g,1}\mathrm{S}"] & (\psi_{g,1}\widetilde{\mathcal{T}})^*(-w)
\end{tikzcd}$$

This is seen by interpreting

- $(\phi_{g,1}\widetilde{\mathcal{T}})^*(-w)$ as $\phi_{g,1}(\widetilde{\mathcal{T}}^*(-w))$,
- $(\psi_{g,1}\widetilde{\mathcal{T}})^*(-w)$ as $\psi_{g,1}(\widetilde{\mathcal{T}}(-1)^*(-w))$
- and can^* as $-\mathrm{var}(\widetilde{\mathcal{T}}^*)$,

and by applying the commutation relations above to $\varphi = \mathrm{S}$.

In particular, $\operatorname{Im}\mathrm{N} = (\operatorname{Im}\mathrm{N}', \operatorname{Coim}\mathrm{N}'', \mathfrak{s}_{|\operatorname{Im}\mathrm{N}'\otimes\overline{\operatorname{Coim}\mathrm{N}''}})$. We also define the near-by/vanishing Lefschetz quiver of $\widetilde{\mathcal{T}}$ as the diagram:

$$\begin{tikzcd}
\psi_{g,1}\widetilde{\mathcal{T}} \arrow[r, bend left, "\mathrm{can}"] & \phi_{g,1}\widetilde{\mathcal{T}} \arrow[l, bend left, "\mathrm{var}", "{(-1)}"' near end]
\end{tikzcd}$$

Notice also that Propositions 9.7.2 and 9.7.5 extend to the present setting, up to replacing “injective” with “monomorphism” and “onto” with “epimorphism”.

12.7.17. The $\mathfrak{sl}_2$-triples attached to $(\psi_{g,\lambda}\widetilde{\mathcal{T}}, \mathrm{N})$ ***and*** $(\phi_{g,1}\widetilde{\mathcal{T}}, \mathrm{N})$

The monodromy filtration of N exists in the abelian category $\widetilde{\mathcal{D}}$-Triples(X), and we have, according to Remark 12.5.16(2) (and similarly for $\phi_{g,1}\widetilde{\mathcal{T}}$ without the twist),

$$\begin{aligned} \mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}} &= (\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}', \mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}''(-1), \mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\mathfrak{s}),\\ \mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}} &= (\mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}', \mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}''(-1), \mathrm{P}_{\ell}\psi_{g,\lambda}\mathfrak{s}) \quad (\ell \geqslant 0). \end{aligned}$$

If $\mathrm{S} : (\widetilde{\mathcal{T}}, \mathrm{N}) \to (\widetilde{\mathcal{T}}, \mathrm{N})^*(-w) := (\widetilde{\mathcal{T}}^*, \mathrm{N}^*)(-w)$ is a pre-polarization of weight w, we define for any $\ell \geqslant 0$, as in Section 3.4.c, the morphisms

$$\begin{aligned} \mathrm{P}_{\ell}\psi_{g,\lambda}\mathrm{S} : \mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}} &\longrightarrow (\mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}})^*(-(w-1+\ell))\\ \mathrm{P}_{\ell}\phi_{g,1}\mathrm{S} : \mathrm{P}_{\ell}\phi_{g,1}\widetilde{\mathcal{T}} &\longrightarrow (\mathrm{P}_{\ell}\phi_{g,1}\widetilde{\mathcal{T}})^*(-(w+\ell)), \end{aligned}$$

which are pre-polarizations of respective weights $(w-1+\ell)$ and $(w+\ell)$, and we set

$$(12.7.17\,*)\qquad \begin{aligned} \mathrm{P}_{\ell}\psi_{g,\lambda}(\widetilde{\mathcal{T}}, \mathrm{S}) &= (\mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}}, (-1)^{\ell}\mathrm{P}_{\ell}\psi_{g,\lambda}\mathrm{S}),\\ \mathrm{P}_{\ell}\phi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) &= (\mathrm{P}_{\ell}\phi_{g,1}\widetilde{\mathcal{T}}, (-1)^{\ell}\mathrm{P}_{\ell}\phi_{g,1}\mathrm{S}), \end{aligned} \qquad (\ell \geqslant 0,\ \text{see 3.2.11}).$$

12.7.18. Middle extension of a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}$-triple. Assume that $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g). We say that it is a *middle extension along* (g) if $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so (see 9.7.3).

If $\widetilde{\mathcal{T}}$ is a middle extension along (g), then $\phi_{g,1}\widetilde{\mathcal{T}} \simeq \mathrm{Im}\,\mathrm{N}$ in the abelian category $\widetilde{\mathcal{D}}$-Triples(X).

12.7.19. S-decomposable $\widetilde{\mathcal{D}}$-triples. We say that a coherent $\widetilde{\mathcal{D}}$-triple $\widetilde{\mathcal{T}}$ is *S-decomposable along* (g) resp. *S-decomposable* if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so.

- If $\widetilde{\mathcal{T}}$ is S-decomposable along (g), it has a decomposition $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_1 \oplus \widetilde{\mathcal{T}}_2$, where $\widetilde{\mathcal{T}}_1$ is a middle extension along (g) and $\widetilde{\mathcal{T}}_2$ is supported on $g^{-1}(0)$.
- If $\widetilde{\mathcal{T}}$ is S-decomposable, then $\widetilde{\mathcal{T}} = \bigoplus_i \widetilde{\mathcal{T}}_{Z_i}$ with $\widetilde{\mathcal{T}}_{Z_i}$ having pure support the irreducible closed analytic subset $Z_i \subset X$ (see Proposition 12.3.9).

12.7.20. Proposition **(Criterion for S-decomposability along (g))**

Assume that $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g). Then $\widetilde{\mathcal{T}}$ is S-decomposable along (g) if and only if $\phi_{g,1}\widetilde{\mathcal{T}} = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$.

Proof. This follows from Proposition 9.7.5 and Lemma 12.3.10. □

12.7.21. Properties along (g^r). If $\widetilde{\mathcal{T}}$ is an object of $\widetilde{\mathcal{D}}$-Triples(X) which is strictly $\mathbb{R}$-specializable along (g), a middle extension along (g), S-decomposable along (g), then it satisfies the corresponding properties along (g^r) for any $r \geqslant 2$. This follows from Proposition 9.9.1 and Exercise 12.12.

12.7.22. Non-characteristic restriction of a $\widetilde{\mathcal{D}}$-triple along a closed submanifold

Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}$-Triples(X) such that the closed submanifold $\iota : Y \hookrightarrow X$ is strictly non-characteristic for its $\widetilde{\mathcal{D}}$-module components (see Section 8.8.d). Then ${}_{\mathrm{T}}\iota^*\widetilde{\mathcal{T}} := ({}_{\mathrm{D}}\iota^{*(0)}\widetilde{\mathcal{M}}', {}_{\mathrm{D}}\iota^{*(0)}\widetilde{\mathcal{M}}'', {}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s})$ (see Section 12.5.19) is a well-defined object of $\widetilde{\mathcal{D}}$-Triples(Y), called the *non-characteristic restriction* of $\widetilde{\mathcal{T}}$ along Y.

