CHAPTER 4

VARIATIONS OF HODGE STRUCTURE ON A COMPLEX MANIFOLD

Summary. The notion of a variation of Hodge structure on a complex manifold is the first possible generalization of a Hodge structure. It naturally occurs when considering holomorphic families of smooth projective varieties. Later, we will identify this notion with the notion of a *smooth Hodge module*. We consider global properties of polarizable variations of Hodge structure on a smooth projective variety. On the one hand, the Hodge theorem asserts that the de Rham cohomology of a polarizable variation of Hodge structure on a smooth projective variety is itself a polarizable $\mathfrak{sl}_2$-Hodge structure. On the other hand, we show that the local system underlying a polarizable variation of Hodge structure on a smooth projective variety is semi-simple, and we classify all such variations with a given underlying semi-simple local system.

4.1. Variations of Hodge structure

4.1.a. Variations of $\mathbb{C}$-Hodge structure. The definition of a variation of $\mathbb{C}$-Hodge structure is modeled on the behaviour of the cohomology of a family of smooth projective varieties parametrized by a smooth algebraic variety, that is, a smooth projective morphism $f : Y \to X$, that we call below the "geometric setting".

Let us first motivate the definition. Let X be a connected (possibly non compact) complex manifold. In such a setting, the generalization of a vector space $\mathcal{H}^o$ is a *locally constant sheaf of vector spaces* $\underline{\mathcal{H}}$ *on* X. Let us choose a universal covering $\widetilde{X} \to X$ of X and let us denote by Π its group of deck-transformations, which is isomorphic to $\pi_1(X, \star)$ for any choice of a base-point $\star \in X$. Let us denote by $\widetilde{\mathcal{H}}$ the space of global sections of the pullback $\underline{\widetilde{\mathcal{H}}}$ of $\underline{\mathcal{H}}$ to $\widetilde{X}$. Then, giving $\underline{\mathcal{H}}$ is equivalent to giving the *monodromy representation* $\Pi \to \mathrm{GL}(\widetilde{\mathcal{H}})$. However, it is known that, in the geometric setting, the Hodge decomposition in each fiber of the family does not give rise to locally constant sheaves, but to C^∞-bundles.

In the geometric setting, to the locally constant sheaf $R^k f_* \mathbb{C}_X$ ($k \in \mathbb{N}$) is associated the *Gauss-Manin connection*, which is a holomorphic vector bundle on Y equipped with a holomorphic flat connection. In such a case, the Hodge filtration can be naturally defined and it is known to produce holomorphic bundles. Therefore, in the general setting of a variation of $\mathbb{C}$-Hodge structure that we intend to define, a better analogue of the complex vector space $\mathcal{H}^o$ is a holomorphic vector bundle $\mathcal{H}'$ equipped with a flat holomorphic connection $\nabla : \mathcal{H}' \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{H}'$, so that the locally constant sheaf $\underline{\mathcal{H}}' = \operatorname{Ker} \nabla$, that we also denote by $\mathcal{H}'^{\nabla}$, is the desired local system. Note that we can recover $(\mathcal{H}', \nabla)$ from $\underline{\mathcal{H}}'$ since the natural morphism of flat bundles

$$(\mathcal{O}_X \otimes_{\mathbb{C}} \underline{\mathcal{H}}', \mathrm{d} \otimes \mathrm{Id}) \longrightarrow (\mathcal{H}', \nabla)$$

is an isomorphism. A filtration is then a finite (exhaustive) decreasing filtration by sub-bundles $F^\bullet\mathcal{H}'$ (recall that a sub-bundle $F^p\mathcal{H}'$ of $\mathcal{H}'$ is a locally free $\mathcal{O}_X$-submodule of $\mathcal{H}'$ such that $\mathcal{H}'/F^p\mathcal{H}'$ is also a locally free $\mathcal{O}_X$-module; $F^\bullet\mathcal{H}'$ is a filtration by sub-bundles if each $F^p\mathcal{H}'/F^{p+1}\mathcal{H}'$ is a locally free $\mathcal{O}_X$-module). The main property, known as *Griffiths transversality property* is that the filtration should satisfy

$$\nabla(F^p\mathcal{H}') \subset \Omega^1_X \otimes_{\mathcal{O}_X} F^{p-1}\mathcal{H}' \quad \forall p \in \mathbb{Z}. \tag{4.1.1}$$

However, the analogue of a bi-filtered vector space is not a bi-filtered holomorphic flat bundle, since one knows in the geometric setting that one of the filtrations should behave holomorphically, while the other one should behave anti-holomorphically. This leads to a presentation by C^∞-bundles.

Let $\mathcal{H} = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} \mathcal{H}'$ be the associated C^∞ bundle and let D be the connection on $\mathcal{H}$ defined, for any C^∞ function φ and any local holomorphic section v of $\mathcal{H}'$, by $D(\varphi \otimes v) = \mathrm{d}\varphi \otimes v + \varphi \otimes \nabla v$ (this is a flat connection which decomposes with respect to types as $D = D' + D''$ and $D'' = \mathrm{d}'' \otimes \mathrm{Id}$). Then D'' is a holomorphic structure on $\mathcal{H}$, i.e., $\operatorname{Ker} D''$ is a holomorphic bundle with connection ∇ induced by D': this is $(\mathcal{H}', \nabla)$ by construction. Each bundle $F^p\mathcal{H}'$ gives rise similarly to a C^∞-bundle $F'^p\mathcal{H}$ which is holomorphic in the sense that $D''F'^p\mathcal{H} \subset \mathcal{E}^{0,1}_X \otimes F'^p\mathcal{H}$ (and thus $(D'')^2 = 0$ on $F'^p\mathcal{H}$).

On the other hand, D' defines an anti-holomorphic structure on $\mathcal{H}$ (see below), and $\operatorname{Ker} D'$ is an anti-holomorphic bundle with a flat anti-holomorphic connection $\overline{\nabla}$ induced by D''. If we wish to work with holomorphic bundle, we can thus consider the conjugate bundle[(1)] $\mathcal{H}'' = \overline{\operatorname{Ker} D'}$, that we equip with the holomorphic flat connection $\nabla = \overline{D''_{|\operatorname{Ker} D'}}$. A filtration of $\mathcal{H}$ by anti-holomorphic sub-bundles is by definition a filtration $F''^\bullet\mathcal{H}$ by C^∞-sub-bundles on which $D' = 0$. It corresponds to a filtration of $\mathcal{H}''$ by holomorphic sub-bundles $F^\bullet\mathcal{H}''$.

Conversely, given a flat C^∞ bundle $(\mathcal{H}, D)$, we decompose the flat connection into its $(1,0)$ part D' and its $(0,1)$ part D''. By considering types, one checks that flatness is equivalent to the three properties

$$(D')^2 = 0, \quad (D'')^2 = 0, \quad D'D'' + D''D' = 0.$$

Since, by flatness, $(D'')^2 = 0$, the Koszul-Malgrange theorem **[KM58]** implies that $\operatorname{Ker} D''$ is a holomorphic bundle $\mathcal{H}'$, that we can equip with the restriction ∇ to

[(1)] The precise definition is as follows. Let $\overline{\mathcal{O}}_X$ denote the sheaf of anti-holomorphic functions on X and regard $\mathcal{O}_X$ as an $\overline{\mathcal{O}}_X$-module: the action of an anti-holomorphic function $\overline{g}$ on a holomorphic function f is by definition $\overline{g} \cdot f := gf$. Then any $\overline{\mathcal{O}}_X$-module E'' determines an $\mathcal{O}_X$-module $\overline{E''}$ by setting $\overline{E''} := \mathcal{O}_X \otimes_{\overline{\mathcal{O}}_X} E''$.

$\operatorname{Ker} D''$ of the connection D'. Flatness of D also implies that ∇ is a flat holomorphic connection on $\mathcal{H}'$.

The conjugate C^∞ bundle $\overline{\mathcal{H}}$ is equipped with the conjugate connection $\overline{D}$, which is also flat. Conjugation exchanges of course the $(1,0)$-part and the $(0,1)$-part, that is, $\overline{D}' = \overline{D''}$ and $\overline{D}'' = \overline{D'}$. The corresponding holomorphic sub-bundle is $\mathcal{H}'' := (\overline{\mathcal{H}})' = \operatorname{Ker} \overline{D}''$. We can also express it as $\mathcal{H}'' = \overline{\operatorname{Ker} D'}$, and it is equipped with the flat holomorphic connection induced by $\overline{D}' = \overline{D''}$.

Similarly, we set $F'^p\overline{\mathcal{H}} = \overline{F''^p\mathcal{H}}$, etc.

***4.1.2. Definition* (Flat sesquilinear pairings).**

(1) A *sesquilinear pairing* $\mathfrak{s}$ on a C^∞ bundle $\mathcal{H}$ is a pairing on $\mathcal{H}$ with values in the sheaf $\mathcal{C}^\infty_X$, which satisfies, for local sections u,v of $\mathcal{H}$ and a C^∞ function g the relation $\mathfrak{s}(gu,\overline{v}) = \mathfrak{s}(u,\overline{gv}) = g\mathfrak{s}(u,\overline{v})$. We regard it as a $\mathcal{C}^\infty_X$-linear morphism

$$\mathfrak{s} : \mathcal{H} \otimes_{\mathcal{C}^\infty_X} \overline{\mathcal{H}} \longrightarrow \mathcal{C}^\infty_X.$$

(2) A *flat sesquilinear pairing* $\mathfrak{s}$ on a flat C^∞ bundle $(\mathcal{H},D)$ is a sesquilinear pairing which satisfies

$$\mathrm{d}\mathfrak{s}(u,\overline{v}) = \mathfrak{s}(Du,\overline{v}) + \mathfrak{s}(u,\overline{Dv});$$

equivalently, decomposing into types,

$$\begin{cases} \mathrm{d}'\mathfrak{s}(u,\overline{v}) = \mathfrak{s}(D'u,\overline{v}) + \mathfrak{s}(u,\overline{D''v}),\\ \mathrm{d}''\mathfrak{s}(u,\overline{v}) = \mathfrak{s}(D''u,\overline{v}) + \mathfrak{s}(u,\overline{D'v}).\end{cases}$$

4.1.3. Lemma. *Giving a flat sesquilinear pairing $\mathfrak{s}$ on $(\mathcal{H},D)$ is equivalent to giving an $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X$-linear morphism $\mathfrak{s} : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}'} \to \mathcal{C}^\infty_X$, that is, which satisfies*

$$\begin{cases} \mathfrak{s}(gu,\overline{v}) = g\mathfrak{s}(u,\overline{v}),\\ \mathfrak{s}(u,\overline{gv}) = \overline{g}\mathfrak{s}(u,\overline{v}),\end{cases} \qquad g \in \mathcal{O}_X,\ u,v \in \mathcal{H}',$$

and

$$\begin{cases} \mathrm{d}'\mathfrak{s}(u,\overline{v}) = \mathfrak{s}(\nabla u,\overline{v}),\\ \mathrm{d}''\mathfrak{s}(u,\overline{v}) = \mathfrak{s}(u,\overline{\nabla v}).\end{cases}$$

Proof. Immediate from the definitions. □

***4.1.4. Definition* (Variation of $\mathbb{C}$-Hodge structure, first definition)**

A variation of $\mathbb{C}$-Hodge structure H of weight w consists of the data of a flat C^∞ bundle $(\mathcal{H},D)$, equipped with a filtration $F'^\bullet\mathcal{H}$ by holomorphic sub-bundles satisfying Griffiths transversality (4.1.1), and with a filtration $F''^\bullet\mathcal{H}$ by anti-holomorphic sub-bundles satisfying anti-Griffiths transversality, such that the restriction of these data at each point $x \in X$ is a $\mathbb{C}$-Hodge structure of weight w (Definition 2.5.2).

A *morphism* $\varphi : H_1 \to H_2$ is a flat morphism of C^∞-bundles compatible with both the holomorphic and the anti-holomorphic filtrations.

A *polarization* S is a morphism $H \otimes \overline{H} \to \mathcal{C}^\infty_X(-w)$ of flat filtered bundles, where $\mathcal{C}^\infty_X(-w)$, is equipped with the natural connection d and w-shifted trivial filtrations, whose restriction to each $x \in X$ is a polarization of the Hodge structure H_x

(see Definition 2.5.11). We usually denote by $\mathcal{S} : \mathcal{H} \otimes \overline{\mathcal{H}} \to \mathcal{C}^\infty$ the underlying flat morphism and by $\underline{\mathcal{S}} : \underline{\mathcal{H}} \otimes \overline{\underline{\mathcal{H}}} \to \mathbb{C}$ its restriction to the local system $\underline{\mathcal{H}}$.

***4.1.5. Definition* (Variation of $\mathbb{C}$-Hodge structure, second definition)**
A variation of $\mathbb{C}$-Hodge structure H of weight w consists of the data of a flat C^∞ bundle $(\mathcal{H}, D)$, equipped with a *Hodge decomposition* by C^∞-sub-bundles

$$\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$$

satisfying *Griffiths transversality*:

$$(4.1.5\,*)\qquad \begin{aligned} D'\mathcal{H}^{p,q} &\subset \Omega^1_X \otimes (\mathcal{H}^{p,q} \oplus \mathcal{H}^{p-1,q+1}),\\ D''\mathcal{H}^{p,q} &\subset \overline{\Omega^1_X} \otimes (\mathcal{H}^{p,q} \oplus \mathcal{H}^{p+1,q-1}).\end{aligned}$$

A *morphism* $H_1 \to H_2$ is a D-flat morphism $(\mathcal{H}_1, D) \to (\mathcal{H}_2, D)$ which is compatible with the Hodge decomposition.

A *polarization* is a C^∞ Hermitian metric h on the C^∞-bundle $\mathcal{H}$ such that

- the Hodge decomposition is orthogonal with respect to h,
- The polarization form $\mathcal{S}$, defined by the property that (see Definition 2.5.15)
  - the decomposition is S-orthogonal and
  - $\mathrm{h}_{|\mathcal{H}^{p,w-p}} := (-1)^q \mathrm{S}_{|\mathcal{H}^{p,w-p}}$,

induces a D-flat $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}$-linear pairing $\mathcal{S} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathcal{C}^\infty_X$.

4.1.6. Lemma. *The two definitions are equivalent.*

Proof. Given a C^∞ flat bundle $(\mathcal{H}, D)$ equipped with a C^∞ Hodge decomposition satisfying (4.1.5 ∗), we define $F'^p\mathcal{H} = \bigoplus_{p' \geqslant p} \mathcal{H}^{p',w-p'}$, and (4.1.5 ∗) implies $D''(F'^p\mathcal{H}) \subset \mathcal{E}^{1,0}_X \otimes F'^{p-1}\mathcal{H}$, so that D'' induces a holomorphic structure on $F'^p\mathcal{H}$ and $\mathrm{gr}^p_F\mathcal{H} \simeq \mathcal{H}^{p,q-p}$ is a C^∞ bundle. We argue similarly to obtain the properties of $F''^\bullet\mathcal{H}$. By construction, the restriction of the filtrations to any point of X is give rise to the Hodge decomposition $\bigoplus \mathcal{H}^{p,w-p}_x$.

