CHAPTER 8

TRAINING ON $\mathcal{D}$-MODULES

Summary. In this chapter, we introduce the fundamental functors on $\mathcal{D}$-modules that we use in order to define supplementary structures, and we also introduce various operations: pullback and pushforward by a holomorphic map between complex manifolds or a morphism between smooth algebraic varieties. Most results are presented as exercises. They mainly rely on Leibniz rule. The main references for this chapter are **[Bjö93]**, **[Kas03]** and **[GM93]**.

8.1. The sheaf of holomorphic differential operators

Let $(X, \mathcal{O}_X)$ be a complex manifold equipped with its sheaf of holomorphic functions. We also denote by $\mathcal{C}_X^\infty$ the sheaf of complex-valued C^∞ functions on the underlying C^∞ manifold $X_\mathbb{R}$. This sheaf is a c-soft sheaf.

8.1.a. Vector fields, derivations, differential forms, contractions. We will denote by Θ_X the sheaf of holomorphic vector fields on X. This is the $\mathcal{O}_X$-locally free sheaf generated in local coordinates by $\partial_{x_1}, \dots, \partial_{x_n}$. It is a sheaf of $\mathcal{O}_X$-Lie algebras, and vector fields act (on the left) on functions by derivation, in a way compatible with the Lie algebra structure: given local vector fields ξ, η acting on functions as a derivations and given a local holomorphic function f,

- $f\xi$ is the vector field acting as $(f\xi)(g) = f \cdot \xi(g)$,
- the bracket $[\xi, \eta]$ defined as the operator $[\xi, \eta](g) := \xi(\eta(g)) - \eta(\xi(g))$ is still a derivation, hence defines a vector field.

We will denote by $\Theta_{X,k}$ the exterior product $\wedge^k \Theta_X$, which is also a locally free $\mathcal{O}_X$-module.

Dually, we denote by Ω_X^1 the sheaf of holomorphic 1-forms on X. We will set $\Omega_X^k = \wedge^k \Omega_X^1$ and $\omega_X = \Omega_X^n$. We denote by $\mathrm{d} : \Omega_X^k \to \Omega_X^{k+1}$ the differential.

The natural nondegenerate pairing $\langle \bullet, \bullet \rangle : \Omega_X^1 \otimes \Theta_X \to \mathcal{O}_X$ extends in a natural way as a nondegenerate pairing $\Omega_X^k \otimes \Theta_{X,k} \to \mathcal{O}_X$. In local coordinates $(x_1, \dots, x_n)$, a basis of Ω_X^k is given by the family $(\mathrm{d}x_I)_I$, where I runs among the subsets of cardinal k of $\{1, \dots, n\}$ and $\mathrm{d}x_I$ is defined as $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, with $I = \{i_1, \dots, i_k\}$

and $i_1 < \cdots < i_k$. Dually, the partial derivatives ∂_{x_i} lead to the basis $(\partial_{x_I})_I$ of $\Theta_{X,k}$, with a similar meaning. Due to anti-commutativity of the wedge product, $(\partial_{x_I})_I$ is the basis dual to $(\mathrm{d}x_I)_I$ up to sign, that is, denoting by δ the Kronecker index,

$$\langle \mathrm{d}x_I, \partial_{x'_I}\rangle = \varepsilon(k)\delta_{I,I'} \quad (\varepsilon(k) := (-1)^{k(k-1)/2}).$$

We can thus regard sections of Ω_X^k as $\mathcal{O}_X$-linear forms on $\Theta_{X,k}$. For a local section η of Ω_X^k, we may denote $\langle \eta, \bullet\rangle$ as $\eta(\bullet)$.

The *contraction* by a vector field ξ is the $\mathcal{O}_X$-linear morphism $\xi \lrcorner \colon \Omega_X^k \to \Omega_X^{k-1}$ defined by $\eta \mapsto \eta(\xi\wedge\bullet)$, where $\bullet$ is local section of $\Theta_{X,k-1}$. More generally, for a local section ξ of $\Theta_{X,j}$, the contraction $\eta \mapsto \eta(\xi \wedge \bullet)$ sends Ω_X^k to Ω_X^{k-j}.

For example, if $k = n = \dim X$, set

$$\mathrm{d}\boldsymbol{x} := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \quad \text{and} \quad \mathrm{d}\boldsymbol{x}_{\widehat{\imath}} := \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n.$$

Then we have

$$\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x} = (-1)^{n-i}\mathrm{d}\boldsymbol{x}_{\widehat{\imath}},$$

since

$$\begin{aligned}(\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x})(\partial_{\boldsymbol{x}_{\widehat{\imath}}}) &= \mathrm{d}\boldsymbol{x}(\partial_{x_i}\wedge\partial_{\boldsymbol{x}_{\widehat{\imath}}}) = (-1)^{i-1}\mathrm{d}\boldsymbol{x}(\partial_{\boldsymbol{x}})\\ &= (-1)^{i-1}\,\varepsilon(n) = (-1)^{n-i}\,\varepsilon(n-1) = (-1)^{n-i}\mathrm{d}\boldsymbol{x}_{\widehat{\imath}}(\partial_{\boldsymbol{x}_{\widehat{\imath}}}).\end{aligned}$$

As a consequence, for $f \in \mathcal{O}_X$, we have $\mathrm{d}(f\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x}) = (-1)^{n-1}\partial f/\partial_{x_i}\cdot \mathrm{d}\boldsymbol{x}$.

The *Lie derivative* of $\mathrm{d}\boldsymbol{x}$ along ξ is defined as $\mathcal{L}_\xi(\mathrm{d}\boldsymbol{x}) := \mathrm{d}(\xi \lrcorner\, \mathrm{d}\boldsymbol{x})$. Similarly, we rename the action of ξ as a derivation on f as $\mathcal{L}_\xi(f) = \partial f/\partial_{x_i}$. Note that $\mathcal{L}_{\partial_{x_i}}(\mathrm{d}\boldsymbol{x}) = 0$. We conclude from these formulas that there is a natural *right* action (in a compatible way with the Lie algebra structure) of Θ_X on ω_X, defined by

$$\omega\cdot\xi = (-1)^n\mathcal{L}_\xi\omega := (-1)^n\mathrm{d}(\xi \lrcorner\, \omega). \tag{8.1.1}$$

Indeed, the relation $\xi(f)\omega = \omega\cdot[\xi,f] = (\omega\cdot\xi)f - (\omega f)\cdot\xi$ holds, as for example, taking $\xi = \partial_{x_i}$, we find $(f\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i} = -(\partial f/\partial x_i)\mathrm{d}\boldsymbol{x}$ and

$$(\partial f/\partial x_i)\mathrm{d}\boldsymbol{x} = (-1)^{n-1}\mathrm{d}(f\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x}) = -(f\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i} \quad\text{and}\quad (\mathrm{d}\boldsymbol{x}\cdot\partial_{x_i})f = 0.$$

Similarly, let us check $\omega\cdot[\xi,\xi'] = (\omega\cdot\xi)\cdot\xi' - (\omega\cdot\xi')\cdot\xi$ with $\omega = \mathrm{d}\boldsymbol{x}$, $\xi = f\partial_{x_i}$, $\xi' = \partial_{x_j}$. We have $[\xi,\xi'] = -(\partial f/\partial_{x_j})\partial_{x_i}$ and $\omega\cdot\xi' = 0$, so we only have to check

$$-(\partial f/\partial x_j\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i} = ((f\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i})\cdot\partial_{x_j},$$

which follows from the commutativity of the partial derivatives of f.

***8.1.2. Definition* (The sheaf of holomorphic differential operators)**

For any open set U of X, the ring $\mathcal{D}_X(U)$ of *holomorphic differential operators on* U is the subring of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U,\mathcal{O}_U)$ generated by

- multiplication by holomorphic functions on U,
- derivation by holomorphic vector fields on U.

The sheaf $\mathcal{D}_X$ is defined by $\Gamma(U,\mathcal{D}_X) = \mathcal{D}_X(U)$ for every open set U of X.

By construction, the sheaf $\mathcal{D}_X$ acts on the left on $\mathcal{O}_X$, i.e., $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

***8.1.3. Definition* (The filtration of $\mathcal{D}_X$ by the order).** The increasing family of subsheaves $F_k\mathcal{D}_X \subset \mathcal{D}_X$ is defined inductively:

- $F_k\mathcal{D}_X = 0$ if $k \leqslant -1$,
- $F_0\mathcal{D}_X = \mathcal{O}_X$ (*via* the canonical injection $\mathcal{O}_X \hookrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$),
- the local sections P of $F_{k+1}\mathcal{D}_X$ are characterized by the fact that $[P, f]$ is a local section of $F_k\mathcal{D}_X$ for any holomorphic function g.

8.1.4. Proposition. *Giving a left $\mathcal{D}_X$-module $\mathcal{M}$ is equivalent to giving an $\mathcal{O}_X$-module $\mathcal{M}$ together with an integrable connection ∇.*

Proof. Exercises 8.1, 8.5 and 8.6. □

8.1.b. Vector fields and differential forms in presence of a filtration

We now apply the constructions of Section 5.1 to the filtered ring $(\mathcal{D}_X, F_\bullet\mathcal{D}_X)$ and its (left or right) modules. We obtain the following properties:

- $\widetilde{\mathcal{O}}_X := R_F\mathcal{O}_X = \mathcal{O}_X[z]$.
- in local coordinates, we have

$$\widetilde{\mathcal{D}}_X := R_F\mathcal{D}_X = \mathcal{O}_X[z]\langle\widetilde{\partial}_{x_1}, \dots, \widetilde{\partial}_{x_n}\rangle, \tag{8.1.5}$$

i.e., any germ of section of $\widetilde{\mathcal{D}}_X$ may be written in a unique way as

$$\sum_\alpha a_\alpha(x,z)\widetilde{\partial}_x^\alpha = \sum_\alpha \widetilde{\partial}_x^\beta b_\alpha(x,z),$$

where $a_\alpha, b_\alpha \in \widetilde{\mathcal{O}}_X$, and where we set

$$\widetilde{\partial}_{x_i} := z\partial_{x_i}. \tag{8.1.6}$$

- The ring $\widetilde{\mathcal{D}}_X$ is equipped with a natural filtration $F_\bullet\widetilde{\mathcal{D}}_X$ by the order in $\widetilde{\partial}_x$. If we write $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, then this filtration is defined by the formula

$$F_k\widetilde{\mathcal{D}}_X = \bigoplus_{j=0}^{k-1} F_j\mathcal{D}_X z^j \oplus F_k\mathcal{D}_X z^k\mathbb{C}[z]. \tag{8.1.7}$$

The graded ring $\mathrm{gr}^F\widetilde{\mathcal{D}}_X$ can be identified with the graded ring $\mathrm{gr}^F\mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[z]$ (with grading only coming from $\mathrm{gr}^F\mathcal{D}_X$) by dividing each $\mathrm{gr}^F_k\widetilde{\mathcal{D}}_X$ by z^k. If we regard $\mathrm{gr}^F\mathcal{D}_X$ as the ring of holomorphic functions on the cotangent space T^*X which are polynomial with respect to the projection $\widetilde{T}^*X := T^*X \to X$, we interpret the ring $\mathrm{gr}^F\widetilde{\mathcal{D}}_X$ as the ring of holomorphic functions on $T^*X \times \mathbb{C}_z$ which are polynomial with respect to the projection to X.

- The sheaf $\widetilde{\Theta}_X$ is the locally free $\widetilde{\mathcal{O}}_X$-module locally generated by $\widetilde{\partial}_{x_1}, \dots, \widetilde{\partial}_{x_n}$ (having degree 1, due to our convention in Section 5.1.3) and we have $[\widetilde{\partial}_{x_i}, f] = z\widetilde{\partial} f/\widetilde{\partial} x_i$ for any local section g of $\widetilde{\mathcal{O}}_X$; we also set $\widetilde{\Theta}_{X,k} = \wedge^k\widetilde{\Theta}_X$;
- $\widetilde{\Omega}^1_X$ is the locally free graded $\widetilde{\mathcal{O}}_X$-module $z^{-1}\mathbb{C}[z] \otimes_{\mathbb{C}} \Omega^1_X$, and $\widetilde{\Omega}^k_X = \wedge^k\widetilde{\Omega}^1_X$; the differential $\widetilde{\mathrm{d}}$ is induced by $1 \otimes \mathrm{d}$ on $\widetilde{\Omega}^k_X = z^{-k}\mathbb{C}[z] \otimes_{\mathbb{C}} \Omega^k_X$; we set $\widetilde{\omega}_X = \widetilde{\Omega}^n_X$; we regard the differential as a graded morphism of degree zero

$$\widetilde{\mathrm{d}} : \widetilde{\Omega}^k_X \longrightarrow \widetilde{\Omega}^{k+1}_X;$$

the local basis $(\widetilde{\mathrm{d}}x_i = z^{-1}\mathrm{d}x_i)_i$ (having degree -1) is dual to the basis $(\widetilde{\partial}_{x_i})_i$ of $\widetilde{\Theta}_X$.

- We also set $\widetilde{\mathcal{C}}^\infty_X := \mathcal{C}^\infty_X[z]$. This is a c-soft sheaf on the underlying C^∞ manifold $X_\mathbb{R}$.
- Contraction of a z-differential form of degree k by a z-vector field is defined as in Section 8.1.a.
- We have natural Lie algebra actions of $\widetilde{\Theta}_X$ on $\widetilde{\mathcal{O}}_X$ (action on the left) and on $\widetilde{\omega}_X$ (action on the right).

8.1.8. Example **(Filtered flat bundles).** Let $(\mathcal{L},\nabla)$ be a flat holomorphic bundle on X and let $F^\bullet\mathcal{L}$ be a decreasing filtration of $\mathcal{L}$ by $\mathcal{O}_X$-locally free sheaves. Then the flat connection ∇ endows $\mathcal{L}$ with the structure of a left $\mathcal{D}_X$-module (Proposition 8.1.4). The *Griffiths transversality property*

$$\nabla F^p\mathcal{L} \subset \Omega^1_X \otimes F^{p-1}\mathcal{L}, \quad \forall\, p \in \mathbb{Z} \tag{8.1.8 $*$}$$

is equivalent to the property that the corresponding increasing filtration $F_\bullet\mathcal{L}$ is an $F\mathcal{D}_X$-filtration of the $\mathcal{D}_X$-module $\mathcal{L}$.

8.1.9. Definition **(Connection).** Let $\widetilde{\mathcal{M}}$ be a graded $\widetilde{\mathcal{O}}_X$-module. A *connection* on $\widetilde{\mathcal{M}}$ is a graded $\widetilde{\mathbb{C}}$-linear morphism $\widetilde{\nabla} : \widetilde{\mathcal{M}} \to \widetilde{\Omega}^1_X \otimes \widetilde{\mathcal{M}}$ (of degree zero) which satisfies the Leibniz rule

$$\forall\, f \in \widetilde{\mathcal{O}}_X, \quad \widetilde{\nabla}(fm) = f\,\widetilde{\nabla}m + \widetilde{\mathrm{d}}\,f \otimes m.$$

Proposition 8.1.4 holds true in this filtered setting (Exercise 8.7).

8.1.10. Example. The fundamental examples of filtered left and right $\mathcal{D}_X$-modules are:

- $(\mathcal{O}_X, F_\bullet\mathcal{O}_X)$ with $\mathrm{gr}^F_p\mathcal{O}_X = 0$ for $p \neq 0$, so $R_F\mathcal{O}_X = \mathcal{O}_X[z]$,
- $(\omega_X, F_\bullet\omega_X)$ with $\mathrm{gr}^F_p\omega_X = 0$ for $p \neq -n$, so $R_F\omega_X = \widetilde{\omega}_X = \widetilde{\Omega}^n_X = z^{-n}\omega_X[z]$.

8.1.11. Convention. We will use the following convention.

(i) $\widetilde{\mathcal{O}}_X$ (resp. $\widetilde{\mathcal{C}}^\infty_X$) denotes either the sheaf rings $\mathcal{O}_X$ (resp. $\mathcal{C}^\infty_X$) or the sheaf of graded rings $\mathcal{O}_X[z] = R_F\mathcal{O}_X$ (resp. $\mathcal{C}^\infty_X[z]$), and $\mathsf{Mod}(\widetilde{\mathcal{O}}_X)$ denotes the category of $\mathcal{O}_X$-modules or that of *graded* $\mathcal{O}_X[z]$-modules.

(ii) The notation $\widetilde{\Theta}_X$, $\widetilde{\Omega}^k_X$, $\wedge^k\widetilde{\Theta}_X$ has a similar double meaning.

(iii) Similarly, $\widetilde{\mathcal{D}}_X$ denotes either the sheaf rings $\mathcal{D}_X$ or the sheaf of graded rings $R_F\mathcal{D}_X$, and $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ denotes the category of $\mathcal{D}_X$-modules or that of *graded* $R_F\mathcal{D}_X$-modules.

(iv) It will also be convenient to denote by $\widetilde{\mathbb{C}}$ either the field $\mathbb{C}$ or the *graded* ring $\mathbb{C}[z]$.

(v) In each of the second cases above, we will usually omit the word "graded", although it is always understood.

(vi) One recovers standard results for $\mathcal{D}_X$-modules by setting $z = 1$ and $\widetilde{\partial} = \partial$.

(vii) The strictness condition that we may consider (see Section 5.1.b) only refers to the second cases above, it is empty in the first cases.

8.2. Left and right

Considering left or right $\widetilde{\mathcal{D}}_X$-modules is not completely symmetric. The main reason is that the *left* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{O}}_X$ is a sheaf of rings, while its right analogue $\widetilde{\omega}_X := \widetilde{\Omega}^n_X$, is not a sheaf of rings. So for example the behaviour with respect to tensor products over $\widetilde{\mathcal{O}}_X$ is not the same for left and right $\widetilde{\mathcal{D}}$-modules. Also, the side changing functor defined below sends $\widetilde{\mathcal{D}}^{\mathrm{left}}_X$ to $\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, and not to $\widetilde{\mathcal{D}}_X$ regarded as a right $\widetilde{\mathcal{D}}_X$-module over itself.

8.2.1. Notation **(The category $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$).** The categories of left (resp. right) $\widetilde{\mathcal{D}}_X$-modules are denoted by $\mathsf{Mod}^{\mathrm{left}}(\widetilde{\mathcal{D}}_X)$ (resp. $\mathsf{Mod}^{\mathrm{right}}(\widetilde{\mathcal{D}}_X)$ with the convention that we implicitly consider graded modules and morphisms of degree zero in the case of $\widetilde{\mathcal{D}} = R_F\mathcal{D}$.

We analyze the relations between both categories in this section. *The main rule to be followed is that the side-changing functor changes a functor in the category of left objects to the functor denoted in the same way in the category of right objects, and conversely.*

Exercises 8.8 and 8.9 give the basic tools for generating left or right $\widetilde{\mathcal{D}}$-modules.

8.2.2. Example **(Example 8.1.10 continued).**

(1) $\widetilde{\mathcal{D}}_X$ is a left and a right $\widetilde{\mathcal{D}}_X$-module.

(2) $\widetilde{\mathcal{O}}_X$ is a left $\widetilde{\mathcal{D}}_X$-module (Exercise 8.10), with grading

$$(\widetilde{\mathcal{O}}_X)_p = \begin{cases} \mathcal{O}_X & \text{if } p \geqslant 0, \\ 0 & \text{if } p < 0. \end{cases}$$

(3) $\widetilde{\omega}_X := \widetilde{\Omega}^{\dim X}_X$ is a right $\widetilde{\mathcal{D}}_X$-module (Exercise 8.11), with grading

$$(\widetilde{\omega}_X)_p = \begin{cases} \omega_X & \text{if } p \geqslant -n, \\ 0 & \text{if } p < -n. \end{cases}$$

8.2.3. Definition **(Side-changing of $\widetilde{\mathcal{D}}_X$-modules).** Any left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{left}}$ gives rise to a right one $\widetilde{\mathcal{M}}^{\mathrm{right}}$ by setting $\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ and, for any vector field ξ and any function g,

$$(\omega \otimes m) \cdot f = f\omega \otimes m = \omega \otimes fm, \quad (\omega \otimes m) \cdot \xi = \omega\xi \otimes m - \omega \otimes \xi m.$$

Conversely, set $\widetilde{\mathcal{M}}^{\mathrm{left}} = \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{M}}^{\mathrm{right}})$, which also has in a natural way the structure of a left $\widetilde{\mathcal{D}}_X$-module (see Exercise 8.13(2)). The grading behaves as follows (see Example 8.1.10 and (5.1.4)):

$$\begin{aligned} \widetilde{\mathcal{M}}^{\mathrm{right}} &= z^{-n}\omega_X \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} = \omega_X \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}(-n), \\ \mathcal{M}^{\mathrm{right}}_p &= \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\mathrm{left}}_{p+n}. \end{aligned} \tag{8.2.3 $*$}$$

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ is the Rees module of a filtration, then the side-changing functor is written as

$$F_p\mathcal{M}^{\mathrm{right}} = F_p(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\mathrm{left}}) = \omega_X \otimes_{\mathcal{O}_X} F_{p+n}\mathcal{M}^{\mathrm{left}}. \tag{8.2.3 $**$}$$

8.2.4. Caveat. Let $\widetilde{\omega}_X^\vee = \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{O}}_X)$ as an $\widetilde{\mathcal{O}}_X$-module. One often finds in the literature the formula $\widetilde{\mathcal{M}}^{\mathrm{left}} = \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\omega}_X^\vee$, which give the $\widetilde{\mathcal{O}}_X$-module structure of $\widetilde{\mathcal{M}}^{\mathrm{left}}$. However, the left $\widetilde{\mathcal{D}}_X$-module structure is not obtained with a "tensor product formula" as in Exercise 8.12, but uses the interpretation as $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{M}}^{\mathrm{right}})$.

On the other hand, let U be a chart of X with coordinates $x_1, \dots, x_n$. Set $\widetilde{\mathrm{d}}\boldsymbol{x} = \widetilde{\mathrm{d}}x_1 \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n$. This is an $\widetilde{\mathcal{O}}_U$-basis of $\widetilde{\omega}_X$. Let $\widetilde{\mathrm{d}}\boldsymbol{x}^\vee$ denote the dual basis of $\widetilde{\omega}_X^\vee$. It is often convenient, for a right $\widetilde{\mathcal{D}}_U$-module $\widetilde{\mathcal{M}}^{\mathrm{right}}$, to write $\widetilde{\mathcal{M}}^{\mathrm{left}} = \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes \widetilde{\mathrm{d}}\boldsymbol{x}^\vee$ with the convention that a local section $\widetilde{\mathrm{d}}\boldsymbol{x}^\vee \otimes m$ is regarded as the morphism sending $\widetilde{\mathrm{d}}\boldsymbol{x}$ to m. In view of the duality between $\widetilde{\Omega}_X^1$ and $\widetilde{\Theta}_X$, one can identify $\widetilde{\omega}_X^\vee$ with $\wedge^n\widetilde{\Theta}_X$ and choose the local basis $\widetilde{\partial}_{\boldsymbol{x}}^{\wedge n} := \widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_n}$ of $\wedge^n\widetilde{\Theta}_U$. Both bases are related by $\widetilde{\mathrm{d}}\boldsymbol{x}^\vee = \varepsilon(n)\widetilde{\partial}_{\boldsymbol{x}}^{\wedge n}$. See also Exercise 8.17.

The following is obvious from Exercises 8.14 and 8.15.

8.2.5. Proposition. *The side-changing functors left-to-right and right-to-left are isomorphisms of between the categories of left and right graded $\widetilde{\mathcal{D}}_X$-modules, which are inverse one another. The left-to-right functor induces a twist $(-n)$, while the right-to-left functor induces a twist (n) ($n = \dim X$).* □

8.2.6. Remark. The ring $\widetilde{\mathcal{D}}_X$ considered as a right $\widetilde{\mathcal{D}}_X$-module over itself *is not equal* to the right $\widetilde{\mathcal{D}}_X$-module associated with $\widetilde{\mathcal{D}}_X$ regarded as a left $\widetilde{\mathcal{D}}_X$-module over itself by the side-changing functor.

***8.2.7. Caveat* (Side-changing and grading).** For a filtered left $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, side-changing and grading are related by the formula (according to example 8.2.2(3))

$$\mathrm{gr}^F(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \omega_X \otimes_{\mathcal{O}_X} \mathrm{gr}^F\mathcal{M}(-n),$$

as $\mathcal{O}_X$-modules. The action of $\mathrm{gr}^F\mathcal{D}_X$ is not exactly preserved by this isomorphism. Indeed, recall that, for a vector field ξ, we have $(\omega \otimes m)\xi = \omega\xi \otimes m - \omega \otimes \xi m$ and, taking classes in the suitable graded piece, we find $[\omega \otimes m][\xi] = -\omega \otimes [\xi m]$. We can thus write, as $\mathrm{gr}^F\mathcal{D}_X$-modules,

$$\mathrm{gr}^F(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \omega_X \otimes_{\mathcal{O}_X} \mathrm{inv}^*\mathrm{gr}^F\mathcal{M}(-n),$$

where $\mathrm{inv}^*\mathrm{gr}^F\mathcal{M}$ denotes the $\mathcal{O}_X$-module $\mathrm{gr}^F\mathcal{M}$ on which the action of $\mathrm{gr}^F\mathcal{D}_X$ is modified in such a way that $\mathrm{gr}^F_k\mathcal{D}_X$ acts by multiplying by $(-1)^k$ the usual action.

8.3. Examples of $\widetilde{\mathcal{D}}$-modules

We list here some classical examples of $\widetilde{\mathcal{D}}$-modules. One can get many other examples by applying various operations on $\widetilde{\mathcal{D}}$-modules.

8.3.1. ***Induced $\widetilde{\mathcal{D}}_X$-modules.*** Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. There is a very simple way to get a right $\widetilde{\mathcal{D}}_X$-module from $\widetilde{\mathcal{L}}$: consider $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ equipped with the natural right action of $\widetilde{\mathcal{D}}_X$. This is called an *induced* $\widetilde{\mathcal{D}}_X$-module. Although this construction is very simple, it is also very useful to get cohomological properties of $\widetilde{\mathcal{D}}_X$-modules. One can also consider the left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}$ (however, this is not the left $\widetilde{\mathcal{D}}_X$-module attached to the right one $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ by the side-changing functor of Definition 8.2.3).

8.3.2. ***(Meromorphic) $\mathcal{O}_X$-modules with integrable connection.*** One of the main geometrical examples of $\mathcal{D}_X$-modules are the vector bundles on X equipped with an *integrable* connection. Recall (Proposition 8.1.4) that left $\mathcal{D}_X$-modules are $\mathcal{O}_X$-modules with an integrable connection. Among them, the coherent $\mathcal{D}_X$-modules are of particular interest. One can show that such modules are $\mathcal{O}_X$-locally free, i.e., correspond to holomorphic vector bundles of finite rank on X.

It may happen that, for some X, such a category does not give any interesting geometric object. Indeed, if for instance X has a trivial fundamental group (e.g. $X = \mathbb{P}^1(\mathbb{C})$), then any vector bundle with integrable connection is isomorphic to the trivial bundle $\mathcal{O}_X$ with the connection d. However, on non simply connected Zariski open sets of X, there exist interesting vector bundles with connections. This leads to the notion of meromorphic vector bundle with connection.

Let D be a divisor in X and denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic functions on X with poles along D at most. This is a sheaf of left $\mathcal{D}_X$-modules, being an $\mathcal{O}_X$-module equipped with the natural connection $\mathrm{d} : \mathcal{O}_X(*D) \to \Omega^1_X(*D)$.

By definition, a *meromorphic bundle* is a locally free $\mathcal{O}_X(*D)$ module of finite rank. When it is equipped with an integrable connection, it becomes a left $\mathcal{D}_X$-module.

8.3.3. ***Twisted connections.*** One can *twist* the previous examples. Assume that ω is a *closed* holomorphic form on X. Define $\nabla : \mathcal{O}_X \to \Omega^1_X$ by the formula $\nabla = \mathrm{d} + \omega$. As ω is closed, ∇ is an integrable connection on the trivial bundle $\mathcal{O}_X$.

Usually, the nonzero closed form on X are meromorphic, with poles on some divisor D. Then ∇ is an integrable connection on $\mathcal{O}_X(*D)$.

If ω is exact, $\omega = \mathrm{d}g$ for some meromorphic function g on X, then ∇ can be written as $e^{-g} \circ \mathrm{d} \circ e^g$.

More generally, if $\mathcal{M}$ is any meromorphic bundle with an integrable connection ∇, then, for any such ω, $\nabla + \omega \,\mathrm{Id}$ defines a new $\mathcal{D}_X$-module structure on $\mathcal{M}$.

8.3.4. ***Filtered flat bundles.*** Contrary to what happens for $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules, which are automatically $\mathcal{O}_X$-locally free and correspond to vector bundles with integrable connection, $\widetilde{\mathcal{O}}_X$-coherent $\widetilde{\mathcal{D}}_X$-modules may not be $\widetilde{\mathcal{O}}_X$-locally free. We are mainly interested in $\widetilde{\mathcal{O}}_X$-locally free such objects. Let $\widetilde{\mathcal{M}}$ be one such. In particular, $\widetilde{\mathcal{M}}$ is $\widetilde{\mathbb{C}}$-flat, hence corresponds to a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$. Furthermore, $\mathcal{M} = \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is $\mathcal{O}_X$-coherent, hence is $\mathcal{O}_X$-locally free with an integrable connection ∇ satisfying the Griffiths transversality property with respect to $F_\bullet\mathcal{M}$. The $\widetilde{\mathcal{O}}_X$-coherency property implies that the (increasing) filtration is stationary and is a filtration by $\mathcal{O}_X$-coherent subsheaves. Lastly, the graded module $\widetilde{\mathcal{M}}/z\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{O}}_X/z\widetilde{\mathcal{O}}_X$-locally free, hence each graded component $\operatorname{gr}_p^F\mathcal{M}$ is $\mathcal{O}_X$-locally free, thus each $F_p\mathcal{M}$ also. For example, *variations of Hodge structure* in their holomorphic description of Section 4.1.a are $\widetilde{\mathcal{O}}_X$-locally free (of finite rank) $\widetilde{\mathcal{D}}_X$-modules.

8.3.5. *Distributions and currents.* Denote by $\mathfrak{Db}_X$ the sheaf of distributions on the complex manifold X of dimension n: given any open set U of X, $\mathfrak{Db}_X(U)$ is the space of distributions on U, which is by definition the weak dual of the space of C^∞ forms with compact support on U, of type (n,n). By Exercise 8.11, there is a right action of $\mathcal{D}_X$ on such forms. The left action of $\mathcal{D}_X$ on distributions is defined by adjunction: denote by $\langle\eta,u\rangle$ the natural pairing between a compactly supported C^∞-form η and a distribution u on U; let P be a holomorphic differential operator on U; define then $P\cdot u$ in such a way that, for every η, on has

$$\langle\eta, P\cdot u\rangle = \langle\eta\cdot P, u\rangle.$$

Given any distribution u on X, the subsheaf $\mathcal{D}_X\cdot u\subset\mathfrak{Db}_X$ is the $\mathcal{D}_X$-module generated by this distribution. Saying that a distribution is a solution of a family $P_1,\dots,P_k$ of differential equation is equivalent to saying that the morphism $\mathcal{D}_X\to\mathcal{D}_X\cdot u$ sending 1 to u induces a surjective morphism $\mathcal{D}_X/(P_1,\dots,P_k)\to\mathcal{D}_X\cdot u$.

Similarly, the sheaf $\mathfrak{C}_X$ of currents of degree 0 on X is defined in such a way that, for any open set $U\subset X$, $\mathfrak{C}_X(U)$ is dual to $C^\infty_{\mathrm{c}}(U)$ with a suitable topology. It is a right $\mathcal{D}_X$-module.

In local coordinates $x_1,\dots,x_n$, a current of degree 0 is nothing but a distribution times the volume form $dx_1\wedge\cdots\wedge dx_n\wedge d\overline{x}_1\wedge\cdots\wedge d\overline{x}_n$.

As we are now working with C^∞ forms or with currents, it is natural not to forget the anti-holomorphic part of these objects. Denote by $\mathcal{O}_{\overline{X}}$ the sheaf of anti-holomorphic functions on X and by $\mathcal{D}_{\overline{X}}$ the sheaf of anti-holomorphic differential operators. Then $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) are similarly left (resp. right) $\mathcal{D}_{\overline{X}}$-modules. Of course, the $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$ actions do commute, and they coincide when considering multiplication by constants.

The *conjugation* exchanges both structures. For example, if u is a distribution on U, its conjugate $\overline{u}$ is defined by the formula

$$\langle\eta,\overline{u}\rangle := \overline{\langle\overline{\eta},u\rangle}\qquad(\eta\in\mathcal{E}^{n,n}_{\mathrm{c}}(U)). \tag{8.3.0$\ast$}$$

This is of course compatible with the usual conjugation of L^1_{loc} functions.

It is therefore natural to introduce the following sheaves of rings:

$$\mathcal{O}_{X,\overline{X}} := \mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}},\qquad \mathcal{D}_{X,\overline{X}} := \mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_{\overline{X}}, \tag{8.3.0$\ast\ast$}$$

and consider $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) as left (resp. right) $\mathcal{D}_{X,\overline{X}}$-modules.

Operations on $\widetilde{\mathcal{D}}_X$-modules. One can construct new examples from old ones by using various operations.

• Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Then $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ has a natural structure of right $\widetilde{\mathcal{D}}_X$-module. Using a resolution $\widetilde{\mathcal{N}}^\bullet$ of $\widetilde{\mathcal{M}}$ by *left* $\widetilde{\mathcal{D}}_X$-modules which are acyclic for $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\bullet,\widetilde{\mathcal{D}}_X)$, one gets a right $\widetilde{\mathcal{D}}_X$-module structure on $\mathcal{E}xt^k_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ for $k\geqslant 0$.

• Given two left (resp. a left and a right) $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$, the same argument enables one to put on the various $\mathcal{T}or_{i,\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{N}},\widetilde{\mathcal{M}})$ a left (resp. a right) $\widetilde{\mathcal{D}}_X$-module structure.

8.4. The de Rham functor

***8.4.1. Definition* (de Rham).** For a *left* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the de Rham complex $\mathrm{DR}\,\widetilde{\mathcal{M}}$ is the bounded complex (with $\bullet$ in degree zero and all nonzero terms in non-negative degrees)

$$\mathrm{DR}\,\widetilde{\mathcal{M}} := \{0\to \underset{\bullet}{\widetilde{\mathcal{M}}} \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^1_X\otimes\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\nabla}} \cdots \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^n_X\otimes\widetilde{\mathcal{M}}\to 0\}.$$

The terms are the $\widetilde{\mathcal{O}}_X$-modules $\widetilde{\Omega}^\bullet_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}^{\mathrm{left}}$ and the differentials the $\widetilde{\mathbb{C}}$-linear morphisms $\widetilde{\nabla}$ defined in Exercise 8.6 or 8.7.

The *shifted* de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is defined as

$${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}} := \{0\to \widetilde{\mathcal{M}} \xrightarrow{(-1)^n\widetilde{\nabla}} \widetilde{\Omega}^1_X\otimes\widetilde{\mathcal{M}} \xrightarrow{(-1)^n\widetilde{\nabla}} \cdots \xrightarrow{(-1)^n\widetilde{\nabla}} \widetilde{\Omega}^n_X\underset{\bullet}{\otimes}\widetilde{\mathcal{M}}\to 0\}.$$

The previous definition produces a *complex* since $\widetilde{\nabla}\circ\widetilde{\nabla}=0$, according to the integrability condition on $\widetilde{\nabla}$, as remarked in Exercise 8.6 or 8.7. The notation ${}^{\mathrm{p}}\mathrm{DR}$ is motivated by the property that, for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, the complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is a perverse sheaf (a theorem of Kashiwara).

***8.4.2. Remark* (Shift of a complex).** Given a complex $(C^\bullet,\widetilde{\delta})$, the shifted complex $(C^\bullet,\widetilde{\delta})[n]$ is the complex $(C^{n+\bullet},(-1)^n\widetilde{\delta})$. Thus the complex ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is equal to $\mathrm{DR}\,\widetilde{\mathcal{M}}[n]$. The shifted de Rham complex is implicitly considered in Formula (8.1.1). *In the following, given a complex $(C^\bullet,\widetilde{\delta})$, we will also denote by $C^{n+\bullet}$ the shifted complex $(C^\bullet,\widetilde{\delta})[n]$ when there is no doubt about the differential.*

***8.4.3. Definition* (Spencer).** The *Spencer complex* $\mathrm{Sp}(\widetilde{\mathcal{M}})$ of a *right* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is the bounded complex (with $\bullet$ in degree zero and all nonzero terms in non-positive degrees; recall also the notation $\widetilde{\Theta}_{X,k}=\wedge^k\widetilde{\Theta}_X$)

$$\mathrm{Sp}(\widetilde{\mathcal{M}}) := \{0\to\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,n}\xrightarrow{\widetilde{\delta}_{\widetilde{\mathcal{M}}}}\cdots\xrightarrow{\widetilde{\delta}_{\widetilde{\mathcal{M}}}}\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,1}\xrightarrow{\widetilde{\delta}_{\widetilde{\mathcal{M}}}}\underset{\bullet}{\widetilde{\mathcal{M}}}\to 0\},$$

where the differential $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ is the $\widetilde{\mathbb{C}}$-linear map given by

$$\begin{aligned} m\otimes(\xi_1\wedge\cdots\wedge\xi_k) \xmapsto{\widetilde{\delta}_{\widetilde{\mathcal{M}}}} &\sum_{i=1}^k(-1)^{i-1}(m\xi_i)\otimes(\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\xi_k)\\ &+\sum_{i<j}(-1)^{i+j}m\otimes\big([\xi_i,\xi_j]\wedge\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\widehat{\xi_j}\wedge\cdots\wedge\xi_k\big), \end{aligned}$$

where $\widehat{\xi_i}$ means that we omit ξ_i in the wedge product.

Of special interest will be, of course, the de Rham or Spencer complex of the ring $\widetilde{\mathcal{D}}_X$, considered as a left or right $\widetilde{\mathcal{D}}_X$-module. Notice that in $\mathrm{DR}(\widetilde{\mathcal{D}}_X)$ the differentials are *right* $\widetilde{\mathcal{D}}_X$-linear, and in $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ they are *left* $\widetilde{\mathcal{D}}_X$-linear. See Exercises 8.21–8.24 for some of their properties.

8.4.4. Remark.

(1) For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the complex $\mathrm{Sp}(\widetilde{\mathcal{M}})$ is isomorphic to $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ (Exercise 8.24). It is then possible to prove some statements on $\mathrm{Sp}(\widetilde{\mathcal{M}})$ by only considering the case where $\widetilde{\mathcal{M}}=\widetilde{\mathcal{D}}_X$.

(2) For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, it is usual to find in the literature the definition of the unshifted de Rham complex $\mathrm{DR}\,\widetilde{\mathcal{M}}$ as $\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X\widetilde{\mathcal{M}})$ (in a suitable derived category). Since $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ by locally free $\widetilde{\mathcal{D}}_X$-modules, this isomorphism amounts to the isomorphism $\mathrm{DR}\,\widetilde{\mathcal{M}}\simeq\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X),\widetilde{\mathcal{M}})$. This is shown in Exercise 8.25.

Side-changing. Given any $k\geqslant 0$, the *contraction* is the morphism (see Section 8.1.a)

$$\begin{aligned}\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}&\xrightarrow{\ \lrcorner\ }\widetilde{\Omega}_X^{n-k}\\ \omega\otimes\xi&\longmapsto(\xi\lrcorner\,\omega)(\bullet)=\omega(\xi\wedge\bullet).\end{aligned}\tag{8.4.5}$$

8.4.6. Example. In local coordinates $(x_1,\dots,x_n)$, let us set $\widetilde{\mathrm{d}}\boldsymbol{x}=\widetilde{\mathrm{d}}x_1\wedge\cdots\wedge\widetilde{\mathrm{d}}x_n$. For $i=1,\dots,k\leqslant n$, let us set $\widetilde{\partial}_{\widehat{x_i}}:=\widetilde{\partial}_{x_1}\wedge\cdots\wedge\widehat{\widetilde{\partial}_{x_i}}\wedge\cdots\wedge\widetilde{\partial}_{x_k}$ (i.e., omitting $\widetilde{\partial}_{x_i}$ in the wedge product) for simplicity. Then the following formulas hold, for $k\leqslant n$:

$$(\widetilde{\partial}_{x_1}\wedge\cdots\wedge\widetilde{\partial}_{x_n})\lrcorner\,\widetilde{\mathrm{d}}\boldsymbol{x}=\varepsilon(n),$$

$$(\widetilde{\partial}_{x_1}\wedge\cdots\wedge\widetilde{\partial}_{x_k})\lrcorner\,\widetilde{\mathrm{d}}\boldsymbol{x}=\varepsilon(n)\,\varepsilon(n-k)\widetilde{\mathrm{d}}x_{k+1}\wedge\cdots\wedge\widetilde{\mathrm{d}}x_n,\tag{8.4.6$*$}$$

$$\widetilde{\partial}_{\widehat{x_i}}\lrcorner\,\widetilde{\mathrm{d}}\boldsymbol{x}=(-1)^{k-i}\,\varepsilon(n)\,\varepsilon(n-k+1)\widetilde{\mathrm{d}}x_i\wedge\widetilde{\mathrm{d}}x_{k+1}\wedge\cdots\wedge\widetilde{\mathrm{d}}x_n.\tag{8.4.6$**$}$$

8.4.7. Lemma. *There exists a natural isomorphism of complexes of right $\widetilde{\mathcal{D}}_X$-modules (i.e., is compatible with the differentials of these complexes)*

$$\iota:\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\xrightarrow{\ \sim\ }{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)$$

which induces the identity

$$\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}^0(\widetilde{\mathcal{D}}_X)=\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X=\mathrm{DR}^n\,\widetilde{\mathcal{D}}_X.$$

It is induced by the isomorphisms of right $\widetilde{\mathcal{D}}_X$-modules

$$\begin{aligned}\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\big(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}\big)&\xrightarrow[\sim]{\ \iota\ }\widetilde{\Omega}_X^{n-k}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X\\ \big[\omega\otimes(1\otimes\xi)\big]\cdot P&\longmapsto(\xi\lrcorner\,\omega)\otimes P\end{aligned}$$

(where the right structure of the right-hand term is the natural one and that of the left-hand term is nothing but that induced by the natural left structure of $\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}$ by side-changing).

Proof. It is enough to prove that the diagram

$$\begin{array}{ccc}\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}\big) & \xrightarrow{\ \iota\ } & \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \\ {\scriptstyle\widetilde{\delta}}\big\downarrow & & \big\downarrow{\scriptstyle(-1)^n\widetilde{\nabla}} \\ \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k-1}\big) & \xrightarrow{\ \iota\ } & \widetilde{\Omega}_X^{n-k+1} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\end{array}$$

commutes. We will make use of the relations satisfied by the function ε (see Notation 0.2). It is also enough to check this locally, and, in local coordinates $(x_1, \dots, x_n)$, we are reduced by right $\widetilde{\mathcal{D}}_X$-linearity to checking this on sections of the form $\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes (\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}))$. We use the notation of Example 8.4.6.