If $Y = H$ has codimension one, then $\mathrm{gr}_V^\beta \widetilde{\mathcal{T}}$ are zero except for $\beta \in \mathbb{N}$ and ${}_{\mathrm{T}}\iota^* \widetilde{\mathcal{T}} = (\mathrm{gr}_V^0 \widetilde{\mathcal{M}}', \mathrm{gr}_V^0 \widetilde{\mathcal{M}}''(-1), \mathrm{gr}_V^0 \mathfrak{s})$. We will see another definition in Section 12.7.25.

12.7.23. Specialization along a strictly non-characteristic divisor with normal crossings

We take up the setting of Sections 9.9.b and 12.5.21 with $g = x_1 x_2$. As a consequence, we obtain the following property.

Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is holonomic, strictly non-characteristic with respect to D_1, D_2, D_{12}, hence strictly $\mathbb{R}$-specializable along (g). Then $\widetilde{\mathcal{T}}$ is a middle extension along (g) and we have $\psi_{g,\lambda}\widetilde{\mathcal{T}} = 0$ for $\lambda \neq 1$. Assume that $\widetilde{\mathcal{T}}$ is equipped with a pre-polarization S of weight w. Then there are isomorphisms

$$\begin{aligned} &\mathrm{P}_0\psi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) \simeq \psi_{x_1,1}(\widetilde{\mathcal{T}}, \mathrm{S}) \oplus \psi_{x_2,1}(\widetilde{\mathcal{T}}, \mathrm{S}), \\ &\mathrm{P}_1\psi_{g,1}(\widetilde{\mathcal{T}}, -\mathrm{S}) \simeq \psi_{x_1,1}\psi_{x_2,1}(\widetilde{\mathcal{T}}, \mathrm{S})(-1) = \psi_{x_2,1}\psi_{x_1,1}(\widetilde{\mathcal{T}}, \mathrm{S})(-1), \\ &\mathrm{P}_0\psi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) = 0 \quad \text{if } \ell \geqslant 2. \end{aligned} \tag{12.7.23$\,*$}$$

Because of the pre-polarization, we can reduce the question to the case of a Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$ of weight w. Let us check the middle line for example. For the Hermitian pair, according to (9.9.3 $*$) and (12.5.21 $**$), we only need to check the sign of the pre-polarization. On the left-hand side, we introduce a minus sign (which is the sign that enters in front of $\mathrm{P}_1\mathrm{S}$ in Section 3.2.11), while on the right-hand side, the Tate twist (-1) introduces a minus sign, as wanted. Let us end by checking the weights. That of the left-hand side is, since $\ell = 1$, $1 + w - 1 = w$, while that of the right-hand side is $0 + w - 2 + 2 = w$, since it is equal to $\mathrm{P}_0\psi_{x_1,1}\mathrm{P}_0\psi_{x_2,1}(-1)(\bullet)$, so the weights also match.

12.7.24. Nearby cycles along a monomial function of a smooth $\widetilde{\mathcal{D}}$-module

We take up the setting of Sections 9.9.c and 12.5.22. As a consequence, we obtain the following property.

Let $\widetilde{\mathcal{T}}$ be a smooth object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, where X is an open set in $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$, and set $g = x_1 \cdots x_r$. Recall that, for $\ell \leqslant r$, we denote by $\mathfrak{J}_{\ell+1}$ the set of subsets $J \subset \{1, \dots, r\}$ with $\#J = \ell + 1$.

We have seen in Proposition 9.9.12 that $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable and a middle extension along (g) and the morphisms $\mathrm{N}, \mathrm{can}, \mathrm{var}$ are strict. Assume that $\widetilde{\mathcal{T}}$ is equipped with a pre-polarization S of weight w. Then, for each $\ell \geqslant 0$ there is an isomorphism

$$\mathrm{P}_\ell\psi_{g,1}(\widetilde{\mathcal{T}}, (-1)^\ell\mathrm{S}) \simeq \bigoplus_{J \in \mathfrak{J}_{\ell+1}} {}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_I^*(\widetilde{\mathcal{T}}, \mathrm{S}))(-\ell) \qquad (I = J^{\mathrm{c}}). \tag{12.7.24$\,*$}$$

The proof is similar to that of the previous example (one can identify ${}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_I^*(\widetilde{\mathcal{T}}, \mathrm{S})$ with the result of the iteration of $\psi_{x_j,1}$, for j varying in J, applied to $(\widetilde{\mathcal{T}}, \mathrm{S})$).

12.7.25. Localization, dual localization and middle extension in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$. Let D be an effective divisor in X and let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is strictly $\mathbb{R}$-specializable and localizable along D (i.e., its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so).

We then set

$$\begin{aligned}\widetilde{\mathcal{T}}[*D] &:= (\widetilde{\mathcal{M}}'[*D], \widetilde{\mathcal{M}}''[!D], \mathfrak{s}^{(*D)}),\\ \widetilde{\mathcal{T}}[!D] &:= (\widetilde{\mathcal{M}}'[!D], \widetilde{\mathcal{M}}''[*D], \mathfrak{s}^{(!D)}).\end{aligned}$$

These functors satisfy obvious identities with respect to Hermitian duality 12.7.4. By Proposition 12.5.39, there are natural morphisms

$$\widetilde{\mathcal{T}}[!D] \xrightarrow{\text{dloc}} \widetilde{\mathcal{T}} \xrightarrow{\text{loc}} \widetilde{\mathcal{T}}[*D].$$

The image of $\text{loc} \circ \text{dloc}$ is equal to $(\text{Im}(\text{loc}' \circ \text{dloc}'), \widetilde{\mathcal{M}}''[!D]/\operatorname{Ker}(\text{loc}'' \circ \text{dloc}''), \mathfrak{s})$, where $\mathfrak{s}$ is induced by $\mathfrak{s}^{(*D)}$ (this is well-defined according to (12.5.39 $*$)). The coimage of $\text{loc} \circ \text{dloc}$ is equal to $(\widetilde{\mathcal{M}}'[!D]/\operatorname{Ker}(\text{loc}' \circ \text{dloc}'), \text{Im}(\text{loc}'' \circ \text{dloc}''), \mathfrak{s})$, where $\mathfrak{s}$ is induced by $\mathfrak{s}^{(!D)}$, and both coincide according to the same argument. We denote this object by $\widetilde{\mathcal{T}}[!{*}D]$. As noticed in Section 11.5, it is possibly not strictly $\mathbb{R}$-specializable along D, and we have given criteria for this property to hold.

In particular, if $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable and a middle extension along D, then $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}[!{*}D]$, dloc is an epimorphism and loc is a monomorphism.

12.7.26. Beilinson extension of a $\widetilde{\mathcal{D}}$-triple. Let $g : X \to \mathbb{C}$ be a holomorphic function, set $D = (g)$ and let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}$-Triples(X) as in 12.7.25. We set (see Propositions 11.6.15 and 12.5.43)

$$\Xi_g \widetilde{\mathcal{T}} = (\Xi_g \widetilde{\mathcal{M}}', \Xi_g \widetilde{\mathcal{M}}'', \Xi_g \mathfrak{s}).$$

The exact sequences (11.6.2 !) and (11.6.2 $*$) exist at the level of $\widetilde{\mathcal{D}}$-triples and the gluing construction of Theorem 11.6.3 together with the equivalence of Corollary 11.6.5 hold at the level of $\widetilde{\mathcal{D}}$-triples.