Conversely, assume that we are given $(\mathcal{H}, D, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$ as in Definition 4.1.4. The dimension of each fiber of $\mathcal{H}^{p,w-p} := F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H}$ is constant, since it is equal to that of the bundle $\mathrm{gr}^p_F\mathcal{H}$. We now use that, if all fibers of the intersection of two sub-bundles of a vector bundle have the same dimension, then this intersection is also a sub-bundle. This implies that $\mathcal{H}^{p,w-p}$ is a sub-bundle of $\mathcal{H}$, and the decomposition follows from the Hodge property in each fiber. In order to obtain (4.1.5 ∗), we notice that $F''^q\mathcal{H}$, being anti-holomorphic, is preserved by D', so

$$D'(F'^p\mathcal{H} \cap F''^q\mathcal{H}) \subset \Omega^1_X \otimes (F'^{p-1}\mathcal{H} \cap F''^q\mathcal{H}).$$

We argue similarly with D''. □

4.1.7. Remark. While it is easy, by using a partition of unity, to construct a Hermitian metric compatible with the Hodge decomposition, the condition of flatness of $\mathcal{S}$ is a true constraint if $\dim X \geqslant 1$. For example, any flat C^∞-bundle $(\mathcal{H}, D)$ can be regarded as a variation of $\mathbb{C}$-Hodge structure of type $(0, 0)$, and it admits many Hermitian metrics, but the polarization condition imposes that the Hermitian metric is flat, which only occurs when the monodromy representation of the flat bundle is (conjugate to) a unitary representation.

4.1.8. Remark **(Polarized variation of Hodge structure as a flat filtered Hermitian pair)**
In analogy with Section 2.5.18, we can describe a polarized variation of Hodge structure by using only one filtration. By a *flat filtered Hermitian pair* we mean the data $((\mathcal{H}', \nabla), F^\bullet\mathcal{H}', \mathcal{S})$, where

(i) $(\mathcal{H}', \nabla)$ is a flat holomorphic vector bundle and $F^\bullet\mathcal{H}'$ is a filtration by holomorphic sub-bundles satisfying Griffiths transversality,
(ii) $\mathcal{S} : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}}' \to \mathcal{C}^\infty_X$ is a ∇-flat Hermitian pairing as defined in Lemma 4.1.3.

This can also be described as C^∞ data $((\mathcal{H}, D), F^\bullet\mathcal{H}, \mathcal{S})$ as in Definition 4.1.2, the equivalence being given by Lemma 4.1.3.

A flat filtered Hermitian pair $((\mathcal{H}', \nabla), F^\bullet\mathcal{H}', \mathcal{S})$ is a *polarized variation of Hodge structure of weight* w if its restriction to every $x \in X$ is a polarized Hodge structure in the sense of 2.5.18(1)–(3).

4.1.9. Definition **(The categories $\mathsf{VHS}(X, \mathbb{C}, w)$ and $\mathsf{pVHS}(X, \mathbb{C}, w)$)**
Definitions 4.1.4 and 4.1.5 produce the category $\mathsf{VHS}(X, \mathbb{C}, w)$ of variations of $\mathbb{C}$-Hodge structures of weight w on X. The category $\mathsf{pVHS}(X, \mathbb{C}, w)$ of polarizable variations of $\mathbb{C}$-Hodge structures of weight w is the full subcategory of $\mathsf{VHS}(X, \mathbb{C}, w)$ whose objects admit a polarization.

We refer to Exercises 4.1 and 4.2 for the following result.

4.1.10. Proposition.

(1) *The category* $\mathsf{VHS}(X, \mathbb{C}, w)$ *is abelian and each morphism is strictly compatible with the Hodge filtration. It is equipped with the operations tensor product, Hom, dual, conjugation and Hermitian dual.*

(2) *The full subcategory* $\mathsf{pVHS}(X, \mathbb{C}, w)$ *is abelian and stable by the previous operations. It is stable by direct summand in* $\mathsf{VHS}(X, \mathbb{C}, w)$ *and is semi-simple.* □

4.1.b. Variations of $\mathbb{Q}$-Hodge structure. We can now mimic the definition of Section 2.5.c. An object of $\mathsf{VHS}(X, \mathbb{Q}, w)$ is a tuple $(\underline{\mathcal{H}}_{\mathbb{Q}}, H, \mathrm{iso})$, where

- $\mathcal{H}_{\mathbb{Q}}$ is a $\mathbb{Q}$-local system on X,
- H is an object of $\mathsf{VHS}(X, \mathbb{C}, w)$,
- iso is an isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} \underline{\mathcal{H}}_{\mathbb{Q}} \xrightarrow{\sim} \underline{\mathcal{H}}$,

with the condition that at each $x \in X$, these data restrict to a $\mathbb{Q}$-Hodge structure. Morphisms are the obvious ones which are compatible with the data. The definition of $\mathsf{pVHS}(X, \mathbb{Q}, w)$ is similar, by imposing that the polarization form $\underline{\mathcal{S}}$ comes, after tensoring with $\mathbb{C}$, from a bilinear form

$$\underline{\mathcal{S}}_{\mathbb{Q}} : \underline{\mathcal{H}}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \underline{\mathcal{H}}_{\mathbb{Q}} \longrightarrow (2\pi \mathrm{i})^{-w}\mathbb{Q}.$$

4.1.11. Proposition.

(1) *The category* $\mathsf{VHS}(X,\mathbb{Q},w)$ *is abelian and each morphism is strictly compatible with the Hodge filtration. It is equipped with the operations tensor product, Hom, and dual.*

(2) *The full subcategory* $\mathsf{pVHS}(X,\mathbb{Q},w)$ *is abelian and stable by the previous operations. It is stable by direct summand in* $\mathsf{VHS}(X,\mathbb{Q},w)$ *and is semi-simple.* □

4.1.12. Remark. One can define the category $\mathsf{VHS}(X,\mathbb{R},w)$ of variations of $\mathbb{R}$-Hodge structure without referring to the Riemann-Hilbert correspondence, i.e., without using the local system $\underline{\mathcal{H}}$, by using instead a complex involution $\kappa:(\mathcal{H},D)\xrightarrow{\sim}(\overline{\mathcal{H}},\overline{D})$.

4.2. The Hodge theorem

4.2.a. The Hodge theorem for unitary representations. We will extend the Hodge theorem (Theorem 2.4.4 and the results indicated after its statement concerning the polarization) to the case of the cohomology with coefficients in a unitary representation.

Let us start with a holomorphic vector bundle $\mathcal{H}'$ of rank d on a complex projective manifold X equipped with a flat holomorphic connection ∇. The local system $\underline{\mathcal{H}}=\mathcal{H}'^{\nabla}$ corresponds to a representation $\pi_1(X,\star)\to\mathrm{GL}_d(\mathbb{C})$, up to conjugation. The unitary assumption means that we can conjugate the given representation in such a way that it takes values in the unitary group.

In other words, there exists a Hermitian metric h on the associated C^∞-bundle $\mathcal{H}=\mathcal{C}^\infty\otimes_{\mathcal{O}_X}\mathcal{H}'$ such that, if we denote as above by D the connection on $\mathcal{H}$ defined by $D(\varphi\otimes v)=\mathrm{d}\varphi\otimes v+\varphi\otimes\nabla v$, the connection D is compatible with the metric h (i.e., is the Chern connection of the metric h).

That D is a connection *compatible with the metric* implies that its formal adjoint (with respect to the metric) is obtained with a suitably defined Hodge $\star$ operator by the formula $D^\star=-\star D\star$. This leads to the decomposition of the space of C^∞ k-forms on X with coefficients in $\mathcal{H}$ (resp. (p,q)-forms) as the orthogonal sum of the kernel of the Laplace operator with respect to D (resp. D' or D''), that is, the space of harmonic sections, and its image.

As the connection D is *flat*, there is a C^∞ de Rham *complex* $(\mathcal{E}^\bullet_X\otimes\mathcal{H},D)$, and standard arguments give

$$H^k(X,\underline{\mathcal{H}})=\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{H}',\nabla))=H^k\big(\Gamma(X,(\mathcal{E}^\bullet_X\otimes\mathcal{H},D))\big).$$

One can also define the Dolbeault cohomology groups by decomposing $\mathcal{E}^\bullet$ into $\mathcal{E}^{p,q}$'s and by decomposing D as $D'+D''$. Then $H^{p,q}_{D''}(X,\mathcal{H})=H^q(X,\Omega^p_X\otimes\mathcal{H}')$.

As the projective manifold X is *Kähler*, we obtain the Kähler identities for the various Laplace operators: $\Delta_D=2\Delta_{D'}=2\Delta_{D''}$.

Then, exactly as in Theorem 2.4.4, we get:

4.2.1. Theorem. *Under these conditions, one has a canonical decomposition*

$$\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{H}',\nabla))=\bigoplus_{p+q=k}H^{p,q}_{D''}(X,\mathcal{H})$$

and $H^{q,p}_{D''}(X,\mathcal{H})$ is identified with $\overline{H^{p,q}_{D''}(X,\mathcal{H}^\vee)}$, where $\mathcal{H}^\vee$ is the dual bundle.[(2)]

The Hard Lefschetz theorem also holds in this context.

4.2.b. Harmonic bundles. If we do not assume anymore that $\underline{\mathcal{H}}$ is unitary, but only assume that it underlies a polarized variation of Hodge structure of some weight w (so that the unitary case is the particular case of a variation of pure type $(0,0)$), we have a flat connection D on the C^∞-bundle $\mathcal{H}$ associated to $\mathcal{H}'$, with $D=D'+\mathrm{d}''$, and we also have a Hermitian metric h on $\mathcal{H}$ associated with S, but D is possibly *not* compatible with the metric. The argument using the Hodge $\star$ operator is not valid anymore. We first consider a general situation.

Let X be a complex manifold, let $(\mathcal{H},D)$ be a flat C^∞ bundle on X, and let h be a Hermitian metric on $\mathcal{H}$. We decompose D into its $(1,0)$ and $(0,1)$ parts: $D=D'+D''$.

4.2.2. Lemma. *Given $(\mathcal{H},D,\mathrm{h})$, there exists a unique connection $D_{\mathrm{h}}=D'_{\mathrm{h}}+D''_{\mathrm{h}}$ on $\mathcal{H}$ and a unique C^∞-linear morphism $\theta=\theta'+\theta'':\mathcal{H}\to\mathcal{E}^1_X\otimes\mathcal{H}$ satisfying the following properties:*

(1) *D_{h} is compatible with* h, *i.e.,* $\mathrm{dh}(u,\overline{v})=\mathrm{h}(D_{\mathrm{h}}u,\overline{v})+\mathrm{h}(u,\overline{D_{\mathrm{h}}v})$, *or equivalently, decomposing into types,*

$$\mathrm{d}'\mathrm{h}(u,\overline{v})=\mathrm{h}(D'_{\mathrm{h}}u,\overline{v})+\mathrm{h}(u,\overline{D''_{\mathrm{h}}v}),\quad \mathrm{d}''\mathrm{h}(u,\overline{v})=\mathrm{h}(D''_{\mathrm{h}}u,\overline{v})+\mathrm{h}(u,\overline{D'_{\mathrm{h}}v}).$$

(2) *θ is self-adjoint with respect to* h, *i.e.,* $\mathrm{h}(\theta u,\overline{v})=\mathrm{h}(u,\overline{\theta v})$, *or equivalently, decomposing into types,*

$$\mathrm{h}(\theta' u,\overline{v})=\mathrm{h}(u,\overline{\theta'' v}),\quad \mathrm{h}(\theta'' u,\overline{v})=\mathrm{h}(u,\overline{\theta' v}).$$

(3) *$D=D_{\mathrm{h}}+\theta$, or equivalently, decomposing into types,*

$$D'=D'_{\mathrm{h}}+\theta',\quad D''=D''_{\mathrm{h}}+\theta''.$$

4.2.3. Remark. In 4.2.2(1), we have extended the metric h in a natural way as a sesquilinear operator $(\mathcal{E}^1_X\otimes\mathcal{H})\otimes\overline{\mathcal{H}}\to\mathcal{E}^1_X$ resp. $\mathcal{H}\otimes\overline{(\mathcal{E}^1_X\otimes\mathcal{H})}\to\mathcal{E}^1_X$.

Proof. Let D_{h} be a connection on $\mathcal{H}$ which is compatible with h. Let A be a $\mathcal{C}^\infty_X$-linear morphism $A:\mathcal{H}\to\mathcal{E}^\infty_X\otimes\mathcal{H}$ which is skew-adjoint with respect to h, that is, such that $\mathrm{h}(Au,\overline{v})=-\mathrm{h}(u,\overline{Av})=0$ for every local sections u,v of $\mathcal{H}$. Then the connection $D_{\mathrm{h}}+A$ is also compatible with the metric. So let us choose any h-compatible connection $\widetilde{D}_{\mathrm{h}}$ (for example the Chern connection, also compatible with the holomorphic structure D'') and let us set $A=D-\widetilde{D}_{\mathrm{h}}$. Let us decompose A as A^++A^-, with A^+ self-adjoint and A^- skew-adjoint. We can thus set $D_{\mathrm{h}}=\widetilde{D}_{\mathrm{h}}+A^-$ and $\theta=A^+$. Uniqueness is seen similarly. $\square$

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(2) When we work with a polarized variation of Hodge structure, the polarization S identifies $(\mathcal{H},D)$ and $(\mathcal{H}^\vee,D^\vee)$ and we recover the usual conjugation relation between $H^{q,p}$ and $H^{p,q}$.

4.2.4. Remarks.

(1) Iterating 4.2.2(2), we find that $\theta''\wedge\theta''$ is h-adjoint to $-\theta'\wedge\theta'$ and $\theta'\wedge\theta''+\theta''\wedge\theta'$ is skew-adjoint. By applying d' or d'' to 4.2.2(1) and (2), we see that D''^2_{h} is adjoint to $-D'^2_{\mathrm{h}}$, $D''_{\mathrm{h}}(\theta')$ is adjoint to $-D'_{\mathrm{h}}(\theta'')$, $D''_{\mathrm{h}}(\theta'')$ is adjoint to $-D'_{\mathrm{h}}(\theta')$, and $D'_{\mathrm{h}}D''_{\mathrm{h}}+D''_{\mathrm{h}}D'_{\mathrm{h}}$ is skew-adjoint with respect to h.

(2) Let us set $\widehat{D}'=D'_{\mathrm{h}}-\theta'$. Then the Chern connection for the metric h on the holomorphic bundle $(\mathcal{H},D'')$ is equal to $\widehat{D}'+D''$ (see Exercise 4.4). Similarly, setting $\widehat{D}''=D''_{\mathrm{h}}-\theta''$, the Chern connection for the anti-holomorphic bundle $(\mathcal{H},D')$ is $D'+\widehat{D}''$. We will set

$$D^{\mathrm{c}}=\widehat{D}''-\widehat{D}'.$$

We refer to Exercise 4.4 for the properties of these operators.