On the one hand, we have $\widetilde{\mathrm{d}}\boldsymbol{x} \cdot \widetilde{\partial}_{x_i} = 0$ and, according to (8.4.6∗∗) we find

$$\begin{aligned}\widetilde{\delta}\big[\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes (\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}))\big] &= \sum_{i=1}^k (-1)^{i-1} \widetilde{\mathrm{d}}\boldsymbol{x} \otimes (\widetilde{\partial}_{x_i} \otimes \widetilde{\partial}_{\widehat{x_i}}) \\ &= \sum_{i=1}^k (-1)^i \big[\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes \widetilde{\partial}_{\widehat{x_i}})\big] \cdot \widetilde{\partial}_{x_i} \\ &\xrightarrow{\ \iota\ } \sum_{i=1}^k (-1)^i (\widetilde{\partial}_{\widehat{x_i}} \lrcorner\, \widetilde{\mathrm{d}}\boldsymbol{x}) \otimes \widetilde{\partial}_{x_i} \\ &= (-1)^k\, \varepsilon(n)\, \varepsilon(n-k+1) \sum_{i=1}^k (\widetilde{\mathrm{d}}x_i \wedge \widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n) \otimes \widetilde{\partial}_{x_i}.\end{aligned}$$

On the other hand, we have, according to (8.4.6∗) (see Exercises 8.5 and 8.7),

$$\begin{aligned}&(-1)^n \widetilde{\nabla}\iota\big[\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes (\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}))\big] \\ &\qquad = (-1)^n \widetilde{\nabla}\big[(\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}) \lrcorner\, \widetilde{\mathrm{d}}\boldsymbol{x}) \otimes 1\big] \\ &\qquad = (-1)^n\, \varepsilon(n)\, \varepsilon(n-k) \widetilde{\nabla}\big[(\widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n) \otimes 1\big] \\ &\qquad = (-1)^n\, \varepsilon(n)\, \varepsilon(n-k) \sum_{i=1}^k (-1)^{n-k} (\widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n \wedge \widetilde{\mathrm{d}}x_i) \otimes \widetilde{\partial}_{x_i} \\ &\qquad = (-1)^n\, \varepsilon(n)\, \varepsilon(n-k) \sum_{i=1}^k (\widetilde{\mathrm{d}}x_i \wedge \widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n) \otimes \widetilde{\partial}_{x_i}.\end{aligned}$$

and the desired equality follows from the relation $\varepsilon(n-k+1) = (-1)^{n-k}\, \varepsilon(n-k)$. □

Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ the associated right module. We will now compare ${}^{\mathrm{p}}\mathrm{DR}_X(\widetilde{\mathcal{M}})$ and $\mathrm{Sp}(\widetilde{\mathcal{M}}^{\mathrm{right}})$. We will denote by ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})$ the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{M}}^{\mathrm{right}})$ and we keep the notation ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})$ for the de Rham complex of a left $\widetilde{\mathcal{D}}_X$-module. Exercise 8.26 gives an isomorphism

$$\text{(8.4.8)}\qquad {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\ \sim\ } {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}).$$

8.4.9. The grading of ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$**.** In the left and right case, ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is a bounded complex of sheaves of graded $\widetilde{\mathbb{C}}$-modules and the isomorphism (8.4.8) is an isomorphism as such (i.e., preserves the grading). Indeed, we note that, for $k \geqslant 0$, $\widetilde{\Omega}_X^k$ (resp. $\widetilde{\Theta}_{X,k}$) is homogeneous of degree $-k$ (resp. k); therefore, the degree p component of ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is the complex of $\mathbb{C}$-vector spaces

$$({}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{left}})_p := \{0 \to \mathcal{M}_p^{\mathrm{left}} \longrightarrow \Omega_X^1 \otimes \mathcal{M}_{p+1}^{\mathrm{left}} \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes \underset{\bullet}{\mathcal{M}_{p+n}^{\mathrm{left}}} \to 0\} \cdot z^p,$$

$$({}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{right}})_p := \{0 \to \mathcal{M}_{p-n}^{\mathrm{right}} \otimes \Theta_{X,n} \to \cdots \to \mathcal{M}_{p-1}^{\mathrm{right}} \otimes \Theta_{X,1} \to \underset{\bullet}{\mathcal{M}_p^{\mathrm{right}}} \to 0\} \cdot z^p,$$

and the side-changing functors preserve the grading (see (8.2.3 $*$)). If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ is the Rees module of an F-filtered $\mathcal{D}_X$-module $\mathcal{M}$, we regard ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ as the Rees complex of the filtered complex

$$F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}} := \{0 \to F_p\mathcal{M}^{\mathrm{left}} \to \Omega_X^1 \otimes F_{p+1}\mathcal{M}^{\mathrm{left}} \to \cdots \to \Omega_X^n \otimes \underset{\bullet}{F_{p+n}}\mathcal{M}^{\mathrm{left}} \to 0\},$$

$$F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{right}} := \{0 \to F_{p-n}\mathcal{M}^{\mathrm{right}} \otimes \Theta_{X,n} \to \cdots \to F_{p-1}\mathcal{M}^{\mathrm{right}} \otimes \Theta_{X,1} \to \underset{\bullet}{F_p\mathcal{M}^{\mathrm{right}}} \to 0\}.$$

Recall that the side-changing functor for filtered $\mathcal{D}_X$-modules (8.2.3 $**$) amounts to

$$F_p\mathcal{M}^{\mathrm{right}} = \omega_X \otimes F_{p+n}\mathcal{M}^{\mathrm{left}}.$$

Exercise 8.24 clearly shows that ${}^{\mathrm{p}}\mathrm{DR}$ is a functor from the category of left (resp. right) $\widetilde{\mathcal{D}}_X$-modules to the category of bounded complex of sheaves of $\widetilde{\mathbb{C}}$-modules. It can be extended to a functor between the corresponding bounded derived categories.

***8.4.10. Definition* (Contraction by a one-form).** The contraction morphism

$$\widetilde{\Theta}_{X,k} \otimes \widetilde{\Omega}_X^1 \xrightarrow{\;\lrcorner\;} \widetilde{\Theta}_{X,k-1}$$

is the unique morphism such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\omega}_X \otimes \widetilde{\Theta}_{X,k} \otimes \widetilde{\Omega}_X^1 & \xrightarrow{\mathrm{Id}\otimes\lrcorner} & \widetilde{\omega}_X \otimes \widetilde{\Theta}_{X,k-1} \\ \downarrow & & \downarrow \\ \widetilde{\Omega}_X^{n-k} \otimes \widetilde{\Omega}_X^1 & \xrightarrow{\;\wedge\;} & \widetilde{\Omega}_X^{n-k+1} \end{array}$$

where the vertical morphisms are induced by (8.4.5), i.e., $\widetilde{\omega}(\xi \lrcorner \eta^1) = \widetilde{\omega}(\xi) \wedge \eta^1$.

8.4.11. Action of a closed one-form on the de Rham complex. Let η be a *closed* holomorphic one-form on X. Then the exterior product by η induces a morphism

$$\eta \wedge \bullet : {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})[1].$$

Indeed, for a local section m of $\widetilde{\mathcal{M}}$ and a k-form ω, we have

$$\widetilde{\nabla}((\eta \wedge \omega) \otimes m) = (\widetilde{\mathrm{d}}\eta \wedge \omega) \otimes m - \eta \wedge \widetilde{\nabla}(\omega \otimes m) = -\eta \wedge \widetilde{\nabla}(\omega \otimes m),$$

so that the morphism $\eta\,\wedge$ commutes with the differentials (see Remark 8.4.2).

According to Lemma 8.4.7, we can define the contraction

$$\bullet \lrcorner\, \eta : \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \longrightarrow \mathrm{Sp}(\widetilde{\mathcal{D}}_X)[1]$$

as the unique morphism which corresponds to $\eta\wedge\bullet$ on ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)$ via ι. According to Remark 8.4.4(1), we can define in a similar way a morphism of complexes

$$\bullet \lrcorner\, \eta : {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})[1]. \tag{8.4.12}$$

Note that, if $\eta = \widetilde{\mathrm{d}}f$ is *exact*, then the induced morphism

$$\eta\wedge\ : H^i\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}) \longrightarrow H^{i+1}\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})$$

is zero. Indeed, if a local section μ of $\widetilde{\Omega}^k_X\otimes\widetilde{\mathcal{M}}^{\mathrm{left}}$ satisfies $\widetilde{\nabla}\mu=0$, then $\widetilde{\mathrm{d}}f\wedge\mu=\widetilde{\nabla}(f\mu)$. In other words, the morphism $\eta\wedge$ on the cohomology only depends on the class of η in $H^1\Gamma(X,(\widetilde{\Omega}^\bullet_X,\widetilde{\mathrm{d}}))$. The same result holds when we make η acting on the complex $\Gamma(X,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}))$, and a similar result holds for the action $\bullet\lrcorner\,\eta$ on ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})$.

8.4.13. C^∞ de Rham and Spencer complexes. Let us denote by $(\widetilde{\mathcal{E}}^{(\bullet,0)}_X,\widetilde{\mathrm{d}}')$ the complex $\widetilde{\mathcal{C}}^\infty_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Omega}^\bullet_X$ with the differential induced by $\widetilde{\mathrm{d}}$ (here, we assume $\bullet\geqslant 0$). More generally, let us set

$$\widetilde{\mathcal{E}}^{(p,q)}_X=\widetilde{\Omega}^p_X\wedge\widetilde{\mathcal{E}}^{(0,q)}_X=\widetilde{\mathcal{E}}^{(p,0)}_X\wedge\widetilde{\mathcal{E}}^{(0,q)}_X$$

and let d'' be the (usual) anti-holomorphic differential. For every p, the complex $(\widetilde{\mathcal{E}}^{(p,\bullet)}_X,\mathrm{d}'')$ is a resolution of $\widetilde{\Omega}^p_X$ (note that, here, d'' is not affected by z, hence is homogeneous of degree zero with respect to the grading). We therefore have a complex $(\widetilde{\mathcal{E}}^\bullet_X,\widetilde{\mathrm{d}})$, which is the single complex associated to the double complex $(\widetilde{\mathcal{E}}^{(\bullet,\bullet)}_X,\widetilde{\mathrm{d}}',\mathrm{d}'')$. In particular, since $\widetilde{\mathcal{D}}_X$ is $\widetilde{\mathcal{O}}_X$-locally free, we have a natural quasi-isomorphism of complexes of right $\widetilde{\mathcal{D}}_X$-modules:

$$(\widetilde{\Omega}^\bullet_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,\widetilde{\nabla})\xrightarrow{\ \sim\ }(\widetilde{\mathcal{E}}^\bullet_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,\widetilde{D})=:\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X),\quad \widetilde{D}:=\mathrm{Id}\otimes\widetilde{\nabla}+\mathrm{d}''\otimes\mathrm{Id},$$

by sending holomorphic k-forms to $(k,0)$-forms. Given a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{left}}$, we can define similarly the C^∞ de Rham complex

$${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}}):=(\widetilde{\mathcal{E}}^{n+\bullet}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}^{\mathrm{left}},\widetilde{D}),\quad \widetilde{D}:=(-1)^n(\mathrm{Id}\otimes\widetilde{\nabla}+\mathrm{d}''\otimes\mathrm{Id}).$$

As in Exercise 8.24(2), by using that $(\widetilde{\mathcal{E}}^{n+\bullet}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,\widetilde{D})$ is a complex of right $\widetilde{\mathcal{D}}_X$-modules, we obtain a quasi-isomorphism:

$${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}^{\mathrm{left}}\xrightarrow{\ \sim\ }{}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}}).$$

From the commutative diagram

$$\begin{array}{ccc}
{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}^{\mathrm{left}} & \xrightarrow{\ \sim\ } & {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})\\
\wr\big\downarrow & & \big\downarrow\\
{}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}^{\mathrm{left}} & \xrightarrow{\ \sim\ } & {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}})
\end{array}$$

we conclude that the right vertical morphism is a quasi-isomorphism.

We can argue similarly for defining the C^∞ Spencer complex of a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{right}}$. We resolve

$$\widetilde{\Theta}_{X,k}\xrightarrow{\ \sim\ }(\widetilde{\Theta}_{X,k}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{E}}^{(0,\bullet)}_X,\mathrm{Id}\otimes\mathrm{d}'').$$

Let us set, for each $\ell \in \mathbb{Z}$,

$$\widetilde{\mathrm{Sp}}_X^{\infty,\ell} = \bigoplus_{j-i=\ell} (\widetilde{\Theta}_{X,i} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}_X^{(0,j)}).$$

For any right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{right}}$, we define

$${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}) := (\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}_X^{\infty,\bullet}, \widetilde{\delta}^\infty_{\widetilde{\mathcal{M}}}),$$

where the differential $\widetilde{\delta}^\infty_{\widetilde{\mathcal{M}}}$ is defined in Exercise 8.28. We will use the notation $\mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X)$ for the C^∞ Spencer complex of $\widetilde{\mathcal{D}}_X$ with its right structure. Then, arguing as in Exercise 8.24(1), we obtain a quasi-isomorphism

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}),$$

from which we deduce as above a quasi-isomorphism

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}).$$

Recall that $\widetilde{\mathcal{C}}_X^\infty$ is flat over $\widetilde{\mathcal{O}}_X$, hence so are $\widetilde{\mathcal{E}}_X^k$ and $\widetilde{\mathrm{Sp}}_X^{\infty,\ell}$. The terms of ${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X)$ and $\mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X)$ are flat over $\widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{D}}_X$, and are c-soft sheaves, so that any short exact sequence $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ gives rise to an exact sequence of the corresponding C^∞ de Rham complexes, which consist of c-soft sheaves.

Moreover, by Exercise 8.28, if $\widetilde{\mathcal{M}}^{\mathrm{right}}$ corresponds to $\widetilde{\mathcal{M}}^{\mathrm{left}}$ by side-changing, then ${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}})$.

8.5. Induced $\widetilde{\mathcal{D}}$-modules

A subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ proves very useful in many places, namely that of *induced right $\widetilde{\mathcal{D}}_X$-modules*. Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. It induces a *right* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, called an *induced right $\widetilde{\mathcal{D}}_X$-module.*

8.5.1. Remark. We note that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has two structures of $\widetilde{\mathcal{O}}_X$-module, one coming from the action on $\widetilde{\mathcal{L}}$ and the other one from the right $\widetilde{\mathcal{D}}_X$-module structure, and they do not coincide. We will mainly use the right one. The "left" $\widetilde{\mathcal{O}}_X$-module structure on $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ will only be used when noticing that some naturally defined sheaves of $\widetilde{\mathbb{C}}$-vector spaces are in fact sheaves of $\widetilde{\mathcal{O}}_X$-modules. On the other hand, $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has a canonical structure of right $\widetilde{\mathcal{D}}_X$-module.

The category $\mathsf{Mod}_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ of right induced differential modules is the full subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ consisting of induced $\widetilde{\mathcal{D}}_X$-modules (i.e., we consider as morphisms all $\widetilde{\mathcal{D}}_X$-linear morphisms). It is an additive category (but not an abelian category).

***8.5.2. Proposition* (The canonical resolution by induced $\widetilde{\mathcal{D}}_X$-modules)**
Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Then the complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is isomorphic to a complex of right induced $\widetilde{\mathcal{D}}_X$-modules which is a resolution of $\widetilde{\mathcal{M}}$ as such.

One should not confuse $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ with $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}(\widetilde{\mathcal{M}})$ as in Exercise 8.24(1), where a tensor product over $\widetilde{\mathcal{D}}_X$ is considered. A good preliminary for the following proof is Exercise 8.29.

Proof. (See Exercise 8.31 for a detailed proof.) That the terms of the complex are induced $\widetilde{\mathcal{D}}_X$-modules follows from Exercise 8.19(4) applied to $\widetilde{\mathcal{L}} = \widetilde{\Theta}_{X,k}$. Since $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a $\widetilde{\mathcal{D}}_X$-module, hence as an $\widetilde{\mathcal{O}}_X$-module, and since the terms of $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ are $\widetilde{\mathcal{O}}_X$-locally free, we conclude that $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{M}}$. □

Let $\mathsf{C}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ the category of $\star$-bounded complexes of the additive category $\mathsf{Mod}_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ and let $\mathsf{K}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ be the corresponding homotopy category. Since $\mathrm{Sp}\,\widetilde{\mathcal{D}}_X$ is a complex of locally free $\widetilde{\mathcal{O}}_X$-modules, the functor $\widetilde{\mathcal{M}}^\bullet \to \widetilde{\mathcal{M}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\,\widetilde{\mathcal{D}}_X$ is a functor of triangulated categories, and sends acyclic complexes to acyclic complexes according to the previous proposition. It induces therefore a functor $\mathsf{D}^\star(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$.

***8.5.3. Corollary* (Equivalence of $\mathsf{D}^\star(\widetilde{\mathcal{D}}_X)$ with $\mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$).** *The natural functor $\mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)\to \mathsf{D}^\star(\widetilde{\mathcal{D}}_X)$ is an equivalence of categories, and the functor $\mathsf{D}^\star(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ induced by $\widetilde{\mathcal{M}}^\bullet \mapsto \widetilde{\mathcal{M}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\,\widetilde{\mathcal{D}}_X$ is a quasi-inverse functor.* □

8.6. Pullback and external product of $\widetilde{\mathcal{D}}$-modules

8.6.a. Pullback of left $\widetilde{\mathcal{D}}$-modules. Let us begin with some relative complements to Section 8.2. Let $f : X \to Y$ be a holomorphic map between analytic manifolds. For any local section ξ of the sheaf $\widetilde{\Theta}_X$ of z-vector fields on X, $Tf(\xi)$ is a local section of $\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\Theta}_Y$. We hence have an $\widetilde{\mathcal{O}}_X$-linear map

$$Tf : \widetilde{\Theta}_X \longrightarrow \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\Theta}_Y,$$

and dually

$$T^*f : \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} \widetilde{\Omega}^1_Y \longrightarrow \widetilde{\Omega}^1_X.$$

Therefore, if $\widetilde{\mathcal{N}}$ is any left $\widetilde{\mathcal{D}}_Y$-module, the connection $\widetilde{\nabla}^Y$ on $\widetilde{\mathcal{N}}$ can be lifted as a connection

$$\widetilde{\nabla}^X : f^*\widetilde{\mathcal{N}} := \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}} \longrightarrow \widetilde{\Omega}^1_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}} = \widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} f^*\widetilde{\mathcal{N}}$$

by setting

$$\widetilde{\nabla}^X = \widetilde{\mathrm{d}} \otimes \mathrm{Id} + (T^*f \otimes \mathrm{Id}_{\widetilde{\mathcal{N}}}) \circ (1 \otimes \widetilde{\nabla}^Y). \tag{8.6.1}$$

8.6.2. Lemma. *The connection $\widetilde{\nabla}^X$ on $f^*\widetilde{\mathcal{N}}$ is integrable and defines the structure of a left $\widetilde{\mathcal{D}}_X$-module on $\widetilde{\mathcal{N}}$.*

Proof. Exercise 8.32(1). □

This leads to the first definition of the pullback functor for $\widetilde{\mathcal{D}}_Y$-modules.

8.6.3. Definition. The left $\widetilde{\mathcal{D}}_X$-module corresponding to $(f^*\widetilde{\mathcal{N}}, \widetilde{\nabla}_X)$ is the pullback of $\widetilde{\mathcal{N}}$ in the sense of $\widetilde{\mathcal{D}}$-modules, and is denoted ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}$.

However, this definition is not suited for considering derived inverse images, since the sheaves $\mathcal{T}or_j^{f^{-1}\widetilde{\mathcal{O}}_Y}(\widetilde{\mathcal{O}}_X, f^{-1}\widetilde{\mathcal{N}})$ are not obviously equipped with an integrable connection. In order to overcome this difficulty, we introduce the transfer modules.

***8.6.4. Definition* (Transfer modules).**

(1) The sheaf
$$\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{D}}_Y$$
is a left-right $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodule when using the natural right $f^{-1}\widetilde{\mathcal{D}}_Y$-module structure and the left $\widetilde{\mathcal{D}}_X$-module introduced above (see Exercise 8.32(2)). It has a canonical section $\mathbf{1}$.

Correspondingly, we have $F_p\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}F_p\mathcal{D}_Y$ and the previous definition reads $R_F\mathcal{D}_{X\to Y} = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}R_F\mathcal{D}_Y$ (with $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z]$ and $\widetilde{\mathcal{O}}_Y = \mathcal{O}_Y[z]$).

(2) The sheaf $\widetilde{\mathcal{D}}_{Y\leftarrow X}$ is obtained from $\widetilde{\mathcal{D}}_{X\to Y}$ by using the usual side-changing functor on both sides:
$$\widetilde{\mathcal{D}}_{Y\leftarrow X} = \mathcal{H}om_{f^{-1}\widetilde{\mathcal{O}}_Y}\big(\widetilde{\omega}_Y, \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}\big).$$
In the filtered/graded setting, this definition reads
$$F_p\mathcal{D}_{Y\leftarrow X} = \mathcal{H}om_{f^{-1}\mathcal{O}_Y}\big(\omega_Y, \omega_X \otimes_{\mathcal{O}_X} F_{p+n-m}\mathcal{D}_{X\to Y}\big).$$

8.6.5. Example.

(1) One recovers $\widetilde{\mathcal{D}}_X$ as $\widetilde{\mathcal{D}}_{X\to X}$ for the identity map $\mathrm{Id} : X \to X$, so that $\widetilde{\mathcal{D}}_{X\leftarrow X}$ is identified with $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}\big(\widetilde{\omega}_X, \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\big)$.

(2) On the other hand, if Y is reduced to a point, so that f^{-1} is the constant map, we have $\widetilde{\mathcal{D}}_{X\to \mathrm{pt}} = \widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{D}}_{X\leftarrow \mathrm{pt}} = \widetilde{\omega}_X$.

We can now give a better definition of the pullback of a left $\widetilde{\mathcal{D}}_Y$-module $\widetilde{\mathcal{N}}$, better in the sense that it is defined inside of the category of $\widetilde{\mathcal{D}}$-modules. It also enables one to give a definition of a derived inverse image. The coincidence between both definitions can be obtained by Exercise 8.39.

***8.6.6. Definition* (of the pullback of a left $\widetilde{\mathcal{D}}_Y$-module).** Let $\widetilde{\mathcal{N}}$ be a left $\widetilde{\mathcal{D}}_Y$-module. The pullback ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}$ is the left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{D}}_{X\to Y} \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{N}}$.

The *derived pullback* ${}_{\mathrm{D}}f^{*}\widetilde{\mathcal{N}}$ is now defined by the usual method, i.e., by taking a flat resolution of $\widetilde{\mathcal{N}}$ as a left $\widetilde{\mathcal{D}}_Y$-module, or by taking a right $f^{-1}\widetilde{\mathcal{D}}_Y$-flat resolution of $\widetilde{\mathcal{D}}_{X\to Y}$ by $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodules. The cohomology modules ${}_{\mathrm{D}}f^{*(j)}\widetilde{\mathcal{N}} := \mathcal{T}or_j^{f^{-1}\widetilde{\mathcal{D}}_Y}(\widetilde{\mathcal{D}}_{X\to Y}, f^{-1}\widetilde{\mathcal{N}})$ are left $\widetilde{\mathcal{D}}_X$-modules.

8.6.7. Remark. If $f : X \to Y$ is a *smooth* morphism, that is, locally expressed as the projection of a product, or more generally a *flat* morphism, i.e., having equidmensional fibers (since both X and Y are smooth), then for any left $\widetilde{\mathcal{D}}_Y$-module $\widetilde{\mathcal{N}}$, we have ${}_{\mathrm{D}}f^{*}\widetilde{\mathcal{N}} = {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}$, i.e., ${}_{\mathrm{D}}f^{*(j)}\widetilde{\mathcal{N}} = 0$ for $j \neq 0$. Indeed, in such a case, $\widetilde{\mathcal{D}}_{X\to Y}$ is $f^{-1}\widetilde{\mathcal{D}}_Y$-flat (Exercise 8.36). Moreover, if $\widetilde{\mathcal{N}}$ is *strict* (see Definition 5.1.6), then so is

${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}$: indeed, the assumption amounts to the injectivity of $z:\widetilde{\mathcal{N}}\to\widetilde{\mathcal{N}}$, which is preserved after flat base change.

We sometimes use the notation ${}_{\mathrm{D}}f^{*}$ instead of ${}_{\mathrm{D}}f^{*(0)}$.

8.6.8. Side-changing and pullback. The pullback for a right $\widetilde{\mathcal{D}}_Y$-module $\widetilde{\mathcal{N}}^{\mathrm{right}}$ is obtained by applying the side-changing functor at the source and the target. Let $\widetilde{\mathcal{N}}^{\mathrm{left}}$ be the left $\widetilde{\mathcal{D}}_Y$-module associated with $\widetilde{\mathcal{N}}^{\mathrm{right}}$, so that $\widetilde{\mathcal{N}}^{\mathrm{right}}=\widetilde{\omega}_Y\otimes\widetilde{\mathcal{N}}^{\mathrm{left}}$. Then we set

$${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}^{\mathrm{right}}:=\widetilde{\omega}_X\otimes{}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}^{\mathrm{left}},$$

and similarly with ${}_{\mathrm{D}}f^{*}$. Notice the change of grading by $\dim Y-\dim X$, due to the grading of $\widetilde{\omega}_X\otimes f^{-1}\widetilde{\omega}_Y^{\vee}$, i.e., we have

$$({}_{\mathrm{D}}f^{*(0)}\mathcal{N}^{\mathrm{right}})_p:=\omega_X\otimes({}_{\mathrm{D}}f^{*(0)}\mathcal{N}^{\mathrm{left}})(m-n)_p=\omega_X\otimes f^{*}\mathcal{N}^{\mathrm{left}}_{p+n-m}.$$

***8.6.9. Example* (Pull-back of a filtered module).** Assume that $\widetilde{\mathcal{N}}$ is the Rees module $R_F\mathcal{N}$ of a filtered left $\mathcal{D}_Y$-module $(\mathcal{N},F_\bullet\mathcal{N})$. Then $f^*\mathcal{N}=\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{N}$ is equipped with the filtration

$$F_pf^*\mathcal{N}=\mathrm{image}\big[\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}F_p\mathcal{N}\to\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{N}\big],$$

and the corresponding Rees module $R_Ff^*\mathcal{N}$ is equal to $f^*\widetilde{\mathcal{N}}/z$-torsion. If for example f is a smooth morphism, so that $\mathcal{O}_X$ is $f^{-1}\mathcal{O}_Y$-flat, then $\widetilde{\mathcal{O}}_X$ is also $f^{-1}\widetilde{\mathcal{O}}_Y$-flat and $f^*\widetilde{\mathcal{N}}=R_Ff^*\mathcal{N}$.

We also have $F_pf^*\mathcal{N}^{\mathrm{right}}=\omega_X\otimes F_{p+n-m}\mathcal{N}^{\mathrm{left}}$, after (8.2.3 $**$).

8.6.b. External product. We start with the case of $\mathcal{D}_X$-modules. Let X,Y be two complex manifolds and let p_X,p_Y be the projections from $X\times Y$ to X and Y respectively. For any pair of sheaves $\mathcal{F}_X,\mathcal{F}_Y$ of $\mathbb{C}$-vector spaces on X and Y respectively, let us set $\mathcal{F}_X\boxtimes_{\mathbb{C}}\mathcal{F}_Y:=p_X^{-1}\mathcal{F}_X\otimes_{\mathbb{C}}p_Y^{-1}\mathcal{F}_Y$.

By using an analogue of Theorem 8.8.7(2), one obtains that $\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y$ and $\mathcal{D}_X\boxtimes_{\mathbb{C}}\mathcal{D}_Y$ are coherent sheaves of rings on $X\times Y$. Moreover, $\mathcal{O}_{X\times Y}$ is flat over $\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y$ (as can be seen by applying [**Ser56**, Prop. 28] to each germ $\mathcal{O}_{X\times Y,(x,y)}$ and the localization of $\mathcal{O}_{X,x}\boxtimes_{\mathbb{C}}\mathcal{O}_{Y,y}$), and we also have

$$\mathcal{D}_{X\times Y}=\mathcal{O}_{X\times Y}\otimes_{(\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y)}(\mathcal{D}_X\boxtimes_{\mathbb{C}}\mathcal{D}_Y)=(\mathcal{D}_X\boxtimes_{\mathbb{C}}\mathcal{D}_Y)\otimes_{(\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y)}\mathcal{O}_{X\times Y}.$$

For an $\mathcal{O}_X$-module $\mathcal{L}_X$ (resp. a $\mathcal{D}_X$-module $\mathcal{M}_X$) and an $\mathcal{O}_Y$-module $\mathcal{L}_Y$ (resp. a $\mathcal{D}_Y$-module $\mathcal{M}_Y$), set

$$\mathcal{L}_X\boxtimes_{\mathcal{O}}\mathcal{L}_Y=(\mathcal{L}_X\boxtimes_{\mathbb{C}}\mathcal{L}_Y)\otimes_{\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y}\mathcal{O}_{X\times Y}$$

resp.

$$\begin{aligned}\mathcal{M}_X\boxtimes_{\mathcal{D}}\mathcal{M}_Y&=(\mathcal{M}_X\boxtimes_{\mathbb{C}}\mathcal{M}_Y)\otimes_{\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y}\mathcal{O}_{X\times Y}\\&=(\mathcal{M}_X\boxtimes_{\mathbb{C}}\mathcal{M}_Y)\otimes_{\mathcal{D}_X\boxtimes_{\mathbb{C}}\mathcal{D}_Y}\mathcal{D}_{X\times Y}.\end{aligned}$$

Clearly, if $\mathcal{L}_X,\mathcal{L}_Y$ are $\mathcal{O}$-coherent, then $\mathcal{L}_X\boxtimes_{\mathbb{C}}\mathcal{L}_Y$ is $\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y$-coherent. It follows that $\mathcal{L}_X\boxtimes_{\mathcal{O}}\mathcal{L}_Y$ is $\mathcal{O}_{X\times Y}$-coherent. Similarly, if $\mathcal{M}_X,\mathcal{M}_Y$ are $\mathcal{D}$-coherent, $\mathcal{M}_X\boxtimes_{\mathcal{D}}\mathcal{M}_Y$ is $\mathcal{D}_{X\times Y}$-coherent.

We now consider the case of $\widetilde{\mathcal{D}}_X$-modules. For any pair of sheaves $\widetilde{\mathcal{F}}_X, \widetilde{\mathcal{F}}_Y$ of $\widetilde{\mathbb{C}}$-modules on X and Y respectively, we set $\widetilde{\mathcal{F}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{F}}_Y := p_X^{-1}\widetilde{\mathcal{F}}_X \otimes_{\widetilde{\mathbb{C}}} p_Y^{-1}\widetilde{\mathcal{F}}_Y$. If we identify $\widetilde{\mathbb{C}} \boxtimes_{\mathbb{C}} \widetilde{\mathbb{C}}$ with $\mathbb{C}[z_1, z_2]$, then $\widetilde{\mathcal{F}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{F}}_Y$ is a $\mathbb{C}[z_1, z_2]$-module and

$$\widetilde{\mathcal{F}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{F}}_Y = \operatorname{Coker}\big[\widetilde{\mathcal{F}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{F}}_Y \xrightarrow{\ z_1 - z_2\ } \widetilde{\mathcal{F}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{F}}_Y\big].$$

As a consequence, we will obtain a behaviour of $\boxtimes_{\widetilde{\mathbb{C}}}$ similar to that of $\boxtimes_{\mathbb{C}}$ only with a supplementary $\mathbb{C}[z]$-flatness (i.e., strictness) condition for $\widetilde{\mathcal{F}}_X, \widetilde{\mathcal{F}}_Y$.

We have $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y = (\mathcal{O}_X \boxtimes_{\mathbb{C}} \mathcal{O}_Y) \otimes_{\mathbb{C}} \mathbb{C}[z]$, therefore $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y$ is a coherent sheaf of rings, and one also checks that $\widetilde{\mathcal{D}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_Y$ is coherent. Moreover, from the above flatness result, we find that $\widetilde{\mathcal{O}}_{X\times Y}$ is flat over $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y$.

For *strict* $\widetilde{\mathcal{O}}$-modules $\widetilde{\mathcal{L}}_X, \widetilde{\mathcal{L}}_Y$ (resp. $\widetilde{\mathcal{D}}$-modules $\widetilde{\mathcal{M}}_X, \widetilde{\mathcal{M}}_Y$), one defines the external product $\widetilde{\mathcal{L}}_X \boxtimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{L}}_Y$ (resp. $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$) as for $\mathcal{O}$-modules (resp. $\mathcal{D}$-modules). In such a case, we have $\widetilde{\mathcal{M}}_X = R_F\mathcal{M}_X$ for some $F_\bullet\mathcal{D}_X$-filtration $F_\bullet\mathcal{M}$, and similarly for Y, according to Proposition 5.1.8(1).

8.6.10. Lemma **(See [Kas03, §4.3]).** *If $F_\bullet\mathcal{M}_X, F_\bullet\mathcal{M}_Y$ are $F_\bullet\mathcal{D}$-filtrations, then*

$$F_j(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y) := \sum_{k+\ell=j} F_k\mathcal{M}_X \boxtimes_{\mathcal{O}} F_\ell\mathcal{M}_Y$$

is an $F_\bullet\mathcal{D}$-filtration of $\mathcal{M}_X \boxtimes_{\widetilde{\mathcal{D}}} \mathcal{M}_Y$ for which

$$\operatorname{gr}^F(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y) = \operatorname{gr}^F\mathcal{M}_X \boxtimes_{\operatorname{gr}^F\mathcal{D}} \operatorname{gr}^F\mathcal{M}_Y.$$

Proof. We set $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. Let us start by considering $\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y$ as a $\mathbb{C}[z_1, z_2]$-module. One checks that multiplication by $z_1 - z_2$ is injective on $\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y$. Its cokernel is identified with $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y$, where the action of z is induced either by that of $z_1 \boxtimes 1$ or that of $1 \boxtimes z_2$. But $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y$ is also $\widetilde{\mathbb{C}}$-torsion free, and defining $F_\bullet(\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y)$ by a formula similar to that of the lemma amounts to setting (due to torsion-freeness)

$$R_F(\mathcal{M}_X \boxtimes_{\mathbb{C}} \mathcal{M}_Y) = \widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y.$$

We have a commutative diagram of short exact sequences

$$\begin{array}{ccccc}
\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y & \xhookrightarrow{\ z_1 - z_2\ } & \widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y & \longrightarrow\!\!\!\!\!\rightarrow & \widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y \\
{\scriptstyle z_1}\big\downarrow & & {\scriptstyle z_1}\big\downarrow & & {\scriptstyle z}\big\downarrow \\
\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y & \xhookrightarrow{\ z_1 - z_2\ } & \widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y & \longrightarrow\!\!\!\!\!\rightarrow & \widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\operatorname{gr}^F\mathcal{M}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y & \xhookrightarrow{\ -z_2\ } & \operatorname{gr}^F\mathcal{M}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y & \longrightarrow\!\!\!\!\!\rightarrow & C
\end{array}$$

and the term C is identified with $\operatorname{gr}^F(\mathcal{M}_X \boxtimes_{\mathbb{C}} \mathcal{M}_Y)$ when considered as the cokernel of the vertical arrow, while it is identified with $\operatorname{gr}^F\mathcal{M}_X \boxtimes_{\mathbb{C}} \operatorname{gr}^F\mathcal{M}_Y$ when considered as the cokernel of the horizontal one.

Once this identification is obtained, the formula of the lemma is simply deduced by tensoring with $\mathcal{O}_{X\times Y}$ over $\mathcal{O}_X \boxtimes_{\mathbb{C}} \mathcal{O}_Y$. □

8.6.11. Remark. We will interpret this property in terms of flatness in Exercise 15.4.

8.7. Pushforward of $\widetilde{\mathcal{D}}$-modules

Let $f : X \to Y$ be a holomorphic map between complex manifolds. The pullback of a C^∞ function on Y is easy to define and, by adjunction, the pushforward of a current of degree 0 is easily defined provided that f is proper. On the other hand, the pullback of a form of maximal degree on Y is usually not of maximal degree on X, so the pushforward of a distribution is not defined in an easy way. This example is an instance of the fact that the pushforward of $\widetilde{\mathcal{D}}_X$-modules by a proper holomorphic map should be defined in a simple way for right $\widetilde{\mathcal{D}}_X$-modules, while for left $\widetilde{\mathcal{D}}_X$-modules one should use the side-changing functors.

8.7.1. Remark.

(1) We will distinguish the usual direct image and the direct image with proper supports for the sake of completeness. However, in the main part of this text, we always assume properness of the map on the support of the object to which it is applied. Therefore, this distinction will not be useful.

(2) The pushforward functor by a map $f : X \to Y$ applied to a $\widetilde{\mathcal{D}}_X$-module takes values in the derived category $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)$.

8.7.a. Definition and examples.

We aim at defining the derived pushforward of a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ by a formula using the transfer module (see Definition 8.6.4(1)) like

$$\boldsymbol{R}f_\star\big(\widetilde{\mathcal{M}} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_{X\to Y}\big).$$

However, the derived tensor product $\widetilde{\mathcal{M}} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is a priori an object of $\mathsf{D}^-(\widetilde{\mathcal{D}}_X)^{\mathrm{right}}$ and we need to argue that f has finite cohomological dimension in order to apply $\boldsymbol{R}f_\star$ to it. In order to avoid such an argument, we will simply make explicit a finite resolution of $\widetilde{\mathcal{D}}_{X\to Y}$ as a $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodule whose terms are $\widetilde{\mathcal{D}}_X$-locally free: this is the *relative Spencer complex* $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ (see Exercise 8.40). Recall also that the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$, which was defined in 8.4.3, is a complex of locally free left $\widetilde{\mathcal{D}}_X$-modules (hence locally free $\widetilde{\mathcal{O}}_X$-modules) and is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module. There is an isomorphism of complexes of bi-modules (see Exercise 8.40)

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}. \tag{8.7.2}$$

On the right-hand term, the left $\widetilde{\mathcal{O}}_X$-structure on each factor is used for the tensor product, and it is a complex of $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodules: the right $f^{-1}\widetilde{\mathcal{D}}_Y$ structure is the trivial one; the left $\widetilde{\mathcal{D}}_X$-structure is that defined by Exercise 8.12(1). It is a resolution of

$$\widetilde{\mathcal{O}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{D}}_{X\to Y}$$

as a left $\widetilde{\mathcal{D}}_X$-module, in a way compatible with the right $f^{-1}\widetilde{\mathcal{D}}_Y$-module structure (see Exercise 8.41).

For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we will use the identification

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$$

(see Exercise 8.40).

***8.7.3. Definition* (Pushforward of a $\widetilde{\mathcal{D}}$-module).** Setting $\star = *$ or $\star = !$, the *direct image* ${}_{\mathrm{D}}f_\star$ is the functor from $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)^{\mathrm{right}}$ to $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)^{\mathrm{right}}$ defined[(1)] by

$$(8.7.3\,*)\qquad {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} := \boldsymbol{R}f_*\big(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big) \simeq \boldsymbol{R}f_*\,\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}).$$

For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we set

$$(8.7.3\,**)\qquad {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} := \big({}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}^{\mathrm{right}}\big)^{\mathrm{left}}.$$

The cohomology modules are objects of $\mathsf{Mod}(\widetilde{\mathcal{D}}_Y)$ (right or left, respectively) and are denoted by

$${}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}} := H^j\,{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}.$$

One can give a formula for the pushforward of left $\widetilde{\mathcal{D}}_X$-modules which looks like that for the right $\widetilde{\mathcal{D}}_X$-modules.

8.7.4. Lemma. *For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we have*

$${}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} \simeq \boldsymbol{R}f_\star\big(\widetilde{\mathcal{D}}_{Y\leftarrow X} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}\big).$$

***Proof*.** See Definition 8.6.4(2) for the transfer module. The meaning of $\widetilde{\mathcal{D}}_{Y\leftarrow X} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}$ is $\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$, with an obvious notation. For the proof, see Exercise 8.43. □

8.7.5. Remarks.

(1) If f is proper, or proper on the support of $\widetilde{\mathcal{M}}$, we have an isomorphism in the category $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)$:

$${}_{\mathrm{D}}f_!\widetilde{\mathcal{M}} \xrightarrow{\sim} {}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}.$$

(2) If $\widetilde{\mathcal{F}}$ is any sheaf on X, we have $R^jf_*\widetilde{\mathcal{F}} = 0$ and $R^jf_!\widetilde{\mathcal{F}} = 0$ for $j \notin [0, 2\dim X]$. Therefore, taking into account the length $\dim X$ of the relative Spencer complex, we find that ${}_{\mathrm{D}}f_*^{(j)}\widetilde{\mathcal{M}}$ and ${}_{\mathrm{D}}f_!^{(j)}\widetilde{\mathcal{M}}$ are zero for $j \notin [-\dim X, 2\dim X]$: we say that ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}, {}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}$ have *bounded amplitude* (see Remark 8.7.13 for a more precise estimate of the amplitude).

(3) See Exercise 8.52 for a simple expression of the pushforward in terms of differential forms.

---

(1) Recall that, if $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, then $\mathsf{Mod}(\widetilde{\mathcal{D}}_X) := \mathsf{Modgr}(R_F\mathcal{D}_X)$.

Let us give natural examples of pushforward of $\widetilde{\mathcal{D}}_X$-modules.

8.7.6. Example **(Pushforward of a $\widetilde{\mathcal{D}}$-module by a closed embedding)**
If ι is a closed embedding, it is proper, so the ordinary pushforward and the pushforward with proper support will be the same. Since $\widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\mathcal{D}}_X$-locally free in this case (Exercise 8.35), we have, for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$,

$$ {}_{\mathrm{D}}\iota_*^{(0)}\widetilde{\mathcal{M}} = \iota_*(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_{X\to Y}), \quad {}_{\mathrm{D}}\iota_*^{(k)}\widetilde{\mathcal{M}} = 0 \text{ if } k \neq 0, $$

so that we will simply denote ${}_{\mathrm{D}}\iota_*^{(0)}$ by ${}_{\mathrm{D}}\iota_*$, and it is a functor $\mathsf{Mod}(\widetilde{\mathcal{D}}_X) \mapsto \mathsf{Mod}(\widetilde{\mathcal{D}}_Y)$. Similarly, for a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ we can write

$$ {}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}} = \iota_*(\widetilde{\mathcal{D}}_{Y\leftarrow X} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}). $$

8.7.7. Example **(Pushforward by a graph inclusion (see also Exercise 8.46))**
Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\iota_g : X \hookrightarrow X \times \mathbb{C}$ denote the graph embedding of g, with coordinate t on the factor $\mathbb{C}$. A special case is when $g \equiv 0$, so that the formulas below can be simplified by replacing every occurrence of g by zero. We denote $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ by $\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$. In order to simplify notation, we also denote the pushforward ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ by $\widetilde{\mathcal{M}}_g$.