12.7.27. Non-characteristic restriction of a $\widetilde{\mathcal{D}}$-triple (second version). We revisit Section 12.7.22. If $D = H$ is smooth and strictly non-characteristic with respect to $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$, then, according to Propositions 11.2.9, 11.2.13 and 12.5.42, there are natural identifications $\operatorname{Ker} \text{dloc} \simeq {}_{\mathrm{T}}\iota_*({}_{\mathrm{T}}\iota^* \widetilde{\mathcal{T}})$ and ${}_{\mathrm{T}}\iota_*({}_{\mathrm{T}}\iota^* \widetilde{\mathcal{T}})(-1) \simeq \operatorname{Coker} \text{loc}$, where $\iota : H \hookrightarrow X$ denotes the inclusion. In such a way, we have two exact sequences extending (11.2.18) to $\widetilde{\mathcal{D}}$-Triples(X):

$$\begin{gathered} 0 \longleftarrow \widetilde{\mathcal{T}} \xleftarrow{\text{dloc}} \widetilde{\mathcal{T}}[!H] \longleftarrow {}_{\mathrm{T}}\iota_*({}_{\mathrm{T}}\iota^* \widetilde{\mathcal{T}}) \longleftarrow 0,\\ 0 \longrightarrow \widetilde{\mathcal{T}} \xrightarrow{\text{loc}} \widetilde{\mathcal{T}}[*H] \longrightarrow {}_{\mathrm{T}}\iota_*({}_{\mathrm{T}}\iota^* \widetilde{\mathcal{T}})(-1) \longrightarrow 0. \end{gathered} \tag{12.7.27 $*$}$$

12.7.c. Pushforward in the category $\widetilde{\mathcal{D}}$-Triples(X)

***12.7.28. Definition* (Proper pushforward).** Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}$-Triples$(X)_{\mathrm{coh}}$ supported on Z and let $f : X \to Y$ be a holomorphic map which is proper on Z. Then the k-th pushforward ${}_{\mathrm{T}}f_*^{(k)} \widetilde{\mathcal{T}}$ is the object

$${}_{\mathrm{T}}f_*^{(k)} \widetilde{\mathcal{T}} := ({}_{\mathrm{D}}f_*^{(k)} \widetilde{\mathcal{M}}', {}_{\mathrm{D}}f_*^{(-k)} \widetilde{\mathcal{M}}'', {}_{\mathrm{T}}f_*^{(k,-k)} \mathfrak{s})$$

of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(Y)_{\mathrm{coh}}$. It satisfies (see (12.4.4 $*$))

(12.7.28 $*$)
$$({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}})^* = {}_{\mathrm{T}}f_*^{(-k)}(\widetilde{\mathcal{T}}^*).$$

It is convenient to consider the pushforward as a graded object ${}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}} = \bigoplus_k {}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$. Then a pre-polarization $\mathrm{S} : \widetilde{\mathcal{T}} \xrightarrow{\sim} \widetilde{\mathcal{T}}^*(-w)$ of weight w induces a pre-polarization
$$ {}_{\mathrm{T}}f_*^{(\bullet)}\mathrm{S} : {}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}} \xrightarrow{\sim} ({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}})^*(-w),$$
which is graded, by taking the grading considered in Section 12.7.6 for $({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}})^*$. More specifically, each ${}_{\mathrm{T}}f_*^{(k)}\mathrm{S}$ is an isomorphism ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}} \xrightarrow{\sim} ({}_{\mathrm{T}}f_*^{(-k)}\widetilde{\mathcal{T}})^*(-w)$.

If we represent $(\widetilde{\mathcal{T}}, \mathrm{S})$ by a Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$ of weight w with $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$, then ${}_{\mathrm{T}}f_*^{(\bullet)}(\widetilde{\mathcal{T}}, \mathrm{S}) = ({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{M}}, {}_{\mathrm{T}}f_*^{(\bullet)}\mathcal{S})$, where ${}_{\mathrm{T}}f_*^{(k)}\mathcal{S} := {}_{\mathrm{T}}f_*^{(k,-k)}\mathcal{S}$ pairs ${}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ and ${}_{\mathrm{D}}f_*^{(-k)}\overline{\mathcal{M}}$.

12.7.29. Kashiwara's equivalence. Let $\iota : Z \hookrightarrow X$ denote the inclusion of a closed submanifold. The functor ${}_{\mathrm{T}}\iota_*$ from $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(Z)$ into itself is fully faithful. Moreover, if $Z = H$ is smooth of codimension one in X, the functor ${}_{\mathrm{T}}\iota_*$ induces an equivalence between the full subcategory of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(H)$ whose objects are strict, and the full subcategory of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ whose objects are strictly $\mathbb{R}$-specializable along H and supported on H. An inverse functor is then $\widetilde{\mathcal{T}} \mapsto \mathrm{gr}_0^V \widetilde{\mathcal{T}}$ (see Remark 12.5.37 for $\mathrm{gr}_0^V \mathfrak{s}$). Indeed, this follows from Propositions 9.6.2 and 12.4.7.

12.7.30. The Lefschetz morphism for triples. From Definition 8.7.20 and (12.4.13 $*$) we can define the Lefschetz morphism attached to a real $(1,1)$-form η by the formula
$$\mathrm{X}_\eta = (\mathrm{X}'_\eta, \mathrm{X}''_\eta) : {}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}} \longrightarrow {}_{\mathrm{T}}f_*^{(k+2)}\widetilde{\mathcal{T}}(1).$$
It is functorial with respect to $\widetilde{\mathcal{T}}$ and satisfies $\mathrm{X}_\eta^* = \mathrm{X}_\eta$. Moreover, the graded object $({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}}, \mathrm{X}_\eta)$ is an $\mathfrak{sl}_2$-structure on the category $\mathsf{A} = \widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ with X_η corresponding to X, in the sense of Definition 3.3.3 (together with Remark 3.3.4 for the twist).

12.7.31. Adjunction and trace in the case of a finite morphism. We consider the setting of Example 8.7.31, with a finite morphism $f : X = \mathbb{C}^n \to Y = \mathbb{C}^n$ defined by $f_i(x_1, \dots, x_n) = x_i^{r_i}$, $i = 1, \dots, n$, and $r_i \geqslant 2$ for $i = 1, \dots, \ell$, and $r_i = 1$ for $i = \ell + 1, \dots, n$. Furthermore, we assume that the object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(Y)$ is such that $\mathfrak{s}$ takes values in $\widetilde{\mathcal{C}}_Y^\infty$. We deduce from Examples 8.7.31 and 12.4.17 two morphisms
$$\widetilde{\mathrm{adj}}_f = (\mathrm{adj}'_f, \mathrm{Tr}''_f) : \widetilde{\mathcal{T}} \longrightarrow {}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}), \quad \widetilde{\mathrm{Tr}}_f = (\mathrm{Tr}'_f, \mathrm{adj}''_f) : {}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}) \longrightarrow \widetilde{\mathcal{T}},$$
whose composition is the identity, making $\widetilde{\mathcal{T}}$ a direct summand of ${}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}})$ in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(Y)$.