***4.2.5. Definition* (Harmonic bundle).** Let $(\mathcal{H},D,\mathrm{h})$ be a flat C^∞-bundle equipped with a Hermitian metric h. We say that $(\mathcal{H},D,\mathrm{h})$ is a *harmonic bundle* if the operator $D''_{\mathrm{h}}+\theta'=\frac{1}{2}(D+D^{\mathrm{c}})$ has square 0. We also set

$$\mathcal{D}=D'_{\mathrm{h}}+\theta'',\quad \overline{\mathcal{D}}=D''_{\mathrm{h}}+\theta', \tag{4.2.5 $*$}$$

so that $D=D'+D''=\mathcal{D}+\overline{\mathcal{D}}$ and $D^{\mathrm{c}}=\overline{\mathcal{D}}-\mathcal{D}$.

By looking at types, the harmonicity condition is equivalent to

$$D''^2_{\mathrm{h}}=0,\quad D''_{\mathrm{h}}(\theta')=0,\quad \theta'\wedge\theta'=0. \tag{4.2.6}$$

By adjunction, this implies

$$D'^2_{\mathrm{h}}=0,\quad D'_{\mathrm{h}}(\theta'')=0,\quad \theta''\wedge\theta''=0.$$

Moreover, the flatness of D implies then

$$D'_{\mathrm{h}}(\theta')=0,\quad D''_{\mathrm{h}}(\theta'')=0,\quad D'_{\mathrm{h}}D''_{\mathrm{h}}+D''_{\mathrm{h}}D'_{\mathrm{h}}=-(\theta'\wedge\theta''+\theta''\wedge\theta').$$

4.2.7. Lemma. *Let $(\mathcal{H},D)$ be a flat bundle and let* h *be a Hermitian metric on $\mathcal{H}$. Then*

$$(\overline{\mathcal{D}})^2=0\implies\begin{cases}DD^{\mathrm{c}}+D^{\mathrm{c}}D=0,\\ \mathcal{D}\overline{\mathcal{D}}+\overline{\mathcal{D}}\mathcal{D}=0,\quad \mathcal{D}^2=0.\end{cases}$$

Proof. Since $D^2=0$, it is a matter of proving $(D^{\mathrm{c}})^2=0$. From the vanishing above, we find $(\widehat{D}')^2=0$, $(\widehat{D}'')^2=0$. We also get $\widehat{D}''\widehat{D}'+\widehat{D}'\widehat{D}''=0$. The properties for $\mathcal{D},\overline{\mathcal{D}}$ are obtained similarly. □

***4.2.8. Definition* (Higgs bundle).** Set $\mathcal{E}=\operatorname{Ker}D''_{\mathrm{h}}:H\to H$. If $(\mathcal{H},D,\mathrm{h})$ is harmonic, then $\mathcal{E}$ is a holomorphic vector bundle equipped with a holomorphic $\operatorname{End}(\mathcal{E})$-valued 1-form θ induced by θ', which satisfies $\theta\wedge\theta=0$. It is called the *Higgs bundle* associated to the harmonic bundle, and θ is its *associated holomorphic Higgs field.* Let us also notice that, by definition, D_{h} is the Chern connection for the Hermitian holomorphic bundle $(\mathcal{E},\mathrm{h})$.

4.2.9. Kähler identities for harmonic bundles. We assume that X is a compact Kähler complex manifold. We can apply the Kähler identities to the holomorphic bundle $(\mathcal{H}, D'')$ with Hermitian metric h and Chern connection $\widehat{D}' + D''$, as well as to the holomorphic bundle $(\mathcal{H}, D''_{\mathrm{h}})$ with Hermitian metric h and Chern connection $D'_{\mathrm{h}} + D''_{\mathrm{h}}$. Denoting by $P^\star$ the formal L^2-adjoint of an operator P, the classical Kähler identities take the form

$$\widehat{D}'^\star = \mathsf{i}[\Lambda, D''], \quad D''^\star = -\,\mathsf{i}[\Lambda, \widehat{D}'], \quad \Delta_{D''} = -\,\mathsf{i}[D'', [\Lambda, \widehat{D}']],$$
$$D'^\star_{\mathrm{h}} = \mathsf{i}[\Lambda, D''_{\mathrm{h}}], \quad D''^\star_{\mathrm{h}} = -\,\mathsf{i}[\Lambda, D'_{\mathrm{h}}], \quad \Delta_{D''_{\mathrm{h}}} = -\,\mathsf{i}[D''_{\mathrm{h}}, [\Lambda, D'_{\mathrm{h}}]].$$

As a consequence, we find

$$D^\star_{\mathrm{h}} := D'^\star_{\mathrm{h}} + D''^\star_{\mathrm{h}} = \mathsf{i}[\Lambda, D^{\mathrm{c}}_{\mathrm{h}}].$$

Note also the following identity (see Exercise 4.10):

$$\Delta_D = 2\Delta_{\mathcal{D}} = 2\Delta_{\overline{\mathcal{D}}}. \tag{4.2.10}$$

4.2.c. Polarized variations of Hodge structure on a compact Kähler manifold: the Hodge-Deligne theorem. Let us come back to the setting of Section 4.2.a.

4.2.11. Proposition. *Let $(\mathcal{H}, D, \mathrm{h})$ be a flat bundle with metric underlying a polarized variation of $\mathbb{C}$-Hodge structure on X. Then $(\mathcal{H}, D, \mathrm{h})$ is a harmonic bundle.*

Proof. This is the content of Exercise 4.3. □

Let us emphasize that the h-compatible connections D'_{h} resp. D''_{h} of Lemma 4.2.2 are given by the first projection in Griffiths' transversality relations (4.1.5 $*$), and θ' resp. θ'' as the second projections. Recall that $(\mathcal{H}, D)$ is the flat C^∞ bundle associated with the flat holomorphic bundle $(\mathcal{H}', \nabla)$. Then D''_{h} defines a holomorphic structure on $\mathcal{H}^{p,w-p}$, with associated holomorphic bundle $\operatorname{Ker} D''_{\mathrm{h}}$ isomorphic to the holomorphic bundle $\mathrm{gr}^p_F\mathcal{H}'$. Moreover, $\theta' : \mathcal{H}^{p,w-p} \to \mathcal{H}^{p-1,w-p+1}$ is the C^∞ morphism associated with the $\mathcal{O}_X$-linear morphism induced by ∇:

$$\theta := \mathrm{gr}^{-1}_F \nabla : \mathrm{gr}^p_F\mathcal{H}' \longrightarrow \Omega^1_X \otimes \mathrm{gr}^{p-1}_F\mathcal{H}'. \tag{4.2.12}$$

The decomposition $D = D' + D''$ is thus replaced with the decomposition $D = \mathcal{D} + \overline{\mathcal{D}}$. The disadvantage is that we loose the decomposition into types $(1,0)$ and $(0,1)$, but we keep the flatness property. On the other hand, we also keep the Kähler identities (4.2.10).

We did not really loose the decomposition into types: the operator $\overline{\mathcal{D}}$ sends a section of $\mathcal{H}^{p,q}$ to a section of $(\Omega^1_X \otimes \mathcal{H}^{p-1,q+1}) + (\overline{\Omega^1_X} \otimes \mathcal{H}^{p,q})$. Counting the total type, we find $(p, q+1)$ for both terms. In other word, taking into account the Hodge type of a section, the operator $\overline{\mathcal{D}}$ is indeed of type $(0,1)$. A similar argument applies to $\mathcal{D}$. By Definition 4.2.5, $(\mathcal{E}^\bullet_X \otimes \mathcal{H}, \overline{\mathcal{D}})$ is a complex. Let us set

$$F^p(\mathcal{E}^m_X \otimes \mathcal{H}) = \bigoplus_{\substack{i+j=m\\ i+k\geqslant p}} \mathcal{E}^{i,j}_X \otimes \mathcal{H}^{k,w-k}.$$

We note that

$$\overline{\mathcal{D}}F^p(\mathcal{E}_X^m\otimes\mathcal{H})\subset F^p(\mathcal{E}_X^{m+1}\otimes\mathcal{H}),$$

since D''_{h} sends $\mathcal{E}_X^{i,j}\otimes\mathcal{H}^{k,w-k}$ to $\mathcal{E}_X^{i,j+1}\otimes\mathcal{H}^{k,w-k}$ and θ' sends $\mathcal{E}_X^{i,j}\otimes\mathcal{H}^{k,w-k}$ to $\mathcal{E}_X^{i+1,j}\otimes\mathcal{H}^{k-1,w-k+1}$. We thus get a filtered complex by setting

$$F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})=\Big\{F^p(\mathcal{E}_X^0\otimes\mathcal{H})\xrightarrow{\overline{\mathcal{D}}}F^p(\mathcal{E}_X^1\otimes\mathcal{H})\longrightarrow\cdots\Big\},$$

and the associated graded complex has the following degree-m term:

$$\bigoplus_{i=0}^m(\mathcal{E}_X^{i,m-i}\otimes\mathcal{H}^{p-i,w-p+i}).$$

On the other hand, we filter $\mathrm{gr}_F\mathcal{H}'$ by setting $F^p\mathrm{gr}_F\mathcal{H}'=\bigoplus_{p'\geqslant p}\mathrm{gr}_F^{p'}\mathcal{H}'$, so that, according to (4.2.12), we obtain the *holomorphic Dolbeault complex*

$$\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta):=\{0\to\mathrm{gr}_F\mathcal{H}'\xrightarrow{\theta}\Omega_X^1\otimes\mathrm{gr}_F\mathcal{H}'\xrightarrow{\theta}\cdots\}.\tag{4.2.13}$$

which is filtered by setting

$$F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)=\Big\{F^p\mathrm{gr}_F\mathcal{H}'\xrightarrow{\theta}\Omega_X^1\otimes F^{p-1}\mathrm{gr}_F\mathcal{H}'\longrightarrow\cdots\Big\},$$

so that

$$\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)=\Big\{\mathrm{gr}_F^p\mathcal{H}'\xrightarrow{\theta}\Omega_X^1\otimes\mathrm{gr}_F^{p-1}\mathcal{H}'\longrightarrow\cdots\Big\}.$$

***4.2.14. Proposition* (Dolbeault resolution).** *For each p, the complex $F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})$ is a resolution of $F^p(\Omega_X^\bullet\otimes\mathrm{gr}_F\mathcal{H}',\theta)$ and $\mathrm{gr}_F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})$ is a resolution of $\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)$.*

***Proof*.** Since the filtration $F^\bullet$ is finite, it is enough to prove the second statement. Due to the relations (4.2.6), we can regard (up to signs) $\mathrm{gr}_F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})$ as the simple complex associated with the double complex

$$\begin{tikzcd}
\mathcal{E}^{i,j}\otimes\mathcal{H}^{p-i,w-p+i} \arrow[r, "\theta'"] \arrow[d, "D''_{\mathrm{h}}"'] & \mathcal{E}^{i+1,j}\otimes\mathcal{H}^{p-i-1,w-p+i+1} \arrow[d, "D''_{\mathrm{h}}"] \\
\mathcal{E}^{i,j+1}\otimes\mathcal{H}^{p-i,w-p+i} \arrow[r, "\theta'"] & \mathcal{E}^{i+1,j+1}\otimes\mathcal{H}^{p-i-1,w-p+i+1}
\end{tikzcd}$$

The i-th vertical complex a resolution of $\Omega_X^i\otimes\mathrm{gr}_F^{p-i}\mathcal{H}'$. ◻

***4.2.15. Corollary* (Dolbeault Lemma).** *We have for each $p,q\in\mathbb{Z}$:*

$$\begin{aligned}\boldsymbol{H}^q\big(X,\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)\big)&\simeq H^q\big(\Gamma(X,\mathrm{gr}_F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})\big),\\ \boldsymbol{H}^q\big(X,F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)\big)&\simeq H^q\big(\Gamma(X,F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})\big).\end{aligned}\qquad ◻$$

This being understood, the arguments of Hodge theory apply to this situation as in the case considered in Section 4.2.a, to get the Hodge-Deligne theorem.

***4.2.16. Theorem* (Hodge-Deligne theorem).** *Let (H, S) be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on a smooth complex projective variety X of pure dimension n and let $\mathcal{L}$ be an ample line bundle on X. Then $\big(H^\bullet(X, \underline{\mathcal{H}}), \mathrm{X}_{\mathcal{L}}\big)$ is naturally equipped with a polarizable $\mathfrak{sl}_2$-Hodge structure with central weight $w+n$ (see Definition 3.2.7). In particular, each $H^k(X, \underline{\mathcal{H}})$ comes equipped with a polarized $\mathbb{C}$-Hodge structure of weight $w+k$. If (H, S) is a polarized variation of $\mathbb{Q}$-Hodge structure of weight w, then each $H^k(X, \underline{\mathcal{H}}_{\mathbb{Q}})$ is equipped with a polarized $\mathbb{Q}$-Hodge structure of weight $w+k$.*

Sketch of proof. We refer to [**Zuc79**, §2] for the detailed adaptation to this setting of the Kähler identities and their consequences. One realizes each cohomology class in $H^\bullet(X, \underline{\mathcal{H}})$ by a unique Δ_D-harmonic section, by the arguments of Hodge theory, which extend if one takes into account the total type, as above.

The polarization is obtained from S on H and Poincaré duality as we did for S in Section 2.4, and from it we cook up the form S on the cohomology. More precisely, the flat pairing

$$\mathcal{S} : \mathcal{H} \otimes \overline{\mathcal{H}} \longrightarrow \mathcal{C}_X^\infty$$

induces a pairing of locally constant sheaves

$$\underline{\mathcal{S}} : \underline{\mathcal{H}} \otimes \underline{\overline{\mathcal{H}}} \longrightarrow \mathbb{C}_X,$$

Then $\mathrm{S} : H^\bullet(X, \underline{\mathcal{H}})^{\mathrm{H}} \otimes \overline{H^\bullet(X, \underline{\mathcal{H}})^{\mathrm{H}}} \to \mathbb{C}^{\mathrm{H}}(-(w+n))$ satisfies $\mathrm{S}(H^{n+k}, \overline{H^{n-\ell}}) = 0$ if $k \neq \ell$ and otherwise induces for every $k \in \mathbb{Z}$ with $|k| \leqslant n$ a pairing S_k of $\mathbb{C}$-Hodge structures

$$H^{n+k}(X, \underline{\mathcal{H}})^{\mathrm{H}} \otimes \overline{H^{n-k}(X, \underline{\mathcal{H}})^{\mathrm{H}}} \longrightarrow H^{2n}(X, \mathbb{C})^{\mathrm{H}}(-(w+n)) = \mathbb{C}^{\mathrm{H}}(-(w+n))$$

by the formula (see Notation (0.2 $*$))

$$\mathrm{S}_k(\bullet, \overline{\bullet}) := \mathrm{Sgn}(n,k) \int_{[X]} \underline{\mathcal{S}}(\bullet, \overline{\bullet}). \tag{4.2.17}$$

Since the Lefschetz operator $\mathrm{X}_{\mathcal{L}}$ only acts on the forms and not on the sections of $\mathcal{H}$, it is self-adjoint with respect to S in the sense that $\mathrm{S}_k(\mathrm{X}_{\mathcal{L}}\eta', \overline{\eta''}) = \mathrm{S}_{k-2}(\eta', \overline{\mathrm{X}_{\mathcal{L}}\eta''})$, since it is so for the modified Poincaré duality pairing $\mathrm{Sgn}(n,k)\langle\bullet, \overline{\bullet}\rangle_{\mathbb{C}}$ (see (2.4.13)). Then S is a sesquilinear pairing on the $\mathfrak{sl}_2$-Hodge structure $H^\bullet(X, \underline{\mathcal{H}})$ (see Section 3.4.c).