(1) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. By Exercise 8.35, we have $\widetilde{\mathcal{D}}_{X\to X\times\mathbb{C}} \simeq \iota_{g*}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t]$. Then $\widetilde{\mathcal{M}}_g := {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}} \simeq \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ with the right $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-action defined locally be the following formulas (recall that for a holomorphic function $h(\boldsymbol{x}, t, z)$, the bracket $[\widetilde{\partial}_t^k, h]$ can be written as $\sum_{j<k} a_{h,j}(\boldsymbol{x}, t, z)\widetilde{\partial}_t^j = \sum_{j<k} \widetilde{\partial}_t^j b_{h,j}(\boldsymbol{x}, t, z)$):

$$ (8.7.7\,*) \qquad \begin{aligned} (m \otimes \widetilde{\partial}_t^k) \cdot \widetilde{\partial}_{x_i} &= (m\widetilde{\partial}_{x_i}) \otimes \widetilde{\partial}_t^k - \Big(m \frac{\partial g}{\partial x_i}\Big) \otimes \widetilde{\partial}_t^{k+1}, \\ (m \otimes \widetilde{\partial}_t^k) \cdot \widetilde{\partial}_t &= m \otimes \widetilde{\partial}_t^{k+1}, \\ (m \otimes \widetilde{\partial}_t^k) \cdot h(\boldsymbol{x}, t, z) &= \sum_{j<k} m a_{h,j}(\boldsymbol{x}, g(\boldsymbol{x}), z) \otimes \widetilde{\partial}_t^j + m h(\boldsymbol{x}, g(\boldsymbol{x}), z) \otimes \widetilde{\partial}_t^k. \end{aligned} $$

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, then the filtration of $\mathcal{M}_g \simeq \mathcal{M}[\partial_t]$ is simply given by

$$ F_p(\mathcal{M}^{\mathrm{right}}[\partial_t]) = \sum_{q+r=p} F_q\mathcal{M}^{\mathrm{right}}\,\partial_t^r. $$

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Since the coordinate t on $\mathbb{C}$ is fixed, a generator $\widetilde{\mathrm{d}}t$ of $\widetilde{\omega}_{\mathbb{C}}$ is also fixed, and we identify (see Caveat 8.2.4 for the notation $\widetilde{\mathrm{d}}t^\vee$)

$$ \widetilde{\mathcal{M}}_g \simeq \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t] \otimes \widetilde{\mathrm{d}}t^\vee, $$

i.e., the remaining right action of $\widetilde{\partial}_t$ is changed to a left action. Note that the term $\widetilde{\mathrm{d}}t^\vee$ also shifts the grading of the right-hand side. In other words, the left-hand side is obtained from $\widetilde{\mathcal{M}}_g^{\mathrm{right}}$ by applying the left-to-right functor on $X\times\widetilde{\mathbb{C}}$, which introduces a twist $(\dim X+1)$, while the right-hand side is obtained from $\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{right}}[\widetilde{\partial}_t]$ by applying the right-to-left functor on X, which introduces a twist $(\dim X)$ (see Proposition 8.2.5). We will usually omit the term $\widetilde{\mathrm{d}}t^\vee$ in the notation. For example, if $\widetilde{\mathcal{M}} =$

$R_F\mathcal{M}$, the right-hand term corresponds to the $\mathcal{D}_X$-module $\iota_{g*}\mathcal{M}[\partial_t]$ equipped with the filtration

$$F_p(\iota_{g*}\mathcal{M}^{\mathrm{left}}[\partial_t]) = \sum_{q+r=p} \iota_{g*}F[1]_q(\mathcal{M}^{\mathrm{left}})\,\partial_t^r = \sum_{q+r=p-1} \iota_{g*}F_q\mathcal{M}^{\mathrm{left}}\,\partial_t^r.$$

The left $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-action is defined locally be the following formulas (by using Exercise 8.17; note the sign at the second line due to (the omitted) $\widetilde{\mathrm{d}}t^\vee$):

$$\begin{aligned}\widetilde{\partial}_{x_i}(m\otimes\widetilde{\partial}_t^k) &= (\widetilde{\partial}_{x_i}m)\otimes\widetilde{\partial}_t^k - \Big(\frac{\partial g}{\partial x_i}\,m\Big)\otimes\widetilde{\partial}_t^{k+1},\\ \widetilde{\partial}_t(m\otimes\widetilde{\partial}_t^k) &= -m\otimes\widetilde{\partial}_t^{k+1},\\ h(\boldsymbol{x},t,z)(m\otimes\widetilde{\partial}_t^k) &= \sum_{j<k}(-1)^{k-1-j}b_{h,j}(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^j + h(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^k.\end{aligned}\tag{8.7.7 $**$}$$

(3) In both left and right cases, we can also consider $\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ as a module over the ring $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$, i.e., algebraically with respect to the variable t, with the action of t given by

$$(m\otimes\widetilde{\partial}_t^k)\cdot t = mg\otimes\widetilde{\partial}_t^k + kzm\otimes\widetilde{\partial}_t^{k-1},\quad \text{resp. } t\cdot(m\otimes\widetilde{\partial}_t^k) = gm\otimes\widetilde{\partial}_t^k - kzm\otimes\widetilde{\partial}_t^{k-1}.$$

This corresponds to the third lines in (8.7.7 $*$) and (8.7.7 $**$), according to the equality $[\widetilde{\partial}_t^k, t] = kz\widetilde{\partial}_t^{k-1}$.

8.7.8. Remark. If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, one can also consider the left $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-module structure on $\iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t] := \iota_{g*}\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ defined by setting (without a sign on the second line)

$$\begin{aligned}\widetilde{\partial}_{x_i}(m\otimes\widetilde{\partial}_t^k) &= (\widetilde{\partial}_{x_i}m)\otimes\widetilde{\partial}_t^k - \Big(\frac{\partial g}{\partial x_i}\,m\Big)\otimes\widetilde{\partial}_t^{k+1},\\ \widetilde{\partial}_t(m\otimes\widetilde{\partial}_t^k) &= m\otimes\widetilde{\partial}_t^{k+1},\\ h(\boldsymbol{x},t,z)(m\otimes\widetilde{\partial}_t^k) &= -\sum_{j<k}b_{h,j}(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^j + h(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^k.\end{aligned}$$

However, there exists a natural $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-linear isomorphism

$$\iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t] \xrightarrow{\ \sim\ } \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t]\otimes\widetilde{\mathrm{d}}t^\vee(-1),\quad m\otimes\widetilde{\partial}_t^k \longmapsto m\otimes(-\widetilde{\partial}_t^k)\otimes\widetilde{\mathrm{d}}t^\vee.$$

8.7.9. Example **(Pushforward by a constant map).** If $Y = \mathrm{pt}$ we denote by a_X the constant map on X. For a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we have (recall that, as a graded complex, ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{right}} \simeq {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{left}}$)

$${}_{\mathrm{D}}a_{X,*}\widetilde{\mathcal{M}} = \boldsymbol{R}\Gamma(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}),\quad {}_{\mathrm{D}}a_{X,!}\widetilde{\mathcal{M}} = \boldsymbol{R}\Gamma_c(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}).$$

These are bounded complexes of $\widetilde{\mathbb{C}}$-modules. If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, then for every $j\in\mathbb{Z}$, ${}_{\mathrm{D}}a_*^{(j)}\mathcal{M}$ is equipped with the filtration

$$F_p({}_{\mathrm{D}}a_*^{(j)}\mathcal{M}) : \mathrm{image}[\boldsymbol{H}^j(X, F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \longrightarrow \boldsymbol{H}^j(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})],$$

where the filtration $F_\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is defined in Remark 8.4.9, and

$$R_F({}_{\mathrm{D}}a_*^{(j)}\mathcal{M}) \simeq ({}_{\mathrm{D}}a_*^{(j)}R_F\mathcal{M})/z\text{-torsion}.$$

8.7.10. Example **(Pushforward by a projection, right case).** If $X = Y \times T$ and f is the projection $Y \times T \to Y$, denote by $\widetilde{\Theta}_{X/Y}$ the sheaf of *relative* tangent vector fields, i.e., which do not contain $\widetilde{\partial}_{y_j}$ in their local expression in coordinates adapted to the product $Y \times T$. It leads to the subsheaf of *relative* differential operators $\widetilde{\mathcal{D}}_{X/Y} \subset \widetilde{\mathcal{D}}_X$. On the other hand, ${}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_Y = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = \widetilde{\mathcal{D}}_{X\to Y}$ can also be regarded as a subsheaf of $\widetilde{\mathcal{D}}_X$ (differential operators only containing $\widetilde{\partial}_{y_j}$ in their expression).

The relative Spencer complex $\widetilde{\mathcal{D}}_{X/Y} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}$ (with $\widetilde{\Theta}_{X/Y,k} := \wedge^{-k}\widetilde{\Theta}_{X/Y}$)is defined in the same way as its absolute analogue, and is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_{X/Y}$-module. As a consequence, $\widetilde{\mathcal{D}}_{X/Y} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$ is also a resolution of $\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = \widetilde{\mathcal{D}}_{X\to Y}$ as a bimodule by locally free left $\widetilde{\mathcal{D}}_X$-modules. By identifying $\widetilde{\mathcal{D}}_X$ with $\widetilde{\mathcal{D}}_{X/Y} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$, we can also write this resolution as $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}$. There is moreover a canonical quasi-isomorphism as bimodules

$$
\begin{aligned}
\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) &= \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\big(\widetilde{\Theta}_{Y,\bullet} \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y\big)\\
&= \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big) \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\big(\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y\big)\\
&\xrightarrow{\ \sim\ } \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big) \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Y}\\
&= \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}.
\end{aligned}
$$

Definition 8.7.3 now reads

$$
{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} = \boldsymbol{R}f_\star\big(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big), \tag{8.7.10 $*$}
$$

where the right $\widetilde{\mathcal{D}}_Y$ structure is naturally induced from that of ${}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_Y \subset \widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{M}}$.

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, the p-th term of the filtration $F_\bullet(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet})$ of the complex $\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet}$ has $F_{p+k}\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,k}$ in degree $-k$ and for every $j \in \mathbb{Z}$,

$$
{}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}}/z\text{-torsion} \simeq R_F({}_{\mathrm{D}}f_\star^{(j)}\mathcal{M})
$$

with

$$
F_p({}_{\mathrm{D}}f_\star^{(j)}\mathcal{M}) = \mathrm{image}[f_\star^{(j)}F_p(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet}) \to f_\star^{(j)}(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet})].
$$

8.7.11. Example **(Pushforward by a projection, left case).** We take up the setting of Example 8.7.10 and we make explicit the formula in the case of left $\widetilde{\mathcal{D}}_X$-modules (See Exercise 8.44). Let us denote by $\widetilde{\Omega}^1_{X/Y}$ the sheaf of *relative* differential forms, i.e., which do not contain $\widetilde{\mathrm{d}}y_j$ in their local expression in coordinates adapted to the product $Y \times T$. If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, we can form the relative de Rham complex ${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\widetilde{\mathcal{M}}$ by mimicking Definition 8.4.1 and by using the relative connection $\widetilde{\nabla}_{X/Y}$. On the other hand, there remains an action of $\widetilde{\nabla}_Y$ on $\widetilde{\mathcal{M}}$. Due to the integrability property of $\widetilde{\nabla}$ on $\widetilde{\mathcal{M}}$, both connections $\widetilde{\nabla}_{X/Y}$ and $\widetilde{\nabla}_Y$ commute, so that the relative de Rham complex ${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\widetilde{\mathcal{M}}$ (the shift is by $d_{X/Y} := \dim X - \dim Y$) is naturally equipped with an $f^{-1}\widetilde{\mathcal{O}}_Y$-connection $\widetilde{\nabla}_Y$. Then we have (Exercise 8.44), for $\star = *$ or $\star = !$,

$$
{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} = (\boldsymbol{R}f_\star\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\widetilde{\mathcal{M}}, \widetilde{\nabla}_Y).
$$

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, the p-th term of the filtration $F_\bullet({}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M})$ of the complex ${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} = \Omega^\bullet_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{M}$ has $\Omega^k_{X/Y}\otimes_{\mathcal{O}_X} F_{p+k}\mathcal{M}$ in cohomological degree $k-\dim X$ and for every $j\in\mathbb{Z}$,

$$ {}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}}/z\text{-torsion} \simeq R_F\,{}_{\mathrm{D}}f_\star^{(j)}\mathcal{M}$$

with

$$F_p\,{}_{\mathrm{D}}f_\star^{(j)}\mathcal{M} = \mathrm{image}[f_\star^{(j)}F_p({}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}) \to f_\star^{(j)}({}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M})].$$

8.7.12. Remark. Since any morphism can be decomposed as a closed embedding followed be a projection, through the graph embedding, we could simply say that the pushforward by a closed embedding (resp. a projection) of a right $\widetilde{\mathcal{D}}_X$-module is obtained by the definition of Example 8.7.6 (resp. Example 8.7.10), and define the pushforward by any holomorphic map f by composing the pushforward functors in these simple cases. Nevertheless, in order to check various other properties, it is useful to have the intrinsic definition 8.7.3 for any holomorphic mapping f.

8.7.13. Remark **(Amplitude of the pushforward).** Formula (8.7.17) below shows that ${}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}} = 0$ for $j\notin[-n,n]$. On the other hand, if f is a closed inclusion, the amplitude is equal to zero, and if f is a projection, the C^∞ resolutions for Examples 8.7.10 or 8.7.11 show that ${}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}} = 0$ for $j\notin[-(n-m),(n-m)]$.

8.7.b. Explicit constructions with the pushforward functor

There are two natural ways (at least) to make explicit the functor $\boldsymbol{R}f_\star$ entering the definition of ${}_{\mathrm{D}}f_\star$: one can use the canonical Godement resolution by flabby sheaves, which is a very general procedure but with few geometric content, or one can replace the relative Spencer or de Rham complexes by their C^∞ counterparts as in Remark 8.4.13. We will mainly use the latter, but it can be useful to have the former at hand.

Godement resolution. Recall that the flabby sheaves are injective with respect to the functor f_* (direct image) in the category of sheaves (of modules over a ring) and, being c-soft, are injective with respect to the functor $f_!$ (direct image with proper support). The Godement canonical resolution is an explicit functorial flabby resolution for any sheaf (see Exercise 8.49 for details).

8.7.14. Definition **(Godement resolution).**

(1) The *Godement functor* $\mathcal{C}^0$ (see [**God64**, p. 167]) associates to any sheaf $\widetilde{\mathcal{L}}$ the *flabby* sheaf $\mathcal{C}^0(\widetilde{\mathcal{L}})$ of its discontinuous sections and to any morphism the corresponding family of germs of morphisms. Then there is a canonical injection $\widetilde{\mathcal{L}}\hookrightarrow\mathcal{C}^0(\widetilde{\mathcal{L}})$.

(2) Set inductively (see [**God64**, p. 168]) $\mathcal{Z}^0(\widetilde{\mathcal{L}}) = \widetilde{\mathcal{L}}$, $\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}) = \mathcal{C}^k(\widetilde{\mathcal{L}})/\mathcal{Z}^k(\widetilde{\mathcal{L}})$, $\mathcal{C}^{k+1}(\widetilde{\mathcal{L}}) = \mathcal{C}^0(\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}))$ and define $\delta:\mathcal{C}^k(\widetilde{\mathcal{L}})\to\mathcal{C}^{k+1}(\widetilde{\mathcal{L}})$ as the composition $\mathcal{C}^k(\widetilde{\mathcal{L}})\to\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}})\to\mathcal{C}^0(\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}))$. This defines a complex $(\mathcal{C}^\bullet(\widetilde{\mathcal{L}}),\delta)$, that we will denote as $(\mathrm{God}^\bullet\widetilde{\mathcal{L}},\delta)$.

(3) Given any sheaf $\widetilde{\mathcal{L}}$, $(\mathrm{God}^\bullet\widetilde{\mathcal{L}},\delta)$ is a resolution of $\widetilde{\mathcal{L}}$ by flabby sheaves. For a complex $(\widetilde{\mathcal{L}}^\bullet,d)$, we regard $\mathrm{God}^\bullet\widetilde{\mathcal{L}}^\bullet$ as a double complex ordered as written, i.e.,

with differential $(\delta_i, (-1)^i d_j)$ on $\mathrm{God}^i \widetilde{\mathcal{L}}^j$, and therefore also as the associated simple complex.

8.7.15. Corollary. *We have, by taking the single complex associated to the double complex, and for $\star = *$ or $\star = !$,*

$$_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} = f_\star \operatorname{God}^\bullet \operatorname{Sp}_{X\to Y}(\widetilde{\mathcal{M}}). \qquad \square$$

C^∞ ***resolution.*** Recall (see Remark 8.4.13) that $\operatorname{Sp}^\infty(\widetilde{\mathcal{D}}_X)$ is a resolution of $\operatorname{Sp}(\widetilde{\mathcal{D}}_X)$ in the category of left $\widetilde{\mathcal{D}}_X$-modules by flat $\widetilde{\mathcal{O}}_X$-modules. Therefore,

$$\operatorname{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X) \simeq \operatorname{Sp}^\infty(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$$

is a resolution of $\operatorname{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ in the category of $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$ bi-modules, so that, for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, (8.7.3 $*$) becomes

$$_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} \simeq f_\star(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \operatorname{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X)) \simeq f_\star \operatorname{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}}). \tag{8.7.16}$$

On the other hand, for a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we can use Exercise 8.52(5) to obtain

$$_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} \simeq f_\star \big[\widetilde{\mathcal{E}}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big]^{\mathrm{left}}. \tag{8.7.17}$$

This expression clearly shows that $_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}$ can be realized by a bounded complex of amplitude n. It can also be used to construct the spectral sequence attached to a filtered $\widetilde{\mathcal{D}}_X$-module.

8.7.18. Corollary. *Let $W_\bullet \widetilde{\mathcal{M}}$ be a finite increasing filtration of $\widetilde{\mathcal{M}}$ by $\widetilde{\mathcal{D}}_X$-submodules. Then there exists a spectral sequence, which is functorial in $(\widetilde{\mathcal{M}}, W_\bullet \widetilde{\mathcal{M}})$:*

$$E_1^{-\ell,k+\ell} = {}_{\mathrm{D}}f_*^{(k)}(\operatorname{gr}^W_\ell \widetilde{\mathcal{M}}) \implies \operatorname{gr}^W_\ell {}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}),$$

where $W_\bullet({}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}))$ is the image filtration $\operatorname{image}[{}_{\mathrm{D}}f_*^{(k)}(W_\bullet \widetilde{\mathcal{M}}) \to {}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}})]$. $\square$

8.7.19. The Lefschetz morphism. As a consequence of Exercise 8.52(5), given a $(1,1)$-form $\widetilde{\eta} \in \Gamma(X, \widetilde{\mathcal{E}}_X^{(1,1)})$ which $\widetilde{\mathrm{d}}$-closed (equivalently, $\widetilde{\mathrm{d}}'$ and d''-closed), there is a well-defined morphism for a left $\widetilde{\mathcal{D}}_X$-module ($\star = *$ or $\star = !$)

$$\widetilde{\eta}\wedge\ : {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} \longrightarrow {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}[2](1),$$

induced by $\widetilde{\eta}\wedge\ : \widetilde{\mathcal{E}}_X^\bullet \to \widetilde{\mathcal{E}}_X^\bullet[2](1)$. (Here, [2] means the shift by 2 of the complex, which occurs since $\widetilde{\eta}$ has total degree 2, while (1) is the Tate twist shift, which occurs since $\widetilde{\eta}$ has a degree-one holomorphic part.) It is clearly functorial with respect to $\widetilde{\mathcal{M}}$, that is, given any morphism $\varphi : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$, the following diagram commutes (where $\star$ is either for $*$ or for $!$):

$$\begin{array}{ccc}
{}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}_1 & \xrightarrow{\widetilde{\eta}\wedge} & {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}_1[2](1) \\
{\scriptstyle {}_{\mathrm{D}}f_\star\varphi}\big\downarrow & & \big\downarrow{\scriptstyle {}_{\mathrm{D}}f_\star\varphi} \\
{}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}_2 & \xrightarrow{\widetilde{\eta}\wedge} & {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}_2[2](1)
\end{array}$$

8.7.20. Definition **(The Lefschetz morphism attached to a closed $(1,1)$-form)**
For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the Lefschetz morphism associated to a (usual) closed $(1,1)$-form η on X is the morphism

$$\mathrm{L}_\eta := \frac{1}{z}\,\eta\wedge\,:\,{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} \longrightarrow {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}[2](1).$$

It is functorial with respect to $\widetilde{\mathcal{M}}$.

8.7.21. The Lefschetz morphism attached to a line bundle. Let $f : X \to Y$ be any morphism between complex manifolds and let $\mathcal{L}$ be a line bundle on X, with Chern class $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$. We will define a Lefschetz morphism

$$\mathrm{L}_{\mathcal{L}} : {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} \longrightarrow {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}[2](1).$$

We can choose a closed $(1,1)$-form η on X whose class in $H^2(X,\mathbb{C})$ is equal to the complexified class $c_1(\mathcal{L})_{\mathbb{C}}$. We regard η as a closed relative $(1,1)$-form with respect to the projection. As noticed in Remark 8.4.11, namely by using a similar argument, the action of L_η given in Definition 8.7.20 only depends on the class of η in $H^2(X,\mathbb{C})$. Notice also that, since η has degree two, wedging (or contracting) with η on the left or on the right gives the same result.

We thus *define* $\mathrm{L}_{\mathcal{L}}$ as L_η. This operator only depends on $c_1(\mathcal{L})_{\mathbb{C}}$. It is functorial with respect to $\widetilde{\mathcal{M}}$.

8.7.22. Remark **(Restriction to $z = 1$ of the Lefschetz morphism)**
It is obvious that the restriction to $z = 1$ of the morphism $\mathrm{L}_{\mathcal{L}}$ is the morphism

$$\mathrm{L}_{\mathcal{L}} : {}_{\mathrm{D}}f_\star\mathcal{M} \longrightarrow {}_{\mathrm{D}}f_\star\mathcal{M}[2].$$

8.7.c. Composition of direct images and the Leray spectral sequence

We compare the result of the pushforward functor by the composition of two maps with the pushforward by the second map of the pushforward by the first map. We find an isomorphism at the level of derived categories, that we will translate as a spectral sequence, which is the $\widetilde{\mathcal{D}}$-module analogue of the Leray spectral sequence (see Section 8.10.c).

8.7.23. Theorem **(Composition of direct images).** *Let*

$$f : X \longrightarrow Y \quad \textit{and} \quad f' : Y \longrightarrow Z$$

be two holomorphic maps. There is a functorial canonical isomorphism of functors

$${}_{\mathrm{D}}(f'\circ f)_!(\bullet) = {}_{\mathrm{D}}f'_!({}_{\mathrm{D}}f_!(\bullet)).$$

If f is proper, we also have

$${}_{\mathrm{D}}(f'\circ f)_*(\bullet) = {}_{\mathrm{D}}f'_*({}_{\mathrm{D}}f_*(\bullet)).$$

Proof. We start from the canonical isomorphism of $(\widetilde{\mathcal{D}}_X, (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules (Exercise 8.37):

$$\widetilde{\mathcal{D}}_{X\to Y} \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z} \xrightarrow{\ \sim\ } \widetilde{\mathcal{D}}_{X\to Z}. \tag{8.7.24}$$

We deduce an isomorphism of complexes of $(\widetilde{\mathcal{D}}_X, (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules

$$\Big[\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}\Big]\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\xrightarrow{\ \sim\ }\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Z}$$

lifting (8.7.24), that is, a natural isomorphism

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\xrightarrow{\ \sim\ }\mathrm{Sp}_{X\to Z}(\widetilde{\mathcal{D}}_X).$$

On the other hand, there exists a natural morphism of complexes

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\longrightarrow \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z},$$

obtained by tensoring the augmentation morphism $\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\to\widetilde{\mathcal{D}}_{Y\to Z}$ with $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$, and the left-hand term is a resolution of $\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}$ (in the category of $(\widetilde{\mathcal{D}}_X,(f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules) by locally free $\widetilde{\mathcal{D}}_X$-modules. Indeed, remark that, as $\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)$ is $\widetilde{\mathcal{D}}_Y$ locally free, one has

$$\begin{aligned}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)&\xrightarrow{\ \sim\ }\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\\ &=\widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\\ &=\widetilde{\mathcal{O}}_X\otimes^{\boldsymbol{L}}_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\\ &=\widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\quad(\widetilde{\mathcal{D}}_{Y\to Z}\text{ is }\widetilde{\mathcal{O}}_Y\text{ locally free})\\ &=\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}.\end{aligned}$$

Altogether, we have found a morphism, lifting (8.7.24),

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\longrightarrow\mathrm{Sp}_{X\to Z}(\widetilde{\mathcal{D}}_X),$$

between two resolutions (in the category of $(\widetilde{\mathcal{D}}_X,(f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules). This morphism is therefore a quasi-isomorphism. We now have, for an object $\widetilde{\mathcal{M}}$ of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ or of $\mathsf{D}^+(\widetilde{\mathcal{D}}_X)$

$$\begin{aligned}{}_{\mathrm{D}}(f'\circ f)_!(\widetilde{\mathcal{M}})&=\boldsymbol{R}(f'\circ f)_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Z}(\widetilde{\mathcal{D}}_X)\big)\\ &\simeq\boldsymbol{R}(f'\circ f)_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\big)\\ &\simeq\boldsymbol{R}f'_!\boldsymbol{R}f_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\,\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\big)\\ &\simeq\boldsymbol{R}f'_!\Big[\boldsymbol{R}f_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{\widetilde{\mathcal{D}}_Y}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big]\\ &={}_{\mathrm{D}}f'_!({}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}).\end{aligned}$$

The above arguments also apply if we replace Sp with Sp^∞ as defined in Remark 8.4.13, according to the $\widetilde{\mathcal{D}}_X$-flatness of $\mathrm{Sp}^\infty_{X\to Y}$ and the $\widetilde{\mathcal{D}}_Y$-flatness of $\mathrm{Sp}^\infty_{Y\to Z}$ (see Exercise 8.51(1)). All terms of the corresponding complexes are c-soft and we have

$$\begin{aligned}{}_{\mathrm{D}}(f'\circ f)_!(\widetilde{\mathcal{M}})&\simeq(f'\circ f)_!\,\mathrm{Sp}^\infty_{X\to Z}(\widetilde{\mathcal{M}})\\ &\simeq f'_!\Big[f_!\big(\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})\big)\otimes_{\widetilde{\mathcal{D}}_Y}\mathrm{Sp}^\infty_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big].\end{aligned}\tag{8.7.25}$$

The same result holds with ${}_{\mathrm{D}}f_*$ if we only assume that f is proper on the support of $\widetilde{\mathcal{M}}$. On the other hand, if f is proper or proper on the support of $\widetilde{\mathcal{M}}$, but f' is possibly not proper, then the same results are valid for $*$ instead of !: indeed, $f_! = f_*$ and $f_! \operatorname{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ is flabby, so the last isomorphism in (8.7.25) still holds with f'_*, and the same reasoning gives ${}_{\mathrm{D}}(f'\circ f)_* = {}_{\mathrm{D}}f'_*\,{}_{\mathrm{D}}f_*$. $\square$

8.7.26. Remark. If f is not proper, we cannot assert in general that ${}_{\mathrm{D}}(f'\circ f)_*(\bullet) = {}_{\mathrm{D}}f'_*({}_{\mathrm{D}}f_*(\bullet))$. However, such an identity still holds when applied to suitable subcategories of $\mathsf{D}^+(\widetilde{\mathcal{D}}_X)$, the main examples being:

- the restriction of f to the support of $\widetilde{\mathcal{M}}$ is proper, as already seen,
- $\widetilde{\mathcal{M}}$ has $\widetilde{\mathcal{D}}_X$-coherent cohomology.

In such cases, the natural morphism coming in the projection formula for f_* is a quasi-isomorphism (see [**MN93**, §II.5.4] for the coherent case).

This theorem reduces the computation of the direct image by any morphism $f : X \to Y$ by decomposing it as $f = p\circ\iota_f$, where $\iota_f : X \hookrightarrow X\times Y$ denotes the graph inclusion $x \mapsto (x, f(x))$. As ι_f is an embedding, it is proper, so we have ${}_{\mathrm{D}}f_* = {}_{\mathrm{D}}p_*{}_{\mathrm{D}}\iota_{f*}$. The following corollary is a direct consequence of Example 8.7.6.

***8.7.27. Corollary* (Composition with a closed embedding).**

(1) *Assume that f is a closed embedding. Then, for each $k\in\mathbb{Z}$, we have a functorial isomorphism* ${}_{\mathrm{D}}(f'\circ f)_!^{(k)} \simeq {}_{\mathrm{D}}f'^{(k)}_! \circ {}_{\mathrm{D}}f_!$.

(2) *Assume that f' is a closed embedding. Then, for each $k\in\mathbb{Z}$, we have a functorial isomorphism* ${}_{\mathrm{D}}(f'\circ f)_!^{(k)} \simeq {}_{\mathrm{D}}f'_! \circ {}_{\mathrm{D}}f^{(k)}_!$. $\square$

The Leray spectral sequence exists in this setting.

***8.7.28. Corollary* (Leray spectral sequence for the composition of maps)**
There exists a bounded spectral sequence with $E_2^{p,q} = {}_{\mathrm{D}}f'^{(p)}_!({}_{\mathrm{D}}f^{(q)}_!\widetilde{\mathcal{M}})$ *which converges to* ${}_{\mathrm{D}}(f'\circ f)^{p+q}_!\widetilde{\mathcal{M}}$. *There are corresponding spectral sequences with* ${}_{\mathrm{D}}f_*$ *and* ${}_{\mathrm{D}}f'_*$ *under the properness assumptions above.*

Proof. Let us consider the expression (8.7.25). First, $f_! \operatorname{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ is a bounded complex having cohomology ${}_{\mathrm{D}}f^{(q)}_!\widetilde{\mathcal{M}}$. The second line of (8.7.25) is a double complex $(K^{\bullet,\bullet},\delta_1,\delta_2)$. The single complex attached to $(K^{\bullet,\bullet},\delta_1,\delta_2)$ has cohomology ${}_{\mathrm{D}}(f'\circ f)^{(k)}_!(\widetilde{\mathcal{M}})$, according to our previous computation. The spectral sequence attached to this double complex has E_2 term

$$E_2^{p,q} = H^p_{\delta_2}(H^q_{\delta_1}(K^{\bullet,\bullet})) = {}_{\mathrm{D}}f'^{(p)}_!({}_{\mathrm{D}}f^{(q)}_!\widetilde{\mathcal{M}}).$$

The spectral sequence degenerates at a finite step. We have a similar result for ${}_{\mathrm{D}}(f'\circ f)^{(k)}_*(\widetilde{\mathcal{M}})$ if f is proper. $\square$

We call this spectral sequence the *Leray spectral sequence* for the composition $f'\circ f$. In such a way, the abutment ${}_{\mathrm{D}}(f'\circ f)^{(k)}_!(\widetilde{\mathcal{M}})$ comes equipped with a natural filtration, that we call the *Leray filtration*, such that

$$E_\infty^{p,q} = \operatorname{gr}^p_{\mathrm{Ler}}\big[{}_{\mathrm{D}}(f'\circ f)^{(p+q)}_!(\widetilde{\mathcal{M}})\big].$$

It is clear that the restriction to $z = 1$ of the Leray spectral sequence is the Leray spectral sequence for $\mathcal{D}_X$-modules.

8.7.29. Behaviour of the Spencer complex by pushforward. In the proof of Theorem 8.7.23, let us set $Z = \mathrm{pt}$, so that $\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y) = \mathrm{Sp}(\widetilde{\mathcal{D}}_Y)$. By the same argument, but not applying the functor $\boldsymbol{R}f'_!$, we obtain

$$\mathrm{Sp}({}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}) \simeq \boldsymbol{R}f_!\,\mathrm{Sp}(\widetilde{\mathcal{M}}).$$

We already have an identification on X as follows: considering the right $\widetilde{\mathcal{D}}_Y$-structure on $\widetilde{\mathcal{D}}_{X\to Y}$, the Spencer complex $\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})$ is well defined, and is nothing but

$$\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y}) = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\,\mathrm{Sp}(\widetilde{\mathcal{D}}_Y) \simeq \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{O}}_Y = \widetilde{\mathcal{O}}_X$$

as a left $\widetilde{\mathcal{D}}_X$-module. Similarly, regarding $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ as a complex of right $f^{-1}(\widetilde{\mathcal{D}}_Y)$-modules, we obtain

$$\begin{aligned}\mathrm{Sp}_Y(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)) &= \mathrm{Sp}_Y((\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y})\\ &\simeq (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})\\ &\simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}(\widetilde{\mathcal{M}}).\end{aligned}$$

We also conclude that, for a left or right $\widetilde{\mathcal{D}}_X$-module, we have

$${}^{\mathrm{p}}\mathrm{DR}({}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}) \simeq \boldsymbol{R}f_!\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}).$$

8.7.d. A morphism of adjunction. There are various adjunction morphisms for $\widetilde{\mathcal{D}}$-modules in the literature (see [**Kas03, HTT08**]). We will give here a simple one, in the case where the source and target of the *proper holomorphic map* $f : X \to Y$ have *the same dimension.* In such a case, the cotangent map T^*f induces a morphism

$$f^{-1}\widetilde{\Omega}^k_Y \longrightarrow \widetilde{\Omega}^k_X$$

for every k, which is compatible with the differential $\widetilde{\mathrm{d}}$, and similarly for C^∞ forms.

8.7.30. Proposition. *Under this assumption, if $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_Y$-module, there is a functorial morphism*

$$\mathrm{adj}_f : \widetilde{\mathcal{M}} \longrightarrow {}_{\mathrm{D}}f^{(0)}_*({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}).$$

Proof. Set $n = \dim X = \dim Y$. The left setting makes easier the definition of ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}$. Nevertheless, we will construct the morphism in the right setting.

Firstly, by using Exercise 8.31(2), we find

$$\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_Y \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}} \xleftarrow{\sim} (\widetilde{\Omega}^{n+\bullet}_Y \otimes \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}} \xrightarrow{\sim} (\widetilde{\mathcal{E}}^{n+\bullet}_Y \otimes \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}}.$$

The cotangent map $f^{-1}\widetilde{\mathcal{E}}^k_Y \to \widetilde{\mathcal{E}}^k_X$ induces, by using the sheaf-theoretic adjunction $\mathrm{Id} \to f_*f^{-1}$, a morphism $\widetilde{\mathcal{E}}^k_Y \to f_*\widetilde{\mathcal{E}}^k_X$ compatible with differentials, hence a morphism

$$(\widetilde{\mathcal{E}}^{n+\bullet}_Y \otimes \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}} \longrightarrow (f_*\widetilde{\mathcal{E}}^{n+\bullet}_X \otimes \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}}.$$

By using the isomorphism of Exercise 8.18(3), we obtain

$$(f_*\widetilde{\mathcal{E}}^{n+\bullet}_X \otimes \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}} \simeq f_*\widetilde{\mathcal{E}}^{n+\bullet}_X \otimes (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y),$$

where, for the left-hand side, the left $\widetilde{\mathcal{D}}_Y$-module structure of $\widetilde{\mathcal{D}}_Y$ is used for the C^∞-complex $(\bullet)$, and the right $\widetilde{\mathcal{D}}_Y$-module structure of $\widetilde{\mathcal{D}}_Y$ is used in the tensor product with $\widetilde{\mathcal{M}}$ in order to obtain the final right $\widetilde{\mathcal{D}}_Y$-module structure (see Exercise 8.18(2)).

By the sheaf-theoretic projection formula, we have a morphism compatible with differentials

$$f_*\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes_{\widetilde{\mathcal{O}}_Y}(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y)\longrightarrow f_*\Big(\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y)\Big),$$

and we identify the latter complex with the complex

$$f_*\Big(\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes_{\widetilde{\mathcal{O}}_X}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\Big),$$

which, by Exercise 8.52 applied to ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}$, is also identified with

$$({}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}))^{\mathrm{right}}.$$

We finally find a morphism between the cohomologies in degree zero:

$$\widetilde{\mathcal{M}}^{\mathrm{right}}\longrightarrow({}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}))^{\mathrm{right}}.\qquad\square$$

***8.7.31. Example* (Case of a finite morphism).** Let us consider the simple case of a finite morphism $f:X\to Y$. Since X and Y are smooth, it is flat. Let us assume that $f_*\widetilde{\mathcal{O}}_X$ is $\widetilde{\mathcal{O}}_Y$-locally free (hence the same holds for any locally free $\widetilde{\mathcal{O}}_X$-module of finite rank, as e.g. $\widetilde{\Omega}_X^k$ for each $k\geqslant 1$). The adjunction morphism reads

$$\begin{aligned}\widetilde{\mathcal{M}}^{\mathrm{right}}&\simeq(\widetilde{\Omega}_Y^{n+\bullet}\otimes\widetilde{\mathcal{D}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}}\\&\longrightarrow(f_*\widetilde{\Omega}_X^{n+\bullet}\otimes\widetilde{\mathcal{D}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}}\simeq f_*\widetilde{\Omega}_X^{n+\bullet}\otimes(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y)\\&\xrightarrow{\ \sim\ }f_*\Big(\widetilde{\Omega}_X^{n+\bullet}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y)\Big),\end{aligned}$$

where the latter isomorphism follows from the local freeness assumption and the fact that f has cohomological dimension zero. In particular, we obtain that

$${}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}})\simeq{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{O}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}},$$

and the adjunction morphism for $\widetilde{\mathcal{M}}$ is induced by that for $\widetilde{\mathcal{O}}_Y$.

Assume that f is locally multi-cyclic, that is, near each point of X and its image by f, there exist coordinates $x_1,\dots,x_n$ on X and coordinates $y_1,\dots,y_n$ on Y such that, in these coordinates, $f=(f_1,\dots,f_n)$ is the finite morphism defined by $f_i(x_1,\dots,x_n)=x_i^{r_i}$ with $r_i\in\mathbb{N}^*$. Then the local freeness property considered above is easily checked. By Exercise 8.58, there exists a trace morphism

$$\mathrm{Tr}_f:{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}})\longrightarrow\widetilde{\mathcal{M}}$$

such that the composition $\mathrm{Tr}_f\circ\mathrm{adj}_f:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}$ is the identity. As a consequence, $\widetilde{\mathcal{M}}$ is a direct summand of ${}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}})$.

8.7.e. Pushforward of $\mathcal{D}_{X,\overline{X}}$-modules. As we will apply the pushforward functor by a holomorphic map $f : X \to Y$ to the sheaf of distributions on X or to the sheaf of currents of maximal degree (see Example 8.3.5), we will make precise the adaptation of the previous properties to the category of $\mathcal{D}_{X,\overline{X}}$-modules, where we recall that $\mathcal{D}_{X,\overline{X}} := \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$ (see (8.3.0 $**$)). We will denote the corresponding pushforward functor by ${}_{\mathrm{D},\overline{\mathrm{D}}}f_!$ or ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*$. This notation was already used, with that meaning, in Section 7.3.17, for the pushforward by a closed inclusion.

We define $\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}$ as $\mathcal{D}_{X\to Y} \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}\to\overline{Y}}$. This sheaf can also be described as $\mathcal{O}_{X,\overline{X}} \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}\mathcal{D}_{Y,\overline{Y}}$. The Spencer complex $\mathrm{Sp}(\mathcal{D}_{X,\overline{X}})$ is the simple complex associated with the double complex $\mathrm{Sp}(\mathcal{D}_X) \otimes_{\mathbb{C}} \mathrm{Sp}(\mathcal{D}_{\overline{X}})$. Defining $\Theta^k_{X,\overline{X}} = \bigoplus_{i+j=k}(\Theta^i_X \otimes_{\mathbb{C}} \Theta^j_{\overline{X}})$, the $(-k)$-th term of the Spencer complex $\mathrm{Sp}(\mathcal{D}_{X,\overline{X}})$ is equal to $\mathcal{D}_{X,\overline{X}} \otimes \Theta^k_{X,\overline{X}}$, which is $\mathcal{D}_{X,\overline{X}}$-locally free of finite rank, and the differentials are expressed in a way similar to that in Definition 8.4.3. It is a $\mathcal{D}_{X,\overline{X}}$-resolution of $\mathcal{O}_{X,\overline{X}}$ by locally free $\mathcal{D}_{X,\overline{X}}$-modules.

The relative Spencer complex is defined similarly to (8.7.2), by

$$\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}}) = \mathrm{Sp}(\mathcal{D}_{X,\overline{X}}) \otimes_{\mathcal{O}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}\to Y,\overline{Y}},$$

and is a resolution of $\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}$ as a $(\mathcal{D}_{X,\overline{X}}, f^{-1}\mathcal{D}_{Y,\overline{Y}})$-bimodule by locally free $\mathcal{D}_{X,\overline{X}}$-modules.

The pushforward functor ${}_{\mathrm{D},\overline{\mathrm{D}}}f_\star$ ($\star = !, *$) is defined, for a right $\mathcal{D}_{X,\overline{X}}$-module $\mathcal{N}$, or a bounded complex of such, by

$${}_{\mathrm{D},\overline{\mathrm{D}}}f_\star(\mathcal{N}) = \boldsymbol{R}f_\star(\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{N})) \simeq \boldsymbol{R}f_\star\big(\mathcal{N} \otimes_{\mathcal{D}_{X,\overline{X}}} \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big).$$

In a way similar to what is done in Theorem 8.7.23 and Corollary 8.7.28, we obtain the following result. In the present setting, it is enough to use the Godement flabby resolution $\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{N})$ when a flabby resolution is needed.

8.7.32. Proposition. *Let*

$$f : X \longrightarrow Y \quad \textit{and} \quad f' : Y \longrightarrow Z$$

be two holomorphic maps. There is a functorial canonical isomorphism of functors

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f' \circ f)_!(\bullet) = {}_{\mathrm{D},\overline{\mathrm{D}}}f'_!({}_{\mathrm{D},\overline{\mathrm{D}}}f_!(\bullet)).$$

If f is proper, we also have

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f' \circ f)_*(\bullet) = {}_{\mathrm{D},\overline{\mathrm{D}}}f'_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*(\bullet)).$$

Furthermore, there exists a bounded spectral sequence with $E_2^{p,q} = {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(p)}_!({}_{\mathrm{D},\overline{\mathrm{D}}}f^{(q)}_!\mathcal{M})$ which converges to ${}_{\mathrm{D},\overline{\mathrm{D}}}(f' \circ f)^{p+q}_!\mathcal{M}$. There are corresponding spectral sequences with ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*$ and ${}_{\mathrm{D},\overline{\mathrm{D}}}f'_*$ under the properness assumptions above. □

8.8. Coherent $\widetilde{\mathcal{D}}_X$-modules and coherent filtrations

Although it would be natural to develop the theory of coherent $\widetilde{\mathcal{D}}_X$-modules in a way similar to that of $\widetilde{\mathcal{O}}_X$-modules, some points of the theory are not known to extend to $\widetilde{\mathcal{D}}_X$-modules (the lemma on holomorphic matrices). The approach which is therefore classically used consists in using the $\widetilde{\mathcal{O}}_X$-theory, and the main tools for that purpose are the coherent filtrations.

8.8.a. Coherence of $\widetilde{\mathcal{D}}_X$. Let us begin by recalling the definition of coherence. Let $\widetilde{\mathcal{A}}$ be a sheaf of rings on a space X.

8.8.1. Definition.

(1) A sheaf of $\widetilde{\mathcal{A}}$-modules $\widetilde{\mathcal{F}}$ is said to be $\widetilde{\mathcal{A}}$-*coherent* if it is locally of finite type:

$$\forall\, x \in X,\ \exists\, U_x,\ \exists\, q, \quad \exists\, \widetilde{\mathcal{A}}^q_{|U_x} \twoheadrightarrow \widetilde{\mathcal{F}}_{|U_x},$$

and if, for any open set U of X and any $\widetilde{\mathcal{A}}$-linear morphism $\varphi : \widetilde{\mathcal{A}}^r_{|U} \to \widetilde{\mathcal{F}}_{|U}$, the kernel of φ is locally of finite type.

(2) The sheaf $\widetilde{\mathcal{A}}$ is a coherent sheaf of rings if it is coherent as a (left and right) module over itself.

8.8.2. Lemma. *Assume $\widetilde{\mathcal{A}}$ coherent. Let $\widetilde{\mathcal{F}}$ be a sheaf of $\widetilde{\mathcal{A}}$-module. Then $\widetilde{\mathcal{F}}$ is $\widetilde{\mathcal{A}}$-coherent if and only if $\widetilde{\mathcal{F}}$ is* locally of finite presentation*: $\forall\, x \in X$, $\exists\, U_x$, $\exists\, p, q$ and an exact sequence*

$$\widetilde{\mathcal{A}}^p_{|U_x} \longrightarrow \widetilde{\mathcal{A}}^q_{|U_x} \longrightarrow \widetilde{\mathcal{F}}_{|U_x} \longrightarrow 0.$$

Classical theorems of Cartan and Oka claim the *coherence of* $\widetilde{\mathcal{O}}_X$, and a theorem of Frisch asserts that, if K is a compact polycylinder, $\widetilde{\mathcal{O}}_X(K)$ is a Noetherian ring. It follows that $\mathrm{gr}^F\widetilde{\mathcal{D}}_X(K)$ is a Noetherian ring, and one deduces that $\widetilde{\mathcal{D}}_X(K)$ is left and right Noetherian. From this one concludes that the sheaf of rings $\widetilde{\mathcal{D}}_X$ is coherent (see [**GM93, Kas03**] for details).

***8.8.3. Remark* (Noetherianity).** It follows from these properties that $\widetilde{\mathcal{D}}_X$ is a *Noetherian sheaf of rings*, in the sense of [**Kas03**, Def. A.7], that is, together with the coherence property, each germ $\widetilde{\mathcal{D}}_{X,x}$ is Noetherian and for any open subset $U \subset X$ and any family $\widetilde{\mathcal{I}}_i$ of coherent (left or right) ideals of $\widetilde{\mathcal{D}}_U$, the ideal $\sum_i \widetilde{\mathcal{I}}_i$ is a coherent $\widetilde{\mathcal{D}}_U$-module.