12.7.32. Pushforward and specialization of $\widetilde{\mathcal{D}}$-triples. Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ which is coherent and strictly $\mathbb{R}$-specializable along $(g) = (g' \circ f)$, where $f : X \to Y$ is proper. Let $h : Y \to \mathbb{C}$ be a holomorphic function and set $g = h \circ f$. Let us assume that, for each k and λ, ${}_{\mathrm{T}}f_*^{(k)}(\psi_{g,\lambda}\widetilde{\mathcal{T}})$ and ${}_{\mathrm{T}}f_*^{(k)}(\phi_{g,1}\widetilde{\mathcal{T}})$ are *strict.*

It follows from Corollary 9.8.9 and Theorem 12.6.1 that there are natural isomorphisms of $\widetilde{\mathfrak{D}}$-Triples(X)-Lefschetz structures

$$\begin{aligned}{}_{\mathrm{T}}f_*^{(k)}(\psi_{g,\lambda}\widetilde{\mathfrak{T}},\mathrm{N}) &\simeq \psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathfrak{T}},\mathrm{N})),\\ {}_{\mathrm{T}}f_*^{(k)}(\phi_{g,1}\widetilde{\mathfrak{T}},\mathrm{N}) &\simeq \phi_{h,1}({}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathfrak{T}},\mathrm{N})),\end{aligned}$$

and of of nearby/vanishing $\widetilde{\mathfrak{D}}$-Triples(X)-Lefschetz quivers

$${}_{\mathrm{T}}f_*^{(k)}\big((\psi_{g,1}\widetilde{\mathfrak{T}},\mathrm{N}),(\phi_{g,1}\widetilde{\mathfrak{T}},\mathrm{N}),\mathrm{can},\mathrm{var}\big)\simeq\big((\psi_{g,1}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathfrak{T}}),\mathrm{N}),(\phi_{g,1}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathfrak{T}}),\mathrm{N}),\mathrm{can},\mathrm{var}\big).$$

12.7.d. The pushforward functor as a cohomological functor

12.7.33. Complexes and double complexes in $\widetilde{\mathfrak{D}}$**-Triples.** Let us start with a general remark which explains the introduction of a sign in Definition 12.4.3(2). Let $(\widetilde{K}'^{\bullet},d')$ and $(\widetilde{K}''^{\bullet},d'')$ be bounded complexes of right $\widetilde{\mathcal{D}}_Y$-modules and let $\mathfrak{s}:K'^{\bullet}\otimes_{\widetilde{\mathbb{C}}}\overline{K''^{\bullet}}\to\mathfrak{C}_Y$ be a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism to the right $\mathcal{D}_{Y,\overline{Y}}$-module $\mathfrak{C}_Y$ (i.e., a complex with $\mathfrak{C}_Y$ as its only nonzero term, placed in degree zero). We wish to transform this set of data to a complex in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathfrak{D}}\text{-Triples}(Y))$.

Let m'_k, resp. m''_ℓ, be a local section of K'^k, resp. K''^ℓ. The differential d of the simple complex associated to $K'^{\bullet}\otimes_{\widetilde{\mathbb{C}}}\overline{K''^{\bullet}}$ satisfies

$$d(m'_k\otimes\overline{m''_\ell})=d'm'_k\otimes\overline{m''_\ell}+(-1)^k m'_k\otimes\overline{d''m''_\ell}.$$

Since $\mathfrak{s}$ is a morphism of complexes, it is compatible with d, and since the differential of the complex $\mathfrak{C}_Y$ is zero, we obtain the relation

$$\mathfrak{s}(d'm'_k,\overline{m''_\ell})=(-1)^{k-1}\mathfrak{s}(m'_k,\overline{d''m''_\ell})$$

for every k,ℓ. Let $\mathfrak{s}_k:K'^k\otimes\overline{K''^{-k}}\to\mathfrak{C}_Y$ denote the pairing induced by $\mathfrak{s}$. The above relation implies that (recall that $\varepsilon(k)=(-1)^{k(k-1)/2}$)

$$(d',d''):(\widetilde{K}'^k,\widetilde{K}''^{-k},\varepsilon(k)\mathfrak{s}_k)\longrightarrow(\widetilde{K}'^{k+1},\widetilde{K}''^{-k-1},\varepsilon(k+1)\mathfrak{s}_{k+1})$$

is a morphism in $\widetilde{\mathfrak{D}}$-Triples(Y). In this way we obtain a differential complex in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathfrak{D}}\text{-Triples}(Y))$:

$$(\widetilde{K}^{\bullet},d)=\bigoplus_k(\widetilde{K}^k,d),\quad \widetilde{K}^k=(\widetilde{K}'^k,\widetilde{K}''^{-k},\varepsilon(k)\mathfrak{s}_k),\quad d=(d',d'').$$

For double complexes, the argument is similar. Given double complexes

$$\big((\widetilde{K}'^{i,j})_{i,j},d'_1,d'_2\big),\quad \big((\widetilde{K}''^{k,\ell})_{k,\ell},d''_1,d''_2\big)$$

of $\widetilde{\mathcal{D}}_Y$-modules and a sesquilinear pairing $\mathfrak{s}$ with values in $\mathfrak{C}_Y$ whose components are $\mathfrak{s}_{ij}:K'^{i,j}\otimes\overline{K''^{-i,-j}}\to\mathfrak{C}_Y$ and $\mathfrak{s}$ is zero from $K'^{i,j}\otimes\overline{K''^{k,\ell}}$ if $i+k\neq0$ or $j+\ell\neq0$, we obtain a double complex in $\widetilde{\mathfrak{D}}$-Triples(Y) as

$$\begin{aligned}(\widetilde{K}^{\bullet,\bullet},d_1,d_2)&=\bigoplus_{i,j}(\widetilde{K}^{i,j},d_1,d_2),\quad \widetilde{K}^{i,j}=(\widetilde{K}'^{i,j},\widetilde{K}''^{-i,-j},\varepsilon(i+j)\mathfrak{s}_{i,j}),\\ d_a&=(d'_a,d''_a),\ a=1,2.\end{aligned}$$

12.7.34. We interpret the functors ${}_{\mathrm{T}}f_*^{(k)}$ as cohomology functors of a pushforward functor ${}_{\mathrm{T}}f_* : \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)) \to \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y))$. This will enable us to treat the Leray spectral sequence for the composition of maps. In order to do so, it is convenient to work with a flabby resolution of $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})$ $(\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'')$. We will use the resolution $\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ by the relative C^∞ Spencer complex (Exercise 8.51). Recall that we set $\mathrm{Sp}_X^{\infty,k} = \bigoplus_\ell \widetilde{\Theta}_{X,\ell} \otimes \widetilde{\mathcal{E}}_X^{(0,k+\ell)}$.

Let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$. We represent the pushforward complex of each component of $\widetilde{\mathcal{T}}$ as $\widetilde{K}^\bullet = \widetilde{K}'^\bullet, \widetilde{K}''^\bullet$ with

$$\widetilde{K}^\bullet = f_!(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}_X^{\infty,\bullet} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \simeq f_!(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}_X^{\infty,\bullet}) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y, \quad \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'',$$

and the pushforward sesquilinear pairing as given by (12.4.5 $*$).

12.7.35. Definition. The pushforward triple ${}_{\mathrm{T}}f_*\widetilde{\mathcal{T}}$ is the complex whose k-th term is

$$\Big(K'^k, K''^{-k}, \varepsilon(k)(\textstyle\int f_* \mathrm{Sp}_Y^\infty(\mathfrak{s}))_k\Big).$$

and whose differentials are $(d', d'') = (f_*\widetilde{\delta}^\infty_{\widetilde{\mathcal{M}}',Y}, f_*\widetilde{\delta}^\infty_{\widetilde{\mathcal{M}}'',Y})$.