Due to the Kähler identities and the commutation of $\mathrm{L}_{\mathcal{L}}$ with $\Delta_{\mathcal{D}}$, a harmonic section of $\mathcal{E}_X^{n-\ell} \otimes \mathcal{H}$ ($\ell \geqslant 0$) is primitive if and only if each of its components with respect to the total bigrading is so, and since $\mathrm{L}_{\mathcal{L}}$ only acts on the differential form part of such a component, this occurs if an only if each of its component on $\mathcal{E}_X^{p,q} \otimes \mathcal{H}^{a,b}$ is primitive, with $p+q = n-\ell$ and $a+b = w$. Fixing an h-orthonormal basis $(v_i)_i$ of $\mathcal{H}^{a,b}$, such a component can be written in a unique way as $\sum_i \eta_i^{p,q} \otimes v_i$ with $\eta_i^{p,q}$ primitive. Then the positivity property of $\mathrm{P}_{-\ell}\mathrm{S}$ defined in 3.2.10 on $\eta_i^{p,q} \otimes v_i$ amounts to the positivity of

$$\mathrm{Sgn}(n, -\ell) \int_X (-1)^{q+b} \mathcal{S}(v_i, \overline{v_i}) \cdot \eta_i^{p,q} \wedge \overline{\mathrm{X}_{\mathcal{L}}^{\ell} \eta_i^{p,q}}.$$

By the positivity for $\mathcal{S}$, there exists a C^∞ function g_i such that $(-1)^b\mathcal{S}(v_i,\overline{v_i}) = |g_i|^2$. Therefore, (2.4.15) applied to $g_i\eta_i^{p,q}$ gives the desired positivity.

Lastly, in the presence of a $\mathbb{Q}$-structure, we define the bilinear form $\mathrm{S}_\mathbb{Q}$ by replacing $\underline{\mathcal{S}}$ with $\underline{\mathcal{S}}_\mathbb{Q}$ in (4.2.17) in order to obtain a pairing of $\mathbb{Q}$-Hodge structures

$$H^{n+k}(X,\underline{\mathcal{H}_\mathbb{Q}})^{\mathrm{H}} \otimes \overline{H^{n-k}(X,\underline{\mathcal{H}_\mathbb{Q}})^{\mathrm{H}}} \longrightarrow H^{2n}(X,\mathbb{Q})(-(w+n))^{\mathrm{H}} = \mathbb{Q}^{\mathrm{H}}(-(w+n)). \quad \square$$

4.2.18. Remarks. Let H be a polarizable variation of $\mathbb{C}$-Hodge structure of weight w on a smooth complex projective variety X.

(1) (*The Hodge filtration*) Consider the de Rham complex $\mathrm{DR}(\mathcal{H}',\nabla)$. According to the Griffiths transversality property, it comes equipped with a filtration, by setting (see Definition 8.4.1):

$$F^p\,\mathrm{DR}(\mathcal{H}',\nabla) = \{0 \to F^p\mathcal{H}' \xrightarrow{\ \nabla\ } \Omega^1_X \otimes F^{p-1}\mathcal{H}' \xrightarrow{\ \nabla\ } \cdots \xrightarrow{\ \nabla\ } \Omega^n_X \otimes F^{p-n}\mathcal{H}' \to 0\}.$$

The natural inclusion of complexes $F^p\,\mathrm{DR}(\mathcal{H}',\nabla) \hookrightarrow \mathrm{DR}(\mathcal{H}',\nabla)$ induces a morphism

$$\boldsymbol{H}^k(X, F^p\,\mathrm{DR}(\mathcal{H}',\nabla)) \longrightarrow \boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{H}',\nabla)) = H^k(X,\underline{\mathcal{H}}), \tag{4.2.18 $*$}$$

whose image is the filtration $F'^pH^k(X,\underline{\mathcal{H}})$. Working anti-holomorphically with the filtration $F''^\bullet\mathcal{H}$ by anti-holomorphic sub-bundles and the anti-holomorphic connection induced by D'' on $\mathrm{Ker}\,D'$, one obtains the filtration $F''^\bullet H^k(X,\underline{\mathcal{H}})$. The Hodge-Deligne theorem implies that these filtrations are $(w+k)$-opposed.

(2) (*Degeneration at E_1 of the Hodge-to-de Rham spectral sequence*)

Moreover, we claim that, for every p,k, *the morphism* (4.2.18 $*$) *is injective.* In other words, the filtered complex $\boldsymbol{R}\Gamma\big(X, F^\bullet\,\mathrm{DR}(\mathcal{H}',\nabla)\big)$ is *strict* (see Section 5.1.b). The graded complex $\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla)$ is the complex

$$\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla) = \{0 \to \mathrm{gr}^p_F\mathcal{H}' \xrightarrow{\ \theta\ } \Omega^1_X \otimes \mathrm{gr}^{p-1}_F\mathcal{H}' \xrightarrow{\ \theta\ } \cdots \xrightarrow{\ \theta\ } \Omega^n_X \otimes \mathrm{gr}^{p-n}_F\mathcal{H}' \to 0\},$$

that is,

$$\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla) = \mathrm{gr}^p_F\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta).$$

Since each term of this complex is $\mathcal{O}_X$-locally free of finite rank and since θ is $\mathcal{O}_X$-linear, the hypercohomology spaces $\boldsymbol{H}^k(X,\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla))$ are finite-dimensional. The strictness property is then equivalent to

$$\forall\, p,k,\quad \mathrm{gr}^p_F H^k(X,\underline{\mathcal{H}}) = \boldsymbol{H}^k(X,\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla)),$$

where $\mathrm{gr}^p_F H^k(X,\underline{\mathcal{H}}) \simeq H^{p,w+k-p}(X,\underline{\mathcal{H}})$. This property is also equivalent to

$$\forall\, k,\quad \dim H^k(X,\underline{\mathcal{H}}) = \dim \boldsymbol{H}^k(X,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H},\theta)).$$

This statement is obtained by standard arguments of Hodge theory applied to the operators $D,\mathcal{D},\overline{\mathcal{D}}$ and their Laplacians. $\square$

(3) For any smooth projective variety X, the space $H^0(X,\underline{\mathcal{H}})$ is primitive (for any $\mathcal{L}$) and, given a polarization $\mathcal{S}$ of H, a polarization of the pure $\mathbb{C}$-Hodge structure of weight w is obtained by taking integral of the polarization function against a volume form (defined from $\mathcal{L}$, in order to match with the Hodge-Deligne theorem).

If X is a compact Riemann surface, then $H^1(X,\underline{\mathcal{H}})$ is also primitive, and there is no need to choose a polarization bundle $\mathcal{L}$ in order to obtain the polarized pure $\mathbb{C}$-Hodge structure on $H^1(X,\underline{\mathcal{H}})$. The polarization (4.2.17) on $H^1(X,\underline{\mathcal{H}})$ is simply written as

$$\mathrm{S}^{(1)} = -\frac{1}{2\pi \mathrm{i}}\int_{[X]} \underline{\mathcal{S}}(\bullet,\overline{\bullet}).$$

(4) (*The fixed-part theorem*) The maximal constant subsheaf of $\underline{\mathcal{H}}$ is the constant subsheaf with stalk $H^0(X,\underline{\mathcal{H}})$ at each point, by means of a natural injective morphism $H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathbb{C}_X\to\underline{\mathcal{H}}$. By the Hodge-Deligne theorem 4.2.16, $H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathbb{C}_X$ is equipped with a constant variation of Hodge structure of weight w. We claim that *the previous morphism is compatible with the Hodge filtrations, i.e., is a morphism in* $\mathsf{VHS}(X,\mathbb{C},w)$, that is, the injective morphism

$$\varphi : H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow \underline{\mathcal{H}}\otimes_{\mathbb{C}}\mathcal{O}_X=\mathcal{H}'$$

is compatible with the Hodge filtration $F^\bullet$ on both terms.

Since X is compact and $F^p\mathcal{H}'$ is $\mathcal{O}_X$-coherent (being $\mathcal{O}_X$-locally free of finite rank), the space $H^0(X,F^p\mathcal{H}')$ is finite dimensional, and we have a natural injective morphism

$$\varphi_p : H^0(X,F^p\mathcal{H}')\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow F^p\mathcal{H}'$$

by sending a global section of $F^p\mathcal{H}'$ to its germ at every point of X. On the other hand, regarding $F^p\mathcal{H}'$ as a complex with only one term in degree zero, we have an obvious morphism of complexes

$$F^p\,\mathrm{DR}\,\mathcal{H}' \longrightarrow F^p\mathcal{H}',$$

which induces a morphism $\boldsymbol{H}^0(X,F^p\,\mathrm{DR}\,\mathcal{H}')\to H^0(X,F^p\mathcal{H}')$, from which, together with φ_p, we obtain a morphism

$$\boldsymbol{H}^0(X,F^p\,\mathrm{DR}\,\mathcal{H}')\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow F^p\mathcal{H}'.$$

For p small enough so that $F^p\mathcal{H}'=\mathcal{H}'$, we recover the morphism φ above. By the degeneration property (2), $\boldsymbol{H}^0(X,F^p\,\mathrm{DR}\,\mathcal{H}')$ is identified with $F^pH^0(X,\underline{\mathcal{H}})$, hence the assertion.

As a consequence, *if a global horizontal section of* $(\mathcal{H}',\nabla)$*, i.e., a global section of* $\underline{\mathcal{H}}$*, regarded as a global section of* $\mathcal{H}'$*, is in* $F^p\mathcal{H}'$ *at one point, it is a global section of* $F^p\mathcal{H}'$.

Arguing similarly with the anti-holomorphic Hodge filtration, and then with the Hodge decomposition of the C^∞-bundle $\mathcal{H}$, we find that the natural injective morphism

$$H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathcal{C}^\infty_X \longrightarrow \underline{\mathcal{H}}\otimes_{\mathbb{C}}\mathcal{C}^\infty_X=\mathcal{H}$$

is compatible with the Hodge decomposition of each term. As a consequence, *for any global horizontal section of* $(\mathcal{H},D)$*, i.e., any global section of* $\underline{\mathcal{H}}$*, regarded as a global section of* $\mathcal{H}$*, the Hodge* (p,q)*-components are also* D*-horizontal.* In particular, if the global section is of type (p,q) at one point, it is of type (p,q) at every point of X.

Concerning the polarization, let us notice that the restriction of the polarization $\underline{\mathcal{S}}$ to the constant sub local system $H^0(X,\underline{\mathcal{H}})\otimes\mathbb{C}_X$ is constant. The polarization on

$H^0(X, \underline{\mathcal{H}})$ is thus equal, up to a positive multiplicative constant, to the restriction of $\underline{\mathcal{S}}$ to $H^0(X, \underline{\mathcal{H}})$ regarded as the stalk of $H^0(X, \underline{\mathcal{H}}) \otimes \mathbb{C}_X$ at any chosen point of X.

4.2.d. The L^2 de Rham and Dolbeault complexes. The compactness assumption in Hodge theory is not mandatory. One can relax it, provided that the metric on the manifold remains *complete* (see e.g. [**Dem96**, §12]). We will indicate the new phenomena that occur in the setting of Section 4.2.a.

One has to work with C^∞ sections v of $\mathcal{E}_X^\bullet \otimes \mathcal{H}$ which are globally L^2 with respect to the metric h and to the complete metric on X, and whose differential Dv is L^2. The analysis of the Laplace operator is now similar to that of the compact case. One uses a L^2 de Rham complex and a L^2 Dolbeault complex (i.e., one puts a L^2 condition on sections and their derivatives), that we analyze in this section.

We assume that the complex manifold X is equipped with a metric, hence a volume form vol, that enables to define the space $L^2(U, \mathrm{vol})$ for each open set $U \subset X$, and we will omit vol in the notation from now on. The locally free sheaves of differential forms are then equipped with a metric.

We can regard the sheaf of C^∞ functions on X at a subsheaf of the sheaf of locally integrable function, simply denoted by $\mathcal{L}_{1,\mathrm{loc}}$ (with respect to any metric on X). For any relatively compact open set $U \subset X$, $\Gamma(U, \mathcal{L}_{1,\mathrm{loc}}) := L^1_{\mathrm{loc}}(U, \mathrm{vol})$. One checks easily that the assignment $U \mapsto L^1_{\mathrm{loc}}(U, \mathrm{vol})$ is a sheaf, which does not depend on the choice of the metric.

Let $(\mathcal{H}, \mathrm{h})$ be a C^∞ vector bundle on X with a Hermitian metric h and let $\mathcal{L}_{1,\mathrm{loc}} \otimes_{\mathcal{C}_X^\infty} \mathcal{H}$ be the associated sheaf of L^1_{loc} sections of $\mathcal{H}$. The h-norm of any local section of the latter is a locally integrable function.

4.2.19. Definition. The space $L^2(X, \mathcal{H}, \mathrm{h})$ is the subspace of $\Gamma(X, \mathcal{L}_{1,\mathrm{loc}} \otimes \mathcal{H})$ consisting of sections whose h-norm belongs to $L^2(X)$.

The following lemmas are standard.

4.2.20. Lemma. *A section $u \in \Gamma(X, \mathcal{L}_{1,\mathrm{loc}} \otimes_{\mathcal{C}_X^\infty} \mathcal{H})$ belongs to $L^2(X, \mathcal{H}, \mathrm{h})$ if and only if there exists a sequence $u_n \in \Gamma(X, \mathcal{H}) \cap L^2(X, \mathcal{H}, \mathrm{h})$ such that $\|u - u_n\|_{2,\mathrm{h}} \to 0$ in $L^2(X, \mathcal{H}, \mathrm{h})$. In such a case, $u_n \to u$ weakly, that is, for any $\chi \in \Gamma_{\mathrm{c}}(X, \mathcal{H})$,*

$$\int_X (\mathrm{h}(u, \chi) - \mathrm{h}(u_n, \chi)) \,\mathrm{vol} \longrightarrow 0. \qquad \square$$

Let $\boldsymbol{\varepsilon}$ be an h-orthonormal frame of $\mathcal{H}$ on X. Then a section $u = \sum_i f_i \varepsilon_i$, with $f_i \in \mathcal{L}^1_{\mathrm{loc}}(X)$, belongs to $L^2(X, \mathcal{H}, \mathrm{h}))$ if and only if each f_i belongs to $L^2(X)$. Orthonormal frames may not be easy to find and in order to check the L^2 property, other frames may be more convenient.