8.8.b. Coherent $\widetilde{\mathcal{D}}$-modules and filtrations

Let $\widetilde{\mathcal{M}}$ be a $\widetilde{\mathcal{D}}_X$-module. From the preliminary reminder on coherence, we know that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-*coherent* if it is locally finitely presented, i.e., if for any $x \in X$ there exists an open neighbourhood U_x of x an an exact sequence $\widetilde{\mathcal{D}}^q_{X|U_x} \to \widetilde{\mathcal{D}}^p_{X|U_x} \to \widetilde{\mathcal{M}}_{|U_x}$.

***8.8.4. Definition* (Coherent filtrations).** Let $F_\bullet\widetilde{\mathcal{M}}$ be a filtration of $\widetilde{\mathcal{M}}$ (see Section 5.1). We say that the filtration is *coherent* if the Rees module $R_F\widetilde{\mathcal{M}}$ is coherent over the coherent sheaf $R_F\widetilde{\mathcal{D}}_X$ (i.e., locally finitely presented).

It is useful to have various criteria for a filtration to be coherent.

***8.8.5. Proposition* (Existence of coherent filtrations).**

(1) *If $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, then it admits* locally on X *a coherent filtration.*

(2) *If* $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$ *and if* $\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_X$*-coherent and strict, it admits* globally on X *a coherent filtration.*

Proof. For (1), see Exercise 8.63. Let us prove (2). By Proposition 5.1.8(1), we have $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ for some filtered $\mathcal{D}_X$-module $\mathcal{M}$ and since $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, $F_\bullet\mathcal{M}$ is a coherent $F_\bullet\mathcal{D}$-filtration. Then one can apply Exercise 8.66. □

The notion of a coherent filtration is the main tool to obtain results on coherent $\widetilde{\mathcal{D}}_X$-modules from theorems on coherent $\widetilde{\mathcal{O}}_X$-modules, and the main results concerning coherent $\widetilde{\mathcal{D}}_X$-modules are obtained from the theorems of Cartan and Oka for $\widetilde{\mathcal{O}}_X$-modules.

***8.8.6. Theorem* (Theorems of Cartan-Oka for $\widetilde{\mathcal{D}}_X$-modules)**
Let $\widetilde{\mathcal{M}}$ *be a left* $\widetilde{\mathcal{D}}_X$*-module and let* K *be a compact polycylinder contained in an open subset* U *of* X*, such that* $\widetilde{\mathcal{M}}$ *has a coherent filtration on* U*. Then,*

(1) $\Gamma(K,\widetilde{\mathcal{M}})$ *generates* $\widetilde{\mathcal{M}}_{|K}$ *as an* $\widetilde{\mathcal{O}}_K$*-module,*

(2) *For every* $i \geqslant 1$, $H^i(K,\widetilde{\mathcal{M}}) = 0$.

Proof. This is easily obtained from the theorems A and B for $\widetilde{\mathcal{O}}_X$-modules, by using inductive limits (for A it is obvious and, for B, see [**God64**, Th. 4.12.1]). □

***8.8.7. Theorem* (Characterization of coherence for $\widetilde{\mathcal{D}}_X$-modules, see [GM93])**

(1) *Let* $\widetilde{\mathcal{M}}$ *be a left* $\widetilde{\mathcal{D}}_X$*-module. Then, for any small enough compact polycylinder* K*, we have the following properties:*

(a) $\widetilde{\mathcal{M}}(K)$ *is a finite type* $\widetilde{\mathcal{D}}(K)$*-module,*

(b) *For every* $x \in K$, $\widetilde{\mathcal{O}}_x \otimes_{\widetilde{\mathcal{O}}(K)} \widetilde{\mathcal{M}}(K) \to \widetilde{\mathcal{M}}_x$ *is an isomorphism.*

(2) *Conversely, if there exists a covering* $\{K_\alpha\}$ *by polycylinders* K_α *such that* X *is the union of the interiors of the* K_α *and that on any* K_α *the properties* (1a) *and* (1b) *are fulfilled, then* $\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_X$*-coherent.* □

A first application of Theorem 8.8.7 is a variant of the classical Artin-Rees lemma:

8.8.8. Corollary. *Let* $\widetilde{\mathcal{M}}$ *be a* $\widetilde{\mathcal{D}}_X$*-module with a coherent filtration* $F_\bullet\widetilde{\mathcal{M}}$ *and let* $\widetilde{\mathcal{N}}$ *be a coherent* $\widetilde{\mathcal{D}}_X$*-submodule of* $\widetilde{\mathcal{M}}$*. Then the filtration* $F_\bullet\widetilde{\mathcal{N}} := \widetilde{\mathcal{N}} \cap F_\bullet\widetilde{\mathcal{M}}$ *is coherent.*

Proof. Let K be a small compact polycylinder for $R_F\widetilde{\mathcal{M}}$. Then $\Gamma(K, R_F\widetilde{\mathcal{M}})$ is finitely generated, hence so is $\Gamma(K, R_F\widetilde{\mathcal{N}})$, as $\Gamma(K, R_F\widetilde{\mathcal{D}}_X)$ is Noetherian. It remains to be proved that, for any $x \in K$ and any k, the natural morphism

$$\widetilde{\mathcal{O}}_x \otimes_{\widetilde{\mathcal{O}}(K)} (F_k\widetilde{\mathcal{M}}(K) \cap \widetilde{\mathcal{N}}(K)) \longrightarrow F_k\widetilde{\mathcal{M}}_x \cap \widetilde{\mathcal{N}}_x$$

is an isomorphism. This follows from the flatness of $\widetilde{\mathcal{O}}_x$ over $\widetilde{\mathcal{O}}(K)$ (see [**Fri67**]). □

8.8.9. Structure of coherent $\widetilde{\mathcal{D}}_X$-modules. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Its z-torsion submodule is the submodule $\widetilde{\mathcal{M}}' := \bigcup_{k\geqslant 1}\operatorname{Ker}[z^k : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}]$. Since each submodule $\operatorname{Ker}[z^k : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}]$ is coherent (see Exercise 8.60) the union is locally finite and $\widetilde{\mathcal{M}}'$ has a locally finite filtration such that each successive quotient is a coherent $\widetilde{\mathcal{D}}_X$-module annihilated by z. The corresponding graded module $\operatorname{gr}\widetilde{\mathcal{M}}'$ is thus a coherent (graded) $(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X) = \operatorname{gr}^F\mathcal{D}_X$-module, on which the z-action is zero.

On the other hand, the quotient module $\widetilde{\mathcal{M}}'' := \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}'$ is strict by definition, hence of the form $R_F\mathcal{M}''$ for some coherent $\mathcal{D}_X$-module equipped with a coherent F-filtration $F_\bullet\mathcal{M}''$.

8.8.c. Support and characteristic variety. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Being a sheaf on X, $\widetilde{\mathcal{M}}$ has a support $\operatorname{Supp}\widetilde{\mathcal{M}}$, which is the closed subset complement to the set of $x \in X$ in the neighbourhood of which $\widetilde{\mathcal{M}}$ is zero.

8.8.10. Lemma. *The support of a coherent $\widetilde{\mathcal{O}}_X$-module is a closed analytic subset of X.*

Proof. This is standard if $\widetilde{\mathcal{O}}_X = \mathcal{O}_X$. On the other hand, if $\widetilde{\mathcal{O}}_X = R_F\mathcal{O}_X$, let $\widetilde{\mathcal{I}}$ be a *graded* ideal of $\widetilde{\mathcal{O}}_X$, locally generated by functions f_jz^j with $f_j \in \mathcal{O}_X$. Then the support of $\widetilde{\mathcal{O}}_X/\widetilde{\mathcal{I}}$ is that of $\mathcal{O}_X/(f_j)_j$. □

Such a property extends to coherent $\widetilde{\mathcal{D}}_X$-modules:

8.8.11. Proposition. *The support* $\operatorname{Supp}\widetilde{\mathcal{M}}$ *of a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is a closed analytic subset of X.*

Proof. The property of being an analytic subset being local, we may assume that $\widetilde{\mathcal{M}}$ is generated over $\widetilde{\mathcal{D}}_X$ by a coherent $\widetilde{\mathcal{O}}_X$-submodule $\widetilde{\mathcal{F}}$ (see Exercise 8.63(4)). Then the support of $\widetilde{\mathcal{M}}$ is equal to the support of $\widetilde{\mathcal{F}}$. □

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module and let Z be a closed analytic subset of X. It follows from Exercise 8.67 that the subsheaf $\Gamma_Z\widetilde{\mathcal{M}}$ consisting of local sections of $\widetilde{\mathcal{M}}$ annihilated by some power of the ideal $\mathcal{I}_Z$ is $\widetilde{\mathcal{D}}_X$-coherent. In particular, let us denote by $\bigcup_j Z_j$ the decomposition of $\operatorname{Supp}\widetilde{\mathcal{M}}$ into its irreducible components. Then $\Gamma_{Z_j}\widetilde{\mathcal{M}}$ is a coherent sub $\widetilde{\mathcal{D}}_X$-module of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}/\Gamma_{Z_j}\widetilde{\mathcal{M}}$ is supported on $\bigcup_{k\neq j} Z_k$. The following lemma is then obvious.

8.8.12. Lemma. *The kernel and cokernel of the natural morphism*

$$\bigoplus_j \Gamma_{Z_j}\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$$

have support everywhere of codimension $\geqslant 1$ *in* $\operatorname{Supp}\widetilde{\mathcal{M}}$. □

The support is usually not the right geometric object attached to a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, as it does not provide enough information on $\widetilde{\mathcal{M}}$. A finer object is the *characteristic variety.* Using the convention 8.1.11, we set $\widetilde{T}^*X = T^*X$ or $\widetilde{T}^*X = T^*X \times \mathbb{C}_z$.

8.8.13. Definition **(Characteristic variety).** Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. The *characteristic variety* $\operatorname{Char}\widetilde{\mathcal{M}}$ is the subset of the cotangent space $\widetilde{T}^*X$ defined locally as the support of $\operatorname{gr}^F\widetilde{\mathcal{M}}$ for some (or any) local coherent filtration of $\widetilde{\mathcal{M}}$.

8.8.14. Structure of the characteristic variety. The characteristic variety is additive (see Exercise 8.68), so by using the notation of Remark 8.8.9 and after Exercise 8.66, we have a decomposition

$$\operatorname{Char}\widetilde{\mathcal{M}} = \operatorname{Char}\widetilde{\mathcal{M}}' \cup (\operatorname{Char}\mathcal{M}'' \times \mathbb{C}_z),$$

where $\operatorname{Char}\widetilde{\mathcal{M}}'$ is contained in $T^*X = T^*X \times \{0\} \subset \widetilde{T}^*X$.

It is known that $\operatorname{Char}\mathcal{M}''$ is involutive in T^*X: the first proof has been given by Sato, Kawai, Kashiwara [**SKK73**]. Next, Malgrange gave a very simple proof in a seminar Bourbaki talk ([**Mal78**], see also [**GM93**, p. 165]). And finally, Gabber gave the proof of a general algebraic version of this theorem (see [**Gab81**], see also [**Bjö93**, p. 473]). A consequence is that any irreducible component of $\operatorname{Char}\mathcal{M}''$ has a dimension $\geqslant \dim X$.

On the other hand, there is no restriction on $\operatorname{Char}\widetilde{\mathcal{M}}'$, which is nothing but the support of the $\operatorname{gr}^F\mathcal{D}_X$-module $\operatorname{gr}\widetilde{\mathcal{M}}'$.

8.8.d. (Strictly) non-characteristic restriction. Let $\iota_Y : Y \hookrightarrow X$ denote the inclusion of a closed submanifold with ideal $\mathcal{I}_Y$ (in local coordinates $(x_1, \dots, x_n)$, $\mathcal{I}_Y$ is generated by $x_1, \dots, x_p$, where $p = \operatorname{codim} Y$). The pullback functor ${}_{\mathrm{D}}\iota_Y^*$ is defined in Section 8.6.a. The case of left $\widetilde{\mathcal{D}}_X$-modules is easier to treat, so we will consider *left $\widetilde{\mathcal{D}}_X$-modules* in this section.

Let us make the construction explicit in the case of a closed inclusion. A local section ξ of $\iota_Y^{-1}\widetilde{\Theta}_X$ (vector field on X, considered at points of Y only; we denote by ι_Y^{-1} the sheaf-theoretic pullback) is said to be tangent to Y if, for every local section g of $\widetilde{\mathcal{I}}_Y$, $\xi(g) \in \widetilde{\mathcal{I}}_Y$. This defines a subsheaf $\widetilde{\Theta}_{X|Y}$ of $\iota_Y^{-1}\widetilde{\Theta}_X$. Then $\widetilde{\Theta}_Y = \widetilde{\mathcal{O}}_Y \otimes_{\iota_Y^{-1}\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X|Y} = \iota_Y^*\widetilde{\Theta}_{X|Y}$ is a subsheaf of $\iota_Y^*\widetilde{\Theta}_X$.

Given a *left* $\widetilde{\mathcal{D}}_X$-module, the action of $\iota_Y^{-1}\widetilde{\Theta}_X$ on $\iota_Y^{-1}\widetilde{\mathcal{M}}$ restricts to an action of $\widetilde{\Theta}_Y$ on $\iota_Y^*\widetilde{\mathcal{M}} = \widetilde{\mathcal{O}}_Y \otimes_{\iota_Y^{-1}\widetilde{\mathcal{O}}_X} \iota_Y^{-1}\widetilde{\mathcal{M}}$. The criterion of Exercise 8.8 is fulfilled since it is fulfilled for $\widetilde{\Theta}_X$ and $\widetilde{\mathcal{M}}$, defining therefore a left $\widetilde{\mathcal{D}}_Y$-module structure on $\iota_Y^*\widetilde{\mathcal{M}}$: this is ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$.

Without any other assumption, coherence is not preserved by ${}_{\mathrm{D}}\iota_Y^*$. For example, ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{D}}_X$ is not $\widetilde{\mathcal{D}}_Y$-coherent if $\operatorname{codim} Y \geqslant 1$. A criterion for coherence of the pullback is given below in terms of the characteristic variety.

The cotangent map to the inclusion defines a natural bundle morphism

$$\varpi : T^*X_{|Y} \times \mathbb{C}_z \longrightarrow T^*Y \times \mathbb{C}_z,$$

the kernel of which is by definition the conormal bundle $T^*_Y X \times \mathbb{C}_z$ of $Y \times \mathbb{C}_z$ in $X \times \mathbb{C}_z$.

8.8.15. Definition **(Non-characteristic property).** Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module with characteristic variety $\operatorname{Char}\widetilde{\mathcal{M}} \subset T^*X \times \mathbb{C}_z$. Let $Y \subset X$ be a submanifold of X. We say that Y is *non-characteristic* with respect to the holonomic $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, or that $\widetilde{\mathcal{M}}$ *is non-characteristic along* Y, if one of the following equivalent conditions is satisfied:

- $(T_Y^*X \times \mathbb{C}_z) \cap \operatorname{Char} \widetilde{\mathcal{M}} \subset T_X^*X \times \mathbb{C}_z$,
- $\varpi : \operatorname{Char} \widetilde{\mathcal{M}}_{|Y\times\mathbb{C}_z} \to T^*Y \times \mathbb{C}_z$ is finite, i.e., proper with finite fibers.

For example, if $\widetilde{\mathcal{M}}$ is holonomic and strict with characteristic variety contained in $\Lambda \times \mathbb{C}_z$, where $\Lambda \subset T^*X$ is Lagrangean (see Section 8.8.g), the condition is achieved if the usual one is, that is, $T_Y^*X \cap \Lambda \subset T_X^*X$.

8.8.16. Theorem **(Coherence of non-characteristic restrictions)**
Assume that $\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_X$*-coherent and that* Y *is non-characteristic with respect to* $\widetilde{\mathcal{M}}$*. Then* ${}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_Y$*-coherent and* $\operatorname{Char} {}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}} \subset \varpi(\operatorname{Char} \widetilde{\mathcal{M}}_{|Y})$.

Sketch of proof. The question is local near a point $x \in Y$. We may therefore assume that $\widetilde{\mathcal{M}}$ has a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$.

(1) Set $F_k({}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}) = \operatorname{image}[\iota_Y^* F_k\widetilde{\mathcal{M}} \to \iota_Y^{*(0)}\widetilde{\mathcal{M}}]$. Then, using Exercise 8.64(2), one shows that $F_\bullet({}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}})$ is a coherent filtration with respect to $F_\bullet({}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{D}}_X)$.

(2) The module $\operatorname{gr}^F {}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}$ is a quotient of $\iota_Y^*\operatorname{gr}^F\widetilde{\mathcal{M}}$, hence its support is contained in $\operatorname{Char} \widetilde{\mathcal{M}}_{|Y}$. By Remmert's Theorem, it is a coherent $\operatorname{gr}^F\widetilde{\mathcal{D}}_Y$-module.

(3) The filtration $F_\bullet {}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}$ is thus a coherent filtration of the $\widetilde{\mathcal{D}}_Y$-module ${}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}$. By Exercise 8.63(1), ${}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_Y$-coherent. Using the coherent filtration above, it is clear that $\operatorname{Char} {}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}} \subset \varpi(\operatorname{Char} \widetilde{\mathcal{M}}_{|Y})$. □

8.8.17. Definition **(Strictly non-characteristic property).** In the setting of Definition 8.8.15, we say that $\widetilde{\mathcal{M}}$ is *strictly non-characteristic along* Y if $\widetilde{\mathcal{M}}$ is non-characteristic along Y and, moreover, the complex ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$ is *strict*, i.e., each of its cohomology modules are strict.

8.8.18. Proposition. *If* $\widetilde{\mathcal{M}}$ *is strictly non-characteristic along* Y*, then* ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_Y^{*(0)}\widetilde{\mathcal{M}}$.

Proof. The result is known to hold for $\mathcal{D}_X$-modules (where the strictness assumption is empty), and therefore it holds after tensoring with $\mathbb{C}[z, z^{-1}]$. As a consequence, ${}_{\mathrm{D}}\iota^{*(j)}\widetilde{\mathcal{M}}$ is a z-torsion module if $j \neq 0$. It is strict if and only if it is zero. □

8.8.19. Remark. Assume that we have inclusions of closed submanifolds $Y \subset H \subset X$ with H of codimension one.

(1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $\mathcal{M}$ *is non-characteristic along* Y *if only if it is non-characteristic along* H *in some neighborhood of* Y *and* ${}_{\mathrm{D}}\iota_H^* = {}_{\mathrm{D}}\iota_H^{*(0)}\mathcal{M}$ *is non-characteristic along* Y.

Indeed, assume that $\mathcal{M}$ is non-characteristic along Y. Then $\varpi_Y : \operatorname{Char} \mathcal{M}_{|Y\times\mathbb{C}_z} \to T^*H|_Y \times \mathbb{C}_z$ is also finite finite, hence $\varpi_H : \operatorname{Char} \mathcal{M}_{|H\times\mathbb{C}_z} \to T^*H \times \mathbb{C}_z$ is finite in some neighborhood of Y, so that $\mathcal{M}$ is non-characteristic along H in this neighborhood. Furthermore, $\operatorname{Char} {}_{\mathrm{D}}\iota_H^{*(0)}\mathcal{M} \subset \varpi_H(\operatorname{Char} \mathcal{M}|_H)$. Therefore, ${}_{\mathrm{D}}\iota_H^{*(0)}\mathcal{M}$ is also non-characteristic along Y. The converse is proved similarly.

(2) For a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ only one direction of the previous equivalence holds, namely, *if* $\widetilde{\mathcal{M}}$ *is strictly non-characteristic along* H *in some neighborhood of* Y

and ${}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_H^{*(0)}\widetilde{\mathcal{M}}$ *is strictly non-characteristic along* Y*, then* $\widetilde{\mathcal{M}}$ *is strictly non-characteristic along* Y*.*

Indeed, the non-characteristic property holds as in (1). Let $\iota_{Y,H} : Y \hookrightarrow H$ the inclusion. Then ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} \simeq {}_{\mathrm{D}}\iota_{Y,H}^*{}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}}$. The assumption implies that ${}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_H^{*(0)}\widetilde{\mathcal{M}}$ and ${}_{\mathrm{D}}\iota_{Y,H}^*{}_{\mathrm{D}}\iota_H*(0)\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{Y,H}^{*(0)}{}_{\mathrm{D}}\iota_H^{*(0)}\widetilde{\mathcal{M}}$, so that

$$ {}_{\mathrm{D}}\iota_H\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_H^{*(0)}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{Y,H}^{*(0)}{}_{\mathrm{D}}\iota_H^{*(0)}\widetilde{\mathcal{M}}, $$

which is strict.

***8.8.20. Definition* ((Strictly) non-characteristic pullback).** Let $f : X' \to X$ be a morphism between complex manifolds and let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. We say that f is *non-characteristic* with respect to $\widetilde{\mathcal{M}}$ if, decomposing f as $p \circ \iota_f : X' \hookrightarrow X' \times X \to X$, the pullback ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}p^{*(0)}\widetilde{\mathcal{M}}$ is non-characteristic along $\iota_f(X')$. We say that f is *strictly non-characteristic* with respect to $\widetilde{\mathcal{M}}$ if, moreover, the complex ${}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}$ is strict.

Due to the chain rule (Exercises 8.37 and 8.39) and to Remark 8.6.7, we note that f is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$ if and only if the pullback ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}p^{*(0)}\widetilde{\mathcal{M}}$ is strictly non-characteristic along $\iota_f(X')$.

8.8.e. Coherence of the pushforward and strictness

***8.8.21. Theorem* (Coherence of the pushforward).** *Let* $f : X \to Y$ *be a holomorphic map between complex manifolds and let* $\widetilde{\mathcal{M}}$ *be a coherent* $\widetilde{\mathcal{D}}_X$*-module. Assume that* $\widetilde{\mathcal{M}}$ *admits a coherent filtration* $F_\bullet\widetilde{\mathcal{M}}$*. Then, if* f *is proper on the support of* $\widetilde{\mathcal{M}}$*, the pushforward complex* ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ *has* $\widetilde{\mathcal{D}}_Y$*-coherent cohomology.*

Proof. Assume first that $\widetilde{\mathcal{M}}$ is an induced right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ where $\widetilde{\mathcal{L}}$ is a coherent $\widetilde{\mathcal{O}}_X$-module such that f is proper on its support. Due to the formula of Exercise 8.53(3), the result follows from Grauert's direct image theorem. As a consequence, the same result holds for any bounded complex of such induced right $\widetilde{\mathcal{D}}_X$-modules.

For $\widetilde{\mathcal{M}}$ arbitrary, it is enough by Remark 8.7.5(2) to prove the coherence of ${}_{\mathrm{D}}f_*^{(j)}\widetilde{\mathcal{M}}$ for $j \in [-\dim X, 2\dim X]$. Since the $\widetilde{\mathcal{D}}_Y$-coherence is a local property on Y, it is enough to prove the coherence property in the neighbourhood of any $y \in Y$, and therefore it is enough to show the existence, in the neighbourhood of the compact set $f^{-1}(y)$, of a resolution of $\widetilde{\mathcal{M}}_{-N-1} \to \cdots \to \widetilde{\mathcal{M}}_0 \to \widetilde{\mathcal{M}} \to 0$ of sufficiently large length $N+2$, such that $\widetilde{\mathcal{M}}_j$ is a coherent induced $\widetilde{\mathcal{D}}_X$-module for $j = -N, \dots, 0$ and f proper on $\operatorname{Supp}\widetilde{\mathcal{M}}_j$.

Since $f^{-1}(y) \cap \operatorname{Supp}\widetilde{\mathcal{M}}$ is compact, there exists p such that $F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ is onto in some neighbourhood of $f^{-1}(y)$ (i.e., the coherent $\widetilde{\mathcal{O}}_X$-module $F_p\widetilde{\mathcal{M}}$ generates $\widetilde{\mathcal{M}}$ as a $\widetilde{\mathcal{D}}_X$-module). Set $F_q(F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X) = F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} F_{q-p}\widetilde{\mathcal{D}}_X$. This is a coherent filtration of $F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, which therefore induces a coherent filtration on $\operatorname{Ker}[F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}]$. Continuing this way $N+2$ times, we obtained a resolution of length $N+2$ of $\widetilde{\mathcal{M}}$ by coherent induced right $\widetilde{\mathcal{D}}_X$-modules on some neighbourhood of $f^{-1}(y)$, all supported on $\operatorname{Supp}\widetilde{\mathcal{M}}$. $\square$

Let us assume that $\widetilde{\mathcal{M}}$ is strict, that is, $\widetilde{\mathcal{M}}$ is the Rees module $R_F\mathcal{M}$ of a coherent filtration $F_\bullet\mathcal{M}$ on a coherent $\mathcal{D}_X$-module $\mathcal{M}$ (see the proof of Proposition 8.8.5). In general, for $f: X\to Y$ proper (or proper on $\operatorname{Supp}\widetilde{\mathcal{M}}$), there is no reason that the pushforward complex ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ is *strict*, i.e., each of its cohomology modules is strict. However, we will see that this property is satisfies when $\widetilde{\mathcal{M}}$ underlies a polarizable Hodge module (Theorem 14.3.2(1)). We make explicit the meaning of this property in terms of filtered complex. For that purpose, we make use of Formulas (8.52 $*$) and (8.52 $**$) for the pushforward.

Let $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ be a strict coherent left $\widetilde{\mathcal{D}}_X$-module. We can write

$$\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = R_F(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y),$$

with

$$F_p(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)=\sum_{i+j=p}(F_i\mathcal{M})\otimes_{f^{-1}\mathcal{O}_Y}(f^{-1}F_j\mathcal{D}_Y),$$

and therefore

$$\widetilde{\Omega}_X^{n+\bullet}\otimes(\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)=R_F\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)$$

with

$$F_p\big(\Omega_X^{n+k}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)=\Omega_X^{n+k}\otimes F_{p-n-k}(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y).$$

There exists a spectral sequence

$$\text{(8.8.22)}\quad E_1^{p,q}=R^{p+q}f_*\Big(\operatorname{gr}^F_{-p}\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)\Big)\\ \Longrightarrow \operatorname{gr}^F_{-p}R^{p+q}f_*\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big).$$

8.8.23. Proposition **(Degeneration at E_1).** *If $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strict and if f is proper on* $\operatorname{Supp}\widetilde{\mathcal{M}}$*, then the complex ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ is strict if and only if the spectral sequence* (8.8.22) *degenerates at E_1, that is, for each $k,p\in\mathbb{Z}$, the natural morphism*

$$R^kf_*\Big(F_p\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)\Big)\longrightarrow R^kf_*\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)$$

is injective.

For example, if Y is reduced to a point, the strictness of ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ is equivalent to the degeneration at E_1 of the *Hodge-to-de Rham spectral sequence*

$$H^k\big(X,\operatorname{gr}^F_{-p}(\Omega_X^{n+\bullet}\otimes\mathcal{M})\big)\implies \operatorname{gr}^F_{-p}H^k\big(X,(\Omega_X^{n+\bullet}\otimes\mathcal{M})\big).$$

Sketch of proof of Proposition **8.8.23.** The image of the morphism in the proposition is by definition $F_pR^kf_*\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)$. Therefore, injectivity is equivalent to the equality

$$R^kf_*\Big(R_F\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big)\Big)=R_FR^kf_*\big(\Omega_X^{n+\bullet}\otimes(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big),$$

which in turn is equivalent to the left-hand side being $\mathbb{C}[z]$-flat. $\square$

8.8.24. Laumon's formula. We give another consequence of strictness of ${}_{\mathrm{D}}f_*\mathcal{M}$. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module. The behaviour of grading with respect to pushforward is treated in Exercises 8.55 and 8.56. For example, for right $\mathcal{D}_X$-modules, Laumon's formula is that, if $f : X \to Y$ is a holomorphic map and *if* ${}_{\mathrm{D}}f_*\mathcal{M}$ *is a strict complex*, then for every i,

$$\mathrm{gr}^F{}_{\mathrm{D}}f_*^{(i)}\mathcal{M} \simeq H^i\boldsymbol{R}f_*\big(\mathrm{gr}^F\mathcal{M}\otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y\big). \tag{8.8.24 $*$}$$

8.8.f. Künneth formula. Assume that X, Y are compact complex manifolds. Let $\widetilde{\mathcal{M}}_X, \widetilde{\mathcal{M}}_Y$ be strict coherent $\widetilde{\mathcal{D}}$-modules. The Künneth formula compares the de Rham cohomology of the external product $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$ with that of the factors.

***8.8.25. Theorem* (Künneth formula).** *Let $\widetilde{\mathcal{M}}_X, \widetilde{\mathcal{M}}_Y$ be coherent $\widetilde{\mathcal{D}}$-modules having a coherent filtration. Assume that $\boldsymbol{R}\Gamma(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y)$ is strict, i.e., has strict cohomologies. Then for each k we have*

$$\boldsymbol{H}^k\big(X\times Y, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y)\big) \simeq \bigoplus_{i+j=k}\boldsymbol{H}^i(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_X)\otimes_{\widetilde{\mathbb{C}}}\boldsymbol{H}^j(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y). \tag{8.8.25 $*$}$$

Note that, if $\widetilde{\mathcal{D}} = R_F\mathcal{D}$, the existence of a coherent filtration for $\widetilde{\mathcal{M}}$ is ensured by Proposition 8.8.5(2). Note also that the roles of $\widetilde{\mathcal{M}}_X$ and $\widetilde{\mathcal{M}}_Y$ can be exchanged.

Proof. We denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let us assume that $\widetilde{\mathcal{M}}_X = \widetilde{\mathcal{L}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ and $\widetilde{\mathcal{M}}_Y = \widetilde{\mathcal{L}}_Y \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y$ are induced $\widetilde{\mathcal{D}}$-modules such that $\widetilde{\mathcal{L}}_X$ is an inductive limit of coherent $\widetilde{\mathcal{O}}_X$-modules and $\widetilde{\mathcal{L}}_Y$ is strict. One computes that

$$\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y \simeq (\widetilde{\mathcal{L}}_X\boxtimes_{\widetilde{\mathcal{O}}}\widetilde{\mathcal{L}}_Y)\otimes_{\widetilde{\mathcal{O}}_{X\times Y}}\widetilde{\mathcal{D}}_{X\times Y},$$

and thus

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y) \simeq \widetilde{\mathcal{L}}_X\boxtimes_{\widetilde{\mathcal{O}}}\widetilde{\mathcal{L}}_Y = p^*\widetilde{\mathcal{L}}_X\otimes_{q^{-1}\widetilde{\mathcal{O}}_Y} q^{-1}\widetilde{\mathcal{L}}_Y.$$

By the projection formula (see e.g. [**KS90**, Prop. 2.6.6]) and using the strictness of $\widetilde{\mathcal{L}}_Y$, we obtain

$$\boldsymbol{R}q_*\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y) \simeq (\boldsymbol{R}q_*p^*\widetilde{\mathcal{L}}_X)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{L}}_Y,$$

and by Exercise 8.73 the latter term is isomorphic to

$$(\widetilde{\mathcal{O}}_Y\otimes_{\widetilde{\mathbb{C}}}\boldsymbol{R}\Gamma(X,\widetilde{\mathcal{L}}_X))\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{L}}_Y \simeq \boldsymbol{R}\Gamma(X,\widetilde{\mathcal{L}}_X)\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathcal{L}}_Y. \tag{8.8.26}$$

Applying once more the projection formula we finally obtain in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathbb{C}})$:

$$\begin{aligned}\boldsymbol{R}\Gamma\big({}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y)\big) &\simeq \boldsymbol{R}\Gamma\big(Y, \boldsymbol{R}q_*\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y)\big)\\ &\simeq \boldsymbol{R}\Gamma\big(Y, \boldsymbol{R}\Gamma(X,\widetilde{\mathcal{L}}_X)\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathcal{L}}_Y\big)\\ &\simeq \boldsymbol{R}\Gamma(X,\widetilde{\mathcal{L}}_X)\overset{\boldsymbol{L}}{\otimes}_{\widetilde{\mathbb{C}}}\boldsymbol{R}\Gamma(Y,\widetilde{\mathcal{L}}_Y)\\ &\simeq \boldsymbol{R}\Gamma(X,{}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_X)\overset{\boldsymbol{L}}{\otimes}_{\widetilde{\mathbb{C}}}\boldsymbol{R}\Gamma(Y,{}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y).\end{aligned} \tag{8.8.27}$$

Let now $\widetilde{\mathcal{M}}_X$ and $\widetilde{\mathcal{M}}_Y$ be as in the theorem. Each term of their canonical resolution (Proposition 8.5.2) satisfies the corresponding assumptions on $\widetilde{\mathcal{L}}_X, \widetilde{\mathcal{L}}_Y$ and thus

(8.8.27) holds for each term of the corresponding resolution of $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$. As a consequence, (8.8.27) holds for $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$. Strictness of $\boldsymbol{R}\Gamma(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y)$ then implies Künneth formula (8.8.25 $*$) (see e.g. [**God64**, Th. 5.5.2]). □

8.8.g. Holonomic $\widetilde{\mathcal{D}}_X$-modules and duality

8.8.28. Definition **(Smooth $\widetilde{\mathcal{D}}_X$-modules).** A coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is said to be *smooth* if it is $\widetilde{\mathcal{O}}_X$-locally free.

In particular, a smooth $\widetilde{\mathcal{D}}_X$-module is strict, and its characteristic variety is equal to $(T^*_X X) \times \widetilde{\mathbb{C}}_z$. (See Exercise 8.69 for the converse.)

8.8.29. Definition **(Holonomic $\widetilde{\mathcal{D}}_X$-modules).** A coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is said to be holonomic if $\mathrm{Char}\,\widetilde{\mathcal{M}} \subset \Lambda \times \mathbb{C}_z$, where Λ is a Lagrangian closed subvariety of T^*X.

8.8.30. Remark. By Remarks 8.8.9 and 8.8.14, this is equivalent to asking that $\mathcal{M}''$ is holonomic and that the support of $\mathcal{M}'$ is Lagrangian in T^*X. In particular, if $\widetilde{\mathcal{M}}$ is strict, holonomicity of $\widetilde{\mathcal{M}}$ is equivalent to that of the underlying $\mathcal{D}_X$-module $\mathcal{M}$.

Such a Lagrangian subvariety is the union of its irreducible components, each of which is usually written as $T^*_Z X$, where Z is a closed irreducible subvariety of X and $T^*_Z X$ means the closure, in the cotangent space T^*X of the conormal bundle $T^*_{Z^o} X$ of the smooth part Z^o of Z. It is also known that, due to the existence of stratifications satisfying Whitney condition (a), there exist a locally finite family $(Z^o_i)_{i\in I}$ of locally closed sub-manifolds Z^o_i of Z, with analytic closure and one of them being Z^o, such that $T^*_Z X \subset \bigsqcup_i T^*_{Z^o_i} X$.

For example, a smooth $\widetilde{\mathcal{D}}_X$-module, or a coherent $\widetilde{\mathcal{D}}_X$-module as in Exercise 8.69 or 8.70, is holonomic.

8.8.31. Pushforward of a holonomic $\widetilde{\mathcal{D}}_X$-module. Assume that the coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ has a coherent filtration. For example, assume that $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$ and $\widetilde{\mathcal{M}}$ is strict (Proposition 8.8.5(2)). Then, the pushforward of $\widetilde{\mathcal{M}}$ by a proper holomorphic map $f : X \to Y$ has coherent cohomology. Moreover, a theorem of Kashiwara [**Kas76**] complements Theorem 8.8.21 with an estimate of the characteristic variety of the pushforward cohomology $\widetilde{\mathcal{D}}_Y$-modules in terms of the characteristic variety of the source $\widetilde{\mathcal{D}}_X$-module. This estimate shows that holonomicity is preserved by proper pushforward. (The theorem of Kashiwara is proved for holonomic $\mathcal{D}_X$-modules, but it extends in a straightforward way to holonomic $\widetilde{\mathcal{D}}_X$-modules.) Therefore, the pushforward by a proper holomorphic map of a strict coherent $\widetilde{\mathcal{D}}_X$-module which is holonomic has holonomic cohomologies when $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$.

The $\widetilde{\mathcal{D}}_X$-modules $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$. Holonomicity is related with the vanishing of the $\widetilde{\mathcal{D}}_X$-modules $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$. If $\widetilde{\mathcal{M}}$ is a right $\widetilde{\mathcal{D}}_X$-module, $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$ is equipped with the left action coming from the left structure of $\widetilde{\mathcal{D}}_X$, and the corresponding right $\widetilde{\mathcal{D}}_X$-module is $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\omega}_X \otimes \widetilde{\mathcal{D}}_X)$ (see Section 8.3.5) by playing with the two right structures (triv and tens) on $\widetilde{\omega}_X \otimes \widetilde{\mathcal{D}}_X$.

8.8.32. Lemma. *For a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we have $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)=0$ for $i\geqslant 2n+1$.*

Proof. One can argue as in [**Bjö93**, §I.7]. □

For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the right $\widetilde{\mathcal{D}}_X$-modules $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\omega}_X\otimes\widetilde{\mathcal{D}}_X)$ are thus the cohomology modules of a complex $\boldsymbol{D}\widetilde{\mathcal{M}}:=\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\omega}_X\otimes\widetilde{\mathcal{D}}_X)$ in the derived category $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$.

The case of $\mathcal{D}_X$-modules is most useful. We will recall some fundamental results.

8.8.33. Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We have*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0\quad\textit{for } i\geqslant n+1.$$ □

8.8.34. Theorem **(see [Kas76]).** *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $x\in\operatorname{Supp}\mathcal{M}$. Then*

$$2n-\dim_x\operatorname{Char}\mathcal{M}=\inf\{i\in\mathbb{N}\mid\mathcal{E}xt^i_{\mathcal{D}_{X,x}}(\mathcal{M}_x,\mathcal{D}_{X,x})\neq0\}.$$ □

8.8.35. Corollary. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $\mathcal{M}$ is holonomic if and only if $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for $i\neq\dim X$.*

If $\mathcal{M}$ is a right holonomic $\mathcal{D}_X$-module, the $\mathcal{D}_X$-module $\mathcal{E}xt^{\dim X}_{\mathcal{D}_X}(\mathcal{M},\omega_X\otimes\mathcal{D}_X)$ is called the *dual* of $\mathcal{M}$, and is the unique nonzero cohomology of the complex $\boldsymbol{D}\mathcal{M}$. We often identify both objects. For a left $\mathcal{D}_X$-module $\mathcal{M}$, we define the left $\mathcal{D}_X$-module or bounded complex $\boldsymbol{D}(\mathcal{M})$ as $\boldsymbol{D}(\mathcal{M}^{\mathrm{right}})^{\mathrm{left}}$.

8.8.36. Theorem **(Bi-duality, see [Kas76]).** *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then its dual module $\boldsymbol{D}\mathcal{M}$ is holonomic and the natural functorial morphism from $\mathcal{M}$ to its bi-dual module $\boldsymbol{D}\boldsymbol{D}\mathcal{M}$ is an isomorphism.* □

Let us now consider holonomicity and duality for strict coherent $\widetilde{\mathcal{D}}_X$-modules. Recall that, for any coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ is also coherent for any i.

8.8.37. Definition **(Strictly holonomic $\widetilde{\mathcal{D}}_X$-modules).** Let $\widetilde{\mathcal{M}}$ be a holonomic $\widetilde{\mathcal{D}}_X$-module. We say that $\widetilde{\mathcal{M}}$ is *strictly holonomic* if $\widetilde{\mathcal{M}}$ is strict and $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ is a strict $\widetilde{\mathcal{D}}_X$-module for every i.

If $\widetilde{\mathcal{M}}$ is strictly holonomic, then $\mathcal{E}xt^n_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ takes the form $R_F\mathcal{M}^\vee$ for some holonomic $\mathcal{D}_X$-module $\mathcal{M}^\vee$ and a unique coherent filtration on it. The complex $\boldsymbol{D}\widetilde{\mathcal{M}}$ has thus a unique nonzero cohomology module, which is the $\widetilde{\mathcal{D}}_X$-module obtained after side-changing from $\mathcal{E}xt^n_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$. For example, if $\widetilde{\mathcal{M}}$ is a right $\widetilde{\mathcal{D}}_X$-module, then $\boldsymbol{D}\widetilde{\mathcal{M}}=\mathcal{E}xt^n_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\omega}_X\otimes\widetilde{\mathcal{D}}_X)$. Then there exists a unique coherent filtration $F_\bullet\boldsymbol{D}\mathcal{M}$ such that $\boldsymbol{D}\widetilde{\mathcal{M}}=R_F\boldsymbol{D}\mathcal{M}$.

We obtain the following results from Exercise 5.2.

8.8.38. Proposition **(Cohen-Macaulay property of the graded module)**

Assume that $\widetilde{\mathcal{M}}$ is strictly holonomic. Then the following properties hold.

(1) $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)=0$ *for* $i\neq n=\dim X$*;*
(2) $\mathcal{M}^\vee$ *is nothing but* $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)$*;*
(3) *if* $\widetilde{\mathcal{M}}$ *is written as* $R_F\mathcal{M}$*, then*

$$\mathrm{gr}^F\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)\simeq\mathcal{E}xt^n_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{M},\mathrm{gr}^F\mathcal{D}_X)=:(\mathrm{gr}^F\mathcal{M})^\vee,$$

(4) *and (Cohen-Macaulay property)* $\mathcal{E}xt^i_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{M},\mathrm{gr}^F\mathcal{D}_X)=0$ *for* $i\neq n$*;*
(5) *if for example* $\widetilde{\mathcal{M}}$ *is a right* $\widetilde{\mathcal{D}}_X$*-module, then we obtain the following isomorphism of right* $\mathrm{gr}^F\mathcal{D}_X$*-module (see Caveat* 8.2.7*)*

$$\mathrm{gr}^F(\boldsymbol{D}\mathcal{M})\simeq\omega_X\otimes\mathrm{inv}^*(\mathrm{gr}^F\mathcal{M})^\vee(-n).$$

Proof. Let us check Properties (3) and (4). Property (2) is obtained by a similar argument, and the other ones are easy to check. There exists a natural morphism (see e.g. [**Kas03**, (A.10)])

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)&\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X)\\&\longrightarrow\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X))=\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\mathrm{gr}^F\mathcal{D}_X),\end{aligned}$$

which is an isomorphism since $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, where z acts by zero on $\mathrm{gr}^F\mathcal{D}_X$ (check this with $\widetilde{\mathcal{M}}=\widetilde{\mathcal{D}}_X$). On the other hand, the "associativity law" of [**Kas03**, p. 241] provides an isomorphism in the derived category $\mathsf{D}^+(\mathrm{gr}^F\mathcal{D}_X)$

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X}&(\widetilde{\mathcal{M}}\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X),\mathrm{gr}^F\mathcal{D}_X)\\&=\boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}(\widetilde{\mathcal{M}}\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\mathrm{gr}^F\mathcal{D}_X,\mathrm{gr}^F\mathcal{D}_X)\\&\simeq\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}\big(\widetilde{\mathcal{M}},\boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{D}_X,\mathrm{gr}^F\mathcal{D}_X)\big)\\&=\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\mathrm{gr}^F\mathcal{D}_X).\end{aligned}$$

Since $\widetilde{\mathcal{M}}$ is assume to be strict (by its strict holonomicity), it follows that

$$\widetilde{\mathcal{M}}\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X)=\mathrm{gr}^F\mathcal{M},$$

and we finally obtain an isomorphism in $\mathsf{D}^+(\mathrm{gr}^F\mathcal{D}_X)$:

$$\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\mathrm{gr}^F\mathcal{D}_X\simeq\boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{M},\mathrm{gr}^F\mathcal{D}_X).$$

Strict holonomicity of $\widetilde{\mathcal{M}}$ also implies that $\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ has nonzero cohomology in degree n at most, and this cohomology is strict. Since the left-hand side also reads $\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)\otimes_{\mathbb{C}[z]}(\mathbb{C}[z]/z\mathbb{C}[z])$, it has thus cohomology in degree n at most, which reads $\mathrm{gr}^F\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)$. Therefore the right-hand side also has cohomology in degree n at most, that is, $\mathcal{E}xt^i_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{M},\mathrm{gr}^F\mathcal{D}_X)=0$ for $i\neq n$, and we obtain the isomorphism

$$\mathrm{gr}^F\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)\simeq\mathcal{E}xt^n_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{M},\mathrm{gr}^F\mathcal{D}_X).\qquad\square$$

8.8.39. Proposition **(Indpendence of strict holonomicity with respect to embeddings)**
Let $\iota:Z\hookrightarrow X$ *be the closed inclusion of a smooth submanifold* Z *of* X *and let* $\widetilde{\mathcal{M}}$

be a holonomic $\widetilde{\mathcal{D}}_Z$-module. Then $\widetilde{\mathcal{M}}$ is strictly holonomic if and only if the holonomic $\widetilde{\mathcal{D}}_X$-module ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$ is so.