The following is then clear.

12.7.36. Lemma. *For each $k \in \mathbb{Z}$, the cohomology $\mathcal{H}^k({}_{\mathrm{T}}f_*\widetilde{\mathcal{T}})$ is isomorphic to ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ of Definition* 12.7.28. ☐

In a way similar to that of Corollary 8.7.18, we deduce:

12.7.37. Corollary. *The pushforward triple ${}_{\mathrm{T}}f_*\widetilde{\mathcal{T}}$ can be represented by a complex of amplitude n. If $\widetilde{\mathcal{T}}$ has a finite filtration $W_\bullet\widetilde{\mathcal{T}}$ in the abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, then there exists a spectral sequence, functorial with respect to $\widetilde{\mathcal{T}}$:*

$$E_1^{-\ell,k+\ell} = {}_{\mathrm{T}}f_*^{(k)}(\mathrm{gr}_\ell^W\widetilde{\mathcal{T}}) \implies \mathrm{gr}_\ell^W {}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathcal{T}}),$$

where $W_\bullet({}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathcal{T}}))$ is the image filtration $\mathrm{image}[{}_{\mathrm{T}}f_*^{(k)}(W_\bullet\widetilde{\mathcal{T}}) \to {}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathcal{T}})]$. ☐

We can now extend Corollary 8.7.28 to the case of the categories $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}$.

***12.7.38. Corollary* (The Leray spectral sequence for the composition of maps)**

Let $f : X \to Y$ and $f' : Y \to Z$ be holomorphic maps and let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$. We assume that $f' \circ f$ is proper on $S = \mathrm{Supp}\,\widetilde{\mathcal{T}}$ (hence so is f, and f' is proper on $f(\mathrm{Supp}\,\widetilde{\mathcal{T}})$). Then there exists a bounded spectral sequence with $E_2^{p,q} = {}_{\mathrm{T}}f_*'^{(p)}({}_{\mathrm{T}}f_*^{(q)}\widetilde{\mathcal{T}})$ which converges to ${}_{\mathrm{T}}(f' \circ f)_*^{(p+q)}\widetilde{\mathcal{T}}$.

The *Leray filtration* $\mathrm{Ler}^\bullet{}_{\mathrm{T}}(f' \circ f)_*^{(k)}\widetilde{\mathcal{T}}$ satisfies

$$\mathrm{gr}^p_{\mathrm{Ler}}\big({}_{\mathrm{T}}(f' \circ f)_*^{(p+q)}\widetilde{\mathcal{T}}\big) = E_\infty^{p,q}.$$

In particular, since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, $\mathrm{gr}^p_{\mathrm{Ler}}\big({}_{\mathrm{T}}(f' \circ f)_*^{(k)}\widetilde{\mathcal{T}}\big)$ vanishes unless $p \in [-\dim Y, \dim Y]$ (see Remark 8.7.13).

***Proof of Corollary* 12.7.38. *Step 1*.** Arguing as in the proof of Theorem 8.7.23, we find a natural quasi-isomorphism

$$\operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\otimes_{f^{-1}\mathcal{D}_{Y,\overline{Y}}} f^{-1}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}(\mathcal{D}_{Y,\overline{Y}}) \xrightarrow{\sim} \operatorname{Sp}_{X,\overline{X}\to Z,\overline{Z}}(\mathcal{D}_{X,\overline{X}}), \tag{12.7.39}$$

leading to a quasi-isomorphism

$$f'_*\Big(f_*\big(\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big)\otimes_{\mathcal{D}_{Y,\overline{Y}}}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}\Big) \xrightarrow{\sim} f'_*\big(\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\operatorname{Sp}_{X,\overline{X}\to Z,\overline{Z}}(\mathcal{D}_{X,\overline{X}})\big).$$

The integration morphism (12.2.12)

$$\int^{\bullet}_f : f_*\big(\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big)\longrightarrow \mathfrak{C}_{Y,f(S)}$$

can be composed with that for f' to yield

$$\int^{\bullet}_{f'}\circ\Big(\int^{\bullet}_f\otimes\mathrm{Id}\Big) : f'_*\Big(f_*\big(\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big)\otimes_{\mathcal{D}_{Y,\overline{Y}}}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}\Big)\longrightarrow \mathfrak{C}_Z\,.$$

On the other hand, we have the integration morphism

$$\int^{\bullet}_{f'\circ f} : f'_*\big(\mathfrak{C}_{X,S}\otimes_{\mathcal{D}_{X,\overline{X}}}\operatorname{Sp}_{X,\overline{X}\to Z,\overline{Z}}(\mathcal{D}_{X,\overline{X}})\big)\longrightarrow \mathfrak{C}_Z\,.$$

We claim that, through the above quasi-isomorphism, both integration morphisms coincide. It is enough to prove that their restrictions to the degree-zero terms of the complexes coincide, on noting that these complexes have nonzero terms only on nonpositive degrees. In degree zero, the inverse of the isomorphism (12.7.39) is induced by the natural morphism

$$\begin{aligned}&\mathcal{O}_{X,\overline{X}}\otimes_{(f'\circ f)^{-1}\mathcal{O}_{Z,\overline{Z}}}(f'\circ f)^{-1}\mathcal{D}_{Z,\overline{Z}}\\ &\quad\xrightarrow{\sim}\big(\mathcal{O}_{X,\overline{X}}\otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}}f^{-1}\mathcal{D}_{Y,\overline{Y}}\big)\otimes_{f^{-1}\mathcal{D}_{Y,\overline{Y}}}\big(f^{-1}\mathcal{O}_{Y,\overline{Y}}\otimes_{(f'\circ f)^{-1}\mathcal{O}_{Z,\overline{Z}}}(f'\circ f)^{-1}\mathcal{D}_{Z,\overline{Z}}\big)\end{aligned}$$

defined by $\varphi\otimes Q\mapsto(\varphi\otimes 1)\otimes(1\otimes Q)$ (see Exercise 8.37).

We are thus led to checking that the following diagram commutes:

$$\begin{tikzcd}
(f'\circ f)_*\,\mathfrak{C}_{X,S}\otimes_{\mathcal{O}_{Z,\overline{Z}}}\mathcal{D}_{Z,\overline{Z}} \arrow[rr] \arrow[dr, "\int_{f'\circ f}\otimes\mathrm{Id}"'] & & f'_*(f_*\,\mathfrak{C}_{X,S}\otimes_{\mathcal{O}_{Y,\overline{Y}}}\mathcal{D}_{Y,\overline{Y}})\otimes_{\mathcal{O}_{Z,\overline{Z}}}\mathcal{D}_{Z,\overline{Z}} \arrow[dl, "\int_{f'}\int_f\otimes\mathrm{Id}"] \\
& \mathfrak{C}_Z\otimes_{\mathcal{O}_{Z,\overline{Z}}}\mathcal{D}_{Z,\overline{Z}} &
\end{tikzcd}$$

This follows from the obvious commutation (Fubini) $\int_{f'\circ f}=\int_{f'}\int_f$ on currents. □