4.2.21. Definition **(L^2-adaptedness).** A frame $\boldsymbol{v}$ of $\mathcal{H}$ on X is said to be *L^2-adapted* if there exists a positive constant $C_{\boldsymbol{v}}$ such that, for any section $u = \sum_i f_i v_i$ in

$\Gamma(X, \mathcal{L}^1_{\mathrm{loc}} \otimes \mathcal{H})$, the following inequality holds
$$\sum_i \|f_i v_i\|_2 \leqslant C_{\boldsymbol{v}} \|u\|_2 \Big(\leqslant C_{\boldsymbol{v}} \sum_i \|f_i v_i\|_2 \Big).$$

In other words, an L^2-adapted frame $\boldsymbol{v}$ gives rise to a decomposition $L^2(X, \mathcal{H}, \mathrm{h}) = \bigoplus_i L^2(X, \mathcal{H}_i, \mathrm{h})$, where $\mathcal{H}_i$ is the C^∞ subbundle generated by v_i with induced metric. Let us state some properties.

4.2.22. Lemma.

(1) *An orthonormal frame $\boldsymbol{\varepsilon}$ is L^2-adapted.*

(2) *If* h *and* h′ *are comparable metrics on $\mathcal{H}$, then a frame of $\mathcal{H}$ is L^2-adapted for* h *if and only if it is so for* h′.

(3) *If a frame $\boldsymbol{v} = (v_1, \dots, v_r)$ is L^2-adapted, any rescaled frame $(h_1 v_1, \dots, h_r v_r)$, with $h_i \in L^1_{\mathrm{loc}}(X)$ nowhere vanishing, is also L^2-adapted.*

(4) *A sufficient condition for a frame $\boldsymbol{v}$ to be L^2-adapted is that the functions $\|v_i\|_{\mathrm{h}}$ are locally bounded and each entry of the matrix $\mathsf{h}_{\boldsymbol{v}}^{-1} := (\mathrm{h}(v_i, v_j)_{i,j})^{-1}$ is locally bounded.*

Proof. The first three points are clear. For the last one, let $C > 0$ denote a bounding constant, let $u = \sum_i f_i v_i$ be a section of $\mathcal{L}^1_{\mathrm{loc}} \otimes \mathcal{H}$ and set $g_i := \mathrm{h}(u, v_i)$. Then $|g_i| \leqslant \|u\|_{\mathrm{h}} \|v_i\|_{\mathrm{h}} \leqslant C\|u\|_{\mathrm{h}}$, since $\|v_i\|_{\mathrm{h}} \leqslant C$. The column vectors G and F (of the g_i's and the f_i's respectively) are related by $G = \mathsf{h}_{\boldsymbol{v}} \cdot F$, so that $F = \mathsf{h}_{\boldsymbol{v}}^{-1} \cdot G$. It follows that, for each i, $|f_i| \leqslant rC^2 \|u\|_{\mathrm{h}}$ and thus $\|f_i v_i\|_{\mathrm{h}} \leqslant rC^3 \|u\|_{\mathrm{h}}$. Then $\boldsymbol{v}$ is L^2-adapted with constant $C_{\boldsymbol{v}} = rC^3$. ☐

Assume now that $\mathcal{H}$ is equipped with a flat connection $D : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$.

4.2.23. Definition. The space $L^2(X, \mathcal{H}, \mathrm{h}, D)$ is the subspace of $L^2(X, \mathcal{H}, \mathrm{h})$ consisting of sections u such that

- Du, considered in the weak sense, (i.e., distributional sense) is a section of $\mathcal{L}_{1,\mathrm{loc}} \otimes (\mathcal{E}^1_X \otimes \mathcal{H})$,
- as such, its h-norm belongs to $L^2(X)$.

C^∞-approximation also holds in this case.

4.2.24. Lemma. *A section $u \in \Gamma(X, \mathcal{L}_{1,\mathrm{loc}} \otimes_{\mathcal{C}^\infty_X} \mathcal{H})$ belongs to $L^2(X, \mathcal{H}, \mathrm{h}, D)$ if and only if there exists a sequence $u_n \in \Gamma(X, \mathcal{H}) \cap L^2(X, \mathcal{H}, \mathrm{h})$ such that*

- *$\|u - u_n\|_{2,\mathrm{h}} \to 0$ in $L^2(X, \mathcal{H}, \mathrm{h})$,*
- *Du_n belongs to $L^2(X, \mathcal{E}^1_X \otimes \mathcal{H}, \mathrm{h})$ and is a Cauchy sequence in this space.* ☐

The L^2 de Rham complex is then well-defined as the complex

$$(4.2.25)\quad 0 \longrightarrow L^2(X, \mathcal{H}, \mathrm{h}, D) \xrightarrow{\ D\ } L^2(X, \mathcal{E}^1_X \otimes \mathcal{H}, \mathrm{h}, D) \xrightarrow{\ D\ } \cdots \xrightarrow{\ D\ } L^2(X, \mathcal{E}^{2n}_X \otimes \mathcal{H}, \mathrm{h}, D) \longrightarrow 0,$$

whose cohomology is denoted by $H^k_{D,L^2}(X, \mathcal{H})$.

4.2.26. Definition. The assignment $U \mapsto L^2(U, \mathcal{H}, \mathrm{h})$ defines a presheaf, which is a sheaf on X, denoted by $\mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h})$. If U is relatively compact in X, we have

$$\Gamma(U, \mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h})) = L^2(U, \mathcal{H}, \mathrm{h}).$$

We define similarly the sheaf $\mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h}, D)$ and we set

$$\mathcal{L}^i_{(2)}(\mathcal{H}, \mathrm{h}, D) := \mathcal{L}_{(2)}(\mathcal{E}^i_X \otimes \mathcal{H}, \mathrm{h}, D).$$

We obtain therefore a complex of sheaves

$$0 \longrightarrow \mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h}) \xrightarrow{D} \mathcal{L}^1_{(2)}(\mathcal{H}, \mathrm{h}, D) \xrightarrow{D} \cdots \xrightarrow{D} \mathcal{L}^{2n}_{(2)}(\mathcal{H}, \mathrm{h}, D) \longrightarrow 0. \tag{4.2.27}$$

4.2.28. Lemma **(L^2 Poincaré lemma).** *The complex* (4.2.27) *is a resolution of the locally constant sheaf* $\underline{\mathcal{H}} = \operatorname{Ker} D$.

Proof. Near a given point of X, we can find a local isomorphism $(\mathcal{H}, D) \simeq (\mathcal{C}^\infty_X)^{\operatorname{rk}\mathcal{H}}, \mathrm{d})$ and the metric h is equivalent to the standard metric on $(\mathcal{C}^\infty_X)^{\operatorname{rk}\mathcal{H}}$ in which the canonical basis is orthonormal. Then both assertions of the theorem need only to be proved for $(\mathcal{C}^\infty_X, \mathrm{d}, \|\cdot\|)$ where $\|1\| = 1$. The proof is then obtained by a standard regularization procedure (see Corollary 12.2.5). $\square$

4.2.29. Definition. By considering the decomposition (4.2.5 $*$) one defines similarly $L^2(X, \mathcal{H}, \mathrm{h}, \overline{\mathcal{D}})$ and the L^2 Dolbeault complexes ($p = 0, \ldots, d_X$)

$$\begin{aligned} 0 \to L^2(X, \operatorname{gr}^p_F(\mathcal{E}^0_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) &\xrightarrow{\overline{\mathcal{D}}} L^2(X, \operatorname{gr}^p_F(\mathcal{E}^1_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \\ &\xrightarrow{\overline{\mathcal{D}}} \cdots \xrightarrow{\overline{\mathcal{D}}} L^2(X, \operatorname{gr}^p_F(\mathcal{E}^n_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \to 0, \end{aligned} \tag{4.2.30}$$

whose m-th cohomology is denoted by $H^{p,m+w-p}_{\overline{\mathcal{D}},L^2}(X, \mathcal{H})$.

The analogue of the C^∞-approximation lemma 4.2.24 also holds in this case, and we define the sheaves $\mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^i_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}})$ in a way similar to Definition 4.2.26. Then the Dolbeault complex (4.2.30) sheafifies as

$$0 \to \mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^0_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \xrightarrow{\overline{\mathcal{D}}} \mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^1_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \xrightarrow{\overline{\mathcal{D}}} \cdots \tag{4.2.31}$$

4.2.32. Lemma **(L^2 Dolbeault lemma).** *The inclusion of complexes*

$$\operatorname{gr}^p_F \operatorname{Dol}(\operatorname{gr}_F \mathcal{H}', \theta) \longrightarrow \big(\mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^\bullet_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}), \overline{\mathcal{D}}\big)$$

is a quasi-isomorphism.

Proof. We write $\overline{\mathcal{D}} = \mathrm{d}''_{\mathrm{h}} - \theta'$. Since θ' is C^∞, it is locally bounded, so the local L^2-condition on the $\overline{\mathcal{D}}$-derivative only concerns d'', and we can consider (up to signs) the complex (4.2.31) as the simple complex associated with the double complex with differentials d'' and θ'. For the complex with differential d'' we can apply the standard L^2-Dolbeault lemma (recalled as Theorem 12.2.6 in Chapter 12). Then the statement is clear. $\square$

4.2.e. Polarized variations of Hodge structure on a complex manifold with a complete metric

One missing point in the general context of noncompact complex manifolds is the finite dimensionality of the L^2-cohomologies involved. In the compact case, it is ensured, for instance, by the finiteness of the Betti cohomology $H^k(X, \underline{\mathcal{H}})$. So the Hodge theorem is stated as

4.2.33. Theorem. *Let (X, ω) be a complete Kähler manifold and let $(H, \mathcal{S})$ be a polarized variation of Hodge structure on X. Let $(\mathcal{H}, D)$ be the associated flat C^∞ bundle. Then, with the assumption that all the terms involved are finite dimensional, one has a canonical isomorphisms*

$$H^k_{D,L^2}(X, \mathcal{H}) \simeq \bigoplus_{p+q=k+w} H^{p,q}_{\mathcal{D}'',L^2}(X, \mathcal{H}), \quad H^{q,p}_{\mathcal{D}'',L^2}(X, \mathcal{H}) \simeq \overline{H^{p,q}_{\mathcal{D}'',L^2}(X, \mathcal{H}^\vee)}. \qquad \square$$

We refer to [**Zuc79**, §7] for a proof of this result. The finiteness assumption can be obtained by relating the L^2 de Rham cohomology with topology. If we are lucky, then this will not only provide a relation with Betti cohomology, but the Betti cohomology will be finite-dimensional and this will also provide finiteness of the L^2 de Rham cohomology.

There is also a need for finiteness of the L^2 Dolbeault cohomology. In the case that will occupy us later, where X is a punctured compact Riemann surface, this will be done by relating L^2 Dolbeault cohomology with the cohomology of a coherent sheaf on the compact Riemann surface.

We will indicate in Sections 6.12 and 6.14 the way to solve these two problems in dimension 1, by means of the L^2 Poincaré lemma and the L^2 Dolbeault lemma.

What about the Lefschetz aspect of Hodge theory in this context? The complete Kähler metric acts as a Lefschetz operator on the L^2 cohomology, and gives rise to a polarization of the Hodge structure. On the other hand, the theory of Hodge modules takes place on smooth complex projective varieties (or for some aspects compact Kähler manifolds). The non-compact manifold that occurs is the complement of divisor with normal crossings. Such a manifold can be equipped with a complete Kähler metric having a controlled behaviour at infinity, that is, in the neighbourhood of the normal crossing divisor of the compactification: the metric has a Poincaré-like behaviour locally at infinity. However, in the theory of Hodge modules, we only consider the Lefschetz operator coming from an ample line bundle on the projective variety.(3)

4.3. Semi-simplicity

4.3.a. A review on completely reducible representations. We review here some classical results concerning the theory of finite-dimensional linear representations. Let Π be a group and let ρ be a linear representation of Π on a finite-dimensional

(3) See [**KK87**, §6.4] for the comparison between both Lefschetz operators.

$\boldsymbol{k}$-vector space V. In other words, ρ is a group homomorphism $\Pi \to \mathrm{GL}(V)$. We will say that V is a Π-module (it would be more correct to introduce the associative algebra $\boldsymbol{k}[\Pi]$ of the group Π, consisting of $\boldsymbol{k}$-linear combinations of the elements of Π, and to speak of a left $\boldsymbol{k}[\Pi]$-module). The subspaces of V stable by $\rho(\Pi)$ correspond thus to the sub-Π-modules of V.

We say that a Π-module V is *irreducible* if it does not admit any nontrivial sub-Π-module. Then, any homomorphism between two irreducible Π-modules is either zero, or an isomorphism (Schur's lemma). If k is algebraically closed, any automorphism of an irreducible Π-module is a nonzero multiple of the identity (consider an eigenspace of the automorphism).

4.3.1. Proposition. *Given a Π-module V, the following properties are equivalent:*

(1) *The Π-module V is semi-simple, i.e., every sub-Π-module has a supplementary sub-Π-module.*

(2) *The Π-module V is completely reducible, i.e., V has a decomposition (in general non unique) into the direct sum of irreducible sub-Π-modules.*

(3) *The Π-module V is generated by its irreducible sub-Π-modules.*

Proof. The only non-obvious point is (3) $\implies$ (1). Let then W be a sub-Π-module of V. We will show the result by induction on $\operatorname{codim} W$, this being clear for $\operatorname{codim} W = 0$. If $\operatorname{codim} W \geqslant 1$, there exists by assumption a nontrivial irreducible sub-Π-module $V_1 \subset V$ not contained in W. Since V_1 is irreducible, we have $W \cap V_1 = \{0\}$, so $W_1 := W \oplus V_1$ is a sub-Π-module of V to which one can apply the induction hypothesis. If W_1' is a supplementary Π-module of W_1, then $W' = W_1' \oplus V_1$ is a supplementary Π-module of W. $\square$

It follows then from Schur's lemma that a completely reducible Π-module has a *unique* decomposition as the direct sum

$$V = \bigoplus_i V_i = \bigoplus_i (V_i^o \otimes E_i),$$

in which the *isotypic components* V_i are sub-Π-modules of the form $V_i^o \otimes E_i$, where V_i^o is an irreducible Π-module, V_i^o is not isomorphic to V_j^o for $i \neq j$, and E_i is a trivial Π-module, i.e., on which Π acts by the identity.

One also notes that if W is a sub-Π-module of a completely reducible Π-module V, then W is completely reducible and its isotypical decomposition is

$$W = \bigoplus_i (W \cap V_i),$$

in which $W \cap V_i = V_i^o \otimes F_i$ for some subspace F_i of E_i. A Π-module supplementary to W can be obtained by choosing for every i a $\boldsymbol{k}$-vector space supplementary to F_i in E_i.

4.3.2. Remarks. The previous properties have easy consequences.

(1) A $\boldsymbol{k}$-vector space V is a semi-simple Π-module *if and only if* the associated complex space $V_{\mathbb{C}} = \mathbb{C} \otimes V$ is a semi-simple Π-module (for the complexified representation).