Proof. The question is local and we can assume that $X = Z \times \mathbb{C}^r$ with coordinates $x_1, \dots, x_r$ on $\mathbb{C}^r$, and that ι is the inclusion induced by $\{0\} \hookrightarrow \mathbb{C}^r$. Using the notation of external product as in Section 8.6.b, we can write ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_{x_1}, \dots, \widetilde{\partial}_{x_r}]$. The proof of the following lemma is the subject of Exercise 8.74.

8.8.h. Duality, filtration and de Rham. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, a celebrated theorem of Kashiwara [**Kas75**] asserts that the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is a complex with constructible cohomology. Furthermore, the de Rham functors transforms duality of holonomic $\mathcal{D}_X$-modules to Poincaré-Verdier duality of constructible complexes: this is the local duality theorem (see [**Nar04**] for an account of various proofs of this theorem). Although there is no reasonable notion of a constructible complex with filtration (for example, one does not expect that ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is a constructible complex of $\mathbb{C}[z]$-modules; see however [**MFS13, MFS19**] for such a notion), one can regard, for a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, the filtered de Rham complex as a filtered differential complex and one has a "local duality theorem" in this context (see [**Sai88**, §2.4]). Furthermore, the notion of perversity is meaningful in this context (see [**BSY98**]).

In this section, we focus on the graded object $\mathrm{gr}^F\mathcal{M}$ attached to a coherently filtered right $\mathcal{D}_X$-module $\mathcal{M}$. Recall (see Section 8.4.9) that the Spencer complex $\mathrm{Sp}(\mathcal{M})$ is naturally filtered, so that we can consider the graded complex $\mathrm{gr}^F \mathrm{Sp}(\mathcal{M})$. We will prove a coherent version of the local duality theorem for $\mathrm{gr}^F \mathrm{Sp}(\mathcal{M})$, that makes use of the Grothendieck-Serre duality functor. We start with the following observation:

8.8.40. Lemma. *For a filtered right $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, the filtered Spencer complex satisfies*

$$\mathrm{gr}^F \mathrm{Sp}(\mathcal{M}) \simeq \mathrm{gr}^F\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{gr}^F\mathcal{D}_X} \mathcal{O}_X.$$

In other words, one can interpret the graded complex $\mathrm{gr}^F \mathrm{Sp}(\mathcal{M})$ as the $\mathcal{O}$-module pullback by the inclusion $X \hookrightarrow T^*X$ (zero section) of $\mathrm{gr}^F\mathcal{M}$. In particular, if $\mathcal{M}$ is coherently filtered, $\mathrm{gr}^F \mathrm{Sp}(\mathcal{M})$ has $\mathcal{O}_X$-coherent cohomology (check this for $\mathcal{M} = \mathcal{D}_X$).

Proof. Let us set $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. By Exercise 8.24 we have a natural isomorphism $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}(\widetilde{\mathcal{M}})$. Since all involved $\mathbb{C}[z]$-modules are flat, we obtain, by tensoring with $\mathbb{C}[z]/z\mathbb{C}[z]$, the isomorphism

$$\mathrm{gr}^F\mathcal{M} \otimes_{\mathrm{gr}^F\mathcal{D}_X} \mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X) \simeq \mathrm{gr}^F \mathrm{Sp}(\mathcal{M}).$$

Recall (see Exercise 8.21) that $\mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X)$ is a resolution of $\mathcal{O}_X$ by locally free $\mathrm{gr}^F\mathcal{D}_X$-modules. The conclusion follows. $\square$

If $(\mathcal{M}, F_\bullet\mathcal{M})$ is a coherently filtered right $\mathcal{D}_X$-module, we define the dual $\mathrm{gr}^F\mathcal{D}_X$-module of $\mathrm{gr}^F\mathcal{M}$ as the object of $\mathsf{D}^+_{\mathrm{coh}}(\mathrm{gr}^F\mathcal{D}_X)$:

$$\begin{aligned}\boldsymbol{D}\mathrm{gr}^F\mathcal{M} &:= \boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}\big(\mathrm{gr}^F\mathcal{M}, \mathrm{gr}^F(\omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X)\big)\\ &\simeq \mathrm{inv}^*\boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}\big(\mathrm{gr}^F\mathcal{M}, \mathrm{gr}^F(\omega_X)\otimes_{\mathrm{gr}^F\mathcal{O}_X}\mathrm{gr}^F(\mathcal{D}_X)\big) \text{ (see Caveat 8.2.7)},\end{aligned}$$

where we recall that $\mathrm{gr}^F\mathcal{O}_X = \mathcal{O}_X$ is graded of degree zero and $\mathrm{gr}^F\omega_X = \omega_X$ is graded of degree $-n$. Therefore, if $\mathcal{M}$ is strictly holonomic we have, according to Proposition 8.8.38(5),

$$\boldsymbol{D}\mathrm{gr}^F\mathcal{M} \simeq \mathrm{gr}^F\boldsymbol{D}\mathcal{M}.$$

We regard $\mathrm{gr}^F\mathrm{Sp}(\mathcal{M})$ as an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ (Lemma 8.8.40) and its Grothendieck-Serre dual object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ is defined as

$$\boldsymbol{D}(\mathrm{gr}^F\mathrm{Sp}\,\mathcal{M}) := \boldsymbol{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{gr}^F\mathrm{Sp}(\mathcal{M}), \mathrm{gr}^F(\omega_X)\big).$$

8.8.41. Proposition. *If $\widetilde{\mathcal{M}}$ is strictly holonomic, we have an isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ which depends functorially on $\widetilde{\mathcal{M}}$:*

$$\boldsymbol{D}(\mathrm{gr}^F\mathrm{Sp}\,\mathcal{M}) \simeq \mathrm{gr}^F\mathrm{Sp}(\boldsymbol{D}\mathcal{M}).$$

Proof. We will use an argument similar to that of the proof of Proposition 8.8.38, by justifying the following sequence of isomorphisms:

$$\begin{aligned}\boldsymbol{D}(\mathrm{gr}^F\mathrm{Sp}\,\mathcal{M}) &= \boldsymbol{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{gr}^F\mathrm{Sp}(\mathcal{M}), \mathrm{gr}^F(\omega_X)\big)\\ &\overset{\text{(a)}}{\simeq} \boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}\big(\mathrm{gr}^F\mathcal{M}, \mathrm{gr}^F(\omega_X\otimes\mathcal{D}_X)\big)\otimes^{\boldsymbol{L}}_{\mathrm{gr}^F\mathcal{D}_X}\mathrm{gr}^F\mathcal{O}_X\\ &\overset{\text{(b)}}{\simeq} \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X)\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X)\otimes^{\boldsymbol{L}}_{\mathrm{gr}^F\mathcal{D}_X}\mathrm{gr}^F\mathcal{O}_X\\ &\overset{\text{(c)}}{\simeq} \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X)\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X/z\widetilde{\mathcal{O}}_X)\\ &\overset{\text{(d)}}{\simeq} (\boldsymbol{D}\widetilde{\mathcal{M}}\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{O}}_X)\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{O}}_X/z\widetilde{\mathcal{O}}_X)\\ &\overset{\text{(e)}}{\simeq} \mathrm{Sp}(\boldsymbol{D}\widetilde{\mathcal{M}})\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{O}}_X/z\widetilde{\mathcal{O}}_X)\\ &\overset{\text{(f)}}{\simeq} \mathrm{gr}^F\mathrm{Sp}(\boldsymbol{D}\mathcal{M}).\end{aligned}$$

For (a), we argue with [**Kas03**, (A.10)] and the "associativity law" to obtain

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{D}_X}&\big(\mathrm{gr}^F\mathcal{M}, \mathrm{gr}^F(\omega_X\otimes\mathcal{D}_X)\big)\otimes^{\boldsymbol{L}}_{\mathrm{gr}^F\mathcal{D}_X}\mathrm{gr}^F\mathcal{O}_X\\ &\simeq \boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{O}_X}\big(\mathrm{gr}^F\mathcal{M}\otimes^{\boldsymbol{L}}_{\mathrm{gr}^F\mathcal{D}_X}\mathrm{gr}^F\mathcal{O}_X, \mathrm{gr}^F(\omega_X\otimes\mathcal{D}_X)\otimes^{\boldsymbol{L}}_{\mathrm{gr}^F\mathcal{D}_X}\mathrm{gr}^F\mathcal{O}_X\big),\end{aligned}$$

and by Lemma 8.8.40, the latter term is identified with

$$\boldsymbol{R}\mathcal{H}om_{\mathrm{gr}^F\mathcal{O}_X}\big(\mathrm{gr}^F\mathrm{Sp}\,\mathcal{M}, \mathrm{gr}^F\mathrm{Sp}(\omega_X\otimes\mathcal{D}_X)\big) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{gr}^F\mathrm{Sp}(\mathcal{M}), \mathrm{gr}^F(\omega_X)\big).$$

For (b), the argument is the same as in the proof of Proposition 8.8.38 and similar to the above. The isomorphism (c) is then clear, and (d) is obtained by identifying $\bullet\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X/z\widetilde{\mathcal{O}}_X)$ with $(\bullet\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{O}}_X)\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{O}}_X/z\widetilde{\mathcal{O}}_X)$. Then (e) follows from

Lemma 8.8.40 and (f) from the fact that each term of $\mathrm{Sp}(\boldsymbol{D}\widetilde{\mathcal{M}})$ is strict, due to the strictness of $\boldsymbol{D}\widetilde{\mathcal{M}}$. ☐

8.8.i. Duality and operations

***8.8.42. Lemma* (Duality and external product).** *For $i = 1,2$, let $\mathcal{M}_i$ be a coherent $\mathcal{D}_{X_i}$-module on the complex manifold X_i. Then there exists a natural isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X_1\times X_2})$:*

$$\boldsymbol{D}(\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2) \simeq (\boldsymbol{D}\mathcal{M}_1) \boxtimes_{\mathcal{D}} (\boldsymbol{D}\mathcal{M}_2).$$

It is thus enough to prove that

$$\boldsymbol{D}(\mathbb{C}[\partial_{x_1},\dots,\partial_{x_r}]) \simeq \mathbb{C}[\partial_{x_1},\dots,\partial_{x_r}]. \tag{8.8.43}$$

Indeed, this implies that, in this local setting, ${}_{\mathrm{D}}\iota_*\boldsymbol{D}\mathcal{M} \simeq \boldsymbol{D}({}_{\mathrm{D}}\iota_*\mathcal{M})$, and one concludes by observing that $\mathcal{M}$ is strict if and only if ${}_{\mathrm{D}}\iota_*\mathcal{M}$ is strict.

For the proof of (8.8.43), one can use Lemma 8.8.42 once more to reduce to the case where $r = 1$. Then we have a simple two-term free resolution of $\mathbb{C}[\partial_x]$ which immediately gives the result. ☐

***8.8.44. Proposition* (Duality and pushforward).** *Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module admitting a coherent F-filtration and let $f : X \to X'$ be a proper morphism. Then there exists a functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})$:*

$${}_{\mathrm{D}}f_*(\boldsymbol{D}\widetilde{\mathcal{M}}) \simeq \boldsymbol{D}({}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}).$$

Indication of proof. For $\mathcal{D}_X$-modules, this is a classical result. One can find a proof in [**Bjö93**, §II.11] and [**Kas03**, §4.9] for the analytic case, and in [**HTT08**, §2.7.2] for the algebraic case, for example. The adaptation to $\widetilde{\mathcal{D}}_X$-modules is straightforward. The main point is to adapt the construction of the trace morphism ${}_{\mathrm{D}}f_*\widetilde{\omega}_X[\dim X] \to \widetilde{\omega}_{X'}[\dim X']$. In the analytic setting that we consider here, we argue as for the C^∞ Spencer complex of Section 8.4.13, by replacing the bicomplex of currents $(\mathfrak{Db}^{p,q}, \mathrm{d}', \mathrm{d}'')$ with the bicomplex $(\widetilde{\mathfrak{Db}}^{p,q}, \widetilde{\mathrm{d}}', \mathrm{d}'')$ having terms $\widetilde{\mathfrak{Db}}^{p,q} = \widetilde{\Omega}^p_X \wedge \mathfrak{Db}^{(0,q)}_X[z]$. ☐

***8.8.45. Corollary* (A criterion for the commutation of $\boldsymbol{D}$ with ${}_{\mathrm{D}}f_*^{(k)}$)**

In the setting of Proposition 8.8.44, assume that

- *the $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is strictly holonomic,*
- *the decomposition theorem holds for the pushforward complex ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$, that is, ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}} \simeq \bigoplus_{k\in\mathbb{Z}} {}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}[-k]$, and*
- *each holonomic $\widetilde{\mathcal{D}}_{X'}$-module ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is strictly holonomic.*

Then there exists an isomorphism, for each $k \in \mathbb{Z}$,

$${}_{\mathrm{D}}f_*^{(-k)}(\boldsymbol{D}\widetilde{\mathcal{M}}) \simeq \boldsymbol{D}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}).$$

Proof. By assumption, $\boldsymbol{D}\widetilde{\mathcal{M}}$ has cohomology in degree zero only, so that the k-th cohomology of ${}_{\mathrm{D}}f_*(\boldsymbol{D}\widetilde{\mathcal{M}})$ is ${}_{\mathrm{D}}f_*^{(k)}(\boldsymbol{D}\widetilde{\mathcal{M}})$. On the other hand, the assumption also implies that the complex $\boldsymbol{D}({}_{\mathrm{D}}f_*\widetilde{\mathcal{M}})$ decomposes as $\bigoplus_{k\in\mathbb{Z}} \boldsymbol{D}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})[k]$, where $\boldsymbol{D}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})$

has cohomology in degree zero only. The isomorphism of Proposition 8.8.44 yields the conclusion after taking the k-th cohomology of both sides. □

For a morphism $f : X' \to X$ of complex manifolds, we denote by $\mathsf{Mod}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X)$ the full subcategory of $\mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ consisting of coherent $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}$ such that ${}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}$ has $\widetilde{\mathcal{D}}_{X'}$-coherent cohomology. We then define $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X)$ as the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ consisting of complexes having cohomology in $\mathsf{Mod}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X)$.

- If f is smooth, then $\mathsf{Mod}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X) = \mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$.
- If f decomposes as $X' \xrightarrow{g} Z \xrightarrow{h} X$, we define $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh},g,h}(\widetilde{\mathcal{D}}_X)$ as the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ consisting of complexes $\widetilde{\mathcal{M}}$ with coherent cohomology such that ${}_{\mathrm{D}}h^*\widetilde{\mathcal{M}}$ has $\widetilde{\mathcal{D}}_Z$-coherent cohomology and ${}_{\mathrm{D}}f^*\widetilde{\mathcal{M}} \simeq {}_{\mathrm{D}}g^*({}_{\mathrm{D}}h^*\widetilde{\mathcal{M}})$ has $\widetilde{\mathcal{D}}_{X'}$-coherent cohomology.

The next theorem is classical in the theory of $\mathcal{D}_X$-modules (see e.g. [**HTT08**, §2.7.1]) and its proof can be adapted in a straightforward way to $\widetilde{\mathcal{D}}_X$-modules, due to the results of Section 8.8.d. We give details in Section 8.9

***8.8.46. Theorem* (Duality and smooth pullback).** *Let $f : X' \to X$ be a morphism of complex manifolds.*

(1) *There exists a canonical morphism of functors $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})$*

$$\alpha_f(\bullet) : \boldsymbol{D}\,{}_{\mathrm{D}}f^*(\bullet) \longrightarrow {}_{\mathrm{D}}f^*\boldsymbol{D}(\bullet).$$

(2) *If f decomposes as $X' \xrightarrow{g} Z \xrightarrow{h} X$, then there exists an isomorphism*

$$\alpha_f(\bullet) \simeq {}_{\mathrm{D}}g^*\alpha_h(\bullet) \circ \alpha_g({}_{\mathrm{D}}h^*(\bullet))$$

of functors $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh},g,h}(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})$.

(3) *If f is a smooth morphism, then $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X) = \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ and for each $\widetilde{\mathcal{M}} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$, $\alpha_f(\widetilde{\mathcal{M}})$ is an isomorphism.*

(4) *In the non-filtered setting, if f is non-characteristic with respect to (each cohomology module of) $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, then $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh},f}(\mathcal{D}_X)$ and $\alpha_f(\mathcal{M})$ is an isomorphism.* □

8.8.47. Corollary. *Assume that f is a smooth morphism and $\widetilde{\mathcal{M}}$ is strictly holonomic. Then ${}_{\mathrm{D}}f^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}$ is also strictly holonomic.*

Proof. The equality ${}_{\mathrm{D}}f^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}$ as well as strictness of the latter, is due to flatness of f and is not related to strict holonomicity, hence we have the same properties for $\boldsymbol{D}\widetilde{\mathcal{M}}$. Since $\alpha_f(\widetilde{\mathcal{M}})$ is an isomorphism by (3), it follows that $\boldsymbol{D}({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}})$ is in degree zero only, and it is strict. □

8.9. Appendix A: Some fundamental results on $\widetilde{\mathcal{D}}_X$-modules

In this section, we provide a proof of Theorem 8.8.46 and take this opportunity to state in the framework of $\widetilde{\mathcal{D}}_X$-modules and their derived categories various classical results for $\mathcal{D}_X$-modules.

8.9.a. Some fundamental identities. We work in the categories of left $\widetilde{\mathcal{D}}$-modules. For a morphism $f : X' \to X$ of complex manifold and an object $\widetilde{\mathcal{M}}$ of $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$, we denote by ${}_{\mathrm{D}}f^* : \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})$ the $\widetilde{\mathcal{D}}$-module derived pullback:

$$
{}_{\mathrm{D}}f^*\widetilde{\mathcal{M}} = \widetilde{\mathcal{D}}_{X'\to X} \otimes^{L}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}.
$$

that is expressed as ${}_{\mathrm{D}}f^*\widetilde{\mathcal{M}} = \mathrm{Sp}_{X'\to X} \otimes_{f^{-1}\widetilde{\mathcal{D}}_X} f^{-1}\widetilde{\mathcal{M}}$ and similarly for a morphism $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$, where $\mathrm{Sp}_{X'\to X}$ is the relative Spencer complex, which is a $f^{-1}\widetilde{\mathcal{D}}_X$-locally free resolution of $\widetilde{\mathcal{D}}_{X'\to X}$ by $(\widetilde{\mathcal{D}}_{X'}, f^{-1}\widetilde{\mathcal{D}}_X)$ bimodules. We thus have a natural morphism

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) &\longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}, {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}})\\
\varphi &\longmapsto {}_{\mathrm{D}}f^*\varphi.
\end{aligned}
\tag{8.9.1}
$$

On the other hand, we denote by $\widetilde{\mathcal{M}} \otimes^{\mathrm{D}} \widetilde{\mathcal{N}}$ the derived tensor product over $\widetilde{\mathcal{O}}_X$. By taking flat $\widetilde{\mathcal{D}}_X$-resolutions (which are thus also $\widetilde{\mathcal{O}}_X$-flat), this defines a bifunctor $\bullet \otimes^{\mathrm{D}} \bullet$

$$
\mathsf{D}^-(\widetilde{\mathcal{D}}_X) \times \mathsf{D}^-(\widetilde{\mathcal{D}}_X) \longrightarrow \mathsf{D}^-(\widetilde{\mathcal{D}}_X).
$$

If the complexes on the left are bounded, their image has bounded cohomology, hence by truncation can be regarded as an object in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$. For a morphism $f : X' \to X$ of complex manifolds, there exists a canonical bi-functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$ (see [**HTT08**, Prop. 1.5.18]), for $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}} \in \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_Y)$,

$$
{}_{\mathrm{D}}f^*(\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{N}}) \simeq ({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}) \overset{\mathrm{D}}{\otimes} ({}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}).
\tag{8.9.2}
$$

Furthermore, we can also consider this bifunctor with the first term and the target being right $\widetilde{\mathcal{D}}_X$-modules.

We denote by $\boldsymbol{D}\widetilde{\mathcal{M}}$ the dual $\widetilde{\mathcal{D}}_X$-module, defined in such a way that

$$
\widetilde{\omega}_X \overset{\mathrm{D}}{\otimes} \boldsymbol{D}\widetilde{\mathcal{M}} = \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)[n],
$$

or equivalently, denoting by $\widetilde{\mathcal{D}}_X^{\mathrm{right}}$ the ring $\widetilde{\mathcal{D}}_X$ with its right structure,

$$
\boldsymbol{D}\widetilde{\mathcal{M}} = \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X^{\mathrm{right}}}(\widetilde{\omega}_X \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X^{\mathrm{right}})[m].
$$

The $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_X^{\mathrm{right}}$-linear Yoneda pairing $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X) \to \widetilde{\mathcal{D}}_X^{\mathrm{right}}$ yields a natural morphism

$$
\widetilde{\mathcal{M}} \longrightarrow \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X^{\mathrm{right}}}\big(\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X), \widetilde{\mathcal{D}}_X^{\mathrm{right}}\big)
$$

which is an isomorphism if $\widetilde{\mathcal{M}} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$, since it is so if $\widetilde{\mathcal{M}} = \widetilde{\mathcal{D}}_X$. One deduces a functorial biduality isomorphism for such an $\widetilde{\mathcal{M}}$:

$$
\widetilde{\mathcal{M}} \xrightarrow{\ \sim\ } \boldsymbol{D}(\boldsymbol{D}(\widetilde{\mathcal{M}})).
$$

There exists a canonical isomorphism of bifunctors

$$
\mathsf{D}^-_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X) \longrightarrow \mathsf{D}^+(\widetilde{\mathbb{C}}_X)
$$

given by (see [**Kas03**, Prop. 3.12 or (A.10)])

$$\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{N}}\xrightarrow{\sim}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}).\tag{8.9.3}$$

There exists a canonical tri-functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathbb{C}}_X)$ for $\widetilde{\mathcal{L}}\in\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X^{\mathrm{right}})$ and $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}\in\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$ (see [**HTT08**, Prop. 1.5.19]):

$$(\widetilde{\mathcal{L}}\overset{\mathrm{D}}{\otimes}\widetilde{\mathcal{M}})\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{N}}\simeq\widetilde{\mathcal{L}}\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}\overset{\mathrm{D}}{\otimes}\widetilde{\mathcal{N}})\simeq(\widetilde{\mathcal{L}}\overset{\mathrm{D}}{\otimes}\widetilde{\mathcal{N}})\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}},\tag{8.9.4}$$

which, for $\widetilde{\mathcal{D}}_X$-modules, is simply obtained by switching the entries of the tensor products (on noting that this is well-defined). Here, we have used $\otimes^{\mathrm{D}}$ in both its configurations.

8.9.5. Example. We illustrate these properties by showing the existence of a canonical bi-functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathbb{C}}_X)$, for $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ (see [**Kas03**, (3.14)]):

$$\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})\xrightarrow{\sim}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\boldsymbol{D}\widetilde{\mathcal{N}},\boldsymbol{D}\widetilde{\mathcal{M}}).$$

Proof. It is obtained as follows:

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})&\simeq(\widetilde{\omega}_X\overset{\mathrm{D}}{\otimes}\boldsymbol{D}\widetilde{\mathcal{M}})\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{N}}[-n]\quad\text{by (8.9.3)}\\&\simeq(\widetilde{\omega}_X\overset{\mathrm{D}}{\otimes}\widetilde{\mathcal{N}})\otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X}\boldsymbol{D}\widetilde{\mathcal{M}}[-n]\quad\text{by (8.9.4)}\\&\simeq\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\boldsymbol{D}\widetilde{\mathcal{N}},\boldsymbol{D}\widetilde{\mathcal{M}})\quad\text{by (8.9.3).}\end{aligned}\tag{8.9.6}$$

□

This morphism induces a natural isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})\longrightarrow\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\boldsymbol{D}\widetilde{\mathcal{N}},\boldsymbol{D}\widetilde{\mathcal{M}}).$$

Let us check for example that, if $\widetilde{\mathcal{N}}=\widetilde{\mathcal{M}}$, then $\mathrm{Id}_{\widetilde{\mathcal{M}}}$ is mapped to $\mathrm{Id}_{\boldsymbol{D}\widetilde{\mathcal{M}}}$. Since the question is local, it is enough to check this for $\widetilde{\mathcal{M}}=\widetilde{\mathcal{D}}_X$. For the sake of simplicity, we use the duality functor such that $\widetilde{\omega}_X\otimes^{\mathrm{D}}\boldsymbol{D}'(\bullet)=\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\bullet,\widetilde{\mathcal{D}}_X)$. We fix a local section $\widetilde{\mathrm{d}}x$ of $\widetilde{\omega}_X$ and its dual section $(\widetilde{\mathrm{d}}x)^{-1}$ of $\widetilde{\omega}_X^{-1}$.

The section 1 of $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X,\widetilde{\mathcal{D}}_X)$ writes $\widetilde{\mathrm{d}}x\otimes^{\mathrm{D}}((\widetilde{\mathrm{d}}x)^{-1}\otimes 1)\in\widetilde{\omega}_X\otimes^{\mathrm{D}}\boldsymbol{D}'(\widetilde{\mathcal{D}}_X)$. It is identified with the section $(\widetilde{\mathrm{d}}x\otimes^{\mathrm{D}}1)\otimes_{\widetilde{\mathcal{D}}_X}((\widetilde{\mathrm{d}}x)^{-1}\otimes 1)$ of $(\widetilde{\omega}_X\otimes^{\mathrm{D}}\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\boldsymbol{D}'(\widetilde{\mathcal{D}}_X)$. The biduality isomorphism $\widetilde{\mathcal{D}}_X\simeq\boldsymbol{D}'\boldsymbol{D}'\widetilde{\mathcal{D}}_X$ identifies the section 1 with the section $(\widetilde{\mathrm{d}}x)^{-1}\otimes s$ of $\widetilde{\omega}_X^{-1}\otimes\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\boldsymbol{D}'\widetilde{\mathcal{D}}_X,\widetilde{\mathcal{D}}_X)$ such that $s((\widetilde{\mathrm{d}}x)^{-1}\otimes 1)=1$. We deduce that the section 1 of $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X,\widetilde{\mathcal{D}}_X)$ is sent, by (8.9.6), to the section $s\otimes_{\widetilde{\mathcal{D}}_X}((\widetilde{\mathrm{d}}x)^{-1}\otimes 1)$ of $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\boldsymbol{D}'\widetilde{\mathcal{D}}_X,\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\boldsymbol{D}'\widetilde{\mathcal{D}}_X$, and it corresponds to the section of $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\boldsymbol{D}'\widetilde{\mathcal{D}}_X,\boldsymbol{D}'\widetilde{\mathcal{D}}_X)$ that sends $(\widetilde{\mathrm{d}}x)^{-1}\otimes 1$ to itself, as wanted.

By means of (8.9.4), one also obtains a canonical bi-functorial isomorphism for $\widetilde{\mathcal{M}}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ and $\widetilde{\mathcal{N}}\in\mathsf{D}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ (see [**HTT08**, Prop. 2.6.14])

$$\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})\simeq\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X,\boldsymbol{D}\widetilde{\mathcal{M}}\overset{\mathrm{D}}{\otimes}\widetilde{\mathcal{N}}).\tag{8.9.7}$$

Indeed, Applying (8.9.4) to (8.9.6) yields

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) &\simeq \widetilde{\omega}_X \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} (\boldsymbol{D}\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{N}})[-n]\\ &\simeq \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X, \boldsymbol{D}\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{N}}).\end{aligned}$$

We deduce a bi-functorial isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{O}}_X, \boldsymbol{D}\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{N}}). \tag{8.9.8}$$

8.9.b. Proof of Theorem 8.8.46

***Proof of Theorem* 8.8.46(1).** For $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh},f}(\widetilde{\mathcal{D}}_X)$, we first construct a morphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}), {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}}), \tag{8.9.9}$$

that we also denote by $(8.9.9)_{\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}}$, as follows:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{O}}_X, \boldsymbol{D}\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{N}}) && \text{by (8.9.8)}\\ &\longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}({}_{\mathrm{D}}f^*\widetilde{\mathcal{O}}_X, {}_{\mathrm{D}}f^*(\boldsymbol{D}\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} \widetilde{\mathcal{N}})) && \text{by (8.9.1)}\\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\widetilde{\mathcal{O}}_{X'}, {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}} \overset{\mathrm{D}}{\otimes} {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}) && \text{by (8.9.2)}\\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\widetilde{\mathcal{O}}_{X'}, {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} \overset{\mathrm{D}}{\otimes} {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}})\\ &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}), {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}}) && \text{by biduality.}\end{aligned} \tag{8.9.10}$$

For $\widetilde{\mathcal{M}}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$, we set

$$\alpha_f(\widetilde{\mathcal{M}}) = (8.9.9)_{\widetilde{\mathcal{M}},\widetilde{\mathcal{M}}}(\mathrm{Id}_{\widetilde{\mathcal{M}}}) \in \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}), {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}}).$$

We will check functoriality. Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism. We aim at proving the relation

$${}_{\mathrm{D}}f^*\boldsymbol{D}\varphi \circ \alpha_f(\widetilde{\mathcal{N}}) = \alpha_f(\widetilde{\mathcal{M}}) \circ \boldsymbol{D}({}_{\mathrm{D}}f^*\varphi). \tag{8.9.11}$$

For that purpose, we consider the following diagram:

$$\begin{array}{ccc}\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{M}}) & \xrightarrow{(8.9.9)_{\widetilde{\mathcal{M}},\widetilde{\mathcal{M}}}} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}), {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}})\\ \Big\downarrow \varphi\circ & & \Big\downarrow \circ\,\boldsymbol{D}({}_{\mathrm{D}}f^*\varphi)\\ \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) & \xrightarrow{(8.9.9)_{\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}}} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}), {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}})\\ \Big\uparrow \circ\,\varphi & & \Big\uparrow \boldsymbol{D}({}_{\mathrm{D}}f^*\varphi)\circ\\ \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)}(\widetilde{\mathcal{N}},\widetilde{\mathcal{N}}) & \xrightarrow{(8.9.9)_{\widetilde{\mathcal{N}},\widetilde{\mathcal{N}}}} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}), {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{N}})\end{array}$$

That it is commutative follows from the bi-functoriality of the morphisms in (8.9.10). Then, since $\varphi\circ\mathrm{Id}_{\widetilde{\mathcal{M}}} = \varphi = \mathrm{Id}_{\widetilde{\mathcal{N}}}\circ\varphi$, this implies that both terms in (8.9.11) are equal to $(8.9.9)_{\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}}(\varphi)$. ∎

***Proof of Theorem* 8.8.46**(3)**.** We will rely on 8.8.46(2) proved below. The question is local, so we can assume that f is the projection $X' = Y \times X \to X$. Furthermore, since $\widetilde{\mathcal{M}}$ admits a bounded local resolution by free $\widetilde{\mathcal{D}}_X$-modules of finite rank, it is enough, by the functoriality property of α_f, to consider the case where $\widetilde{\mathcal{M}} = \widetilde{\mathcal{D}}_X$. We are thus reduced to proving that $\alpha_f(\widetilde{\mathcal{D}}_X)$ is an isomorphism.

For $y_o \in Y$, we consider a decomposition $\mathrm{Id} : X \xhookrightarrow{g_{y_o}} Y \times X \xrightarrow{f} X$, where g_{y_o} is the inclusion $X \simeq \{y_o\} \times X \hookrightarrow Y \times X$. We conclude from 8.8.46(2) that ${}_{\mathrm{D}}g_{y_o}^* \alpha_f(\widetilde{\mathcal{D}}_X)$: ${}_{\mathrm{D}}g_{y_o}^* \boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_X) \to {}_{\mathrm{D}}g_{y_o}^*({}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{D}}_X)$ is onto for any $y_o \in Y$. In local coordinates, we identify up to the same shift both $\boldsymbol{D}({}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_X)$ and ${}_{\mathrm{D}}f^*(\boldsymbol{D}\widetilde{\mathcal{D}}_X)$ with $\widetilde{\mathcal{O}}_{X'}\langle\widetilde{\partial}_x\rangle \simeq \widetilde{\mathcal{D}}_{X'}/\widetilde{\mathcal{D}}_{X'}\widetilde{\partial}_y$ (with $\widetilde{\partial}_x = (\widetilde{\partial}_{x_1}, \dots, \widetilde{\partial}_{x_m})$ and $\widetilde{\partial}_y = (\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_p})$). The morphism $\alpha_f(\widetilde{\mathcal{D}}'_X)$ is the right multiplication by the operator $\alpha_f(\widetilde{\mathcal{D}}'_X)(1) = \sum_{a,k} c_{a,k}(y,x) z^k \widetilde{\partial}_x^a$. Since any surjective morphism $\widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{D}}_X$ must send 1 to an invertible holomorphic function in $\mathcal{O}_X$, $c_{a,k}(y_o, x) = 0$ unless $z = 0$ and $a = 0$, and $c_{0,0}(y_o, x)$ is an invertible holomorphic function for any y_o, hence the same properties hold for $c_{a,k}(y,x)$. As a consequence, $\alpha_f(\widetilde{\mathcal{D}}_X)$ is an isomorphism. □

***Proof of Theorem* 8.8.46**(2)**.** The proof uses the existence of an isomorphism

$$ {}_{\mathrm{D}}f^* \simeq {}_{\mathrm{D}}g^* \circ {}_{\mathrm{D}}h^*. $$

One notices that the following diagram, where the horizontal morphisms are obtained by applying ${}_{\mathrm{D}}g^*$, is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_Z)}({}_{\mathrm{D}}h^*\widetilde{\mathcal{O}}_X, {}_{\mathrm{D}}h^*(\boldsymbol{D}\widetilde{\mathcal{M}} \otimes^{\mathrm{D}} \widetilde{\mathcal{N}})) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}({}_{\mathrm{D}}f^*\widetilde{\mathcal{O}}_X, {}_{\mathrm{D}}f^*(\boldsymbol{D}\widetilde{\mathcal{M}} \otimes^{\mathrm{D}} \widetilde{\mathcal{N}})) \\
\big\downarrow \wr & & \big\downarrow \wr \\
\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_Z)}(\widetilde{\mathcal{O}}_Z, {}_{\mathrm{D}}h^*\boldsymbol{D}\widetilde{\mathcal{M}} \otimes^{\mathrm{D}} {}_{\mathrm{D}}h^*\widetilde{\mathcal{N}}) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\widetilde{\mathcal{O}}_{X'}, {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}} \otimes^{\mathrm{D}} {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}) \\
\big\downarrow \wr & & \big\downarrow \wr \\
\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_Z)}(\widetilde{\mathcal{O}}_Z, {}_{\mathrm{D}}h^*\widetilde{\mathcal{N}} \otimes^{\mathrm{D}} {}_{\mathrm{D}}h^*\boldsymbol{D}\widetilde{\mathcal{M}}) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X'})}(\widetilde{\mathcal{O}}_{X'}, {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} \otimes^{\mathrm{D}} {}_{\mathrm{D}}f^*\boldsymbol{D}\widetilde{\mathcal{M}})
\end{array}$$

and this leads to the desired isomorphism. □

8.10. Appendix B: Differential complexes and the Gauss-Manin connection

In this section we switch to the case of $\mathcal{D}_X$-modules as in Section 8.1 (see Remark 8.10.9). Let $\mathcal{M}$ be a *left* $\mathcal{D}_X$-module and let $f : X \to Y$ be a holomorphic mapping. On the one hand, we have defined the direct images ${}_{\mathrm{D}}f_*\mathcal{M}$ or ${}_{\mathrm{D}}f_!\mathcal{M}$ of $\mathcal{M}$ viewed as $\mathcal{D}_X$-modules. These are objects in $\mathsf{D}^+(\mathcal{D}_Y)^{\mathrm{left}}$. On the other hand, *when f is a smooth holomorphic mapping*, a flat connection called the *Gauss-Manin connection* is defined on the relative de Rham cohomology of $\mathcal{M}$. We will compare both constructions, when f is smooth. Such a comparison has essentially already been done *when f is the projection of a product $X = Y \times T \to Y$* (see Examples 8.7.10 and 8.7.11).

In this section we also introduce the derived category of differential complexes on a complex manifold X, that is, complexes of $\mathcal{O}_X$-modules with differential morphisms as differential. This derived category is shown to be equivalent to that of $\mathcal{D}_X$-modules (Theorem 8.10.15). It is sometimes useful to work in this category (see e.g. the proof of Theorem 8.10.21).

8.10.a. Differential complexes. Given an $\mathcal{O}_X$-module $\mathcal{L}$, there is a natural $\mathcal{O}_X$-linear morphism (with the right structure on the right-hand term)

$$\mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \quad \ell \longmapsto \ell \otimes 1.$$

There is also a (only) $\mathbb{C}$-linear morphism

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{L} \tag{8.10.1}$$

defined at the level of local sections by $\ell \otimes P \mapsto P(1)\ell$, where $P(1)$ is the result of the action of the differential operator P on 1, which is equal to the degree 0 coefficient of P if P is locally written as $\sum_\alpha a_\alpha(x)\partial_x^\alpha$. In an intrinsic way, consider the natural augmentation morphism $\mathcal{D}_X \to \mathcal{O}_X$, which is left $\mathcal{D}_X$-linear, hence left $\mathcal{O}_X$-linear; then apply $\mathcal{L} \otimes_{\mathcal{O}_X} \bullet$ to it. Notice however that (8.10.1) is an $\mathcal{O}_X$-linear morphism by using the left $\mathcal{O}_X$-module structure on $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

Let $\mathcal{L}, \mathcal{L}'$ be two $\mathcal{O}_X$-modules. A (right) $\mathcal{D}_X$-linear morphism

$$v : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \tag{8.10.2}$$

is uniquely determined by the $\mathcal{O}_X$-linear morphism

$$w : \mathcal{L} \longrightarrow \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \tag{8.10.3}$$

that it induces (where the right $\mathcal{O}_X$-module structure is chosen on $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$). In other words, the natural morphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \longrightarrow \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$$

is an isomorphism. We also have, at the sheaf level,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X). \tag{8.10.4}$$

Notice that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is naturally equipped with an $\mathcal{O}_X$-module structure by using the left $\mathcal{O}_X$-module structure on $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$ (see Remark 8.5.1), and similarly $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is a $\Gamma(X, \mathcal{O}_X)$-module.

Now, w induces a $\mathbb{C}$-linear morphism

$$u : \mathcal{L} \longrightarrow \mathcal{L}', \tag{8.10.5}$$

by composition with (8.10.1): $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{L}'$. By Exercise 8.75, u is nothing but the morphism

$$H^0({}^{\mathrm{p}}\mathrm{DR}(v)) : H^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X)\big) \longrightarrow H^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)\big).$$

***8.10.6. Definition* (Differential operators between two $\mathcal{O}_X$-modules)**

The $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$ of *differential operators from $\mathcal{L}$ to $\mathcal{L}'$* is the image of the morphism $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}')$.

Similarly we define the sheaf of $\mathbb{C}$-vector spaces $\mathcal{H}om_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$.

8.10.7. Definition **(The category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$).** We denote by $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ the category whose objects are $\mathcal{O}_X$-modules and morphisms are differential operators between $\mathcal{O}_X$-modules (this is justified by Exercise 8.76(4)).

In particular, $\mathsf{Mod}(\mathcal{O}_X)$ is a subcategory of $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$, since any $\mathcal{O}_X$-linear morphism is a differential operator (of degree zero).

We will now show that the correspondence $\mathcal{L} \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ induces a functor $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X)$. In order to do so, one first needs to show that to any differential morphism u corresponds at most one v.

8.10.8. Lemma. *The morphism*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}') \\ v &\longmapsto u \end{aligned}$$

is injective.

Proof. Recall that, for any multi-index β, we have $\partial_x^\alpha(x^\beta) = 0$ if $\beta_i < \alpha_i$ for some i, and $\partial_x^\alpha(x^\alpha) = \alpha!$. Assume that $u = 0$. Let ℓ be a local section of $\mathcal{L}$ and, using local coordinates $(x_1, \dots, x_n)$, write in a unique way $w(\ell) = \sum_\alpha w(\ell)_\alpha \otimes \partial_x^\alpha$, where the sum is taken on multi-indices α and w is as in (8.10.3). If $w(\ell) \neq 0$, let β be minimal (with respect to the usual partial ordering on $\mathbb{N}^n$) among the multi-indices α such that $w(\ell)_\alpha \neq 0$. Then,

$$0 = u(x^\beta \ell) = \sum_\alpha \partial_x^\alpha(x^\beta) w(\ell)_\alpha = \beta! w(\ell)_\beta,$$

a contradiction. □

8.10.9. Remark. A similar lemma would not hold in the category of induced graded $R_F\mathcal{D}_X$-modules because of possible z-torsion: one would only get that $z^k u(x^\beta \ell) = 0$ for some k. One thus cannot just replace $\mathcal{D}_X$ with $\widetilde{\mathcal{D}}_X$ in this section. On the other hand, it is possible to restrict to $\widetilde{\mathcal{O}}_X$-modules which have no z-torsion, in other words, to filtered $\mathcal{O}_X$-modules. This leads to considering derived categories in the framework of exact but non abelian categories. We will need such a construction in Chapter 10.

According to Lemma 8.10.8, the following definition is meaningful.

8.10.10. Definition **(The inverse de Rham functor).** The functor

$${}^{\mathrm{diff}}\mathrm{DR}^{-1} : \mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \longrightarrow \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X)$$

is defined by ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ and ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(u) = v$.

8.10.11. Remarks.

(1) The notation is justified by the fact that ${}^{\mathrm{p}}\mathrm{DR}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{L}$ (see Exercise 8.29(5)).

(2) By the isomorphism of Exercise 8.79, $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L},\mathcal{L}')$ is equipped with the structure of a $\Gamma(X,\mathcal{O}_X)$-module. Similarly,

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X,\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)\longrightarrow\mathcal{H}om_{\mathbb{C}}(\mathcal{L},\mathcal{L}')$$

is injective, and and this equips the image sheaf $\mathcal{H}om_{\mathrm{Diff}}(\mathcal{L},\mathcal{L}')$ with the structure of an $\mathcal{O}_X$-module.

(3) When considered as taking values in $\mathsf{Mod}(\mathcal{D}_X)$, the functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ is not, however, an equivalence of categories, i.e., is not essentially surjective. The reason is that, first, not all $\mathcal{D}_X$-modules are isomorphic to some $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X$ and, next, its natural quasi-inverse would be the de Rham functor ${}^{\mathrm{p}}\mathrm{DR}$ which takes values in a category of complexes. Nevertheless, if one extends suitably these functors to categories of complexes, they become equivalences (see below Theorem 8.10.15).

8.10.b. The de Rham complex as a differential complex. Given an induced $\mathcal{D}$-module, its de Rham complex gives enough information to recover it, according to Remark 8.10.11(1). On the other hand, given a bounded complex of induced $\mathcal{D}$-modules, its de Rham complex does not give enough information to recover its differentials. We will refine the functor ${}^{\mathrm{p}}\mathrm{DR}$ to a functor ${}^{\mathrm{diff}}\mathrm{DR}$, which takes values in differential complexes, and has quasi-inverse induced by ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$.