***Proof of Corollary* 12.7.38. *Step 2*.** To compute the iterated pushforward ${}_{\mathrm{T}}f'_*({}_{\mathrm{T}}f_*(\widetilde{\mathcal{T}}))$, we consider the double complexes

$$\begin{aligned}\widetilde{K}'^{\bullet,\bullet}&=f'_*\Big[f_*\big(\widetilde{\mathcal{M}}'\otimes_{\widetilde{\mathcal{D}}_X}\operatorname{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{\widetilde{\mathcal{D}}_Y}\operatorname{Sp}^{\infty}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big]\\ \widetilde{K}''^{\bullet,\bullet}&=f'_*\Big[f_*\big(\widetilde{\mathcal{M}}''\otimes_{\widetilde{\mathcal{D}}_X}\operatorname{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{\widetilde{\mathcal{D}}_Y}\operatorname{Sp}^{\infty}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big],\end{aligned}$$

and the morphism defined from the sesquilinear pairing $\mathfrak{s}$ with values in the double complex

$$\begin{aligned} f'_*\Big[f_*\big(\mathfrak{C}_{X,S}\otimes_{\widetilde{\mathcal{D}}_{X,\overline{X}}}\mathrm{Sp}^{\infty}_{X,\overline{X}\to Y,\overline{Y}}(\widetilde{\mathcal{D}}_{X,\overline{X}})\big)\otimes_{\widetilde{\mathcal{D}}_{Y,\overline{Y}}}\mathrm{Sp}^{\infty}_{Y,\overline{Y}\to Z,\overline{Z}}(\widetilde{\mathcal{D}}_{Y,\overline{Y}})\Big]& \\ \simeq f'_*\Big[f_*\big(\mathfrak{C}_{X,S}\otimes_{\widetilde{\mathcal{D}}_{X,\overline{X}}}\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\widetilde{\mathcal{D}}_{X,\overline{X}})\big)\otimes_{\widetilde{\mathcal{D}}_{Y,\overline{Y}}}\mathrm{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}(\widetilde{\mathcal{D}}_{Y,\overline{Y}})\Big]& \end{aligned}$$

with $\mathrm{Sp}^{\infty}_{X,\overline{X}\to Y,\overline{Y}}(\widetilde{\mathcal{D}}_{X,\overline{X}}) := \mathrm{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathbb{C}}}\mathrm{Sp}^{\infty}_{\overline{X}\to\overline{Y}}(\widetilde{\mathcal{D}}_{\overline{X}})$. Composing with the double integration morphism in Step 1 yields a morphism $\mathfrak{s}: K'^{i,j}\otimes K''^{k,\ell}\to\mathfrak{C}_Z$ of double complexes, which is thus zero if $i+k\neq 0$ or $j+\ell\neq 0$. As explained in Section 12.7.33, we obtain a double complex in $\widetilde{\mathcal{D}}$-Triples$(\widetilde{\mathcal{D}}_Z)$. The computation of Step 1 shows that the associated simple complex in $\widetilde{\mathcal{D}}$-Triples$(\widetilde{\mathcal{D}}_Z)$ is quasi-isomorphic to the complex computing ${}_{\mathrm{T}}(f'\circ f)_*(\widetilde{\mathcal{T}})$. We conclude that the spectral sequence of the double complex, with $E_2^{p,q}={}_{\mathrm{T}}f'^{(p)}_*({}_{\mathrm{T}}f^{(q)}_*\widetilde{\mathcal{T}})$, converges to ${}_{\mathrm{T}}(f'\circ f)^{(p+q)}_*\widetilde{\mathcal{T}}$. ☐

12.7.40. The restriction and Gysin morphisms in **$\widetilde{\mathcal{D}}$-Triples.** We take up the setting of Section 11.2.d, in which H is a smooth hypersurface in the compact manifold X. We consider an object $\widetilde{\mathcal{T}}=(\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}'',\mathfrak{s})$ of $\widetilde{\mathcal{D}}$-Triples(X) and we assume that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}''$.

We denote by a_X the constant map on X. Recall that

$${}_{\mathrm{T}}a^{(k)}_{X*}\widetilde{\mathcal{T}}=({}_{\mathrm{D}}a^{(k)}_{X*}\widetilde{\mathcal{M}}',a^{(-k)}_{X*}\widetilde{\mathcal{M}}'',{}_{\mathrm{T}}a^{(k,-k)}_{X*}\mathfrak{s}).$$

By applying ${}_{\mathrm{T}}a^{(\bullet)}_{X*}$ to the exact sequences (12.7.27 $*$) and noticing the identification ${}_{\mathrm{T}}a^{(\bullet)}_{X*}{}_{\mathrm{T}}\iota_{H*}\simeq{}_{\mathrm{T}}a^{(\bullet)}_{H*}$ following from (12.4.14 $*$) (a special case of Corollary 12.7.38), we define

$$\mathrm{restr}_H:{}_{\mathrm{D}}a^{(k)}_{X*}\widetilde{\mathcal{T}}\longrightarrow{}_{\mathrm{D}}a^{(k+1)}_{H*}\widetilde{\mathcal{T}}_H\quad\text{and}\quad\mathrm{Gys}_H:{}_{\mathrm{D}}a^{(-k-1)}_{H*}\widetilde{\mathcal{T}}_H(-1)\longrightarrow{}_{\mathrm{D}}a^{(-k)}_{X*}\widetilde{\mathcal{T}}$$

as the connecting morphisms in the corresponding long exact sequences in $\widetilde{\mathbb{C}}$-Triples.

12.7.41. Proposition. *We have a commutative diagram*

$$\begin{tikzcd} & {}_{\mathrm{T}}a^{(k)}_{X*}\widetilde{\mathcal{T}} \arrow[rr, "\mathrm{X}_{\mathcal{L}}"] \arrow[dr, "\mathrm{restr}_H"] & & {}_{\mathrm{T}}a^{(k+2)}_{X*}\widetilde{\mathcal{T}}(1) \\ {}_{\mathrm{T}}a^{(k-1)}_{H*}\widetilde{\mathcal{T}}_H(-1) \arrow[ur, "\mathrm{Gys}_H"] \arrow[rr, "\mathrm{X}_{\mathcal{L}}"] & & {}_{\mathrm{T}}a^{(k+1)}_{H*}\widetilde{\mathcal{T}}_H \arrow[ur, "\mathrm{Gys}_H"'] & \end{tikzcd}$$

Proof. We notice that the "prime" component of this diagram is the diagram of Proposition 11.2.20, while the "double-prime" component is the similar diagram after changing the exponents and the Tate twists to their opposite value, taking the arrows in the other direction and exchanging restr_H and Gys_H, and this is a diagram shifted from that of Proposition 11.2.20. Commutativity follows then from the commutativity proved in that proposition (since the sesquilinear pairing is not concerned for commutativity). ☐

12.8. Exercises

Exercise 12.1 **(Pushforward of the sheaf of currents as a right $\mathcal{D}_{X,\overline{X}}$-module)**
Using the definition of Section 8.7.e, show that

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X = f_!(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})). $$

Exercise 12.2. Assume that X is a closed submanifold of Y and denote by $\iota : X \hookrightarrow Y$ the embedding (which is a proper map). Denote by $\mathbf{1}$ the canonical section of $\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}$. Show that the natural map

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\,\mathfrak{C}_X = \iota_*\big(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}\big) \longrightarrow \mathfrak{C}_Y, \quad u\otimes\mathbf{1} \longmapsto \int_\iota u $$

induces an isomorphism of the right $\mathcal{D}_{Y,\overline{Y}}$-module ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\,\mathfrak{C}_X$ with the submodule of $\mathfrak{C}_Y$ consisting of currents supported on X. [*Hint*: Use a local computation.]