Indeed, let us first recall that the group $\mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$ acts on $V_{\mathbb{C}}$: if $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ is some $\boldsymbol{k}$-basis of V then, for $a_1, \dots, a_n \in \mathbb{C}$ and $\sigma \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$, one sets $\sigma\bigl(\sum_i a_i\varepsilon_i\bigr) = \sum_i \sigma(a_i)\varepsilon_i$. A subspace $W_{\mathbb{C}}$ of $V_{\mathbb{C}}$ is "defined over $\boldsymbol{k}$", i.e., of the form $\mathbb{C} \otimes W$ for some sub-space W of V, if and only if it is stable by any automorphism $\sigma \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$: indeed, if $d = \dim_{\mathbb{C}} W_{\mathbb{C}}$, one can find, up to renumbering the basis $\boldsymbol{\varepsilon}$, a basis $e_1, \dots, e_d$ of $W_{\mathbb{C}}$ such that

$$
\begin{aligned}
e_1 &= \varepsilon_1 + a_{1,2}\varepsilon_2 + \cdots + a_{1,d}\varepsilon_d + \cdots + a_{1,n}\varepsilon_n \\
e_2 &= \qquad\qquad \varepsilon_2 + \cdots + a_{1,d}\varepsilon_d + \cdots + a_{2,n}\varepsilon_n \\
&\ \vdots \\
e_d &= \qquad\qquad\qquad\qquad\quad \varepsilon_d + \cdots + a_{d,n}\varepsilon_n,
\end{aligned}
$$

with $a_{i,j} \in \mathbb{C}$; one then shows by descending induction on $i \in \{d, \dots, 1\}$ that, if $W_{\mathbb{C}}$ is stable by $\mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$, then $a_{i,j}$ are invariant by any automorphism of $\mathbb{C}$ over $\boldsymbol{k}$, i.e., belong to $\boldsymbol{k}$ since $\mathbb{C}$ is separable over $\boldsymbol{k}$.

Let us now prove the assertion. Let us first assume that V is irreducible and let us consider the subspace $W_{\mathbb{C}}$ of $V_{\mathbb{C}}$ generated by the sub-Π-modules of minimal dimension (hence irreducible). Since the representation of Π is defined over $\boldsymbol{k}$, if $E_{\mathbb{C}}$ is a Π-module, so is $\sigma(E_{\mathbb{C}})$ for every $\sigma \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$; therefore the space $W_{\mathbb{C}}$ is invariant by $\mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$, in other words takes the form $\mathbb{C} \otimes_{\boldsymbol{k}} W$ for some subspace W of V. It is clear that W is a sub-Π-module of V, hence $W = V$. According to 4.3.1(3), $V_{\mathbb{C}}$ is semi-simple.

Conversely, let us assume that $V_{\mathbb{C}}$ is semi-simple. Let us choose a $\boldsymbol{k}$-linear form $\ell : \mathbb{C} \to \boldsymbol{k}$ such that $\ell(1) = 1$. It defines a $\boldsymbol{k}$-linear map $L : V_{\mathbb{C}} \to V$ which is Π-invariant and which induces the identity on V. Let W be a sub-Π-module of V. We have a Π-invariant projection $V_{\mathbb{C}} \to W_{\mathbb{C}}$, hence a composed projection p which is Π-invariant:

$$
\begin{tikzcd}
V_{\mathbb{C}} \arrow[r] & W_{\mathbb{C}} \arrow[r, "L"] & W \\
V \arrow[u, hook] \arrow[urr, "p"']
\end{tikzcd}
$$

from which one obtains a Π-module supplementary to W in V.

(2) If $\Pi'' \to \Pi$ is a surjective group-homomorphism and ρ'' is the composed representation, then V is a semi-simple Π-module if and only if it is a semi-simple Π''-module. Indeed, the Π-module structure only depends on the image $\rho(\Pi) \subset \mathrm{GL}(V)$.

(3) Let $\Pi' \lhd \Pi$ be a *normal* subgroup, and let V be a Π-module. Then, if V is semi-simple as a Π-module, it so as a Π'-module. Indeed, if V' is an irreducible sub-Π'-module of V, then $\rho(\pi)V'$ remains so for every $\pi \in \Pi$. If V is Π-irreducible and if V' is a nonzero irreducible sub-Π'-module, the sub-Π'-module generated by the $\rho(\pi)V'$ is a Π-module, hence coincides with V. As a consequence, V is generated by its irreducible sub-Π'-modules, hence is Π'-semi-simple, according to 4.3.1(3).

(4) A real representation $\Pi \to \mathrm{Aut}(V_{\mathbb{R}})$ is simple if and only if the associated complexified representation $\Pi \to \mathrm{Aut}(V_{\mathbb{C}})$ has at most two simple components. [*Hint*: Any simple component of the complexified representation can be summed with its conjugate to produce a sub-representation of the real representation.]

4.3.b. The semi-simplicity theorem. Let X be a smooth projective variety. Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}, D, \mathrm{S})$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on X (see Definition 4.1.4), and let $\underline{\mathcal{H}} = \operatorname{Ker} D$ be the associated complex local system.

4.3.3. Theorem. *Under these assumptions, the complex local system $\underline{\mathcal{H}}$ is semi-simple.*

Let us already note that the result is easy for unitary local systems (underlying thus polarized variations of type $(0,0)$, as in Section 4.2.a). The general case will use the objects introduced in Section 4.2.b, and will not be specific to polarized variations of Hodge structure. The proof of the semi-simplicity theorem will apply to these more general objects called harmonic bundle (see Section 4.2.b). Moreover, we can relax the property that the smooth variety is projective, and only assume that it is a compact Kähler manifold, since we will only use the Kähler identities.

4.3.4. Remark. If $\underline{\mathcal{H}}$ is obtained from a local system $\underline{\mathcal{H}}_{\mathbb{Q}}$ defined over $\mathbb{Q}$, then $\underline{\mathcal{H}}_{\mathbb{Q}}$ is also semi-simple as such, according to Remark 4.3.2(1).

Let $\mathcal{H}$ be a C^∞-bundle with metric h. The group of C^∞ automorphisms g of $\mathcal{H}$ acts on a given connection D by the formula ${}_gD := g \circ D \circ g^{-1} = D - D(g) \circ g^{-1}$, where we have extended the action of D in a natural way on the bundle $\mathcal{E}nd(\mathcal{H})$. If D is flat, then so is ${}_gD$. We then set ${}_gD = {}_gD_{\mathrm{h}} + {}_g\theta$. Let us also set $\widehat{D} = D_{\mathrm{h}} - \theta$ (see Lemma 4.2.2).

4.3.5. Lemma. *We have ${}_g\theta = \theta - \frac{1}{2}\big(D(g)g^{-1} + g^{*-1}\widehat{D}(g^*)\big)$, where g^* is the* h*-adjoint of g.*

Proof. We have

$$ {}_gD = D_{\mathrm{h}} + \theta - \big(D_{\mathrm{h}}(g)g^{-1} + [\theta, g]g^{-1}\big) = D_{\mathrm{h}} - D_{\mathrm{h}}(g)g^{-1} + g^{-1}\theta g. $$

It follows that ${}_g\theta$ is the self-adjoint part of $-D_{\mathrm{h}}(g)g^{-1} + g^{-1}\theta g$, that is, taking into account that the adjoint of $D_{\mathrm{h}}(g)$ is $D_{\mathrm{h}}(g^*)$ (by working in a local h-orthonormal basis),

$$ {}_g\theta = \frac{1}{2}\big(-(D_{\mathrm{h}}(g)g^{-1} + g^{*-1}D_{\mathrm{h}}(g^*)) + g^{-1}\theta g + g^*\theta g^{*-1}\big). \tag{4.3.6} $$

The lemma follows from a straightforward computation. $\square$

If we fix a metric on X, we deduce with h a metric on $\mathcal{E}^1_X \otimes \mathcal{H}$ and then a metric $\|\cdot\|$ on $\mathcal{H}om(\mathcal{H}, \mathcal{E}^1_X \otimes \mathcal{H})$ with associated scalar product $(\cdot\,,\cdot)$. We then denote by $\langle\cdot\,,\cdot\rangle$ the integrated product using the volume form on X:

$$ \langle\cdot\,,\cdot\rangle = \int_X (\cdot\,,\cdot)\,\mathrm{vol}\,. $$

4.3.7. Definition. The *energy* of $g \in \operatorname{Aut}(\mathcal{H})$ with respect to $(\mathcal{H}, D, \mathrm{h})$ is defined as

$$\mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g) := \|{}_g\theta\|^2 = \langle {}_g\theta, {}_g\theta\rangle.$$

Let $\xi \in \operatorname{End}(\mathcal{H})$ and let $\xi = \xi^+ + \xi^-$ be its decomposition into its self-adjoint part $\xi^+ = \frac{1}{2}(\xi + \xi^*)$ and its skew-adjoint part $\xi^- = \frac{1}{2}(\xi - \xi^*)$.

4.3.8. Proposition. *For t varying in $\mathbb{R}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(e^{t\xi})\big|_{t=0} = 2\langle D_{\mathrm{h}}\xi^+, \theta\rangle.$$

Proof. We have $D(e^{t\xi})e^{-t\xi} = tD\xi \bmod t^2$ and $e^{-t\xi^*}\widehat{D}(e^{t\xi^*}) = t\widehat{D}\xi^* \bmod t^2$. From Lemma 4.3.5 we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(e^{t\xi})\big|_{t=0} &= -\langle D\xi + \widehat{D}\xi^*, \theta\rangle - \langle \theta, D\xi + \widehat{D}\xi^*\rangle\\ &= -\langle D_{\mathrm{h}}\xi^+ + [\theta, \xi^-], \theta\rangle - \langle \theta, D_{\mathrm{h}}\xi^+ + [\theta, \xi^-]\rangle\\ &= -2\operatorname{Re}\langle D_{\mathrm{h}}\xi^+, \theta\rangle = -2\langle D_{\mathrm{h}}\xi^+, \theta\rangle,\end{aligned}$$

since $\langle \theta\xi^-, \theta\rangle = -\langle \theta, \theta\xi^-\rangle$, $\langle \xi^-\theta, \theta\rangle = -\langle \theta, \xi^-\theta\rangle$, and both θ and $D_{\mathrm{h}}\xi^+$ are self-adjoint. □

The property of being semi-simple or not for $(\mathcal{H}, D)$ is seen on the energy functional.

4.3.9. Proposition. *Let $(\mathcal{H}, D)$ be a flat bundle. Assume that there exists a metric* h *such that the energy functional $g \mapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g)$ has a critical point at $g = \mathrm{Id}$. Then $(\mathcal{H}, D)$ is semi-simple.*

Proof. Let us argue by contraposition and let us assume that $(\mathcal{H}, D)$ is not semi-simple. Let h be any metric on $\mathcal{H}$. We will prove that Id is not a critical point for $g \mapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g)$. It is enough to prove that there exists $\xi \in \operatorname{End}(\mathcal{H})$ such that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(e^{t\xi})$$

has no critical point at $t = 0$. By assumption, there exists a sub-bundle $\mathcal{H}_1$ of $\mathcal{H}$ stable by D such that its orthogonal $\mathcal{H}_2$ is not stable by D. Set $n_i = \operatorname{rk}\mathcal{H}_i$ $(i = 1, 2)$. With respect to this decomposition we have

$$D = \begin{pmatrix} D_1 & 2\eta\\ 0 & D_2\end{pmatrix},$$

with $\eta : \mathcal{H}_2 \to \mathcal{E}^1_X \otimes \mathcal{H}_1$ nonzero. Set $\xi = n_2\operatorname{Id}_{\mathcal{H}_1} - n_1\operatorname{Id}_{\mathcal{H}_2}$ and $g = e^{t\xi}$ $(t \in \mathbb{R})$. We have

$$\begin{aligned}D(e^{t\xi})e^{-t\xi} &= \left[\begin{pmatrix} 0 & 2\eta\\ 0 & 0\end{pmatrix}, \begin{pmatrix} e^{n_2t} & 0\\ 0 & e^{-n_1t}\end{pmatrix}\right] \cdot \begin{pmatrix} e^{-n_2t} & 0\\ 0 & e^{n_1t}\end{pmatrix}\\ &= \begin{pmatrix} 0 & 2\eta\\ 0 & 0\end{pmatrix} - \begin{pmatrix} e^{-n_2t} & 0\\ 0 & e^{n_1t}\end{pmatrix}\begin{pmatrix} 0 & 2\eta\\ 0 & 0\end{pmatrix}\begin{pmatrix} e^{n_2t} & 0\\ 0 & e^{-n_1t}\end{pmatrix}\\ &= \begin{pmatrix} 0 & 2(1 - e^{-(n_1+n_2)t})\eta\\ 0 & 0\end{pmatrix},\end{aligned}$$

so

$$ {}_gD = D - D(e^{t\xi})e^{-t\xi} = \begin{pmatrix} D_1 & 2e^{-(n_1+n_2)t}\eta \\ 0 & D_2 \end{pmatrix}, $$

and

$$ {}_g\theta = \begin{pmatrix} \theta_1 & e^{-(n_1+n_2)t}\eta \\ e^{-(n_1+n_2)t}\eta^* & \theta_2 \end{pmatrix}. $$

It follows that

$$ f(t) = c_0 + c_1 e^{-(n_1+n_2)t}, \quad c_0 \geqslant 0,\ c_1 > 0, $$

and it is clear that $f'(0) \neq 0$. $\square$

Proof of the semi-simplicity theorem. In view of Propositions 4.2.11 and 4.3.9, the semi-simplicity theorem 4.3.3 is a consequence of the following.

4.3.10. Proposition. *Assume that X is compact Kähler and that $(\mathcal{H}, D, \mathrm{h})$ is a harmonic bundle. Then $g \mapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g)$ has a critical point at $g = \mathrm{Id}$.*

Proof. According to Proposition 4.3.8, it is enough to show that $D_{\mathrm{h}}^\star(\theta) = 0$, where $D_{\mathrm{h}}^\star$ denotes the formal adjoint of D_{h}. Setting $D_{\mathrm{h}}^{\mathrm{c}} := D''_{\mathrm{h}} - D'_{\mathrm{h}}$, the Kähler identities for a Hermitian vector bundle (see Section 4.2.9) imply that $D_{\mathrm{h}}^\star$ is a multiple of $[\Lambda, D_{\mathrm{h}}^{\mathrm{c}}]$. Since θ is a matrix of 1-forms and Λ is an operator of type $(-1,-1)$, we have $\Lambda(\theta) = 0$. On the other hand, by the properties after Definition 4.2.5, we have $D_{\mathrm{h}}^{\mathrm{c}}(\theta) = 0$. $\square$

4.3.c. Structure of polarized variations of $\mathbb{C}$-Hodge structure

Let X be a complex manifold. We will say that two polarized variations of $\mathbb{C}$-Hodge structures are *equivalent* if one is obtained from the other one by a twist (k,ℓ) (see Exercise 2.10) and by multiplying the polarization form by a positive constant.