According to Exercise 8.78, one may consider the category $\mathsf{C}^\star(\mathcal{O}_X,\mathrm{Diff}_X)$ of $\star$-bounded complexes of objects of $\mathsf{Mod}(\mathcal{O}_X,\mathrm{Diff}_X)$ (with $\star=\varnothing,+,-,\mathrm{b}$), and the category $\mathsf{K}^\star(\mathcal{O}_X,\mathrm{Diff}_X)$ of $\star$-bounded complexes up to homotopy (see [**KS90**, Def. 1.3.4]). These are called $\star$-bounded *differential complexes*.

There is a natural forgetful functor Forget from $\mathsf{Mod}(\mathcal{O}_X,\mathrm{Diff}_X)$ to $\mathsf{Mod}(\mathbb{C}_X)$, and by extension a functor Forget at the level of $\mathsf{C}^\star$ and $\mathsf{K}^\star$. Exercise 8.80 shows that we can decompose the ${}^{\mathrm{p}}\mathrm{DR}$ functor as

$$\begin{tikzcd}\mathsf{Mod}(\mathcal{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{C}^{\mathrm{b}}(\mathcal{O}_X,\mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{C}^{\mathrm{b}}(\mathbb{C}_X)\end{tikzcd}$$

and

$$\begin{tikzcd}\mathsf{K}^\star(\mathcal{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{K}^\star(\mathcal{O}_X,\mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{K}^\star(\mathbb{C}_X)\end{tikzcd}$$

In order to define the "derived category" of the additive category $\mathsf{Mod}(\mathcal{O}_X,\mathrm{Diff}_X)$, one needs to define the notion of null system in $\mathsf{K}^\star(\mathcal{O}_X,\mathrm{Diff}_X)$ and localize the category with respect to the associated multiplicative system. A possible choice would be to say that an object belongs to the null system if it belongs to the null system of $\mathsf{C}^\star(\mathbb{C}_X)$ when forgetting the Diff structure, i.e., which has zero cohomology when considered as a complex of sheaves of $\mathbb{C}$-vector spaces. This is not the choice made below. One says that a differential morphism $u:\mathcal{L}\to\mathcal{L}'$ as in (8.10.5) is a

Diff-quasi-isomorphism if the corresponding v as in (8.10.2) is a quasi-isomorphism of right $\mathcal{D}_X$-modules.

The functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ of Definition 8.10.10 extends as a functor $\mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{C}^\star_{\mathrm{i}}(\mathcal{D}_X)$ and $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{K}^\star_{\mathrm{i}}(\mathcal{D}_X)$ in a natural way, and is a functor of triangulated categories on K. Moreover, according to the last part of Exercise 8.79, it is an equivalence of triangulated categories.

We wish now to define *acyclic objects* in the triangulated category $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, and show that they form a *null system* in the sense of [**KS90**, Def. 1.6.6].

8.10.12. Definition. We say that a object $\mathcal{L}^\bullet$ of $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ is Diff-*acyclic* if ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^\bullet)$ is acyclic in $\mathsf{K}^\star_{\mathrm{i}}(\mathcal{D}_X)$ (equivalently, in $\mathsf{K}^\star(\mathcal{D}_X)$).

Define, as in [**KS90**, (1.6.4)], the family $S(\mathsf{N})$ as the family of morphisms which can be embedded in a distinguished triangle of $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, with the third term being an object of N. We call such morphisms Diff-*quasi-isomorphisms*. Clearly, they correspond exactly *via* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ to quasi-isomorphisms in $\mathsf{K}^\star(\mathcal{D}_X)$.

We now may localize the category $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ with respect to the null system N and get a category denoted by $\mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$. By construction, we still get a functor

$$ {}^{\mathrm{diff}}\mathrm{DR}^{-1} : \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \longrightarrow \mathsf{D}^\star_{\mathrm{i}}(\mathcal{D}_X) \longrightarrow \mathsf{D}^\star(\mathcal{D}_X). \tag{8.10.13} $$

We note that the first component is an equivalence by definition of the null system (since we have an equivalence at the level of the categories $\mathsf{K}^\star$). The second component is also an equivalence, according to Corollary 8.5.3. We will show below (Theorem 8.10.15) that ${}^{\mathrm{diff}}\mathrm{DR}$ is a quasi-inverse functor.

8.10.14. Remark. The category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ is also naturally a subcategory of the category $\mathsf{Mod}(\mathbb{C}_X)$ of sheaves of $\mathbb{C}$-vector spaces because $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$ is a subset of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}')$. We therefore have a natural functor $\mathsf{Forget} : \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \to \mathsf{K}^\star(\mathbb{C}_X)$, forgetting that the differentials of a complex are differential operators, and forgetting also that the homotopies should be differential operators too. As a consequence of Theorem 8.10.15, we will see in Exercise 8.85 that any object in the null system N defined above is sent to an object in the usual null system of $\mathsf{K}^\star(\mathbb{C}_X)$, i.e., objects with zero cohomology. In other words, a Diff-quasi-isomorphism is sent into a usual quasi-isomorphism. But there may exist morphisms in $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ which are quasi-isomorphisms when viewed in $\mathsf{K}^\star(\mathbb{C}_X)$, but are not Diff-quasi-isomorphisms.

8.10.15. Theorem. *The functors* ${}^{\mathrm{diff}}\mathrm{DR}$ *and* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ *induce quasi-inverse and induce equivalences of categories*

$$ \mathsf{D}^\star(\mathcal{D}_X) \underset{{}^{\mathrm{diff}}\mathrm{DR}^{-1}}{\overset{{}^{\mathrm{diff}}\mathrm{DR}}{\rightleftarrows}} \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X). $$

8.10.16. Lemma. *There is an isomorphism of functors* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\circ{}^{\mathrm{diff}}\mathrm{DR}\xrightarrow{\sim}\mathrm{Id}$ *from* $\mathsf{D}^{\star}(\mathcal{D}_X)$ *(right* $\mathcal{D}_X$*-modules) to itself.*

This lemma enables one to attach to each object of $\mathsf{D}^{\star}(\mathcal{D}_X)$ a canonical resolution by induced $\mathcal{D}_X$-modules since ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ takes values in $\mathsf{D}^{\star}_{\mathrm{i}}(\mathcal{D}_X)$.

Proof. Let us recall that there exists an explicit side-changing isomorphism of complexes ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}}\simeq{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{right}}$ which is given by termwise $\mathcal{O}_X$-linear morphisms. If we regard these complexes as objects of $\mathsf{C}^{\mathrm{b}}(\mathcal{O}_X,\mathrm{Diff})$, we deduce that the side-changing isomorphism is an isomorphism in this category. In other words, we have ${}^{\mathrm{diff}}\mathrm{DR}(\mathcal{M}^{\mathrm{left}})\simeq{}^{\mathrm{diff}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}})$.

For the proof of the lemma, start with a left $\mathcal{D}_X$-module $\mathcal{M}^{\mathrm{left}}$. By definition, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}}=(\Omega_X^{n+\bullet}\otimes\mathcal{M}^{\mathrm{left}})\otimes\mathcal{D}_X$ with differential ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla)$. This is nothing but the complex $\Omega_X^{n+\bullet}\otimes(\mathcal{M}^{\mathrm{left}}\otimes\mathcal{D}_X)$ where the differential is the connection on the left $\mathcal{D}_X$-module $(\mathcal{M}^{\mathrm{left}}\otimes\mathcal{D}_X)_{\mathrm{tens}}$. Furthermore, this identification is right $\mathcal{D}_X$-linear with respect to the $(\mathrm{right})_{\mathrm{triv}}$ structure on both terms.

We note that $\big[(\mathcal{M}^{\mathrm{left}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)^{\mathrm{right}}\big]_{\mathrm{tens}}\simeq(\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)_{\mathrm{tens}}$, i.e., both with the tensor structure, respectively left and right, and this isomorphism is compatible with the right $\mathcal{D}_X$-structure $(\mathrm{right})_{\mathrm{triv}}$ on both terms. By side-changing we find

$$\big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{left}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}}\simeq\big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}},$$

and by using the involution of Exercise 8.19,

$$\big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}}\simeq\big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)_{\mathrm{triv}}\big]_{\mathrm{tens}}.$$

Lastly, we have

$${}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathcal{D}_X)_{\mathrm{triv}}=\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathrm{Sp}^{\bullet}(\mathcal{D}_X)\simeq\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{O}_X}\mathcal{O}_X=\mathcal{M}^{\mathrm{right}},$$

and the remaining right $\mathcal{D}_X$-structure is deduced from the tens one, which is the natural right structure on $\mathcal{M}^{\mathrm{right}}$. We conclude that, functorially, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}}\simeq\mathcal{M}^{\mathrm{right}}$. Since ${}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}}\simeq{}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{right}}$, the lemma follows. □

***Proof of Theorem* 8.10.15.** From the previous lemma, it is now enough to show that, if $\mathcal{L}^{\bullet}$ is a complex in $\mathsf{C}^{\star}(\mathcal{O}_X,\mathrm{Diff}_X)$, there exists a a Diff-quasi-isomorphism ${}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^{\bullet}\to\mathcal{L}^{\bullet}$, and, by definition, this is equivalent to showing the existence of a quasi-isomorphism ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^{\bullet}\to{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^{\bullet}$, that we know from the previous result applied to $\mathcal{M}={}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^{\bullet}$. □

8.10.17. Remark. The functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}$, regarded as a functor $\mathsf{D}^{\star}(\mathcal{D}_X)\to\mathsf{D}^{\star}_{\mathrm{i}}(\mathcal{D}_X)$, is nothing but that of Corollary 8.5.3.

***8.10.18. Remark* (The Godement resolution of a differential complex)**

Let $\mathcal{L}^{\bullet}$ be an object of $\mathsf{C}^{+}(\mathcal{O}_X,\mathrm{Diff}_X)$. Then $\mathrm{God}^{\bullet}\mathcal{L}^{\bullet}$ is maybe not a differential complex (see Exercise 8.49(2)). However, $\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^{\bullet}$ is a differential complex, being equal to ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^{\bullet}$. Therefore, the composite functor $\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}$ plays the role of Godement resolutions in the category of differential complexes.

8.10.c. The Gauss-Manin connexion. We assume in this section that $f : X \to Y$ is a *smooth* holomorphic map. The cotangent map $T^*f : f^*\Omega^1_Y \to \Omega^1_X$ is then injective, and we will identify $f^*\Omega^1_Y$ with its image. We set $n = \dim X$, $m = \dim Y$ and $d = n - m$ (we assume that X and Y are pure dimensional, otherwise one works on each connected component of X and Y).

Consider the *Leray filtration* $\mathrm{Ler}^\bullet$ on the complex $(\Omega^\bullet_X, \mathrm{d})$, defined by

$$\mathrm{Ler}^p\Omega^i_X = \mathrm{Im}(f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{i-p}_X \longrightarrow \Omega^i_X).$$

[With this notation, $\mathrm{Ler}^p\Omega^i_X$ can be nonzero only when $i \in [0, n]$ and $p \in [0, \min(i, m)]$.]

Then, as f is smooth, we have (by computing with local coordinates adapted to f),

$$\mathrm{gr}^p_{\mathrm{Ler}}\Omega^i_X = f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{i-p}_{X/Y},$$

where $\Omega^k_{X/Y}$ is the sheaf of relative differential forms: $\Omega^k_{X/Y} = \wedge^k\Omega^1_{X/Y}$ and $\Omega^1_{X/Y} = \Omega^1_X\big/f^*\Omega^1_Y$. Notice that $\Omega^k_{X/Y}$ is $\mathcal{O}_X$-locally free.

Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module or an object of $\mathsf{D}^+(\mathcal{D}_X)^{\mathrm{left}}$. As f is smooth, the sheaf $\mathcal{D}_{X/Y}$ of relative differential operators is well-defined and by composing the flat connection $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ with the projection $\Omega^1_X \to \Omega^1_{X/Y}$ we get a relative flat connection $\nabla_{X/Y}$ on $\mathcal{M}$, and thus the structure of a left $\mathcal{D}_{X/Y}$-module on $\mathcal{M}$. In particular, the relative de Rham complex is defined as

$$^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} = (\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}, \nabla_{X/Y}).$$

We have $^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = (\Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{M}, \nabla)$ (see Definition 8.4.1) and the Leray filtration $\mathrm{Ler}^p\Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is preserved by the differential ∇. We can therefore induce the filtration $\mathrm{Ler}^\bullet$ on the complex $^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$. We then have an equality of complexes

$$\mathrm{gr}^p_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = f^*\Omega^p_Y \otimes_{\mathcal{O}_X} {}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}[-p].$$

Notice that the differential of these complexes are $f^{-1}\mathcal{O}_Y$-linear.

The complex $f_*\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ (resp. the complex $f_!\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$) is filtered by subcomplexes $f_*\,\mathrm{God}^\bullet\,\mathrm{Ler}^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ (resp. $f_!\,\mathrm{God}^\bullet\,\mathrm{Ler}^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$). We therefore get a spectral sequence (the Leray spectral sequence in the category of sheaves of $\mathbb{C}$-vector spaces, see, e.g. [**God64**]). Using the projection formula for $f_!$ and the fact that Ω^p_Y is $\mathcal{O}_Y$-locally free, one obtains that the E_1 term for the complex $f_!\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is given by

$$E^{p,q}_{1,!} = \Omega^p_Y \otimes_{\mathcal{O}_Y} R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}, \tag{8.10.19}$$

and the spectral sequence converges to (a suitable graded object associated with) $R^{p+q}f_!\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$. If f is proper on $\mathrm{Supp}\,\mathcal{M}$ or if $\mathcal{M}$ has $\mathcal{D}_X$-coherent cohomology, one can also apply the projection formula to f_* (see [**MN93**, §II.5.4]).

By definition of the spectral sequence, the differential $d_1 : E^{p,q}_1 \to E^{p+1,q}_1$ is the connecting morphism (see Exercise 8.88 below) in the long exact sequence associated to the short exact sequence of complexes

$$0 \longrightarrow \mathrm{gr}^{p+1}_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \longrightarrow \mathrm{Ler}^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}\big/\mathrm{Ler}^{p+2}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \longrightarrow \mathrm{gr}^p_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \longrightarrow 0$$

after applying $f_!\,\mathrm{God}^\bullet$ (or $f_*\,\mathrm{God}^\bullet$ if one of the previous properties is satisfied).

***8.10.20. Lemma* (The Gauss-Manin connection).** *The morphism*

$$\nabla^{\mathrm{GM}} := d_1 : R^q f_! \,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} = E_1^{0,q} \longrightarrow E_1^{1,q} = \Omega_Y^1 \otimes_{\mathcal{O}_Y} R^q f_! \,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}$$

is a flat connection on $R^q f_! \,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}$*, called the* Gauss-Manin connection *and the complex* $(E_1^{\bullet,q}, d_1)$ *is equal to the de Rham complex* ${}^{\mathrm{diff}}\mathrm{DR}_Y(R^q f_! \,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}, \nabla^{\mathrm{GM}})$.

***Sketch of proof of Lemma* 8.10.20.** Instead of using the Godement resolution, one can use the C^∞ de Rham complex $\mathcal{E}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$, with the differential D defined by

$$D(\eta \otimes m) = \mathrm{d}\eta \otimes m + (-1)^k \eta \wedge \nabla m,$$

if η is C^∞ differential k-form, that is, a local section of $\mathcal{E}_X^k$ $(k \leqslant 0)$. By a standard argument (Dolbeault resolution) analogous to that of Exercise 8.52(5), this C^∞ de Rham complex is quasi-isomorphic to the holomorphic one, and is equipped with the Leray filtration. The quasi-isomorphism is strict with respect to $\mathrm{Ler}^\bullet$. One can therefore compute with the C^∞ de Rham complex. Moreover, the assertion is local with respect to Y.

Assume first that, in the neighbourhood of $f^{-1}(y)$, X is diffeomorphic to a product $X \simeq Z \times Y$. This occurs for example if f is proper (Ehresmann's theorem). Then we identify $\mathcal{E}_X^{p+q}$ with $\mathcal{E}_Y^p \otimes \mathcal{E}_{X/Y}^q$ and the differential D decomposes accordingly as $D_Y + D_{X/Y}$. The flatness of D implies the flatness of $D_{X/Y}$ and D_Y. Given a section μ of $f_!\big(\mathcal{E}_Y^p \otimes (\mathcal{E}_{X/Y}^q \otimes \mathcal{M})\big)$ which is closed with respect to $D_{X/Y}$, we can identify it with its lift $\widetilde{\mu}$ (see Exercise 8.88), and $d_1\mu$ is thus the class of $D_Y\mu$, so the C^∞ Gauss-Manin connection D^{GM} in degree zero induces d_1 in any degree.

In general, choose a partition of unity (χ_α) such that for every α, when restricted to some open neighbourhood of $\operatorname{Supp}\chi_\alpha$, f is locally the projection from a product to one of its factors. We set $D = \sum_\alpha \chi_\alpha D = \sum_\alpha D^{(\alpha)}$ and we apply the previous argument to each $D^{(\alpha)}$. □

8.10.21. Theorem. *Let* $f : X \to Y$ *be a smooth holomorphic map and let* $\mathcal{M}$ *be left* $\mathcal{D}_X$*-module—or more generally an object of* $\mathsf{D}^+(\mathcal{D}_X)^{\mathrm{left}}$*. Then there is a functorial isomorphism of left* $\mathcal{D}_Y$*-modules*

$$R^k f_! \,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} \longrightarrow {}_{\mathrm{D}}f_!^{(k)}\mathcal{M}$$

when one endows the left-hand term with the Gauss-Manin connection ∇^{GM}*. The same result holds for* ${}_{\mathrm{D}}f_*$ *instead of* ${}_{\mathrm{D}}f_!$ *if* f *is proper on* $\operatorname{Supp}\mathcal{M}$ *or* $\mathcal{M}$ *is* $\mathcal{D}_X$*-coherent (or has coherent cohomology).*

Proof. Recall (Exercise 8.26) that, for a left $\mathcal{D}_X$-module $\mathcal{M}$, we have

$$\mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{D}_X} \mathrm{Sp}^\bullet_{X\to Y}(\mathcal{D}_X) \simeq \Omega_X^\bullet(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)[n],$$

so that the direct image of $\mathcal{M}$, regarded as a right $\mathcal{D}_Y$-module, is

$$({}_{\mathrm{D}}f_!\mathcal{M})^{\mathrm{right}} = \boldsymbol{R}f_! \,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)[m], \tag{8.10.22}$$

by Exercise 8.52(3). We conclude that

$${}^{\mathrm{diff}}\mathrm{DR}_Y\, {}_{\mathrm{D}}f_!\mathcal{M} \simeq {}^{\mathrm{diff}}\mathrm{DR}_Y\big(\boldsymbol{R}f_! \,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\big).$$

There is a Leray filtration $\mathrm{Ler}^{\bullet}\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)$. Notice that the graded complex $\mathrm{gr}^p_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)$ is equal to the complex

$$f^{-1}\Omega^p_Y\otimes_{f^{-1}\mathcal{O}_Y}{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y[-p],$$

with differential induced by $\nabla_{X/Y}$ on $\mathcal{M}$ (remark that the part of the differential involving T^*f is killed by taking $\mathrm{gr}^p_{\mathrm{Ler}}$). The differential is now $f^{-1}\mathcal{O}_Y$-linear.

The filtered complex $\boldsymbol{R}f_!\mathrm{Ler}^{\bullet}\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)$ gives rise to a spectral sequence in the category of right $\mathcal{D}_Y$-modules. By the previous computation, the $E_1^{p,q}$ term of this spectral sequence is the right $\mathcal{D}_Y$-module

$$\begin{aligned}R^{p+q}f_!\big(f^{-1}\Omega^p_Y\otimes_{f^{-1}\mathcal{O}_Y}{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y[-p]\big)&\\ \simeq \Omega^p_Y\otimes_{\mathcal{O}_Y} R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{D}_Y&,\end{aligned}$$

which is an induced $\mathcal{D}_Y$-module, whose ${}^{\mathrm{diff}}\mathrm{DR}_Y$ is equal to the corresponding Gauss-Manin term (8.10.19). We claim, as will show below, that the differential d_1 becomes the Gauss-Manin d_1 after applying ${}^{\mathrm{diff}}\mathrm{DR}_Y$. This will prove that the Gauss-Manin E_1 complex is equal to ${}^{\mathrm{diff}}\mathrm{DR}_Y$ of the E_1 complex of right $\mathcal{D}_Y$-modules.

Notice now that Lemma 8.10.20 shows in particular that the E_1 complex considered there is a complex in $\mathsf{C}^+(\mathcal{O}_Y,\mathrm{Diff}_Y)$, and

$${}^{\mathrm{diff}}\mathrm{DR}_Y^{-1}(E_1^{\bullet,q},d_1)\simeq(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M},\nabla^{\mathrm{GM}})^{\mathrm{right}}[-m],$$

since, for a left $\mathcal{D}_Y$-module $\mathcal{N}$, we have, according to Theorem 8.10.15,

$${}^{\mathrm{diff}}\mathrm{DR}_Y^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}_Y(\mathcal{N})={}^{\mathrm{diff}}\mathrm{DR}_Y^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}_Y(\mathcal{N}^{\mathrm{right}})[-m]\simeq\mathcal{N}^{\mathrm{right}}[-m].$$

The claim above, together with Lemma 8.10.16, implies that the E_1 complex of the $\mathcal{D}_Y$-Leray spectral sequence has cohomology in degree m only, hence this spectral sequence degenerates at E_2, this cohomology being isomorphic to $(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M},\nabla^{\mathrm{GM}})^{\mathrm{right}}[-m]$. But the spectral sequences converges (the Leray filtration is finite) and its limit is $\bigoplus_p \mathrm{gr}^p({}_{\mathrm{D}}f_!^{(q-m)}\mathcal{M})^{\mathrm{right}}$ for the induced filtration on $({}_{\mathrm{D}}f_!\mathcal{M}^{(q-m)})^{\mathrm{right}}$, according to (8.10.22). We conclude that this implicit filtration is trivial and that $({}_{\mathrm{D}}f_!^{(q)}\mathcal{M})^{\mathrm{right}}=(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M},\nabla^{\mathrm{GM}})^{\mathrm{right}}$, as wanted, after side changing.

Let us now compare the d_1 of both spectral sequences. As the construction is clearly functorial with respect to $\mathcal{M}$, we can replace $\mathcal{M}$ by the flabby sheaf $\mathrm{God}^{\ell}\mathcal{M}$ for every ℓ. We then have

$$\begin{aligned}\boldsymbol{R}f_!\big(\Omega^{\bullet}_X&\otimes_{\mathcal{O}_X}\mathrm{God}^{\ell}\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\big)\\ &=\boldsymbol{R}f_!\big(\mathrm{God}^{\ell}(\Omega^{\bullet}_X\otimes_{\mathcal{O}_X}\mathcal{M})\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\big)\quad\text{(Exercise 8.49)}\\ &=\boldsymbol{R}f_!\big(\mathrm{God}^{\ell}(\Omega^{\bullet}_X\otimes_{\mathcal{O}_X}\mathcal{M})\big)\otimes_{\mathcal{O}_Y}\mathcal{D}_Y\quad\text{(projection formula)}\\ &=f_!\big(\mathrm{God}^{\ell}(\Omega^{\bullet}_X\otimes_{\mathcal{O}_X}\mathcal{M})\big)\otimes_{\mathcal{O}_Y}\mathcal{D}_Y\quad\text{(flabbiness of }\mathrm{God}^{\ell})\\ &=f_!\big(\Omega^{\bullet}_X\otimes_{\mathcal{O}_X}\mathrm{God}^{\ell}\mathcal{M}\big)\otimes_{\mathcal{O}_Y}\mathcal{D}_Y\quad\text{(Exercise 8.49)}\\ &=f_!\big(\Omega^{\bullet}_X\otimes_{\mathcal{O}_X}\mathrm{God}^{\ell}\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\big)\quad\text{(projection formula).}\end{aligned}$$

It is also enough to make the computation locally on Y, so that we can write $f = (f_1, \dots, f_m)$, using local coordinates $(y_1, \dots, y_m)$. If μ is a section of $\Omega^k_X \otimes \mathcal{M}$ and $\mathbf{1}_Y$ is the unit of $\mathcal{D}_Y$, then (8.6.1) can be written as

$$\nabla^X(\mu \otimes \mathbf{1}_Y) = (\nabla\mu) \otimes \mathbf{1}_Y + \sum_{j=1}^m \mu \wedge df_j \otimes \partial_{y_j}.$$

Using the definition of d_1 given by Exercise 8.88 and an argument similar to that of Exercise 8.86, one gets the desired assertion. □

8.11. Exercises

8.11.a. Exercises for Section 8.1

Exercise 8.1. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank d and let $\mathcal{E}^\vee$ be its dual. Show that, given any local basis $\boldsymbol{e} = (e_1, \dots, e_d)$ of $\mathcal{E}$ with dual basis $\boldsymbol{e}^\vee$, the section $\sum_{i=1}^d e_i \otimes e_i^\vee$ of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$ does not depend on the choice of the local basis $\boldsymbol{e}$ and extends as a global section of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$. Show that it defines, up to a constant, an $\mathcal{O}_X$-linear section $\mathcal{O}_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$ of the natural duality pairing $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathcal{O}_X$. Conclude that we have a natural global section of $\Omega^1_X \otimes_{\mathcal{O}_X} \Theta_X$ given, in local coordinates, by $\sum_i \mathrm{d}x_i \otimes \partial_{x_i}$.

Exercise 8.2. Show that a differential operator P *of order* $\leqslant 1$ satisfying $P(1) = 0$ is a derivation of $\mathcal{O}_X$, i.e., a section of Θ_X.

***Exercise 8.3* (Local computations).** Let U be an open set of $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$. Denote by $\partial_{x_1}, \dots, \partial_{x_n}$ the corresponding vector fields.

(1) Show that the following relations are satisfied in $\mathcal{D}(U)$:

$$[\partial_{x_i}, f] = \frac{\partial f}{\partial x_i}, \quad \forall\, f \in \mathcal{O}(U),\ \forall\, i \in \{1, \dots, n\},$$
$$[\partial_{x_i}, \partial_{x_j}] = 0 \quad \forall\, i, j \in \{1, \dots, n\}.$$

with standard notation concerning multi-indices α, β.

(2) Show that any element $P \in \mathcal{D}(U)$ can be written in a unique way as $\sum_\alpha a_\alpha \partial_x^\alpha$ or $\sum_\alpha \partial_x^\alpha b_\alpha$with $a_\alpha, b_\alpha \in \mathcal{O}(U)$. Conclude that $\mathcal{D}_X$ is a locally free module over $\mathcal{O}_X$ with respect to the action on the left and that on the right.

(3) Show that $\max\{|\alpha| \ ; \ a_\alpha \neq 0\} = \max\{|\alpha| \ ; \ b_\alpha \neq 0\}$. It is denoted by $\mathrm{ord}_x P$.

(4) Show that $\mathrm{ord}_x P$ does not depend on the coordinate system chosen on U.

(5) Show that $PQ = 0$ in $\mathcal{D}(U) \implies P = 0$ or $Q = 0$.

(6) Identify $F_k\mathcal{D}_X$ with the subsheaf of local sections of $\mathcal{D}_X$ having order $\leqslant k$ (in some or any local coordinate system). Show that it is a locally free $\mathcal{O}_X$-module of finite rank.

(7) Show that the filtration $F_\bullet\mathcal{D}_X$ is exhaustive (i.e., $\mathcal{D}_X = \bigcup_k F_k\mathcal{D}_X$) and that it satisfies

$$F_k\mathcal{D}_X \cdot F_\ell\mathcal{D}_X = F_{k+\ell}\mathcal{D}_X.$$

(The left-hand term consists by definition of all sums of products of a section of $F_k\mathcal{D}_X$ and a section of $F_\ell\mathcal{D}_X$.)

(8) Show that the bracket $[P,Q]:=PQ-QP$ induces for every k,ℓ a $\mathbb{C}$-bilinear morphism $F_k\mathcal{D}_X\otimes_{\mathbb{C}}F_\ell\mathcal{D}_X\to F_{k+\ell-1}\mathcal{D}_X$.

(9) Conclude that the graded ring $\mathrm{gr}^F\mathcal{D}_X$ is commutative.

Exercise 8.4 **(The graded sheaf $\mathrm{gr}^F\mathcal{D}_X$).** We consider the sheaf $\mathcal{D}_X$ of holomorphic differential operators on X with its order filtration $F_\bullet\mathcal{D}_X$ (Definition 8.1.3). The goal of this exercise is to show that the sheaf of commutative graded $\mathcal{O}_X$-algebras $\mathrm{gr}^F\mathcal{D}_X$ can be canonically identified with the sheaf of graded $\mathcal{O}_X$-algebra $\mathrm{Sym}\,\Theta_X$.

(1) Identify Θ_X with the sheaf of functions on the cotangent space T^*X which are linear in the fibers, and $\mathrm{Sym}\,\Theta_X$ with the sheaf of functions on T^*X which are polynomial in the fibers.

(2) Show that $\mathrm{gr}_1^F\mathcal{D}_X=\Theta_X$ and that $\mathrm{gr}^F\mathcal{D}_X$ is a sheaf of commutative graded $\mathcal{O}_X$-algebras. [*Hint*: Use Exercise 8.3.]

(3) Deduce the existence of a unique morphism of commutative graded $\mathcal{O}_X$-algebras $\mathrm{Sym}\,\Theta_X\to\mathrm{gr}^F\mathcal{D}_X$ which extends the identity $\mathcal{O}_X\oplus\Theta_X\xrightarrow{\sim}\mathrm{gr}_0^F\mathcal{D}_X\oplus\mathrm{gr}_1^F\mathcal{D}_X$.

(4) Show that this morphism is an isomorphism. [*Hint*: Check this in local coordinates.]

Exercise 8.5 **(The universal connection).**

(1) Show that the natural left multiplication of Θ_X on $\mathcal{D}_X$ can be written as a *connection*

$$\nabla:\mathcal{D}_X\longrightarrow\Omega_X^1\otimes_{\mathcal{O}_X}\mathcal{D}_X,$$

i.e., as a $\mathbb{C}$-linear morphism satisfying the *Leibniz rule* $\nabla(fP)=\mathrm{d}f\otimes P+f\nabla P$, where g is any local section of $\mathcal{O}_X$ and P any local section of $\mathcal{D}_X$. [*Hint*: $\nabla(1)$ is the global section of $\Omega_X^1\otimes_{\mathcal{O}_X}\Theta_X$ considered in Exercise 8.1.]

(2) Extend this connection for every $k\geqslant1$ as a $\mathbb{C}$-linear morphism

$${}^{(k)}\nabla:\Omega_X^k\otimes_{\mathcal{O}_X}\mathcal{D}_X\longrightarrow\Omega_X^{k+1}\otimes_{\mathcal{O}_X}\mathcal{D}_X$$

satisfying the Leibniz rule written as

$${}^{(k)}\nabla(\omega\otimes P)=d\omega\otimes P+(-1)^k\omega\wedge\nabla P.$$

(3) Show that ${}^{(k+1)}\nabla\circ{}^{(k)}\nabla=0$ for every $k\geqslant0$ (i.e., ∇ is *integrable* or *flat*).

(4) Show that the morphisms ${}^{(k)}\nabla$ are *right* $\mathcal{D}_X$-linear (but not left $\mathcal{O}_X$-linear).

Exercise 8.6. More generally, show that a left $\mathcal{D}_X$-module $\mathcal{M}$ is nothing but an $\mathcal{O}_X$-module with an *integrable* connection $\nabla:\mathcal{M}\to\Omega_X^1\otimes_{\mathcal{O}_X}\mathcal{M}$. [*Hint*: To get the connection, tensor the left $\mathcal{D}_X$-action $\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{M}\to\mathcal{M}$ by Ω_X^1 on the left and compose with the universal connection to get $\mathcal{D}_X\otimes\mathcal{M}\to\Omega_X^1\otimes\mathcal{M}$; compose it on the left with $\mathcal{M}\to\mathcal{D}_X\otimes\mathcal{M}$ given by $m\mapsto1\otimes m$.] Define similarly the iterated connections ${}^{(k)}\nabla:\Omega_X^k\otimes_{\mathcal{O}_X}\mathcal{M}\to\Omega_X^{k+1}\otimes_{\mathcal{O}_X}\mathcal{M}$. Show that ${}^{(k+1)}\nabla\circ{}^{(k)}\nabla=0$.

Exercise 8.7.

(1) Show that $\widetilde{\mathcal{D}}_X$ has a universal connection $\widetilde{\nabla}$ for which $\widetilde{\nabla}(1)=\sum_i\widetilde{\mathrm{d}}x_i\otimes\widetilde{\partial}_{x_i}$.

(2) Show the equivalence between graded left $\widetilde{\mathcal{D}}_X$-modules and graded $\widetilde{\mathcal{O}}_X$-modules equipped with an integrable connection.

(3) Extend the properties shown in Exercises 8.5 and 8.6 to the present case.

8.11.b. Exercises for Section 8.2

Exercise 8.8 **(Generating left $\widetilde{\mathcal{D}}_X$-modules).** Let $\widetilde{\mathcal{M}}$ be an $\widetilde{\mathcal{O}}_X$-module and let $\varphi^{\mathrm{left}}: \widetilde{\Theta}_X \otimes_{\widetilde{\mathbb{C}}_X} \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ be a $\widetilde{\mathbb{C}}$-linear morphism such that, for any local sections g of $\widetilde{\mathcal{O}}_X$, ξ, ξ' of $\widetilde{\Theta}_X$ and m of $\widetilde{\mathcal{M}}$, one has

(1) $\varphi^{\mathrm{left}}(g\xi \otimes m) = g\varphi^{\mathrm{left}}(\xi \otimes m)$,
(2) $\varphi^{\mathrm{left}}(\xi \otimes gm) = g\varphi^{\mathrm{left}}(\xi \otimes m) + \xi(g)m$,
(3) $\varphi^{\mathrm{left}}([\xi, \xi'] \otimes m) = \varphi^{\mathrm{left}}(\xi \otimes \varphi^{\mathrm{left}}(\xi' \otimes m)) - \varphi^{\mathrm{left}}(\xi' \otimes \varphi^{\mathrm{left}}(\xi \otimes m))$.

Show that there exists a unique structure $\widetilde{\mathcal{M}}^{\mathrm{left}}$ of left $\widetilde{\mathcal{D}}_X$-module on $\widetilde{\mathcal{M}}$ such that $\xi m = \varphi^{\mathrm{left}}(\xi \otimes m)$ for every ξ, m.

Exercise 8.9 **(Generating right $\widetilde{\mathcal{D}}_X$-modules).** Let $\widetilde{\mathcal{M}}$ be an $\widetilde{\mathcal{O}}_X$-module and let $\varphi^{\mathrm{right}}: \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}_X} \widetilde{\Theta}_X \to \widetilde{\mathcal{M}}$ be a $\widetilde{\mathbb{C}}$-linear morphism such that, for any local sections g of $\widetilde{\mathcal{O}}_X$, ξ, ξ' of $\widetilde{\Theta}_X$ and m of $\widetilde{\mathcal{M}}$, one has

(1) $\varphi^{\mathrm{right}}(mg \otimes \xi) = \varphi^{\mathrm{right}}(m \otimes g\xi)$ (φ^{right} is in fact defined on $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_X$),
(2) $\varphi^{\mathrm{right}}(m \otimes g\xi) = \varphi^{\mathrm{right}}(m \otimes \xi)g - m\xi(g)$,
(3) $\varphi^{\mathrm{right}}(m \otimes [\xi, \xi']) = \varphi^{\mathrm{right}}(\varphi^{\mathrm{right}}(m \otimes \xi) \otimes \xi') - \varphi^{\mathrm{right}}(\varphi^{\mathrm{right}}(m \otimes \xi') \otimes \xi)$.

Show that there exists a unique structure $\widetilde{\mathcal{M}}^{\mathrm{right}}$ of right $\widetilde{\mathcal{D}}_X$-module on $\widetilde{\mathcal{M}}$ such that $m\xi = \varphi^{\mathrm{right}}(m \otimes \xi)$ for every ξ, m.

Exercise 8.10 **($\mathcal{O}_X$ is a *simple* left $\mathcal{D}_X$-module).** We consider here the setting of Section 8.1.

(1) Use the left action of Θ_X on $\mathcal{O}_X$ to define on $\mathcal{O}_X$ the structure of a left $\mathcal{D}_X$-module.

(2) Let g be a nonzero holomorphic function on $\mathbb{C}^n$. Show that there exists a multi-index $\alpha \in \mathbb{N}^n$ such that $(\partial^\alpha gm)(0) \neq 0$.

(3) Conclude that $\mathcal{O}_X$ is a simple left $\mathcal{D}_X$-module, i.e., does not contain any proper non trivial $\mathcal{D}_X$-submodule. Is it simple as a left $\mathcal{O}_X$-module?

(4) Show that $R_F\mathcal{O}_X$ is not a simple graded $R_F\mathcal{D}_X$-module. [*Hint*: Consider $zR_F\mathcal{O}_X \subset R_F\mathcal{O}_X$.]

Exercise 8.11 **(ω_X is a *simple* right $\mathcal{D}_X$-module).** Same setting as in Exercise 8.10.

(1) Use the right action of Θ_X on ω_X to define on ω_X the structure of a right $\mathcal{D}_X$-module.

(2) Show that it is simple as a right $\mathcal{D}_X$-module.

(3) Show that $R_F\omega_X$ is not a simple graded right $R_F\mathcal{D}_X$-module.

Exercise 8.12 **(Tensor products over $\widetilde{\mathcal{O}}_X$).**

(1) Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{left}}$ be two left $\widetilde{\mathcal{D}}_X$-modules.

(a) Show that the $\widetilde{\mathcal{O}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}$ has the structure of a *left* $\widetilde{\mathcal{D}}_X$-module when setting, by analogy with the Leibniz rule,

$$\xi \cdot (m \otimes n) = \xi m \otimes n + m \otimes \xi n.$$

(b) If $\widetilde{\mathcal{M}}^{\text{left}}$ and $\widetilde{\mathcal{N}}^{\text{left}}$ are regarded as $\widetilde{\mathcal{O}}_X$-modules with connection (Proposition 8.1.4 and Exercise 8.7), show that the connection on $\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\text{left}}$ coming from the left $\widetilde{\mathcal{D}}_X$-module structure above is equal to $\widetilde{\nabla} \otimes \mathrm{Id}_{\widetilde{\mathcal{N}}} + \mathrm{Id}_{\widetilde{\mathcal{M}}} \otimes \widetilde{\nabla}$.

(c) Notice that, in general, $m \otimes n \mapsto (\xi m) \otimes n$ (or $m \otimes n \mapsto m \otimes (\xi n)$) does not define a left $\widetilde{\mathcal{D}}_X$-action on the $\widetilde{\mathcal{O}}_X$-module $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}$.

(d) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ and $\psi : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}'$ be $\widetilde{\mathcal{D}}_X$-linear morphisms. Show that $\varphi \otimes \psi$ is $\widetilde{\mathcal{D}}_X$-linear.

(e) Show the associativity

$$(\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\text{left}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\text{left}} = \widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{N}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\text{left}}).$$

(2) Let $\widetilde{\mathcal{M}}^{\text{left}}$ be a left $\widetilde{\mathcal{D}}_X$-module and $\widetilde{\mathcal{N}}^{\text{right}}$ be a *right* $\widetilde{\mathcal{D}}_X$-module.

(a) Show that $\widetilde{\mathcal{N}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}$ has the structure of a right $\widetilde{\mathcal{D}}_X$-module by setting

$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m,$$

and prove the analogue of (1d).

Remark: one can define a right $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\text{right}}$ by using the natural involution $\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\text{right}} \xrightarrow{\sim} \widetilde{\mathcal{N}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}$, so this brings no new structure.

(b) Show the associativity

$$(\widetilde{\mathcal{N}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\text{left}} = \widetilde{\mathcal{N}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\text{left}}).$$

(3) Assume that $\widetilde{\mathcal{M}}^{\text{right}}$ and $\widetilde{\mathcal{N}}^{\text{right}}$ are right $\widetilde{\mathcal{D}}_X$-modules. Does there exist a (left or right) $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\text{right}}$ defined with analogous formulas?

Exercise 8.13 **($\mathcal{H}om$ over $\widetilde{\mathcal{O}}_X$).**

(1) Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ be left $\widetilde{\mathcal{D}}_X$-modules. Show that $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ has a natural structure of left $\widetilde{\mathcal{D}}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \xi \cdot (\varphi(m)) + \varphi(\xi \cdot m),$$

for any local sections ξ of $\widetilde{\Theta}_X$, m of $\widetilde{\mathcal{M}}$ and φ of $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$.

(2) Similarly, if $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ are right $\widetilde{\mathcal{D}}_X$-modules, then $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ has a natural structure of left $\widetilde{\mathcal{D}}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi.$$

Exercise 8.14 **(Compatibility of side-changing functors).** Show that the natural morphisms

$$\widetilde{\mathcal{M}}^{\text{left}} \longrightarrow \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}), \quad \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{M}}^{\text{right}}) \longrightarrow \widetilde{\mathcal{M}}^{\text{right}}$$

are isomorphisms of graded $\widetilde{\mathcal{D}}_X$-modules.

Exercise 8.15 **(Side-changing on morphisms).** To any left $\widetilde{\mathcal{D}}_X$-linear morphism $\varphi^{\text{left}} : \widetilde{\mathcal{M}}_1^{\text{left}} \to \widetilde{\mathcal{M}}_2^{\text{left}}$ is associated the $\widetilde{\mathcal{O}}_X$-linear morphism $\varphi^{\text{right}} = \mathrm{Id}_{\widetilde{\omega}_X} \otimes \varphi^{\text{left}} : \widetilde{\mathcal{M}}_1^{\text{right}} \to \widetilde{\mathcal{M}}_2^{\text{right}}$.

(1) Show that φ^{right} is right $\widetilde{\mathcal{D}}_X$-linear.

(2) Define the reverse correspondence $\varphi^{\text{right}} \mapsto \varphi^{\text{left}}$.

(3) Conclude that the left-right correspondence $\mathsf{Mod}^{\text{left}}(\widetilde{\mathcal{D}}_X) \mapsto \mathsf{Mod}^{\text{right}}(\widetilde{\mathcal{D}}_X)$ is a functor, as well as the right-left correspondence $\mathsf{Mod}^{\text{right}}(\widetilde{\mathcal{D}}_X) \mapsto \mathsf{Mod}^{\text{left}}(\widetilde{\mathcal{D}}_X)$.

Exercise 8.16 **(Compatibility of side-changing functors with tensor product)**

Let $\widetilde{\mathcal{M}}^{\text{left}}$ and $\widetilde{\mathcal{N}}^{\text{left}}$ be two left $\widetilde{\mathcal{D}}_X$-modules and denote by $\widetilde{\mathcal{M}}^{\text{right}}, \widetilde{\mathcal{N}}^{\text{right}}$ the corresponding right $\widetilde{\mathcal{D}}_X$-modules (see Definition 8.2.3). Show that there is a natural isomorphism of graded right $\widetilde{\mathcal{D}}_X$-modules (by using the right structure given in Exercise 8.12(2)):

$$\widetilde{\mathcal{N}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}} \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\text{left}}$$
$$(\omega \otimes n) \otimes m \longmapsto (\omega \otimes m) \otimes n$$

and that this isomorphism is functorial in $\widetilde{\mathcal{M}}^{\text{left}}$ and $\widetilde{\mathcal{N}}^{\text{left}}$.

Exercise 8.17 **(Local expression of the side-changing functors)**

Let U be an open set of $\mathbb{C}^n$.

(1) Show that there exists a unique $\widetilde{\mathbb{C}}$-linear involution $P \mapsto {}^tP$ from $\widetilde{\mathcal{D}}(U)$ to itself such that

- $\forall\, g \in \widetilde{\mathcal{O}}(U)$, ${}^tg = g$,
- $\forall\, i \in \{1, \dots, n\}$, ${}^t\widetilde{\partial}_{x_i} = -\widetilde{\partial}_{x_i}$,
- $\forall\, P, Q \in \widetilde{\mathcal{D}}(U)$, ${}^t(PQ) = {}^tQ \cdot {}^tP$.