For example, consider the case $\iota : X = X\times\{0\} \hookrightarrow X\times\mathbb{C}$, with coordinate t on $\mathbb{C}$ and identify ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\,\mathfrak{C}_X$ with $\iota_*\,\mathfrak{C}_X[\partial_t, \partial_{\overline{t}}]$.

Exercise 12.3. Extend the result of Exercise 8.54 to the case of right $\mathcal{D}_{X,\overline{X}}$-modules and show that the composed map

$$ f_*\,\mathfrak{C}_X \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_X \longrightarrow \mathfrak{C}_Y $$

is the integration of currents of Definition 12.2.9.

Exercise 12.4. Let $f : X \to Y$ be a holomorphic map and let $Z \subset X$ be a closed subset on which f is proper.

(1) Define the sub-$\mathcal{D}_{X,\overline{X}}$-module $\mathfrak{C}_{X,Z}$ of $\mathfrak{C}_X$ consisting of currents supported on Z.

(2) Show that the integration of currents $\int_f$ induces a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism of complexes

$$ \int_f : {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,Z} \longrightarrow \mathfrak{Db}_Y^{\bullet}[2m] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}} \simeq \mathfrak{C}_Y\,. $$

[*Hint*: In Formula (12.2.9 $*$), let K be the compact support of η; choose a compact neighbourhood K' of $f^{-1}(K)\cap Z$, and use a partition of the unity relative to the covering made by the complement of K' in X and the interior of K'.]

Exercise 12.5. If f is a a projection $X = Y\times T \to Y$ with $\dim T = p = n - m$, show that there exists a morphism

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X \longrightarrow \mathfrak{C}_Y \tag{12.5 $*$} $$

which does not make use of the integration morphism (more precisely, it only uses integration of constant functions).

(1) Consider the morphism

$$ \Theta_{X/Y,\overline{X}/\overline{Y},\bullet} := \Theta_{X/Y,\bullet} \otimes_{\mathbb{C}} \Theta_{\overline{X}/\overline{Y},\bullet}. $$

and, following the same line as for (8.7.10 $*$), show that

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X \simeq \boldsymbol{R}f_!(\mathfrak{C}_X \otimes_{\mathcal{O}_{X,\overline{X}}} \Theta_{X/Y,\overline{X}/\overline{Y},\bullet}). $$

(2) By applying an argument similar to that of Exercise 8.26(1), prove that

$$\mathfrak{C}_X \otimes_{\mathcal{O}_{X,\overline{X}}} \Theta_{X/Y,\overline{X}/\overline{Y},\bullet} \simeq f^{-1}\mathcal{E}_Y^{m,m} \otimes_{f^{-1}\mathcal{C}^\infty_Y} \mathcal{E}^\bullet_{X/Y} \otimes_{\mathcal{C}^\infty_X} \mathfrak{Db}_X[2p].$$

(3) Note that a distribution on X annihilated by $\mathrm{d}_{X/Y}$ is locally a distribution on Y, and deduce that the complex $\mathcal{E}^\bullet_{X/Y} \otimes_{\mathcal{C}^\infty_X} \mathfrak{Db}_X$ is a resolution of $f^{-1}\mathfrak{Db}_Y$.

(4) Deduce an isomorphism

$$\mathfrak{C}_X \otimes_{\mathcal{O}_{X,\overline{X}}} \Theta_{X/Y,\overline{X}/\overline{Y},\bullet} \simeq f^{-1}\,\mathfrak{C}_Y[2p],$$

and thus

$$_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X \simeq \boldsymbol{R}f_!f^{-1}\,\mathfrak{C}_Y[2p].$$

(5) Since f is smooth of relative real dimension $2p$, there exists a natural morphism

$$\boldsymbol{R}f_!f^{-1}\,\mathfrak{C}_Y[2p] = \boldsymbol{R}f_!f^!\,\mathfrak{C}_Y \simeq \boldsymbol{R}f_!f^!\mathbb{C}_X \otimes_{\mathbb{C}_Y} \mathfrak{C}_Y \longrightarrow \mathfrak{C}_Y,$$

according to Verdier duality (see e.g. [**KS90**, Chap. 3]). Conclude the existence of (12.5 ∗).

(6) Compare with the morphism (12.2.12).

Exercise 12.6 **(Trace of a C^∞ function).** We consider the setting of Example 8.7.31 with the finite map $f : X = \mathbb{C}^n \to Y = \mathbb{C}^n$ defined by $f_i(x_1,\dots,x_n) = x_i^{r_i}$, $r_i \in \mathbb{N}^*$ and $r_i \geqslant 2 \Leftrightarrow 1 \leqslant i \leqslant \ell$. The goal of this exercise is to prove that the trace $\mathrm{Tr}_f(\varphi)$ of a C^∞ function φ on X is a C^∞ function on Y.(2)

(1) Use the Malgrange preparation theorem to show that the germ $\mathcal{C}^\infty_{X,0}$ is a module of finite type over $\mathcal{C}^\infty_{Y,0}$ generated by monomials $x^a\overline{x}^b$ with $0 \leqslant a_i, b_i \leqslant r_i - 1$ for all $i = 1,\dots,n$.

(2) Show that $\mathrm{Tr}_f(x^a\overline{x}^b) = 0$ if there exists i such that $a_i - b_i$ is not a multiple of r_i.

(3) Show that, otherwise, $\mathrm{Tr}_f(x^a\overline{x}^b)$ is a monomial in $y_i, \overline{y}_i$, $i = 1,\dots,n$.

(4) Conclude that, for any C^∞ function φ on X, $\mathrm{Tr}_f(\varphi)$ is a C^∞ function on Y.

(5) Show that, for any test function φ on X, we have the equality

$$\int_X \varphi \cdot f^*(\mathrm{d}y \wedge \mathrm{d}\overline{y}) = \int_Y \mathrm{Tr}_f(\varphi) \cdot \mathrm{d}y \wedge \mathrm{d}\overline{y}.$$

[Hint: Use the Fubini theorem.]

Exercise 12.7 **(Trace of a C^∞ form of maximal degree and integral of currents)**

We keep the setting of Exercise 12.6. In analogy with the trace of holomorphic forms of maximal degree (Exercise 8.57), we define the trace of a form of maximal degree $h\,\mathrm{d}x \wedge \mathrm{d}\overline{x}$ as

$$\mathrm{Tr}_f(h\,\mathrm{d}x \wedge \mathrm{d}\overline{x}) := \frac{1}{\prod_i r_i^2} \cdot \frac{\mathrm{Tr}_f\big((\prod_i |x_i|^2)h\big)}{\prod_i |y_i|^2} \cdot \mathrm{d}y \wedge \mathrm{d}\overline{y}.$$

Show that $\mathrm{Tr}_f(h\,\mathrm{d}x \wedge \mathrm{d}\overline{x})$ is C^∞ on Y and is equal to the current $\int_f(h\,\mathrm{d}x \wedge \mathrm{d}\overline{x})$. [*Hint*: Use Exercise 12.6(5).]

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(2) This property is specific to the finite map we consider; it would not be true for a general finite map; see [**Bar83**].