4.3.11. Lemma. *There exists at most one equivalence class of polarized variations of $\mathbb{C}$-Hodge structure on a simple (i.e., irreducible) $\mathbb{C}$-local system $\underline{\mathcal{H}}$ on a compact complex manifold X.*

4.3.12. Remark. A criterion for the *existence* of a polarized variation of $\mathbb{C}$-Hodge structure on a simple $\mathbb{C}$-local system $\underline{\mathcal{H}}$ is given in [**Sim92**, §4] in terms of rigidity.

Proof. If we are given two polarizable variations of $\mathbb{C}$-Hodge structure on an irreducible local system $\underline{\mathcal{H}}$, we deduce such a polarizable variation on $\mathcal{E}nd(\underline{\mathcal{H}})$ (Remark 5.4.5), and the dimension 1 vector space $\mathrm{End}(\underline{\mathcal{H}}) := H^0(X, \mathcal{E}nd(\underline{\mathcal{H}}))$ is equipped with a $\mathbb{C}$-Hodge structure of some type (k,ℓ) by the Hodge-Deligne theorem 4.2.16. The identity morphism $\mathrm{Id}_{\mathcal{H}} \in \mathrm{End}(\underline{\mathcal{H}})$ defines thus a morphism of type (k,ℓ) between the two variations. Therefore, the first one is obtained from the second one by a twist (k,ℓ). It remains to check that, on a given polarizable variation of $\mathbb{C}$-Hodge structure on an irreducible local system $\underline{\mathcal{H}}$, there exists exactly one polarization up to a positive multiplicative constant. Note that such a polarization is an isomorphism $\underline{\mathcal{H}} \xrightarrow{\sim} \underline{\mathcal{H}}^*$, so one polarization is obtained from another one by multiplying by a nonzero constant. This constant must be positive, by the positivity property of the associated Hermitian form. $\square$

Let X be a compact complex manifold and let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}, D, \mathcal{S})$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on X. If the associated local system $\underline{\mathcal{H}}$ is semi-simple, which is the case when X is Kähler according to Theorem 4.3.3, it decomposes as $\underline{\mathcal{H}} = \bigoplus_{\alpha\in A} \underline{\mathcal{H}}_{\alpha}^{n_\alpha}$, where $\underline{\mathcal{H}}_\alpha$ are irreducible and pairwise non isomorphic, and $\underline{\mathcal{H}}_{\alpha}^{n_\alpha}$ means the direct sum of n_α copies of $\underline{\mathcal{H}}_\alpha$. Similarly, $(\mathcal{H}, D) = \bigoplus_{\alpha\in A}(\mathcal{H}_\alpha, D)^{n_\alpha}$, and the polarization $\mathcal{S}$, being D-horizontal, decomposes with respect to $\alpha \in A$ as $\mathcal{S} = \bigoplus \mathcal{S}_{\alpha,n_\alpha}$. Let us set $\mathcal{H}^o_\alpha := \mathbb{C}^{n_\alpha}$ and let us write $\underline{\mathcal{H}} = \bigoplus_{\alpha\in A} \mathcal{H}^o_\alpha \otimes \underline{\mathcal{H}}_\alpha$. If we are given a basis $\mathcal{S}_\alpha$ of the dimension 1 vector space $\mathrm{Hom}(\underline{\mathcal{H}}_\alpha, \underline{\mathcal{H}}_\alpha^*)$, there exists a unique morphism $\mathrm{S}^o_\alpha \in \mathrm{Hom}(\mathcal{H}^o_\alpha, \mathcal{H}^{o*}_\alpha)$ such that $\mathcal{S}_{\alpha,n_\alpha} = \mathrm{S}^o_\alpha \otimes \mathcal{S}_\alpha$.

4.3.13. Theorem. *Under these conditions, the following holds:*

(1) *For every $\alpha \in A$, there exists a unique equivalence class of polarized variation of $\mathbb{C}$-Hodge structure of weight w on $\underline{\mathcal{H}}_\alpha$.*

(2) *For every $\alpha \in A$, let us fix a representative $H_\alpha = (\mathcal{H}_\alpha, F'^{\bullet}\mathcal{H}_\alpha, F''^{\bullet}\mathcal{H}_\alpha, D, \mathcal{S}_\alpha)$ of such an equivalence class. There exists then a polarized $\mathbb{C}$-Hodge structure*

$$H^o_\alpha = (\mathcal{H}^o_\alpha, F'^{\bullet}\mathcal{H}^o_\alpha, F''^{\bullet}\mathcal{H}^o_\alpha, \mathrm{S}^o_\alpha)$$

of weight 0 with $\dim \mathcal{H}^o_\alpha = n_\alpha$ such that

$$H = \bigoplus_{\alpha\in A} (H^o_\alpha \otimes_{\mathbb{C}} H_\alpha). \tag{4.3.13 $*$}$$

4.3.14. Remark. Let us emphasize the following statement: given a polarized variation of Hodge structure on a compact complex manifold X such that the underlying local system is semi-simple (which is the case if X is Kähler), then each irreducible component of this local system underlies a polarized variation of Hodge structure (which is essentially unique). The proposition also explains how to reconstruct the original variation from its irreducible components.

***Proof of Theorem* 4.3.13.**

(1) The uniqueness statement is given by Lemma 4.3.11. In order to prove the existence in 4.3.13(1), it is enough to exhibit for every $\alpha \in A$ a sub-variation of Hodge structure of H of weight w with underlying local system $\underline{\mathcal{H}}_\alpha$. The polarization $\mathcal{S}$ will then induce a polarization $\mathcal{S}_\alpha$, according to Exercise 4.2(1). For that purpose, it is enough to exhibit $\underline{\mathcal{H}}_\alpha$ as the image of an endomorphism $\underline{\mathcal{H}} \to \underline{\mathcal{H}}$ which is compatible with the Hodge structures: by abelianity (Proposition 4.1.10), this image is an object of $\mathsf{VHS}(X, \mathbb{C}, w)$. Let us therefore analyze $\mathrm{End}(\underline{\mathcal{H}}) = H^0(X, \mathcal{E}nd(\underline{\mathcal{H}}))$.

If we set $\mathcal{H}^o_\alpha = \mathbb{C}^{n_\alpha}$, so that $\underline{\mathcal{H}} = \bigoplus_\alpha(\mathcal{H}^o_\alpha \otimes_{\mathbb{C}} \underline{\mathcal{H}}_\alpha)$, we have an algebra isomorphism $\mathrm{End}(\underline{\mathcal{H}}) \simeq \prod_\alpha \mathrm{End}(\mathcal{H}^o_\alpha)$ (where $x_\alpha x_\beta = 0$ if $x_\alpha \in \mathcal{H}^o_\alpha$, $x_\beta \in \mathcal{H}^o_\beta$ and $\alpha \neq \beta$). We know that the local system $\mathcal{E}nd(\underline{\mathcal{H}})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure of weight 0. Therefore, $\mathrm{End}(\underline{\mathcal{H}}) = \Gamma(X, \mathcal{E}nd(\underline{\mathcal{H}}))$ underlies a $\mathbb{C}$-Hodge structure of weight 0 by the Hodge-Deligne theorem 4.2.16. It is then enough to show that *each $\mathcal{H}^o_\alpha$ underlies a $\mathbb{C}$-Hodge structure H^o_α of weight 0 such that the equality $\mathrm{End}(\underline{\mathcal{H}}) = \prod_\alpha \mathrm{End}(\mathcal{H}^o_\alpha)$ is compatible with the Hodge structures on both terms.* Indeed, choose then any rank 1 endomorphism p_α of some nonzero vector space $\mathcal{H}_\alpha^{o,(k,-k)}$. Extend it as a rank 1 endomorphism of $\mathcal{H}_\alpha^o$ of type $(0,0)$ by mapping every other summand $\mathcal{H}_\alpha^{o,(\ell,-\ell)}$ to zero, and extend it similarly as a rank 1 endomorphism of $\bigoplus_\beta \mathcal{H}_\beta^o$ of type $(0,0)$. One obtains thus a rank 1 endomorphism in $\mathrm{End}(\underline{\mathcal{H}})^{0,0}$. With respect to this identification, its image is $(\operatorname{Im} p_\alpha) \otimes_{\mathbb{C}} \underline{\mathcal{H}}_\alpha \simeq \underline{\mathcal{H}}_\alpha$, as wanted.

Let us prove the assertion, which reduces to proving the existence of a grading of each $\mathcal{H}_\alpha^o$ giving rise to the Hodge grading of $\mathrm{End}(\underline{\mathcal{H}})$. By the product formula above, the $\mathbb{C}$-algebra $\mathrm{End}(\underline{\mathcal{H}})$ is semi-simple, with center $Z = \prod_\alpha \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\alpha^o}$. An algebra automorphism φ of $\mathrm{End}(\underline{\mathcal{H}})$ induces an automorphism of the ring Z, whose matrix in the basis above only consists of zeros and ones. By the Skolem-Noether theorem (see e.g. [**Bou12**, §14, N°5, Th. 4]), algebra automorphisms for which the corresponding matrix is the identity are interior automorphisms, that is, products of interior automorphisms of each $\mathrm{End}(\mathcal{H}_\alpha^o)$. Any algebra automorphism can be composed with an automorphism with matrix having block entries Id or 0 in order that the matrix on Z is the identity. As a consequence, the identity component of the group of algebra automorphisms $\mathrm{Aut}^{\mathrm{alg}}(\mathrm{End}(\underline{\mathcal{H}}))$ is identified with $\prod_{\alpha\in A}(\mathrm{Aut}(\mathcal{H}_\alpha^o)/\mathbb{C}^* \,\mathrm{Id}_\alpha)$.

As in Section 2.5.9, the $\mathbb{C}$-Hodge structure of weight 0 on $\mathrm{End}(\underline{\mathcal{H}})$ defines a continuous representation $\rho : \mathbb{S}^1 \to \mathrm{Aut}(\mathrm{End}(\underline{\mathcal{H}}))$, such that $\rho(\lambda) = \lambda^p$ on $\mathrm{End}(\underline{\mathcal{H}})^{p,-p}$. Since the grading is compatible with the algebra structure, the continuous representation ρ takes values in the group of algebra automorphisms $\mathrm{Aut}^{\mathrm{alg}}(\mathrm{End}(\underline{\mathcal{H}}))$. Since $\rho(1) = \mathrm{Id}$, it takes values in the identity component of $\mathrm{Aut}^{\mathrm{alg}}(\mathrm{End}(\underline{\mathcal{H}}))$, i.e., in $\prod_{\alpha\in A}(\mathrm{Aut}(\mathcal{H}_\alpha^o)/\mathbb{C}^* \,\mathrm{Id}_\alpha)$. By the argument given in Section 2.5.9, it defines a grading, up to a shift, on each $\mathcal{H}_\alpha^o$, as wanted.

(2) Let us now equip $\mathcal{H}_\alpha^o$ with a polarized $\mathbb{C}$-Hodge structure of weight 0 so that (4.3.13 $*$) holds. We already have obtained a grading, i.e., a $\mathbb{C}$-Hodge structure of weight 0. In order to obtain a polarization of this $\mathbb{C}$-Hodge structure satisfying (4.3.13 $*$), we note the equality $\mathcal{H}_\alpha^o = \mathrm{Hom}(\underline{\mathcal{H}}_\alpha, \underline{\mathcal{H}})$, and since $\mathcal{H}om(\underline{\mathcal{H}}_\alpha, \underline{\mathcal{H}})$ underlies a polarized variation of Hodge structure of weight 0 according to 4.3.13(1), $\mathcal{H}_\alpha^o$ comes equipped with a polarized Hodge structure of weight 0 by the Hodge-Deligne theorem 4.2.16. By definition, the natural morphism $\mathcal{H}_\alpha^o \otimes \underline{\mathcal{H}}_\alpha \to \underline{\mathcal{H}}$ underlies a morphism of polarized variations of Hodge structure. □

4.4. Exercises

Exercise 4.1.

(1) Show that the category $\mathsf{VHS}(X, \mathbb{C}, w)$ is abelian. [*Hint*: Use that, according to Definition 4.1.5, any morphism is bigraded with respect to the Hodge decomposition, hence so are its kernel, image and cokernel.]

(2) Define the tensor product

$$\mathsf{VHS}(X, \mathbb{C}, w_1) \otimes \mathsf{VHS}(X, \mathbb{C}, w_2) \longrightarrow \mathsf{VHS}(X, \mathbb{C}, w_1 + w_2)$$

and the external Hom

$$\mathsf{VHS}(X, \mathbb{C}, w_1) \otimes \mathsf{VHS}(X, \mathbb{C}, w_2) \longrightarrow \mathsf{VHS}(X, \mathbb{C}, w_2 - w_1).$$

(3) Show that these operations preserve the subcategories of polarizable objects.

***Exercise 4.2* (Abelianity and semi-simplicity).** Let $\big(H, \mathrm{S}\big)$ be a polarized variation of Hodge structure of weight w on X.

(1) Show that any subobject of H in $\mathsf{VHS}(X, \mathbb{C}, w)$ is a direct summand of the given variation, and that the polarization S induces a polarization. $[$*Hint*: Use Exercise 2.12.$]$

(2) Conclude that the full subcategory $\mathsf{pVHS}(X, \mathbb{C}, w)$ of polarizable variations of Hodge structure is abelian and semi-simple (i.e., any object decomposes as the direct sum of its irreducible components). $[$*Hint*: Use the C^∞ interpretation of Definition 4.1.5.$]$

Exercise 4.3. Let $(H, \mathcal{S})$ be a polarized variation of Hodge structure of weight w on X (see Definition 4.1.4). Let h be the Hermitian metric deduced from $\mathcal{S}$ and let $D = D' + D''$ be the flat C^∞ connection. Let $\mathcal{H} = \bigoplus_{p+q=w} \mathcal{H}^{p,q}$ be the Hodge decomposition (which is h-orthogonal by construction). Show the following properties.

(1) In the Griffiths transversality relations (4.1.5 $*$), the composition of D' (resp. D'') with the projection on the first summand defines a $(1,0)$ (resp. $(0,1)$)-connection D'_{h} (resp. D''_{h}), and that the projection to the second summand defines a C^∞-linear morphism θ' (resp. θ'').

(2) The connection $D_{\mathrm{h}} := D'_{\mathrm{h}} + D''_{\mathrm{h}}$ is compatible with the metric h, but is possibly not flat.

(3) The morphism θ'' is the h-adjoint of θ'.

(4) The connection $\overline{\mathcal{D}} := D''_{\mathrm{h}} + \theta'$ has square zero, as well as the connection $\mathcal{D} := D'_{\mathrm{h}} + \theta''$.

For each $p \in \mathbb{Z}$, set $\theta'_p : \mathcal{H}^{p,w-p} \to \mathcal{E}_X^{(1,0)} \otimes \mathcal{H}^{p-1,w-p+1}$ be the component of θ' on $\mathcal{H}^{p,w-p}$ and set θ''_p similarly.

(5) Show that θ''_p is the h-adjoint of θ'_{p-1}.