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let ${}^t\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{M}}$ equipped with the right $\widetilde{\mathcal{D}}_X$-module structure

$$m \cdot P := {}^tPm.$$

Show that $z^{-n}\,{}^t\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\text{right}}$, that is, ${}^t\widetilde{\mathcal{M}}(n) \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\text{right}}$. [*Hint*: Use that $F_p{}^t\mathcal{O}_X = F_{p-n}\omega_X$, hence $R_F{}^t\mathcal{O}_X = R_{F[n]}\omega_X$, so ${}^t\widetilde{\mathcal{O}}_X = \widetilde{\omega}_X(-n)$, according to Remark 5.1.5(2).] Argue similarly starting with a right $\widetilde{\mathcal{D}}_X$-module.

Exercise 8.18 **(Tensor product of a left $\widetilde{\mathcal{D}}_X$-module with $\widetilde{\mathcal{D}}_X$)**

Let $\widetilde{\mathcal{M}}^{\text{left}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Notice that $\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has two commuting structures of $\widetilde{\mathcal{O}}_X$-module. Similarly $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}$ has two such structures. The goal of this exercise is to extend them as $\widetilde{\mathcal{D}}_X$-structures and examine their relations.

(1) Show that $\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has the structure of a left and of a right $\widetilde{\mathcal{D}}_X$-module *which commute*, given by the formulas:

$$\text{(left)} \qquad (\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\text{tens}} : \quad \begin{cases} f \cdot (m \otimes P) = (fm) \otimes P = m \otimes (fP), \\ \xi \cdot (m \otimes P) = (\xi m) \otimes P + m \otimes \xi P, \end{cases}$$

$$\text{(right)} \qquad (\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\text{triv}} : \quad \begin{cases} (m \otimes P) \cdot f = m \otimes (Pf), \\ (m \otimes P) \cdot \xi = m \otimes (P\xi), \end{cases}$$

for any local vector field ξ and any local holomorphic function g. Show that a left $\widetilde{\mathcal{D}}_X$-linear morphism $\varphi : \widetilde{\mathcal{M}}_1^{\text{left}} \to \widetilde{\mathcal{M}}_2^{\text{left}}$ extends as a bi-$\widetilde{\mathcal{D}}_X$-linear morphism $\varphi \otimes 1 : \widetilde{\mathcal{M}}_1^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}_2^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$.

(2) Similarly, show that $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}$ also has such structures *which commute* and are functorial, given by formulas:

$$\text{(left)} \qquad (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}})_{\text{triv}} : \quad \begin{cases} f \cdot (P \otimes m) = (fP) \otimes m, \\ \xi \cdot (P \otimes m) = (\xi P) \otimes m, \end{cases}$$

$$\text{(right)} \qquad (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}})_{\text{tens}} : \quad \begin{cases} (P \otimes m) \cdot f = P \otimes (fm) = (Pf) \otimes m, \\ (P \otimes m) \cdot \xi = P\xi \otimes m - P \otimes \xi m. \end{cases}$$

(3) Show that both morphisms

$$\begin{aligned} \widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X &\longrightarrow \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}} \\ m \otimes P &\longmapsto (1 \otimes m) \cdot P \end{aligned} \qquad \begin{aligned} \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}} &\longrightarrow \widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \\ P \otimes m &\longmapsto P \cdot (m \otimes 1) \end{aligned}$$

are left and right $\widetilde{\mathcal{D}}_X$-linear, induce the identity $\widetilde{\mathcal{M}}^{\text{left}} \otimes 1 = 1 \otimes \widetilde{\mathcal{M}}^{\text{left}}$, and their composition is the identity of $\widetilde{\mathcal{M}}^{\text{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ or $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\text{left}}$, hence both are reciprocal isomorphisms. Show that this correspondence is functorial (i.e., compatible with morphisms).

(4) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. Justify the following isomorphisms of left $\widetilde{\mathcal{D}}_X$-modules and $\widetilde{\mathcal{O}}_X$-modules for the action on the right:

$$\begin{aligned} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) &\simeq (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} \\ &\simeq (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} \simeq \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}). \end{aligned}$$

Assume moreover that $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{L}}$ are $\widetilde{\mathcal{O}}_X$-locally free. Show that $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})$ is $\widetilde{\mathcal{D}}_X$-locally free.

***Exercise 8.19* (Tensor product of a right $\widetilde{\mathcal{D}}_X$-module with $\widetilde{\mathcal{D}}_X$)**

Let $\widetilde{\mathcal{M}}^{\text{right}}$ be a right $\widetilde{\mathcal{D}}_X$-module.

(1) Show that $\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has two structures of right $\widetilde{\mathcal{D}}_X$-module denoted triv and tens (tensor; the latter defined by using the left structure on $\widetilde{\mathcal{D}}_X$ and Exercise 8.12(2)), given by:

$$\text{(right)} \qquad (\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\text{triv}} : \quad \begin{cases} (m \otimes P) \cdot_{\text{triv}} f = m \otimes (Pf), \\ (m \otimes P) \cdot_{\text{triv}} \xi = m \otimes (P\xi), \end{cases}$$

$$\text{(right)} \qquad (\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\text{tens}} : \quad \begin{cases} (m \otimes P) \cdot_{\text{tens}} f = mf \otimes P = m \otimes fP, \\ (m \otimes P) \cdot_{\text{tens}} \xi = m\xi \otimes P - m \otimes (\xi P). \end{cases}$$

(2) Show that there is a unique involution $\iota : \widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ which exchanges both structures and is the identity on $\widetilde{\mathcal{M}}^{\text{right}} \otimes 1$, given by $(m \otimes P)_{\text{triv}} \mapsto (m \otimes 1) \cdot_{\text{tens}} P$. [*Hint*: Show first the properties of ι by using local coordinates, and glue the local constructions by uniqueness of ι.]

(3) For every $p \geqslant 0$, consider the p-th term $F_p\widetilde{\mathcal{D}}_X$ of the filtration of $\widetilde{\mathcal{D}}_X$ by the order (see Exercise 8.1.3) with both structures of $\widetilde{\mathcal{O}}_X$-module (one on the left, one on

the right) and equip similarly $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} F_p\widetilde{\mathcal{D}}_X$ with two structures of $\widetilde{\mathcal{O}}_X$-modules. Show that, for every p, ι preserves $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} F_p\widetilde{\mathcal{D}}_X$ and exchanges the two structures of $\widetilde{\mathcal{O}}_X$-modules.

(4) Let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ be a right $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. By considering the natural $\widetilde{\mathcal{O}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}$, we define an induced right $\widetilde{\mathcal{D}}_X$-module $\big[(\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\big]_{\mathrm{triv}}$. Here, the $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{right}}$ is not used.

On the other hand, considering the canonical left $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}$ and using Exercise 8.12(2), we obtain a right $\widetilde{\mathcal{D}}_X$-module structure $\big[\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})\big]_{\mathrm{tens}}$. Here, the $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{right}}$ is used in an essential way.

Prove that the canonical $\widetilde{\mathcal{O}}_X$-linear morphism

$$\begin{aligned}\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} &\longrightarrow \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})\\ m \otimes \ell &\longmapsto m \otimes (1 \otimes \ell)\end{aligned}$$

extends in a unique way as a $\widetilde{\mathcal{D}}_X$-linear morphism

$$\Big[(\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\Big]_{\mathrm{triv}} \longrightarrow \Big[\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})\Big]_{\mathrm{tens}}$$

which is an isomorphism. [*Hint*: Argue as in (2).]

8.11.c. Exercises for Section 8.4

Exercise 8.20. Check that $\mathrm{Sp}(\widetilde{\mathcal{M}})$ is indeed a complex, i.e., that $\widetilde{\delta} \circ \widetilde{\delta} = 0$.

***Exercise 8.21* ($\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module)**

The natural surjective morphism $\widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{O}}_X$ of left $\widetilde{\mathcal{D}}_X$-modules has kernel the image of $\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_X \to \widetilde{\mathcal{D}}_X$. In other words, we have a morphism of complexes of left $\widetilde{\mathcal{D}}_X$-modules

$$\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \longrightarrow \widetilde{\mathcal{O}}_X$$

(where $\widetilde{\mathcal{O}}_X$ is regarded as a complex with a nonzero term in degree zero only), which induces an isomorphism

$$H^0\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \xrightarrow{\ \sim\ } \widetilde{\mathcal{O}}_X.$$

In this exercise, one proves that $H^k(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) = 0$ for $k \neq 0$, so that the morphism above is a quasi-isomorphism.

Let $F_\bullet\widetilde{\mathcal{D}}_X$ be the filtration of $\widetilde{\mathcal{D}}_X$ by the order of differential operators. Filter the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ by the subcomplexes $F_p(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))$ defined as

$$\cdots \xrightarrow{\ \widetilde{\delta}\ } F_{p-k}\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k} \xrightarrow{\ \widetilde{\delta}\ } F_{p-k+1}\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k-1} \xrightarrow{\ \widetilde{\delta}\ } \cdots$$

(1) Show that, locally on X, using coordinates $x_1, \dots, x_n$, the graded complex $\mathrm{gr}^F\mathrm{Sp}(\widetilde{\mathcal{D}}_X) := \bigoplus_p \mathrm{gr}^F_p\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is equal to the Koszul complex of the ring $\widetilde{\mathcal{O}}_X[\xi_1, \dots, \xi_n]$ with respect to the regular sequence $\xi_1, \dots, \xi_n$.

(2) Conclude that $\mathrm{gr}^F\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$.

(3) Check that $F_p \operatorname{Sp}(\widetilde{\mathcal{D}}_X) = 0$ for $p < 0$, $F_0 \operatorname{Sp}(\widetilde{\mathcal{D}}_X) = \operatorname{gr}_0^F \operatorname{Sp}(\widetilde{\mathcal{D}}_X)$ is isomorphic to $\widetilde{\mathcal{O}}_X$ and deduce that the complex

$$\operatorname{gr}_p^F \operatorname{Sp}(\widetilde{\mathcal{D}}_X) := \{\cdots \xrightarrow{\widetilde{\delta}} \operatorname{gr}_{p-k}^F \widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k} \xrightarrow{\widetilde{\delta}} \operatorname{gr}_{p-k+1}^F \widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k-1} \xrightarrow{\widetilde{\delta}} \cdots\}$$

is acyclic (i.e., quasi-isomorphic to 0) for $p > 0$.

(4) Show that the inclusion $F_0 \operatorname{Sp}(\widetilde{\mathcal{D}}_X) \hookrightarrow F_p \operatorname{Sp}(\widetilde{\mathcal{D}}_X)$ is a quasi-isomorphism for every $p \geqslant 0$ and deduce, by passing to the inductive limit, that the Spencer complex $\operatorname{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module by locally free left $\widetilde{\mathcal{D}}_X$-modules.

Exercise 8.22 **(${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\omega}_X$ as a right $\widetilde{\mathcal{D}}_X$-module)**
Show similarly that the natural morphism of right $\widetilde{\mathcal{D}}_X$-modules

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\omega}_X$$

defined as the right action of $\widetilde{\mathcal{D}}_X$ on $\widetilde{\omega}_X$ extends as a morphism of complexes of right $\widetilde{\mathcal{D}}_X$-modules

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \longrightarrow \widetilde{\omega}_X.$$

Show that $H^k(\mathrm{DR}\,\widetilde{\mathcal{D}}_X) = 0$ for $k \neq n$, so that the shifted complex $\mathrm{DR}(\widetilde{\mathcal{D}}_X)[n]$ is a resolution of $\widetilde{\omega}_X$ as a right $\widetilde{\mathcal{D}}_X$-module by locally free right $\widetilde{\mathcal{D}}_X$-modules.

Exercise 8.23 **(Tensor product over $\widetilde{\mathcal{D}}_X$).** Let $\widetilde{\mathcal{M}}^{\mathrm{left}}, \widetilde{\mathcal{N}}^{\mathrm{left}}$ be two left $\widetilde{\mathcal{D}}_X$-modules. One can consider the tensor products $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$. Both are bi-functors with values in the category of sheaves of $\widetilde{\mathbb{C}}$-vector spaces (a priori they do not have any other structure). Show that there is a natural $\widetilde{\mathbb{C}}$-linear isomorphism $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}} \xrightarrow{\sim} \widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ induced by

$$(\widetilde{\omega} \otimes_{\widetilde{\mathcal{O}}} m) \otimes_{\widetilde{\mathcal{D}}} n \longmapsto (\widetilde{\omega} \otimes_{\widetilde{\mathcal{O}}} n) \otimes_{\widetilde{\mathcal{D}}} m.$$

[*Hint*: Show that, for any holomorphic vector field ξ, one has the equality $(\widetilde{\omega} \otimes m) \otimes \xi n = (\widetilde{\omega} \otimes n) \otimes \xi m$.]

Exercise 8.24 **(The Spencer complex: tensoring over $\widetilde{\mathcal{D}}_X$ with $\operatorname{Sp}(\widetilde{\mathcal{D}}_X)$)**

(1) Let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that the natural morphism

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \longrightarrow \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}$$

defined by $m \otimes (P \otimes \xi) \mapsto mP \otimes \xi$ induces an isomorphism of complexes

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \operatorname{Sp}(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}).$$

[*Hint*: The point is to check that the differential $\operatorname{Id} \otimes \widetilde{\delta}_{\widetilde{\mathcal{D}}}$ on the left corresponds to the differential $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ on the right.]

(2) Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Similar question for

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}).$$

Exercise 8.25 **(The unshifted de Rham complex: $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X),\widetilde{\mathcal{M}})$)**

For left $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}$, the sheaf $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})$ is a priori only a $\widetilde{\mathbb{C}}$-module. If $\widetilde{\mathcal{N}}$ is also a right $\widetilde{\mathcal{D}}_X$-module, like $\widetilde{\mathcal{D}}_X$, $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})$ comes equipped with the structure of right $\widetilde{\mathcal{D}}_X$-module inherited from that of $\widetilde{\mathcal{N}}$. In particular, for each k, $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k},\widetilde{\mathcal{D}}_X)$ is a right $\widetilde{\mathcal{D}}_X$-module and $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X),\widetilde{\mathcal{D}}_X)$ is a complex of right $\widetilde{\mathcal{D}}_X$-modules whose term in degree k is $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k},\widetilde{\mathcal{D}}_X)$.

(1) Identify the complex of right $\widetilde{\mathcal{D}}_X$-modules $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X),\widetilde{\mathcal{D}}_X)$ (where the right structure comes from the second term $\widetilde{\mathcal{D}}_X$) with the unshifted complex $\mathrm{DR}\,\widetilde{\mathcal{D}}_X$ up to changing the sign of the differential in the latter complex. [*Hint*:

(a) Identify first the right $\widetilde{\mathcal{D}}_X$-module $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k},\widetilde{\mathcal{D}}_X)$ with $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\Theta}_{X,k},\widetilde{\mathcal{D}}_X)$, then to $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\Theta}_{X,k},\widetilde{\mathcal{O}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$, hence to $\widetilde{\Omega}^k_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$;

(b) In local coordinates, for $I,I'\subset\{1,\dots,n\}$ such that $\#I,\#I'=k$, set $\widetilde{\mathrm{d}}x_I=\widetilde{\mathrm{d}}x_{i_1}\wedge\cdots\wedge\widetilde{\mathrm{d}}x_{i_k}$ with $i_1<\cdots<i_k$, and similarly for $\widetilde{\partial}_{x_{I'}}$; consider the pairing $\langle\widetilde{\mathrm{d}}x_I,\widetilde{\partial}_{x_{I'}}\rangle=(-1)^{k(k-1)/2}$ if $I=I'$, and $=0$ otherwise (see Section 8.1.a); recall that $\widetilde{\mathrm{d}}(\widetilde{\mathrm{d}}x_I\otimes 1)=\sum_{j\notin I}\widetilde{\mathrm{d}}x_I\wedge\widetilde{\mathrm{d}}x_j\otimes\widetilde{\partial}_{x_j}$ in $\widetilde{\Omega}^{k+1}_X\otimes\widetilde{\mathcal{D}}_X$, and if $J=\{i_1,\dots,i_{k+1}\}$, $\widetilde{\delta}(1\otimes\widetilde{\partial}_{x_J})=\sum_{j=1}^{k+1}(-1)^j\widetilde{\partial}_{x_{i_j}}\otimes\widetilde{\partial}_{J\smallsetminus i_j}$ in $\widetilde{\mathcal{D}}_X\otimes\widetilde{\Theta}_{X,k}$; then show that for any such I,J, one has $\langle\widetilde{\mathrm{d}}x_I\otimes1,\widetilde{\delta}(1\otimes\widetilde{\partial}_{x_J})\rangle=-\langle\widetilde{\mathrm{d}}(\widetilde{\mathrm{d}}x_I\otimes1),1\otimes\widetilde{\partial}_{x_J}\rangle$ and conclude.]

(2) Conclude that, for a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, one has

$$\begin{aligned}\mathrm{DR}\,\widetilde{\mathcal{M}}\simeq\mathrm{DR}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}&\simeq\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X),\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}\\&\simeq\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X),\widetilde{\mathcal{M}}).\end{aligned}$$

Exercise 8.26 **(Side-changing for the de Rham functors).**

(1) If $\widetilde{\mathcal{M}}$ is any left $\widetilde{\mathcal{D}}_X$-module and $\widetilde{\mathcal{M}}^{\mathrm{right}}=\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}$ is the associated right $\widetilde{\mathcal{D}}_X$-module, show that ι defined in Lemma 8.4.7 induces an isomorphism

$$\widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\xrightarrow{\sim}{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$$

which is termwise $\widetilde{\mathcal{O}}_X$-linear. [*Hint*: Use Exercise 8.23 to identify $\widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ with $(\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X))\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$.]

(2) Interpret the isomorphism ι of Lemma 8.4.7 as the composition of the inverse of the isomorphism

$$\big[(\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k})\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X\big]_{\mathrm{triv}}\xrightarrow{\sim}\big[\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k})\big]_{\mathrm{tens}}$$

of Exercise 8.19(4), with $\lrcorner$.

(3) Argue as in Lemma 8.4.7 (with the interpretation above) to show that the $\widetilde{\mathcal{O}}_X$-linear isomorphism

$$\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}\xrightarrow{\sim}\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}\xrightarrow{\sim}\widetilde{\Omega}^{n-k}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}$$

given on $\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}$ by

$$\omega\otimes m\otimes\xi\longmapsto\omega(\xi\wedge\bullet)\otimes m$$

induces a functorial isomorphism ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}})$ for any left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, which is termwise $\widetilde{\mathcal{O}}_X$-linear.

Exercise 8.27 **(Interior product with a 1-form).** Let $x_1, \dots, x_n$ be local coordinates. Fix $k \geqslant 1$ and set $\widetilde{\partial}_x = \widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}$ and, for $i \in \{1, \dots, k\}$, $\widetilde{\partial}_{x_{\hat\imath}} = \widetilde{\partial}_{x_1} \wedge \cdots \wedge \widehat{\widetilde{\partial}_{x_i}} \wedge \cdots \wedge \widetilde{\partial}_{x_k}$. Show the following equalities for $i \neq j \in \{1, \dots, k\}$:

$$\widetilde{\partial}_x \lrcorner \widetilde{\mathrm{d}}x_i = (-1)^{k-i} \widetilde{\partial}_{x_{\hat\imath}}, \qquad \widetilde{\partial}_{x_{\hat\jmath}} \lrcorner \widetilde{\mathrm{d}}x_i = \begin{cases} (-1)^{k-i+1} \widetilde{\partial}_{x_{\hat\imath\hat\jmath}} & \text{if } i < j, \\ (-1)^{k-i} \widetilde{\partial}_{x_{\hat\imath\hat\jmath}} & \text{if } i > j. \end{cases}$$

[*Hint*: Use (8.4.6∗∗) and (8.4.6∗).]

Exercise 8.28 **(The C^∞ Spencer complex).** Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module and let us denote by $\widetilde{\delta}'_{\widetilde{\mathcal{M}}}$ the differential of the Spencer complex ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}})$.

(1) Show that, for each j, the formula (for $i, j \geqslant 0$)

$$\begin{aligned} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,i} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}^{(0,j)} &\xrightarrow{\widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}}} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,i-1} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}^{(0,j)} \\ m \otimes \boldsymbol{\xi}_i \otimes \varphi &\longmapsto \widetilde{\delta}'_{\widetilde{\mathcal{M}}}(m \otimes \boldsymbol{\xi}_i) \otimes \varphi + m \otimes \boldsymbol{\xi}_i \lrcorner \widetilde{\mathrm{d}}'\varphi \end{aligned}$$

defines the differential of a complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}^{(0,j)}$. Show that

$$\widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}} \mathrm{d}'' + \mathrm{d}'' \widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}} = 0,$$

and deduce a complex ${}^{\mathrm{p}}\mathrm{DR}^{\infty}(\widetilde{\mathcal{M}}) := (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}^{\infty,\bullet}_X, \widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}} + \mathrm{d}'')$ (notation of Section 8.4.13).

(2) Show that the natural morphism

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}^{\infty}(\widetilde{\mathcal{M}})$$

is a quasi-isomorphism.

(3) Argue as in Exercise 8.24(1) to define an isomorphism of complexes

$$\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}^{\infty}(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^{\infty}(\widetilde{\mathcal{M}}).$$

(4) Argue as in Exercise 8.26 to define the side-changing isomorphism

$${}^{\mathrm{p}}\mathrm{DR}^{\infty}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^{\infty}(\widetilde{\mathcal{M}}^{\mathrm{left}}).$$

8.11.d. Exercises for Section 8.5

Exercise 8.29. Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module.

(1) Show that, for every k, we have a (termwise) exact sequence of complexes

$$0 \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} F_{k-1}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} F_k(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F_k(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \to 0.$$

[*Hint*: Use that the terms of the complexes $F_j(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))$ and $\mathrm{gr}^F_k(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))$ are $\widetilde{\mathcal{O}}_X$-locally free.]

(2) Show that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{L}}$ as an $\widetilde{\mathcal{O}}_X$-module.

(3) Show that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{L}}$ as an $\widetilde{\mathcal{O}}_X$-module.

(4) Identify the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X)$ with $\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ as complexes of "left" $\widetilde{\mathcal{O}}_X$-modules.

(5) Conclude that ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X)\simeq\widetilde{\mathcal{L}}$.

***Exercise 8.30* (A local resolution of a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$).** Assume that $X=\mathbb{C}^n$ with coordinates $x_1,\dots,x_n$. Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Equip $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$ with the trivial right $\widetilde{\mathcal{D}}_X$-module structure and, for $i=1,\dots,n$, consider the $\widetilde{\mathcal{D}}_X$-linear morphisms $\cdot_{\mathrm{tens}}\widetilde{\partial}_{x_i}$, recalling that the right tens structure commutes with the right triv structure (see Exercise 8.19). Show that the Koszul complex $K((\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X)_{\mathrm{triv}},(\cdot_{\mathrm{tens}}\widetilde{\partial}_{x_i})_{i=1,\dots,n})$ is a resolution of $\widetilde{\mathcal{M}}$ with the following steps. Recall that $\mathrm{gr}^F\widetilde{\mathcal{D}}_X\simeq\widetilde{\mathcal{O}}_X[\xi_1,\dots,\xi_n]$ with $\xi_i=[\widetilde{\partial}_{x_i}]\in\mathrm{gr}_1^F\widetilde{\mathcal{D}}_X$.

(1) Show that the morphism induced by $\widetilde{\partial}_{x_i}$ on $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{gr}^F\widetilde{\mathcal{D}}_X$ is $1\otimes\xi_i$.

(2) Deduce that the Koszul complex $K(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{gr}^F\widetilde{\mathcal{D}}_X,(\xi_i)_{i=1,\dots,n})$ is exact in negative degrees.

(3) Deduce that the Koszul complex $K(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,(\widetilde{\partial}_{x_i})_{i=1,\dots,n})$ is exact in negative degrees, and conclude.

***Exercise 8.31* (Canonical resolution of $\widetilde{\mathcal{M}}$: tensoring over $\widetilde{\mathcal{O}}_X$ with $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$)**

This is an intrinsic version of Exercise 8.30.

(1) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Regarding $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ as a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module, the complex $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is regarded as a complex of right $\widetilde{\mathcal{D}}_X$-module, by using the tensor right $\widetilde{\mathcal{D}}_X$-module structure on each term.

(a) Show that $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{M}}$. [*Hint*: use the functoriality of the tensor right $\widetilde{\mathcal{D}}_X$-module structure and the local $\widetilde{\mathcal{O}}_X$-freeness of each term of $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$.]

(b) Show that the differential of this complex is expressed as follows, for local sections m of $\widetilde{\mathcal{M}}$, ξ_i of $\widetilde{\Theta}_X$ and P of $\widetilde{\mathcal{D}}_X$, and setting

$$\widehat{\boldsymbol{\xi}}_i=\xi_1\wedge\cdots\wedge\xi_{i-1}\wedge\xi_{i+1}\wedge\cdots\wedge\xi_k,$$

and a similar meaning for $\widehat{\boldsymbol{\xi}}_{i,j}$:

$$\begin{aligned}(\mathrm{Id}\otimes\widetilde{\delta})\big[(m\otimes(1\otimes\boldsymbol{\xi}))\cdot_{\mathrm{tens}}P\big]&=\big[(\mathrm{Id}\otimes\widetilde{\delta})(m\otimes(1\otimes\boldsymbol{\xi}))\big]\cdot_{\mathrm{tens}}P\\&=\Big[m\otimes\Big[\sum_{i=1}^k(-1)^{i-1}\xi_i\otimes\widehat{\boldsymbol{\xi}}_i+\sum_{i<j}(-1)^{i+j}1\otimes([\xi_i,\xi_j]\wedge\widehat{\boldsymbol{\xi}}_{i,j})\Big]\Big]\cdot_{\mathrm{tens}}P.\end{aligned}$$

(c) Consider the involution

$$\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k})\simeq(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k})\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$$

exchanging the tens structure on the left-hand side with the triv structure on the right hand side. Show that the differential becomes $\widetilde{\delta}_{\mathrm{triv}}$, with

$$\widetilde{\delta}_{\mathrm{triv}}\big[(m\otimes\boldsymbol{\xi})\otimes P\big]=\widetilde{\delta}_{\mathrm{triv}}\big[(m\otimes\boldsymbol{\xi})\otimes1\big]\cdot_{\mathrm{triv}}P$$

and

$$\begin{aligned}\widetilde{\delta}_{\mathrm{triv}}\big[(m\otimes\boldsymbol{\xi})\otimes 1\big] &= \sum_{i=1}^{k}(-1)^{i-1}(m\xi_i\otimes\widehat{\boldsymbol{\xi}}_i)\otimes 1\\ &\quad-\sum_{i=1}^{k}(-1)^{i-1}(m\otimes\widehat{\boldsymbol{\xi}}_i)\otimes\xi_i+\sum_{i<j}(-1)^{i+j}(m\otimes([\xi_i,\xi_j]\wedge\widehat{\boldsymbol{\xi}}_{i,j}))\otimes 1\\ &= \big[\widetilde{\delta}_{\widetilde{\mathcal{M}}}(m\otimes\boldsymbol{\xi})\big]\otimes 1-\sum_{i=1}^{k}(-1)^{i-1}(m\otimes\widehat{\boldsymbol{\xi}}_i)\otimes\xi_i,\end{aligned}$$

where $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ is the differential occurring in the complex $\operatorname{Sp}\widetilde{\mathcal{M}}$. [*Hint*: write

$$m\otimes(\xi_i\otimes\widehat{\boldsymbol{\xi}}_i)=m\xi_i\otimes(1\otimes\widehat{\boldsymbol{\xi}}_i)-[m\otimes(1\otimes\widehat{\boldsymbol{\xi}}_i)]\cdot\xi_i.]$$

(d) Conclude that the complex of induced $\widetilde{\mathcal{D}}_X$-modules

$$\big((\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,\bullet})\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,\widetilde{\delta}_{\mathrm{triv}}\big)$$

is a resolution of $\widetilde{\mathcal{M}}$.

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Show that the complex

$$^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}})$$

is a resolution of $\widetilde{\mathcal{M}}^{\mathrm{right}}=\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}$ by right $\widetilde{\mathcal{D}}_X$-modules, where the left and right structures of $\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}$ are those of Exercise 8.18(2), and the left one is used to compute the de Rham complex.

8.11.e. Exercises for Section 8.6

Exercise 8.32 **(Definition of the pullback of a left $\widetilde{\mathcal{D}}_X$-module)**

(1) Show that the connection $\widetilde{\nabla}^X$ on $f^*\widetilde{\mathcal{N}}:=\widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{N}}$ is integrable.

(2) Show that, if $\widetilde{\mathcal{N}}$ also has a right $\widetilde{\mathcal{D}}_Y$-module structure commuting with the left one, then $\widetilde{\nabla}^X$ is right $f^{-1}\widetilde{\mathcal{D}}_Y$-linear, and ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}$ is a right $f^{-1}\widetilde{\mathcal{D}}_Y$-module.

Exercise 8.33.

(1) Express the previous connection in local coordinates on X and Y.

(2) Show that, if $\widetilde{\mathcal{M}}$ is any left $\widetilde{\mathcal{D}}_X$-module and $\widetilde{\mathcal{N}}$ any left $f^{-1}\widetilde{\mathcal{D}}_Y$-module, then $\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{N}}$ may be equipped with a left $\widetilde{\mathcal{D}}_X$-module structure: if ξ is a local z-vector field on X, i.e., a local section of $\widetilde{\Theta}_X$, set

$$\xi\cdot(m\otimes n)=(\xi m)\otimes n+Tf(\xi)(m\otimes n).$$

[*Hint*: Identify $\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{N}}$ with $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}{}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$ and use Exercise 8.32.]

Exercise 8.34 **(Local computation of $\widetilde{\mathcal{D}}_{X\to Y}$).**

(1) Show that ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{D}}_Y$ is a locally free $\widetilde{\mathcal{O}}_X$-module. [*Hint*: Use that $\widetilde{\mathcal{D}}_Y$ is a locally free $\widetilde{\mathcal{O}}_Y$-module.]

(2) Choose local coordinates $x_1, \dots, x_n$ on X and $y_1, \dots, y_m$ on Y. Show that $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{O}}_X[\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_m}]$ and, with this identification, the left $\widetilde{\mathcal{D}}_X$-structure is given by

$$\widetilde{\partial}_{x_i} \cdot \sum_\alpha a_\alpha(x)\widetilde{\partial}_y^\alpha = \sum_\alpha \Big(z\frac{\partial a_\alpha}{\partial x_i} + \sum_{j=1}^m a_\alpha(x)\frac{\partial f_j}{\partial x_i}\widetilde{\partial}_{y_j}\Big)\widetilde{\partial}_y^\alpha.$$

Exercise 8.35 **($\widetilde{\mathcal{D}}_{X\to Y}$ for a closed embedding).** Assume that $\iota : X \hookrightarrow Y$ is the closed immersion of a complex submanifold of Y of codimension d.

(1) Show that the canonical section $\mathbf{1}$ of $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{O}}_X \otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y} \iota^{-1}\widetilde{\mathcal{D}}_Y$ is a generator of $\widetilde{\mathcal{D}}_{X\to Y}$ as a right $\iota^{-1}\widetilde{\mathcal{D}}_Y$-module.

(2) Assume that X is defined by $g_1 = \cdots = g_d = 0$, where the g_i are holomorphic functions on Y. Show that

$$\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{D}}_Y \big/ \textstyle\sum_{i=1}^d g_i\widetilde{\mathcal{D}}_Y$$

with its natural right $\widetilde{\mathcal{D}}_Y$ structure. In local coordinates $(x_1, \dots, x_n, y_1, \dots, y_d)$ such that $g_i = y_i$, show that $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_d}]$.

Conclude that, if f is an embedding, the sheaves $\widetilde{\mathcal{D}}_{X\to Y}$ and $\widetilde{\mathcal{D}}_{Y\leftarrow X}$ are locally free over $\widetilde{\mathcal{D}}_X$.

Exercise 8.36 **($\widetilde{\mathcal{D}}_{X\to Y}$ for a flat morphism).** Let $0 \to \widetilde{\mathcal{N}}' \to \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}'' \to 0$ be an exact sequence of left $\widetilde{\mathcal{D}}_Y$-modules.

(1) Show that the sequence ${}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}' \to {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}} \to {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}'' \to 0$ is exact.

(2) Assume that $f : X \to Y$ is flat, i.e., $\widetilde{\mathcal{O}}_X$ is $f^{-1}\widetilde{\mathcal{O}}_Y$-flat (for example, a smooth map, i.e., locally isomorphic to the projection of a product, is flat). Show that the sequence $0 \to {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}' \to {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}} \to {}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}}'' \to 0$ is exact. Conclude that $\widetilde{\mathcal{D}}_{X\to Y}$ is $f^{-1}\widetilde{\mathcal{D}}_Y$-flat.

Exercise 8.37 **(The chain rule).** Consider holomorphic maps $f : X \to Y$ and $f' : Y \to Z$.

(1) Construct a canonical isomorphism $\widetilde{\mathcal{D}}_{X\to Y} \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z} \xrightarrow{\sim} \widetilde{\mathcal{D}}_{X\to Z}$ as right $(f' \circ f)^{-1}\widetilde{\mathcal{D}}_Z$-modules. [*Hint*: Show that the contraction morphisms

$$\begin{aligned}(\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} (f^{-1}\widetilde{\mathcal{O}}_Y \otimes_{(f'\circ f)^{-1}\widetilde{\mathcal{O}}_Z} (f' \circ f)^{-1}\widetilde{\mathcal{D}}_Z) \\ \longrightarrow \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} (f^{-1}\widetilde{\mathcal{O}}_Y \otimes_{(f'\circ f)^{-1}\widetilde{\mathcal{O}}_Z} (f' \circ f)^{-1}\widetilde{\mathcal{D}}_Z) \\ \longrightarrow \widetilde{\mathcal{O}}_X \otimes_{(f'\circ f)^{-1}\widetilde{\mathcal{O}}_Z} (f' \circ f)^{-1}\widetilde{\mathcal{D}}_Z\end{aligned}$$

yield such an isomorphism, whose inverse is the morphism $\varphi \otimes Q \mapsto (\varphi \otimes 1) \otimes (1 \otimes Q)$.]

(2) Use the chain rule to show that this isomorphism is left $\widetilde{\mathcal{D}}_X$-linear.

Exercise 8.38 **(Restriction to $z = 1$).** Show that

$$({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}})/(z-1){}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{N}} = {}_{\mathrm{D}}f^{*(0)}(\widetilde{\mathcal{N}}/(z-1)\widetilde{\mathcal{N}}).$$

Exercise 8.39.

(1) Show that Definition 8.6.6 coincides with that of Exercise 8.32(1).

(2) Let $f: X \to Y$, $f': Y \to Z$ be holomorphic maps and let $\widetilde{\mathcal{N}}$ be a left $\widetilde{\mathcal{D}}_Z$-module. Show that ${}_{\mathrm{D}}(f' \circ f)^*\widetilde{\mathcal{N}} \simeq {}_{\mathrm{D}}f^*({}_{\mathrm{D}}f'^*\widetilde{\mathcal{N}})$.

8.11.f. Exercises for Section 8.7

***Exercise 8.40* (The relative Spencer complex $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})$).** Let $f: X \to Y$ be a holomorphic map and let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. The goal of this exercise is to identify the complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ entering in the definition of the pushforward with the complex

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) := \big((\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y, \widetilde{\delta}_{\widetilde{\mathcal{M}},Y}\big),$$

where $\widetilde{\delta}_{\widetilde{\mathcal{M}},Y}$ is given by the formula

$$\widetilde{\delta}_{\widetilde{\mathcal{M}},Y}((m \otimes \boldsymbol{\xi}) \otimes Q) = \widetilde{\delta}_{\widetilde{\mathcal{M}}}(m \otimes \boldsymbol{\xi}) \otimes Q + \sum_{i=1}^{k} (-1)^i (m \otimes \boldsymbol{\xi}_{\widehat{i}}) \otimes Tf(\xi_i)Q.$$

Here, $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ is given by the formula of Definition 8.4.3 and we use the notation of Exercise 8.31. The first part concerns the complex $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_Y)$.

(1) Let $\widetilde{\mathcal{L}}$ be a locally free $\widetilde{\mathcal{O}}_X$-module. Consider on $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ the following $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$ bi-module structures:

(a) $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$ also called the tens structure, where the right $f^{-1}\widetilde{\mathcal{D}}_Y$ is the trivial one and the left $\widetilde{\mathcal{D}}_X$-module structure is the left tensor one on $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$ (see Exercise 8.12(1)). In particular, $f^{-1}\widetilde{\mathcal{O}}_Y$ acts on the left on $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})$.

(b) $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$ also called the triv structure, where we use the trivial $f^{-1}\widetilde{\mathcal{D}}_Y$-module structure on the right and the trivial $\widetilde{\mathcal{D}}_X$-module structure on the left (on the other hand, the right $\widetilde{\mathcal{O}}_X$-module structure is used on $\widetilde{\mathcal{D}}_X$ for the tensor product).

Show that there exists a unique isomorphism of $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$ bi-modules

$$\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \xrightarrow{\ \sim\ } (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$$

inducing the identity on $\widetilde{\mathcal{L}} = \widetilde{\mathcal{O}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{O}}_Y$. [*Hint*: Show that the morphism $P \otimes \ell \otimes Q \mapsto P \cdot_{\mathrm{tens}} (1 \otimes \ell \otimes Q)$ is well-defined by using that $\widetilde{\mathcal{D}}_X$ is locally free over $\widetilde{\mathcal{O}}_X$, and is an isomorphism by considering the top degree part of P.]

(2) Recall that the differential on the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\delta} \otimes \mathrm{Id}$, with $\widetilde{\delta} = \widetilde{\delta}_{\widetilde{\mathcal{D}}}$ (see Exercise 8.12(1d)). Show that $\widetilde{\delta}_{\widetilde{\mathcal{D}},Y}$ is linear with respect to the triv $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodule structure, and that the following diagram commutes, by

checking first on $\mathbf{1}\otimes\widetilde{\Theta}_{X,\bullet}\otimes f^{-1}\widetilde{\mathcal{D}}_Y$:

$$\begin{array}{ccc}\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}(\widetilde{\Theta}_{X,k}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y) & \xrightarrow{\ \sim\ } & (\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k})\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y\\ {\scriptstyle\widetilde{\delta}_{\widetilde{\mathcal{D}},Y}}\downarrow & & \downarrow{\scriptstyle\widetilde{\delta}\otimes\mathrm{Id}}\\ \widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}(\widetilde{\Theta}_{X,k-1}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y) & \xrightarrow{\ \sim\ } & (\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k-1})\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y\end{array}$$

and conclude that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\to\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$ is an isomorphism.

(3) Deduce that the terms of the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$ are locally free left $\widetilde{\mathcal{D}}_X$-modules. [*Hint*: Check this for the complex $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.]

(4) Conclude that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})\to\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y})$ is an isomorphism. [*Hint*: Check that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})\simeq\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.]

***Exercise 8.41* (The relative Spencer complex of $\widetilde{\mathcal{D}}_X$).**

(1) Let $\widetilde{\mathcal{L}}^\bullet$ be a bounded resolution by left $\widetilde{\mathcal{D}}_X$-modules of $\widetilde{\mathcal{O}}_X$ (as a left $\widetilde{\mathcal{D}}_X$-module). Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Show that, if the terms $\widetilde{\mathcal{L}}^k$ are $\widetilde{\mathcal{O}}_X$-locally free, $\widetilde{\mathcal{L}}^\bullet\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}}$ (with the tensor product structure of left $\widetilde{\mathcal{D}}_X$-module) is a resolution of $\widetilde{\mathcal{M}}$ as a $\widetilde{\mathcal{D}}_X$-module.

(2) Deduce that $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$ is a resolution of $\widetilde{\mathcal{D}}_{X\to Y}$ as a bimodule.

(3) Let $\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})$ be the Spencer complex of $\widetilde{\mathcal{D}}_{X\to Y}$ considered as a right $\widetilde{\mathcal{D}}_Y$-module. Show that $\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module.

(4) Show that $\mathrm{gr}^F\mathcal{D}_{X\to Y}=R_F\mathcal{D}_{X\to Y}/zR_F\mathcal{D}_{X\to Y}$ is identified with $\pi^*\operatorname{Sym}\Theta_Y$ as a graded $(\operatorname{Sym}\Theta_X)$-module (see Exercise 8.4). For example, if $Y=\mathrm{pt}$, so that $\mathcal{D}_{X\to Y}=\mathcal{O}_X$, $\mathrm{gr}^F\mathcal{O}_X=\mathcal{O}_X$ is regarded as a $(\operatorname{Sym}\Theta_X)$-module: in local coordinates, we have $\operatorname{Sym}\Theta_X=\mathbb{C}\{x_1,\dots,x_n\}[\xi_1,\dots,\xi_n]$ and

$$\mathbb{C}\{x_1,\dots,x_n\}=\mathbb{C}\{x_1,\dots,x_n\}[\xi_1,\dots,\xi_n]/(\xi_1,\dots,\xi_n).$$

(5) For $f=\mathrm{Id}:X\to X$, the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to X}=\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$ is a resolution of $\widetilde{\mathcal{D}}_{X\to X}=\widetilde{\mathcal{D}}_X$ as a left and right $\widetilde{\mathcal{D}}_X$-module (notice that the left structure of $\widetilde{\mathcal{D}}_X$ is used for the tensor product).

(6) For $f:X\to\mathrm{pt}$, the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to\mathrm{pt}}=\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{D}}_{X\to\mathrm{pt}}=\widetilde{\mathcal{O}}_X$.

Exercise 8.42. Extend ${}_{\mathrm{D}}f_*$ and ${}_{\mathrm{D}}f_!$ as functors from $\mathsf{D}^+(\widetilde{\mathcal{D}}_X)$ (or $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$) to $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)$. [*Hint*: Replace first $\widetilde{\mathcal{M}}^\bullet\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ with the associated simple complex.]

As in Remark 8.7.5(2), show that if $\widetilde{\mathcal{M}}^\bullet$ has bounded amplitude, then so has ${}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}^\bullet$.

Exercise 8.43. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module.

(1) Show that

$$\big[\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}\big]\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}\simeq(\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}})\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_{X\to Y}$$

as right $f^{-1}\widetilde{\mathcal{D}}_Y$-modules. [*Hint*: Use Exercise 8.16 and show that the corresponding isomorphism is compatible with the right $f^{-1}\widetilde{\mathcal{D}}_Y$-action.]

(2) Same question by replacing $\widetilde{\mathcal{D}}_{X\to Y}$ with $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.

(3) Conclude that

$$\begin{aligned}\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}&\simeq \mathcal{H}om_{f^{-1}\widetilde{\mathcal{O}}_Y}\big(f^{-1}\widetilde{\omega}_Y,\widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\\&\simeq\big(\widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\omega}_Y.\end{aligned}$$

(4) Deduce from the first line, by using that f^{-1} is left adjoint to $\boldsymbol{R}f_*$, that

$$\boldsymbol{R}f_*(\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}})\simeq\big[{}_{\mathrm{D}}f_*(\widetilde{\mathcal{M}}^{\mathrm{right}})\big]^{\mathrm{left}},$$

and deduce from the second line (and justify the identification of the $\widetilde{\mathcal{D}}_Y$-module structures), by the projection formula for $f_!$, that

$$\boldsymbol{R}f_!(\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}})\simeq\big[{}_{\mathrm{D}}f_!(\widetilde{\mathcal{M}}^{\mathrm{right}})\big]^{\mathrm{left}}.$$

Exercise 8.44. Show that the formula for the pushforward in Example 8.7.11 is obtained by side-changing from that of Example 8.7.10. [*Hint*: Adapt Exercise 8.26 in the relative case of a projection.]

Exercise 8.45 **(Pushforward by a closed inclusion).** Assume that $\iota: X\hookrightarrow Y$ is a closed inclusion. For a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, show that ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$ is generated by $\widetilde{\mathcal{M}}\otimes\mathbf{1}$ over $\widetilde{\mathcal{D}}_Y$. [*Hint*: Use Exercise 8.35.]