Exercise 12.8 **(Kashiwara's equivalence).** We keep notation of Proposition 12.4.7. Let $\mathfrak{s} : {}_{\mathrm{D}}\iota_*\mathcal{M}' \otimes \overline{{}_{\mathrm{D}}\iota_*\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing.

(1) Show that $\mathfrak{s}$ takes values in $\mathfrak{C}_{X,Z}$.

(2) Show that $\mathfrak{s}$ is determined by its values on $\iota_*(\mathcal{M}' \otimes \mathbf{1}) \otimes \overline{\iota_*(\mathcal{M}'' \otimes \mathbf{1})}$.

(3) Show that, for a test function η on X, $\langle \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})}), \eta\rangle$ only depends on $\eta_{|Z}$. [*Hint*: Write locally $X = Z \times \mathbb{C}_r$ with coordinates $x_1, \dots, x_r$ on $\mathbb{C}^r$ and use that $\mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})})\partial_{x_i} = \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})})\partial_{\overline{x}_i} = 0$.]

(4) Deduce that the correspondence $\eta_Z \mapsto \langle \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})}), \eta\rangle$, for some (or any) η with $\eta_{|Z} = \eta_Z$, defines a current of maximal degree on Z.

(5) Conclude the proof of Proposition 12.4.7.

Exercise 12.9 **(Pushforward of a sesquilinear pairing by a projection)**

Let $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing between left $\mathcal{D}_X$-modules. Assume that $X = Y \times T$ and that the projection $f : X = Y \times T \to Y$ is proper on $\operatorname{Supp} \mathcal{M}' \cap \operatorname{Supp} \mathcal{M}''$. Set $p = \dim T = n - m$.

(1) Let U be an open set in Y, and let

$$n'^{k}_{\infty} \in \Gamma(U, f_*(\mathcal{E}^{p+k}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}')), \quad n''^{-k}_{\infty} \in \Gamma(U, f_*(\mathcal{E}^{p-k}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}'')).$$

Show that $f_*\mathfrak{s}(n'^{k}_{\infty}, \overline{n''^{-k}_{\infty}})$ is a section on U of $f_*(\mathcal{E}^{2p}_{X/Y} \otimes_{\mathcal{C}^\infty_X} \mathfrak{Db}_X)$.

(2) Deduce that the integration of currents produces a section in $\Gamma(U, \mathfrak{Db}_Y)$:

$$\mathcal{E}^{2m}_Y(U) \ni \eta \xmapsto{\int_f f_*\mathfrak{s}(n'^{k}_{\infty}, \overline{n''^{-k}_{\infty}})} \int_f f_*\Big(\eta \wedge \mathfrak{s}(n'^{k}_{\infty}, \overline{n''^{-k}_{\infty}})\Big).$$

(3) Show that the signed pushforward

$$(-1)^{m(p+k)} \operatorname{Sgn}(p,k) \int_f f_*\mathfrak{s}(n'^{k}_{\infty}, \overline{n''^{-k}_{\infty}}) \in \Gamma(U, \mathfrak{Db}_Y)$$

corresponds, via side-changing, to the pushforward $({}_{\mathrm{T}}f^{(k,-k)}_*\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$ of Proposition 12.4.12. [*Hint*: Use Formula (0.2 $**$).]

Exercise 12.10. Show that, if $\mathcal{M}'$ or $\mathcal{M}''$ is supported on H, the right-hand side of (12.5.4 $**$) is always zero, and the residue formula (12.5.10 $**$) returns the value zero for every $\alpha \in \mathbb{R}$.

Exercise 12.11 **(see Remark 9.4.9).** Show that $\mathrm{gr}^V_\alpha(\mathfrak{s})$ induces pairings ($\ell \in \mathbb{Z}$):

$$\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^V_\alpha(\mathfrak{s}) := \mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^V_\alpha \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{gr}^{\mathrm{M}}_{-\ell} \mathrm{gr}^V_\alpha \mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

and, for $\ell \geqslant 0$,

$$\mathrm{P}_\ell \mathrm{gr}^V_\alpha(\mathfrak{s}) := \mathrm{P}_\ell \mathrm{gr}^V_\alpha \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{P}_\ell \mathrm{gr}^V_\alpha \mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

by composing with N^ℓ on any side.

Exercise 12.12. In the setting of Proposition 9.9.1, show that, with respect to the corresponding isomorphisms,

$$\psi_{h,\lambda}\mathfrak{s} = \psi_{g,\lambda^r}\mathfrak{s} \quad \text{and} \quad \phi_{h,1}\mathfrak{s} = \phi_{g,1}\mathfrak{s}.$$

[*Hint*: Use (12.5.16 $*$) and (12.5.27 $*$).]

12.9. Comments

Complex conjugation of a locally constant sheaf of $\mathbb{C}$-vector spaces can be defined in a straightforward way by considering the conjugate vector space of each fiber. Complex conjugation of a constructible complex of $\mathbb{C}$-vector spaces can be defined similarly, and the complex conjugate of a constructible complex remains a constructible complex. Assume that this complex takes the form of the de Rham complex of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Is it possible to define another holonomic $\mathcal{D}_X$-module $\mathcal{N}$ whose de Rham complex is the complex conjugate of that of $\mathcal{M}$? A solution to this question has been given by M. Kashiwara [**Kas86a**] when $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, and by T. Mochizuki [**Moc11b**, §4.4] (see also [**Sab00**, §II.3.1]) for any holonomic $\mathcal{D}_X$-module. However, the idea of M. Kashiwara is that it is easier to find $\mathcal{N}$ whose de Rham complex is the *Verdier dual* of the complex conjugate of the de Rham complex of $\mathcal{M}$. Namely, $\mathcal{N}$ is defined as the complex conjugate (in the sense of passing from $\mathcal{D}_{\overline{X}}$-modules to $\mathcal{D}_X$-modules) of the $\mathcal{D}_{\overline{X}}$-module $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X)$. In other words, when working with $\mathcal{D}_X$-modules, it is easier to handle the *Hermitian dual* $\mathcal{D}_X$-module than the conjugate $\mathcal{D}_X$-module, since the duality functor is not needed.

This explains why, when considering complex Hodge structures and having in mind the extension to $\mathcal{D}_X$-modules, instead of considering pairs of vector spaces together with an isomorphism of one space with the complex conjugate of the other one, we consider pairs of vector spaces together with an isomorphism of one space with the Hermitian dual of the other one, that is, pairs of vector spaces together with a non-degenerate sesquilinear pairing between them. Furthermore, the notion of non-degeneracy is difficult to manipulate under various operations on $\mathcal{D}_X$-modules, and this explains why this property is relaxed in the definition of the category $\mathcal{D}$-Triples.

The idea of considering poles of Mellin transforms with kernel $|f|^{2s}$ for some holomorphic function f, in order to analyze its nearby cycles and the monodromy on them, goes back to the work of D. Barlet ([**Bar82**] and the subsequent works [**Bar84, Bar85, Bar86, BM87, BM89**]). It has been instrumental in order to define nearby cycles in the theory of twistor $\mathcal{D}$-modules ([**Sab05**]), where the idea of taking a residue of such Mellin transforms has been introduced. Many aspects of this theory have then been much improved in the works of T. Mochizuki [**Moc11a, Moc15**] and we have taken advantage of these improvements in the presentation of this chapter. Such an approach has recently been used in Representation theory by Davis and Vilonen [**DV22**], who have emphasized the use of sesquilinear pairings with the Beilinson functor.