(6) Show that the Hermitian holomorphic bundle $(F^p\mathcal{H}, D'')$ has Chern connection equal to $(D'_{\mathrm{h}} + \sum_{p' \geqslant p+1} \theta'_{p'}) + (D''_{\mathrm{h}} + \sum_{p' \geqslant p} \theta''_{p'})$. $[$*Hint*: Recall that each $\mathcal{H}^{p,w-p}$ is stable by D_{h} and write the holomorphic structure D'' on $F^p\mathcal{H}$ as $D_{\mathrm{h}} + \sum_{p' \geqslant p} \theta''_{p'}$.$]$

Exercise 4.4. Let $(\mathcal{H}, D)$ be a flat bundle and let h be a Hermitian metric on $\mathcal{H}$.

(1) Show that there exist a unique $(1,0)$-connection $\widehat{D}'$ and a unique $(0,1)$-connection $\widehat{D}''$ such that $D' + \widehat{D}''$ and $\widehat{D}' + D''$ preserve the metric h.

(2) Show that $\widehat{D}' = D'_{\mathrm{h}} - \theta'$ and $\widehat{D}'' = D''_{\mathrm{h}} - \theta''$.

(3) Conclude that $\widehat{D}' + D''$ is the Chern connection of the Hermitian holomorphic bundle $(\mathcal{H}, D'', \mathrm{h})$.

(4) We set $D^{\mathrm{c}} := \widehat{D}'' - \widehat{D}'$. Show that $D^{\mathrm{c}} = \overline{\mathcal{D}} - \mathcal{D}$ and $\frac{1}{2}(D + D^{\mathrm{c}}) = D''_{\mathrm{h}} + \theta' = \overline{\mathcal{D}}$.

(5) Let $R_{\mathrm{h}} = (\widehat{D}' + D'')^2$ be the curvature of the Hermitian holomorphic bundle $(\mathcal{H}, \mathrm{h}, D'')$. Show that

$$R_{\mathrm{h}} = -2(\overline{\mathcal{D}}^2 + \theta' \wedge \theta'' + \theta'' \wedge \theta').$$

(6) Show that, *if* $(\mathcal{H}, D, \mathrm{h})$ *is harmonic*, then $\operatorname{tr} R_{\mathrm{h}} = 0$. $[$*Hint*: Use that, in any case, $\operatorname{tr}(\theta' \wedge \theta'' + \theta'' \wedge \theta') = 0$.$]$

(7) In the setting of Exercise 4.3, show that, for each $p \in \mathbb{Z}$, the curvature R_{h}^{p} of the Hermitian holomorphic bundle $(F^p\mathcal{H}, D'')$ satisfies $\|R_{\mathrm{h}}^{p}\|_{\mathrm{h}} \leqslant C_p\|\theta'\|_{\mathrm{h}}^2$ for a suitable positive constant C_p. Here, the curvature R_{h}^{p} is regarded as a section of $\mathcal{E}nd(F^p\mathcal{H}) \otimes \mathcal{E}_X^2$ and its norm is computed with respect to the metric on $\mathcal{E}nd(F^p\mathcal{H})$ induced by h and any fixed norm on differential forms. A similar definition holds for $\|\theta'\|_{\mathrm{h}}$.

Exercise 4.5 **(Example of a harmonic flat bundle).** Let ω be a holomorphic one-form on X which is closed. Show that the trivial rank-one bundle $\mathcal{H} = \mathcal{C}_X^\infty$, equipped with the connexion $D = \mathrm{d} + \omega$ and the trivial metric h (i.e., such that $\mathrm{h}(1, \overline{1}) = 1$), is a flat harmonic bundle, with $D'_{\mathrm{h}} = \mathrm{d}' + \frac{1}{2}\omega$, $D''_{\mathrm{h}} = \mathrm{d}'' - \frac{1}{2}\overline{\omega}$ and $\theta' = \frac{1}{2}\omega$, $\theta'' = \frac{1}{2}\overline{\omega}$. [*Hint*: With the notation of the proof of Lemma 4.2.2, take $\widetilde{D}_{\mathrm{h}} = \mathrm{d}$ and decompose $A = D - \widetilde{D}_{\mathrm{h}} = \omega$ as $A^- + A^+$ with $A^- = \frac{1}{2}(\omega - \overline{\omega})$ and $A^+ = \frac{1}{2}(\omega + \overline{\omega})$; show that harmonicity follows from the relation $\omega \wedge \overline{\omega} + \overline{\omega} \wedge \omega = 0$.]

Exercise 4.6 **(Norm of horizontal sections).** Let $(\mathcal{H}, D, \mathrm{h})$ be a harmonic flat bundle with Higgs fields θ', θ''. Let v be a horizontal section of $(\mathcal{H}, D)$ (equivalently, a horizontal section of the associated holomorphic flat bundle (V, ∇)). Show that the h-norm of v satisfies

$$\mathrm{d}'\|v\|_{\mathrm{h}}^2 = -2\mathrm{h}(\theta' v, \overline{v}), \quad \mathrm{d}''\|v\|_{\mathrm{h}}^2 = -2\mathrm{h}(\theta'' v, \overline{v}).$$

[*Hint*: Use that $D'_{\mathrm{h}}v = -\theta' v$ and $D''_{\mathrm{h}}v = -\theta'' v$ by horizontality.]

Exercise 4.7 **(Rescaling the Higgs field).**

(1) Let $(\mathcal{H}, D, \mathrm{h})$ be a harmonic flat bundle with Higgs fields θ', θ''. Show that, for any nonzero complex number t, there exists a harmonic flat bundle $(\mathcal{H}, D_t, \mathrm{h})$ whose Higgs fields are $(t\theta', \overline{t}\theta'')$. [*Hint*: Set $D'_t = D'_{\mathrm{h}} + t\theta'$, $D''_t = D''_{\mathrm{h}} + \overline{t}\theta''$ and show that $D_t = D'_t + D''_t$ is flat.]

(2) In case $(\mathcal{H}, D, \mathrm{h})$ is attached to a polarized variation of Hodge structure of weight w as in Exercise 4.3, show that $(\mathcal{H}, D_t, \mathrm{h}) \simeq (\mathcal{H}, D, \mathrm{h})$ if $|t| = 1$. [*Hint*: Compare with Section 2.5.9.]

Exercise 4.8.

(1) Given C^∞ bundles with flat connection and Hermitian metric $(\mathcal{H}_1, D_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, D_2, \mathrm{h}_2)$, equip the C^∞ bundles $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ with a natural flat connection D and a natural Hermitian metric h, and identify connection and metric on $\mathcal{H}_1^* \otimes \mathcal{H}_2$ and $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$. [*Hint*: Use Exercise 2.2.]

(2) Show that, for a section φ of $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$, we have $D(\varphi) = D_2 \circ \varphi - \varphi \circ D_1$.

(3) Prove that if $(\mathcal{H}_1, D_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, D_2, \mathrm{h}_2)$ are harmonic, then $(\mathcal{H}_1 \otimes \mathcal{H}_2, D, \mathrm{h})$ and $(\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2), D, \mathrm{h})$ are also harmonic.

(4) Show that θ on $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ is given by the following formula. For an open set U and a local section $\varphi : \mathcal{H}_{1|U} \to \mathcal{H}_{2|U}$ of $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ on U,

$$\forall\, V \subset U,\ \forall\, u_1 \in \Gamma(V, \mathcal{H}_1), \quad [\theta(\varphi)](u_1) = \theta_2(\varphi(u_1)) - \varphi(\theta_1(u_1)) \in \Gamma(V, \mathcal{E}_X \otimes \mathcal{H}_2).$$

Exercise 4.9 **(Formal adjoint).** Let $(\mathcal{H}, \mathrm{h})$ be a Hermitian vector bundle on a complex manifold X equipped with a Hermitian metric on its tangent bundle, which induces a

Hermitian metric on the sheaves $\mathcal{E}_X^k$, simply denoted by $\langle\,,\rangle$. Then the sheaves $\mathcal{E}_X^k\otimes\mathcal{H}$ are equipped with a natural Hermitian metric denoted by $\langle\,,\rangle_{\mathrm{h}}$ (see Exercise 2.2). For the differential operator of order 1

$$D:\mathcal{E}_X^k\otimes\mathcal{H}\longrightarrow\mathcal{E}_X^{k+1}\otimes\mathcal{H}$$

induced by a connection $D:\mathcal{H}\to\mathcal{E}_X^1\otimes\mathcal{H}$, the formal adjoint $D^\star:\mathcal{E}_X^{k+1}\otimes\mathcal{H}\to\mathcal{E}_X^k\otimes\mathcal{H}$ is the operator defined by

$$\int_X\langle Du,\overline{v}\rangle_{\mathrm{h}}\ \mathrm{vol}_X=\int_X\langle u,\overline{D^\star v}\rangle_{\mathrm{h}}\ \mathrm{vol}_X$$

for any pair of local sections with compact support of $\mathcal{E}_X^k\otimes\mathcal{H}$ and $\mathcal{E}_X^{k+1}\otimes\mathcal{H}$.

(1) Show that the formal adjoint $\varphi^\star$ of a $\mathcal{C}_X^\infty$-linear morphism $\varphi\in\mathcal{H}om(\mathcal{H}_1,\mathcal{H}_2)$ is nothing but its h-adjoint φ^*.

(2) Show that, if φ is self-adjoint with respect to h, then regarding $D(\varphi)=[D,\varphi]$ as a $\mathcal{C}_X^\infty$-linear morphism $\mathcal{E}_X^k\otimes\mathcal{H}\to\mathcal{E}_X^{k+1}\otimes\mathcal{H}$, we have $D(\varphi)^*=-[D^\star,\varphi]$.

Exercise 4.10. The goal of this exercise is to prove the identity (4.2.10). Recall $\mathcal{D}=D'_{\mathrm{h}}+\theta''$ and $\overline{\mathcal{D}}=D''_{\mathrm{h}}+\theta'$.

(1) Prove that $\theta'^\star=-\mathsf{i}[\Lambda,\theta'']$ and $\theta''^\star=\mathsf{i}[\Lambda,\theta']$.

(2) Deduce from the Kähler identities of Section 4.2.9 that

$$\widehat{D}''^\star=-\mathsf{i}[\Lambda,D'],\quad \mathcal{D}^\star=\mathsf{i}[\Lambda,\overline{\mathcal{D}}],\quad \overline{\mathcal{D}}^\star=-\mathsf{i}[\Lambda,\mathcal{D}].$$

(3) Conclude by using Lemma 4.2.7 and the computation of standard Kähler identities.

(4) Show also that $D^\star=\mathsf{i}[\Lambda,D^{\mathrm{c}}]$.

Exercise 4.11 **(Formulas for a holomorphic subbundle).** We keep the setting of Exercise 4.4. Let $(\mathcal{H}_1,D_1)$ be a flat holomorphic subbundle of $(\mathcal{H},D)$, i.e., $\mathcal{H}_1$ is stable by D' and D''. Let $\pi:\mathcal{H}\to\mathcal{H}_1$ be the orthogonal projection (so that $\pi\circ\pi=\pi$). We still denote by D the connection on $\mathcal{E}nd(\mathcal{H})$, so that $D\pi-\pi D=D(\pi)$.

(1) Show the following relations for D,D_1 and π:

(a) $\pi D(\pi)=D(\pi)$ and $D(\pi)\pi=0$. [*Hint*: for the first one, use that $\pi\circ D\circ\pi=D\circ\pi$; for the second one, use that, for a section v of $\mathcal{H}_1$, $D(v)$ is a section of $\mathcal{E}_X^1\otimes\mathcal{H}_1$, so that $\pi(D(v))=D(v)$.]

(b) $D_1=\pi\circ D\circ\pi=D\circ\pi=\pi\circ D+D(\pi)$.

(2) Show that $(\widehat{D}'_1+D''_1)=\pi\circ(\widehat{D}'+D'')\circ\pi$. [*Hint*: recall that $(\widehat{D}'+D'')$ is the Chern connection for $(\mathcal{H},\mathrm{h},D'')$ and use [**GH78**, Lem. p. 73]).]

(3) Show a similar relation for $(\widehat{D}''+D')$ and deduce a similar relation for D^{c}.

(4) Conclude that $D_1D_1^{\mathrm{c}}+D_1^{\mathrm{c}}D_1=\pi(DD^{\mathrm{c}}+D^{\mathrm{c}}D)\pi+D(\pi)D^{\mathrm{c}}(\pi)$.

4.5. Comments

Although one can trace back the notion of variation of Hodge structure to the study of the Legendre family of elliptic curves in the nineteenth century, the modern approach using the Gauss-Manin connection goes back to the fundamental work of

Griffiths [**Gri68, Gri70a, Gri70b**] motivated by the properties of the period domain (see also [**Del71c**], [**CMSP03**]), a subject that is not considered in the present text. In the work of Griffiths, the transversality property (4.1.1) has been emphasized. From the point of view of $\mathcal{D}$-modules, this property is now encoded in the notion of a coherent filtration, and is at the heart of the notion of filtered $\mathcal{D}$-module, which is part of a Hodge module as defined by Saito.

The notion of a polarized variation of Hodge structure can be regarded as equivalent to the notion of a smooth polarized Hodge module. However, this equivalence is not obvious since the definition of a polarized Hodge module imposes properties on nearby cycles along any germ of holomorphic function, while the notion of variation only requires to consider coordinate functions.

The C^∞ approach as in Definition 4.1.5 proves useful for extending the Hodge theorem on smooth complex projective varieties and constant coefficients to the case when the coefficient system is a unitary local system (see [**Dem96**]) and the more general case when it underlies a polarized variation of Hodge structure (Hodge-Deligne theorem 4.2.16 explained in the introduction of [**Zuc79**]). It is also well-adapted to the extension of this theorem to harmonic bundles, as explained by Simpson in [**Sim92**]. In this smooth context, the flat sesquilinear pairing $\mathfrak{s}$ gives rise in a natural way to the (non-flat in general) Hermitian Hodge metric. The fixed-part theorem, proved in Remark 4.2.18(4), is originally due to Griffiths [**Gri70a**] in a geometric setting, and has been proved in a more general context by Deligne [**Del71b**, Cor. 4.1.2], and also by Schmid [**Sch73**, Th. 7.22].

We have also mentioned the case of complete Kähler manifolds, going back to Andreotti and Vesentini [**AV65**] and Hörmander [**Hör65, Hör66**]. Theorem 4.2.33 is taken from [**Dem96**, §12B]. They are useful for understanding the L^2 approach as in Zucker's theorem 6.11.1 of [**Zuc79**].

It is remarkable that the local system underlying a polarized variation of Hodge structure on a smooth complex projective variety (or a compact Kähler manifold) is semi-simple. This property, proved by Deligne in the presence of a $\mathbb{Z}$-structure (see [**Del71b**, Th. 4.2.6]), can be regarded as a special case of a result of Corlette [**Cor88**] and [**Sim92**], since the Hodge metric is a pluri-harmonic metric on the corresponding flat holomorphic bundle. These articles are at the source of Sections 4.2.b and 4.3.b. Exercises 4.4 and 4.11 are extracted from [**Sim90**] and [**Sim92**].

Lastly, the structure theorem for polarized variations of Hodge structure (Theorem 4.3.13) is nothing but [**Del87**, Prop. 1.13].