Exercise 8.46 **(Pushforward by a graph inclusion (see Example 8.7.7))**

Let $f: X\to Y$ a holomorphic map and let $\iota_f: X\hookrightarrow X\times Y$ be the graph inclusion. In local coordinates $y_1,\dots,y_m$ on Y, set $f_j=y_j\circ f$.

(1) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that ${}_{\mathrm{D}}\iota_{f*}\widetilde{\mathcal{M}}\simeq\iota_{f*}\widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_m}]$ with right $\widetilde{\mathcal{D}}_{X\times Y}$ structure given locally by

$$\begin{aligned}\mu\widetilde{\partial}_y^\alpha\cdot\widetilde{\partial}_{y_j}&=\mu\widetilde{\partial}_y^{\alpha+\mathbf{1}_j},\\\mu\widetilde{\partial}_y^\alpha\cdot\widetilde{\partial}_{x_i}&=(\mu\widetilde{\partial}_{x_i})\widetilde{\partial}_y^\alpha-\sum_{j=1}^m\mu\,\frac{\partial f_j}{\partial x_i}\,\widetilde{\partial}_y^{\alpha+\mathbf{1}_j}.\end{aligned}$$

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Show that ${}_{\mathrm{D}}\iota_{f*}\widetilde{\mathcal{M}}\simeq\iota_{f*}\widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_m}](m)$ with left $\widetilde{\mathcal{D}}_{X\times Y}$ structure given locally by (omitting $\widetilde{\mathrm{d}}\boldsymbol{y}^\vee$ in the notation)

$$\begin{aligned}\widetilde{\partial}_{y_j}\cdot\mu\widetilde{\partial}_y^\alpha&=-\mu\widetilde{\partial}_y^{\alpha+\mathbf{1}_j},\\\widetilde{\partial}_{x_i}\cdot\mu\widetilde{\partial}_y^\alpha&=(\widetilde{\partial}_{x_i}\mu)\widetilde{\partial}_y^\alpha-\sum_{j=1}^m\frac{\partial f_j}{\partial x_i}\,\mu\,\widetilde{\partial}_y^{\alpha+\mathbf{1}_j}.\end{aligned}$$

[*Hint*: For the shift (m) of the grading, use Remark (8.2.3 $*$).]

Exercise 8.47 **(Compatibility of Spencer with ${}_{\mathrm{D}}\iota_*$ (right case))**

Let $\iota: X\hookrightarrow Y$ be a closed embedding. The goal of this exercise is to make explicit the isomorphism $\mathrm{Sp}_Y({}_{\mathrm{D}}\iota_!\widetilde{\mathcal{M}})\simeq\iota_*\mathrm{Sp}_X\widetilde{\mathcal{M}}$ (equivalently, $\iota^{-1}\mathrm{Sp}_Y({}_{\mathrm{D}}\iota_!\widetilde{\mathcal{M}})\simeq\mathrm{Sp}_X\widetilde{\mathcal{M}}$) for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$.

(1) By using that $\widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\mathcal{D}}_X$-locally free (Exercise 8.35), show that

$$\begin{aligned}\iota^{-1}\operatorname{Sp}_Y({}_{\mathrm{D}}\iota_!\widetilde{\mathcal{M}}) &\simeq (\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_{X\to Y})\otimes_{\iota^{-1}\widetilde{\mathcal{D}}_Y}\iota^{-1}\operatorname{Sp}_Y(\widetilde{\mathcal{D}}_Y)\\ &\simeq \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}\widetilde{\mathcal{D}}_Y)\otimes_{\iota^{-1}\widetilde{\mathcal{D}}_Y}\iota^{-1}\operatorname{Sp}_Y(\widetilde{\mathcal{D}}_Y)\\ &\simeq \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}\operatorname{Sp}_Y(\widetilde{\mathcal{D}}_Y)).\end{aligned}$$

(2) By using the natural $\widetilde{\mathcal{O}}_X$-linear injective morphism $\widetilde{\Theta}_X\to\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}\widetilde{\Theta}_Y$, deduce a natural $\widetilde{\mathcal{O}}_X$-linear injective morphism for each $k\geqslant 0$:

$$\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}\longrightarrow\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}(\widetilde{\mathcal{D}}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\Theta}_{Y,k}).$$

(3) In local coordinates $(x_1,\dots,x_n,y_1,\dots,y_p)$ where X is defined by $y_1=\cdots=y_p=0$, and for multi-indices $\alpha\in\mathbb{N}^n$ and $\beta\in\mathbb{N}^p$, we use the notation $\widetilde{\partial}_x^{\wedge\alpha}$ for $\widetilde{\partial}_{x_1}^{\alpha_1}\wedge\cdots\wedge\widetilde{\partial}_{x_n}^{\alpha_n}$, and similarly for $\widetilde{\partial}_y^{\wedge\beta}$. Then express the above morphism as the composition of the two natural inclusions

$$\bigoplus_{|\alpha|=k}\widetilde{\mathcal{D}}_X\otimes\widetilde{\partial}_x^{\wedge\alpha}\hookrightarrow\bigoplus_{|\alpha|=k}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes\widetilde{\partial}_x^{\wedge\alpha}\subset\bigoplus_{|\alpha|+|\beta|=k}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes(\widetilde{\partial}_x^{\wedge\alpha}\wedge\widetilde{\partial}_y^{\wedge\beta})$$

(4) Show that the left action of $\widetilde{\mathcal{D}}_X$ on the right-hand side of the morphism in (2) comes from the standard left action on $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes(\widetilde{\partial}_x^{\wedge\alpha}\wedge\widetilde{\partial}_y^{\wedge\beta})$.

(5) Show that the following diagram commutes:

$$\begin{array}{ccc}\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k} & \longrightarrow & \widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}(\widetilde{\mathcal{D}}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\Theta}_{Y,k})\\ {\scriptstyle\widetilde{\delta}_X}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Id}\otimes\widetilde{\delta}_Y}\\ \widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k-1} & \longrightarrow & \widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}(\widetilde{\mathcal{D}}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\Theta}_{Y,k-1})\end{array}$$

[*Hint*: Use the local expression of (3) for the horizontal morphisms.]

(6) Show similarly that for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the natural quasi-isomorphism of complexes $\operatorname{Sp}_X(\widetilde{\mathcal{M}})\to\iota^{-1}\operatorname{Sp}_Y({}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}})$ is locally termwise described as

$$\bigoplus_{|\alpha|=k}\widetilde{\mathcal{M}}\otimes\widetilde{\partial}_x^{\wedge\alpha}\hookrightarrow\bigoplus_{|\alpha|=k}\widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes\widetilde{\partial}_x^{\wedge\alpha}\subset\bigoplus_{|\alpha|+|\beta|=k}\widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes(\widetilde{\partial}_x^{\wedge\alpha}\wedge\widetilde{\partial}_y^{\wedge\beta}).$$

Exercise 8.48 **(Compatibility of Spencer with ${}_{\mathrm{D}}\iota_*$ (left case))**

The setting is as in Exercise 8.47. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module.

(1) Show that ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}\simeq\iota_*\widetilde{\mathcal{M}}[\widetilde{\partial}_y]\otimes\widetilde{\mathrm{d}}\boldsymbol{y}^\vee$. [*Hint*: let $\widetilde{\mathcal{N}}$ be the RHS; prove that $(\widetilde{\mathrm{d}}\boldsymbol{x}\wedge\widetilde{\mathrm{d}}\boldsymbol{y})\otimes\widetilde{\mathcal{N}}\simeq(\widetilde{\mathrm{d}}\boldsymbol{x}\otimes\iota_*\widetilde{\mathcal{M}})[\widetilde{\partial}_y]$.]

(2) Show that the isomorphism ${}^{\mathrm{p}}\mathrm{DR}_X(\widetilde{\mathcal{M}})\simeq\iota^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_Y({}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}})$ is given termwise, for any local section η_x of $\widetilde{\Omega}_X^{n+k}$ by

$$\widetilde{\Omega}_X^{n+k}\otimes\widetilde{\mathcal{M}}\ni\eta_x\otimes m\longmapsto(\eta_x\wedge\widetilde{\mathrm{d}}\boldsymbol{y})\otimes(m\otimes\widetilde{\mathrm{d}}\boldsymbol{y}^\vee)\in\widetilde{\Omega}_X^{n+p+k}\otimes{}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}.$$

[*Hint*: Apply Exercise 8.47(6) to $\widetilde{\mathcal{N}}$ considered in (1), and then the side-changing formula of Lemma 8.4.7.]

Exercise 8.49 **(Compatibility with the Godement functor).**

(1) Show by induction on k that, for every $k \geqslant 0$, the functor God^k is exact (see [**God64**, p. 168]). Given an exact sequence $0 \to \widetilde{\mathcal{L}}' \to \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}}'' \to 0$ of sheaves, show that we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{God}^\bullet \widetilde{\mathcal{L}}' \longrightarrow \mathrm{God}^\bullet \widetilde{\mathcal{L}} \longrightarrow \mathrm{God}^\bullet \widetilde{\mathcal{L}}'' \longrightarrow 0.$$

Similarly, show that the functors $f_\star \mathrm{God}^k$ are exact (with $\star = *$ or $\star = !$) and deduce an exact sequence of complexes

$$0 \longrightarrow f_\star \mathrm{God}^\bullet \widetilde{\mathcal{L}}' \longrightarrow f_\star \mathrm{God}^\bullet \widetilde{\mathcal{L}} \longrightarrow f_\star \mathrm{God}^\bullet \widetilde{\mathcal{L}}'' \longrightarrow 0.$$

Deduce also that, for every $k \geqslant 0$ and a complex $\widetilde{\mathcal{L}}^\bullet$, we have

$$H^i(f_\star \mathrm{God}^k \widetilde{\mathcal{L}}^\bullet) \simeq f_\star \mathrm{God}^k H^i \widetilde{\mathcal{L}}^\bullet.$$

(2) Show that, if $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{F}}$ are $\widetilde{\mathcal{O}}_X$-modules and if $\widetilde{\mathcal{F}}$ is locally free, then we have a natural inclusion $\mathcal{C}^0(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}} \hookrightarrow \mathcal{C}^0(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}})$, which is an equality if $\widetilde{\mathcal{F}}$ has finite rank. More generally, show by induction that we have a natural morphism $\mathcal{C}^k(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}} \to \mathcal{C}^k(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}})$, which is an equality if $\widetilde{\mathcal{F}}$ has finite rank.

(3) With the same assumptions, show that both complexes $\mathrm{God}^\bullet(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}}$ and $\mathrm{God}^\bullet(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}})$ are resolutions of $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}}$. Conclude that the natural morphism of complexes $\mathrm{God}^\bullet(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}} \to \mathrm{God}^\bullet(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{F}})$ is a quasi-isomorphism, and an equality if $\widetilde{\mathcal{F}}$ has finite rank.

(4) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that the natural morphism of complex

$$(\mathrm{God}^\bullet \widetilde{\mathcal{M}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\, \widetilde{\mathcal{D}}_X \longrightarrow \mathrm{God}^\bullet(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\, \widetilde{\mathcal{D}}_X)$$

is a quasi-isomorphism.

(5) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that

$$\mathrm{Sp}(\mathrm{God}^\bullet \widetilde{\mathcal{M}}) = \mathrm{God}^\bullet \mathrm{Sp}\, \widetilde{\mathcal{M}}.$$

(6) If $f : X = Y \times T \to Y$ is the projection, show that, for $\star = *, !$,

$${}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} = f_\star \mathrm{God}^\bullet\big(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big).$$

[*Hint*: Use Example 8.7.10.]

Exercise 8.50 **(Restriction to $z = 1$).**

(1) Show that the Godement functor applied to sheaves of $\widetilde{\mathbb{C}}$-modules restricts, for $z = 1$, to the Godement functor applied to sheaves of $\mathbb{C}$-vector spaces.

(2) Show that $\mathrm{Sp}_{X\to Y}(\mathcal{D}_X) = \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)/(z-1)\,\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.

(3) Conclude that ${}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}/(z-1){}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} = {}_{\mathrm{D}}f_\star(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ and, for every i, ${}_{\mathrm{D}}f_\star^{(i)} \widetilde{\mathcal{M}}/(z-1){}_{\mathrm{D}}f_\star^{(i)} \widetilde{\mathcal{M}} = {}_{\mathrm{D}}f_\star^{(i)}(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ ($\star = *, !$).

Exercise 8.51 **(Computation of the pushforward with the C^∞ Spencer complex)**

We take up the notation of Exercise 8.28. Let $f : X \to Y$ be a holomorphic map. For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we define on

$$\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}}) := \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}{}^{\infty,\bullet}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y} \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}{}^{\infty,\bullet}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$$

the right $f^{-1}\widetilde{\mathcal{D}}_Y$-linear differential

$$\widetilde{\delta}^{\infty}_{\widetilde{\mathcal{M}},Y}\big((m\otimes\xi)\otimes\varphi\otimes Q\big) := \widetilde{\delta}'_{\widetilde{\mathcal{M}},Y}\big((m\otimes\xi)\otimes\varphi\otimes Q\big)+(m\otimes(\xi\lrcorner\,\widetilde{\mathrm{d}}'\varphi))\otimes Q+(m\otimes\xi)\otimes\widetilde{\mathrm{d}}''\varphi\otimes Q,$$

where the first term is naturally defined from the formula in Exercise 8.40, and the second and third terms are as in Exercise 8.28.

(1) Show that each term of the complex $\mathrm{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{D}}_X)=\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathrm{Sp}}^{\infty,\bullet}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\mathcal{D}}_X$-flat.

(2) Show that $\widetilde{\delta}^{\infty}_{\widetilde{\mathcal{M}},Y}$ is indeed a differential and that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})\to\mathrm{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{M}})$ is a quasi-isomorphism.

(3) Show that $\mathrm{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{M}})\to\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}^{\infty}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ is an isomorphism. [*Hint*: Argue as in Exercises 8.24 and 8.28.]

Exercise 8.52 **(Computation of the pushforward with differential forms)**

Let $f:X\to Y$ be a holomorphic map. The formula for the pushforward has a simpler expression when we regard it as producing, from a *left* $\widetilde{\mathcal{D}}_X$-module, a complex of *right* $\widetilde{\mathcal{D}}_Y$-modules. This exercise gives such a formula.

Let $\widetilde{\mathcal{M}}$ be a *left* $\widetilde{\mathcal{D}}_X$-module. As $\widetilde{\mathcal{D}}_{X\to Y}$ is a left $\widetilde{\mathcal{D}}_X$-module,

$$\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}=\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y$$

has a natural structure of left $\widetilde{\mathcal{D}}_X$-module (by setting $\xi(\mu\otimes\mathbf{1}_Y)=\xi\mu\otimes\mathbf{1}_Y+\mu\otimes Tf(\xi)$, see Exercise 8.12(2)) and of course a compatible structure of right $f^{-1}\widetilde{\mathcal{D}}_Y$-module.

(1) Show that the de Rham complex

$$\widetilde{\Omega}^{n+\bullet}_X\otimes(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y})=\widetilde{\Omega}^{n+\bullet}_X\otimes(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}f^*\widetilde{\mathcal{D}}_Y)=\widetilde{\Omega}^{n+\bullet}_X\otimes(\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)$$

is isomorphic to $\widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$, as a complex of right $f^{-1}\widetilde{\mathcal{D}}_Y$-modules, by using the isomorphism (see Lemma 8.4.7)

$$\omega\otimes\mu\otimes\xi\otimes\mathbf{1}_Y\longmapsto\omega(\xi\wedge\bullet)\otimes\mu\otimes\mathbf{1}_Y\quad(\xi\in\wedge^k\widetilde{\Theta}_X).$$

[*Hint*: see Exercise 8.26.]

(2) Check that the connection induced on $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}f^*\widetilde{\mathcal{D}}_Y$ by the left $\widetilde{\mathcal{D}}_X$-module structure is $\widetilde{\nabla}\otimes\mathrm{Id}+\mathrm{Id}_{\widetilde{\mathcal{M}}}\otimes\widetilde{\nabla}^X$, where $\widetilde{\nabla}^X$ is obtained from the universal connection $\widetilde{\nabla}^Y$ on $\widetilde{\mathcal{D}}_Y$ by the formula (8.6.1).

(3) Conclude that, for $\star=*,!$,

$$(8.52*)\qquad {}_{\mathrm{D}}f_\star(\widetilde{\mathcal{M}}^{\mathrm{right}})=\boldsymbol{R}f_\star\big[\widetilde{\Omega}^{n+\bullet}_X\otimes(\widetilde{\mathcal{M}}^{\mathrm{left}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\big],$$

$$(8.52**)\qquad {}_{\mathrm{D}}f_\star(\widetilde{\mathcal{M}}^{\mathrm{left}})=\boldsymbol{R}f_\star\big[\widetilde{\Omega}^{n+\bullet}_X\otimes(\widetilde{\mathcal{M}}^{\mathrm{left}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\big]^{\mathrm{left}},$$

where (8.52 $*$) is the complex of right $\widetilde{\mathcal{D}}_Y$-modules associated to the double complex

$$f_\star\,\mathrm{God}^\bullet\big[\widetilde{\Omega}^{n+\bullet}_X\otimes(\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\big].$$

Show that this complex is quasi-isomorphic to the complex

$$f_\star\big[\widetilde{\Omega}^{n+\bullet}_X\otimes(\mathrm{God}^\bullet\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\big].$$

[*Hint*: Use Exercise 8.49.]

(4) Show that the latter complex is the single complex associated with the double complex having terms $f_\star(\widetilde{\Omega}_X^{n+i}\otimes \mathrm{God}^j\,\widetilde{\mathcal{M}})\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y$ and first differential $(-1)^n f_\star(\widetilde{\nabla}\otimes\mathrm{Id}+\mathrm{Id}_{\widetilde{\mathcal{M}}}\otimes\widetilde{\nabla}^X)$ (the second differential is induced by the Godement differential).

(5) It is often more convenient to replace the Godement resolution by a Dolbeault resolution. Prove that

$$({}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}})^{\mathrm{right}}\simeq f_\star\big[\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes(\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\big],$$
$${}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}\simeq f_\star\big[\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes(\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\big]^{\mathrm{left}},$$

where the differential in the latter complexes is obtained in the usual way from the holomorphic differential of 8.52(1) and the anti-holomorphic differential d''.

Other properties of the pushforward functor

Exercise 8.53 **(Pushforward of induced $\widetilde{\mathcal{D}}$-modules).** Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module and let $\widetilde{\mathcal{M}}=\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$ be the associated induced right $\widetilde{\mathcal{D}}_X$-module. Let $f:X\to Y$ be a holomorphic map.

(1) Show that $\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\to\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$ is a quasi-isomorphism. [*Hint*: Use that $\widetilde{\mathcal{D}}_X$ is $\widetilde{\mathcal{O}}_X$-locally free.]

(2) Deduce that

$$\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)=\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_{X\to Y}=\widetilde{\mathcal{L}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y.$$

(3) Show that ${}_{\mathrm{D}}f_!(\widetilde{\mathcal{L}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X)$ is quasi-isomorphic to $(\boldsymbol{R}f_!\widetilde{\mathcal{L}})\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y$. [*Hint*: Use the projection formula.]

Exercise 8.54 **(Pushforward of $\widetilde{\mathcal{D}}$-modules and pushforward of $\widetilde{\mathcal{O}}$-modules)**

Let $f:X\to Y$ be a holomorphic map and let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. It is also an $\widetilde{\mathcal{O}}_X$-module. The goal of this exercise is to exhibit natural $\widetilde{\mathcal{O}}_Y$-linear morphisms ($\star=*,!$)

$$R^if_\star\widetilde{\mathcal{M}}\longrightarrow {}_{\mathrm{D}}f_\star^{(i)}\widetilde{\mathcal{M}}.$$

(1) Show that $\widetilde{\mathcal{D}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y$ has a natural global section $\mathbf{1}$.

(2) Show that there is a natural $f^{-1}\widetilde{\mathcal{O}}_Y$-linear morphism of complexes

$$\widetilde{\mathcal{M}}\longrightarrow\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X),\quad m\longmapsto m\otimes\mathbf{1},$$

where $\widetilde{\mathcal{M}}$ is considered as a complex with $\widetilde{\mathcal{M}}$ in degree 0 and all other terms equal to 0, so the differential are all equal to 0. [*Hint*: Use Exercise 8.18(3) to identify $\mathrm{Sp}^0_{X\to Y}(\widetilde{\mathcal{D}}_X)=\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$ with its twisted left $\widetilde{\mathcal{D}}_X$-structure (denoted by $\widetilde{\mathcal{D}}_{X\to Y}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$ in loc. cit.) with $\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}$, where the tensor product uses the right $\widetilde{\mathcal{O}}_X$-structure on $\widetilde{\mathcal{D}}_X$ and the left $\widetilde{\mathcal{D}}_X$ structure is the trivial one, and then with $\widetilde{\mathcal{D}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{O}}_Y$ with trivial left $\widetilde{\mathcal{D}}_X$-structure and tensor product using the right $\widetilde{\mathcal{O}}_X$-structure of $\widetilde{\mathcal{D}}_X$. Identify then $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y})$ with $\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y$.]

(3) Conclude with the existence of the desired morphisms.

***Exercise 8.55* (Grading and pushforward, right case).** Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a filtered right $\mathcal{D}_X$-module. Set $\mathcal{M} = R_F\mathcal{M}$, so that $\mathrm{gr}^F\mathcal{M} = \mathcal{M}/z\mathcal{M}$.

(1) Show that

$$(\mathcal{M} \otimes_{R_F\mathcal{D}_X} \mathrm{Sp}\, R_F\mathcal{D}_{X\to Y}) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]/z\mathbb{C}[z] \simeq \mathrm{gr}^F\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y.$$

[*Hint*: Use the associativity of $\otimes$ and Exercise 8.41(4).]

(2) Assume that ${}_{\mathrm{D}}f_\star\mathcal{M}$ is *strict* (i.e., the complex of Corollary 8.7.15 is strict in the sense of Definition 5.1.6 or 10.2.2). Show that, for every i, we have, as graded modules

$$\mathrm{gr}^F{}_{\mathrm{D}}f_\star^{(i)}\mathcal{M} \simeq H^i\boldsymbol{R}f_\star\big(\mathrm{gr}^F\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y\big).$$

***Exercise 8.56* (Grading and pushforward, left case).** With the assumptions as in Exercise 8.55(2), but assuming that $\mathcal{M}$ is a left $\mathcal{D}_X$-module, show that

$$\mathrm{gr}^F{}_{\mathrm{D}}f_\star^{(i)}\mathcal{M} \simeq H^i\boldsymbol{R}f_\star\big(\omega_{X/Y} \otimes_{\mathcal{O}_X} \mathrm{gr}^F_{\bullet+n-m}\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y\big),$$

where $\omega_{X/Y} := \omega_X \otimes_{\mathcal{O}_X} f^*\omega_Y^\vee$, and we have set $n = \dim X$, $m = \dim Y$. (Notice the shift of the filtration, which comes from $\widetilde{\omega}_{X/Y} = z^{n-m}\omega_{X/Y}$.) For example, if $Y = \mathrm{pt}$, deduce that

$$\mathrm{gr}^F\boldsymbol{H}^i(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \simeq \boldsymbol{H}^i\big(X, \omega_X \otimes (\mathrm{gr}^F_{\bullet+n}\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} \mathcal{O}_X)\big).$$

***Exercise 8.57* (Trace for a finite map, preliminaries).** We take up the notation of Example 8.7.31, so that $f : X = \mathbb{C}^n \to Y = \mathbb{C}^n$ is defined by $f_i(x_1, \dots, x_n) = x_i^{r_i}$, with $r_i \in \mathbb{N}^*$ and $r_i \geqslant 2$ if and only if $i = 1, \dots, \ell$. We set $D = \{\prod_{i=1}^\ell x_i = 0\}$ and we have $f(D) = \{\prod_{i=1}^\ell y_i = 0\}$.

(1) Define $\mathrm{Tr}_f : f_*\widetilde{\mathcal{O}}_X \to \widetilde{\mathcal{O}}_Y$ as an $\widetilde{\mathcal{O}}_Y$-linear morphism such that, composed with $\mathrm{adj}_f : \widetilde{\mathcal{O}}_Y \to f_*\widetilde{\mathcal{O}}_X$, it yields the identity $\widetilde{\mathcal{O}}_Y \to \widetilde{\mathcal{O}}_Y$. [*Hint*: Set $\mathrm{Tr}_f(g)(y) = (1/\#g^{-1}(y))\sum_{x\in f^{-1}(y)} g(x)$.]

(2) Show that for any holomorphic function g on X, there exists a holomorphic function g' on Y such that $\widetilde{\mathrm{d}}g/g = f^*(\widetilde{\mathrm{d}}g'/g')$ (where f^* means T^*f).

(3) Show that there exists an $\widetilde{\mathcal{O}}_Y$-linear morphism

$$\mathrm{Tr}_f : f_*\widetilde{\Omega}^1_X(\log D) \longrightarrow \widetilde{\Omega}^1_Y(\log f(D))$$

satisfying the following properties:

(a) $\mathrm{Tr}_f(\widetilde{\mathrm{d}}x_i/x_i) = (1/r_i)\widetilde{\mathrm{d}}y_i/y_i$ for $i = 1, \dots, \ell$ and $\mathrm{Tr}_f(\widetilde{\mathrm{d}}x_j) = \widetilde{\mathrm{d}}x_j$ for $j \geqslant \ell + 1$,

(b) $\mathrm{Tr}_f(\widetilde{\mathrm{d}}g/g) = \widetilde{\mathrm{d}}g'/g'$, with g' as above,

(c) $\mathrm{Tr}_f(h \cdot \widetilde{\mathrm{d}}g/g) = \mathrm{Tr}_f(h) \cdot \widetilde{\mathrm{d}}g'/g'$.

(4) Deduce that there exists an $\widetilde{\mathcal{O}}_Y$-linear morphism $\mathrm{Tr}_f : f_*\widetilde{\Omega}^1_X \to \widetilde{\Omega}^1_Y$ such that the composition

$$\widetilde{\Omega}^1_Y \xrightarrow{\;f_*(T^*f)\;} f_*\widetilde{\Omega}^1_X \xrightarrow{\;\mathrm{Tr}_f\;} \widetilde{\Omega}^1_Y$$

is the identity, and satisfies $\widetilde{\mathrm{d}}\,\mathrm{Tr}_f(g) = \mathrm{Tr}_f(\widetilde{\mathrm{d}}g)$ for any holomorphic function g on X.

(5) Extend Tr_f as a morphism of complexes $(f_*(\widetilde{\Omega}^\bullet_X), f_*(\widetilde{\mathrm{d}})) \to (\widetilde{\Omega}^\bullet_Y, \widetilde{\mathrm{d}})$ such that the composition

$$(\widetilde{\Omega}^\bullet_Y, \widetilde{\mathrm{d}}) \xrightarrow{f_*(T^*f)} (f_*(\widetilde{\Omega}^\bullet_X), f_*(\widetilde{\mathrm{d}})) \xrightarrow{\mathrm{Tr}_f} (\widetilde{\Omega}^\bullet_Y, \widetilde{\mathrm{d}})$$

is the identity.

Exercise 8.58 **(Trace for a finite map).** Let $f: X \to Y$ be as in Exercise 8.57 and let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_Y$-module. Show that

$$({}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}))^{\mathrm{right}} \simeq (f_*\widetilde{\Omega}^{n+\bullet}_X \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}}.$$

[*Hint*: Use that $R^if_*(\bullet) = 0$ for $i \geqslant 0$ and argue as in the proof of Proposition 8.7.30.]

Deduce that there exist morphisms whose composition is the identity:

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \xrightarrow{\mathrm{adj}_f} ({}_{\mathrm{D}}f^{(0)}_*({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}))^{\mathrm{right}} \xrightarrow{\mathrm{Tr}_f} \widetilde{\mathcal{M}}^{\mathrm{right}},$$

and conclude that $\widetilde{\mathcal{M}}^{\mathrm{right}}$ is a direct summand in $({}_{\mathrm{D}}f^{(0)}_*({}_{\mathrm{D}}f^{*(0)}\widetilde{\mathcal{M}}))^{\mathrm{right}}$.

8.11.g. Exercises for Section 8.8

Exercise 8.59.

(1) Prove the coherence of the sheaf of rings $\mathrm{gr}^F\widetilde{\mathcal{D}}_X$ in a way similar to that of $\widetilde{\mathcal{D}}_X$.

(2) Let $D \subset X$ be a hypersurface and let $\widetilde{\mathcal{O}}_X(*D)$ be the sheaf of meromorphic functions on X with poles on D at most (with arbitrary order). Prove similarly that $\widetilde{\mathcal{O}}_X(*D)$ is a coherent sheaf of rings.

(3) Prove that $\widetilde{\mathcal{D}}_X(*D) := \widetilde{\mathcal{O}}_X(*D) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ is a coherent sheaf of rings.

(4) Let $\iota: Y \hookrightarrow X$ denote the inclusion of a smooth submanifold. Show that $i^*\widetilde{\mathcal{D}}_X := \widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ is a coherent sheaf of rings on Y.

(5) Let $Y \subset X$ be a smooth hypersurface of X. Show that $V_0\widetilde{\mathcal{D}}_X$ (see Section 9.2) is a coherent sheaf of rings.

Exercise 8.60.

(1) Let $\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{N}}$ be a $\widetilde{\mathcal{D}}_X$-submodule of a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$. Show that, if $\widetilde{\mathcal{M}}$ is locally finitely generated, then it is coherent.

(2) Let $\phi: \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism between coherent $\widetilde{\mathcal{D}}_X$-modules. Show that $\mathrm{Ker}\,\phi$ and $\mathrm{Coker}\,\phi$ are coherent.

Exercise 8.61 **(Non-validity of Cartan Theorem B for $\mathcal{D}$-modules)**

(1) Let X be an open disc with coordinate x, of radius r (possibly ∞) in $\mathbb{C}$, and let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points in X such that $\lim_i(r - |x_i|) = 0$. Let $\varphi: \mathcal{O}_X^{\mathbb{N}} \to \mathcal{O}_X^{\mathbb{N}}$ be the diagonal morphism equal to $(x - x_i)$ on the i-th component. Let $\mathbb{C}_{x_i}$ denote the skyscraper sheaf supported on x_i. Show that

(a) $\mathrm{Coker}\,\varphi = \bigoplus_i \mathbb{C}_{x_i}$ and $H^0(X, \mathrm{Coker}\,\varphi) = \prod_i H^0(X, \mathbb{C}_{x_i})$;

(b) $\mathrm{Coker}[H^0\varphi : H^0(X, \mathcal{O}_X^{\mathbb{N}}) \to H^0(X, \mathcal{O}_X^{\mathbb{N}})] = \bigoplus_i H^0(X, \mathbb{C}_{x_i})$.

(2) Deduce that $H^1(X, \mathcal{O}_X^{\mathbb{N}}) \neq 0$.

(3) Let $\mathcal{O}_X(*0)$ be the sheaf of meromorphic functions on X with poles at $x = 0$ at most. Show that $H^1(X, \mathcal{O}_X(*0)) \neq 0$. [*Hint*: Use Cartan Theorem B for $\mathcal{O}_X$ and apply the previous result to $\mathcal{O}_X(*0)/\mathcal{O}_X$.]

Exercise 8.62 **(Characterization of coherent filtrations).**

(1) Show that the following properties are equivalent:

(a) $F_\bullet\widetilde{\mathcal{M}}$ is a coherent filtration;

(b) for every $k \in \mathbb{Z}$, $F_k\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{O}}_X$-coherent, and, for every $x \in X$, there exists a neighbourhood U of x and $k_0 \in \mathbb{Z}$ such that, for every $k \geqslant 0$, $F_k\widetilde{\mathcal{D}}_{X|U} \cdot F_{k_0}\widetilde{\mathcal{M}}_{|U} = F_{k+k_0}\widetilde{\mathcal{M}}_{|U}$;

(c) the graded module $\operatorname{gr}^F\widetilde{\mathcal{M}}$ is $\operatorname{gr}^F\widetilde{\mathcal{D}}_X$-coherent.

(2) Conclude that, if $F_\bullet\widetilde{\mathcal{M}}, G_\bullet\widetilde{\mathcal{M}}$ are two coherent filtrations of $\widetilde{\mathcal{M}}$, then, locally on X, there exists k_0 such that, for every k, we have

$$F_{k-k_0}\widetilde{\mathcal{M}} \subset G_k\widetilde{\mathcal{M}} \subset F_{k+k_0}\widetilde{\mathcal{M}}.$$

Exercise 8.63 **(Local existence of coherent filtrations).** Let $F_\bullet\widetilde{\mathcal{M}}$ be a filtration of $\widetilde{\mathcal{M}}$.

(1) Write $R_F\widetilde{\mathcal{M}} = \bigoplus_k F_k\widetilde{\mathcal{M}}\zeta^k$, where ζ is a new variable, and show that, if $\widetilde{\mathcal{M}}$ has a coherent filtration, then it is $\widetilde{\mathcal{D}}_X$-coherent. [*Hint*: Use that the tensor product $\mathbb{C}[\zeta]/(\zeta - 1) \otimes_{\mathbb{C}[\zeta]} \bullet$ is right exact.]

(2) Conversely, show that any coherent $\widetilde{\mathcal{D}}_X$-module admits locally a coherent filtration. [*Hint*: Choose a local presentation $\widetilde{\mathcal{D}}^q_{X|U} \xrightarrow{\varphi} \widetilde{\mathcal{D}}^p_{X|U} \to \widetilde{\mathcal{M}}_{|U} \to 0$, and show that the filtration induced on $\widetilde{\mathcal{M}}_{|U}$ by $F_\bullet\widetilde{\mathcal{D}}^p_{X|U}$ is coherent by using Exercise 8.62: Set $\widetilde{\mathcal{K}} = \operatorname{Im}\varphi$ and reduce the assertion to showing that $F_j\widetilde{\mathcal{D}}_X \cap \widetilde{\mathcal{K}}$ is $\widetilde{\mathcal{O}}_X$-coherent; prove that, up to shrinking U, there exists $k_o \in \mathbb{N}$ such that $\varphi(F_k\widetilde{\mathcal{D}}^q_{X|U}) \subset F_{k+k_o}\widetilde{\mathcal{D}}^p_{X|U}$ for every k; deduce that $\varphi(F_k\widetilde{\mathcal{D}}^q_{X|U})$, being locally of finite type and contained in a coherent $\widetilde{\mathcal{O}}_X$-module, is $\widetilde{\mathcal{O}}_X$-coherent for every k; conclude by using the fact that an increasing sequence of coherent $\widetilde{\mathcal{O}}_X$-modules in a coherent $\widetilde{\mathcal{O}}_X$-module is locally stationary.]

(3) Show that a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$ satisfies $F_p\widetilde{\mathcal{M}} = 0$ for $p \ll 0$ locally [*Hint*: Use that this holds for the filtration constructed in (2) and apply Exercise 8.62(2).]

(4) Show that, locally, any coherent $\widetilde{\mathcal{D}}_X$-module is generated over $\widetilde{\mathcal{D}}_X$ by a coherent $\widetilde{\mathcal{O}}_X$-submodule.

(5) Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{F}}$ be an $\widetilde{\mathcal{O}}_X$-submodule which is locally finitely generated. Show that $\widetilde{\mathcal{F}}$ is $\widetilde{\mathcal{O}}_X$-coherent. [*Hint*: Choose a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$ and show that, locally, $\widetilde{\mathcal{F}} \subset F_k\widetilde{\mathcal{M}}$ for some k; apply then the analogue of Exercise 8.60(1) for $\widetilde{\mathcal{O}}_X$-modules.]

Exercise 8.64.

(1) Show statements similar to those of Theorem 8.8.7 for $R_F\widetilde{\mathcal{D}}_X$-modules, $\operatorname{gr}^F\widetilde{\mathcal{D}}_X$-modules, $\widetilde{\mathcal{O}}_X(*D)$-modules, $\widetilde{\mathcal{D}}_X(*D)$-modules and $i^*\widetilde{\mathcal{D}}_X$-modules (see Exercise 8.59).

(2) Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Show that $\widetilde{\mathcal{D}}_X(*D) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X(*D)$-coherent and that $i^*\widetilde{\mathcal{M}}$ is $i^*\widetilde{\mathcal{D}}_X$-coherent.

Exercise 8.65. Similarly to Corollary 8.8.8, prove that if $\varphi:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ is a surjective morphism of coherent $\widetilde{\mathcal{D}}_X$-modules and if $F_\bullet\widetilde{\mathcal{M}}$ is coherent, then $F_\bullet\widetilde{\mathcal{N}}:=\varphi(F_\bullet\widetilde{\mathcal{M}})$ is coherent as well.

Exercise 8.66.

(1) Show that $R_F\mathcal{D}_X$ is naturally filtered by locally free graded $\mathcal{O}_X[z]$-modules of finite rank by setting (locally)

$$F_k(R_F\mathcal{D}_X)=\sum_{|\alpha|\leqslant k}\mathcal{O}_X[z]\partial_x^\alpha.$$

(2) Show that $\mathrm{gr}^F(R_F\mathcal{D}_X)=\mathbb{C}[z]\otimes_{\mathbb{C}}\mathrm{gr}^F\mathcal{D}_X$ with the tensor product grading.

(3) For a filtered $\mathcal{D}_X$-module $(\mathcal{M},F_\bullet\mathcal{M})$, show that, if one defines the filtration

$$F_k(R_F\mathcal{M})=\sum_{j\leqslant k}F_j\mathcal{M}\otimes_{\mathbb{C}}z^j\mathbb{C}[z],$$

then $F_\bullet(R_F\mathcal{M})$ is an $F_\bullet(R_F\mathcal{D}_X)$-filtration and $\mathrm{gr}^F(R_F\mathcal{M})$ can be identified with $\mathbb{C}[z]\otimes_{\mathbb{C}}\mathrm{gr}^F\mathcal{M}$, equipped with the tensor product grading.

Exercise 8.67. Recall (see e.g. [**ST71**, Prop. 1.9]) that, for a coherent sheaf $\mathcal{F}$ of $\widetilde{\mathcal{O}}_X$-modules and a closed analytic subset $Z\subset X$, the sheaf $\Gamma_Z\mathcal{F}$ consisting of local sections which vanish away from Z is also the sheaf of local sections annihilated by some power of $\mathcal{I}_Z$, and is a coherent sheaf of $\widetilde{\mathcal{O}}_X$-modules. Deduce a similar property for coherent $\widetilde{\mathcal{D}}_X$-modules. [*Hint*: Prove that the assertion is local and apply the result for $\widetilde{\mathcal{O}}_X$-modules for a large step of a coherent filtration of $\widetilde{\mathcal{M}}$.]

Exercise 8.68. Let $0\to\widetilde{\mathcal{M}}'\to\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}''\to 0$ be an exact sequence of $\widetilde{\mathcal{D}}_X$-modules. Show that $\mathrm{Char}\,\widetilde{\mathcal{M}}=\mathrm{Char}\,\widetilde{\mathcal{M}}'\cup\mathrm{Char}\,\widetilde{\mathcal{M}}''$. [*Hint*: Take a coherent filtration on $\widetilde{\mathcal{M}}$ and induce it on $\widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{M}}''$.]

***Exercise 8.69* (Coherent $\widetilde{\mathcal{D}}_X$-modules with characteristic variety T^*_XX)**

Assume that $\widetilde{\mathcal{M}}$ is coherent with characteristic variety contained in $T^*_XX\times\mathbb{C}_z$.

(1) Show that, for any local coherent filtration $F_\bullet\widetilde{\mathcal{M}}$, the graded module $\mathrm{gr}^F\widetilde{\mathcal{M}}$ is locally of finite type, hence coherent (see Exercise 8.63(5)) over $\widetilde{\mathcal{O}}_X$.

(2) Deduce that, locally on X, there exists p_o such that $\mathrm{gr}^F_p\widetilde{\mathcal{M}}=0$ for $p\geqslant p_o$.

(3) For a $\mathcal{D}_X$-module $\mathcal{M}$, deduce that $\mathcal{M}$ is locally free of finite rank.

***Exercise 8.70* (Coherent $\mathcal{D}_X$-modules with characteristic variety contained in T^*_YX)**

In this exercise, we switch to the case of $\mathcal{D}_X$-modules. Let $\iota:Y\hookrightarrow X$ be the inclusion of a smooth codimension p closed submanifold. Define the p-th algebraic local cohomology with support in Y by $R^p\Gamma_{[Y]}\mathcal{O}_X=\varinjlim_k\mathcal{E}xt^p(\mathcal{O}_X/\mathcal{I}_Y^k,\mathcal{O}_X)$, where $\mathcal{I}_Y$ is the ideal defining Y. $R^p\Gamma_{[Y]}\mathcal{O}_X$ has a natural structure of $\mathcal{D}_X$-module. In local coordinates $(x_1,\dots,x_n)$ where Y is defined by $x_1=\cdots=x_p=0$, we have

$$R^p\Gamma_{[Y]}\mathcal{O}_X\simeq\frac{\mathcal{O}_{\mathbb{C}^n}[1/x_1\cdots x_n]}{\sum_{i=1}^p\mathcal{O}_{\mathbb{C}^n}(x_i/x_1\cdots x_n)}.$$

Denote this $\mathcal{D}_X$-module by $\mathcal{B}_YX$.

(1) Show that $\mathcal{B}_Y X$ has support contained in Y and characteristic variety equal to $T^*_Y X$.

(2) Identify $\mathcal{B}_Y X$ with ${}_{\mathrm{D}}\iota_*\mathcal{O}_Y$.

(3) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with characteristic variety equal to $T^*_Y X$. Show that $\mathcal{M}$ is locally isomorphic to $(\mathcal{B}_Y X)^d$ for some d.

Exercise 8.71. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module equipped with a coherent filtration $F_\bullet\mathcal{M}$. Set $\mathcal{M} = R_F\mathcal{M}$.

(1) Show that $\mathrm{Char}(R_F\mathcal{M}) = (\mathrm{Char}\,\mathcal{M}) \times \mathbb{C}_z$, so that $\mathcal{M}$ is holonomic (in the sense of Definition 8.8.29) if and only if $\mathcal{M}$ is holonomic. (In other words, for a strict coherent $\widetilde{\mathcal{D}}_X$-module $\mathcal{M}$, $\mathcal{M}/(z-1)\mathcal{M}$ is holonomic if and only if $\mathcal{M}$ itself is holonomic.)

(2) In such a case, show that $\mathcal{E}xt^i_{R_F\mathcal{D}_X}(R_F\mathcal{M}, R_F\mathcal{D}_X)$ consists of z-torsion if $i \neq \dim X$.

Exercise 8.72 **(Characteristic variety of the external product, see [Kas03, §4.3])** Consider the setting of Lemma 8.6.10. Assume moreover that the filtrations $F_\bullet\mathcal{M}_X, F_\bullet\mathcal{M}_Y$ are coherent. Show that $F_\bullet(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y)$ is coherent. Conclude that $\mathrm{Char}(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y) = \mathrm{Char}\,\mathcal{M}_X \times \mathrm{Char}\,\mathcal{M}_Y$.

Exercise 8.73 **(Projection formula for $\widetilde{\mathcal{O}}$-modules).** Let X, Y be complex manifolds, X being compact, let $\widetilde{\mathcal{L}}_X$ be an $\widetilde{\mathcal{O}}_X$-module and let us denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections.

(1) Show that there exists a natural morphism $\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \to \boldsymbol{R}q_*p^*\widetilde{\mathcal{L}}_X$. [*Hint*: Justify the following composition of morphisms

$$\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \xrightarrow{\sim} \widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}q_*p^{-1}\widetilde{\mathcal{L}}_X \simeq \boldsymbol{R}q_*(q^{-1}\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} p^{-1}\widetilde{\mathcal{L}}_X) \longrightarrow \boldsymbol{R}q_*p^*\widetilde{\mathcal{L}}_X$$

and conclude.]

The goal of the remaining part is to prove (8.8.26), that is, if $\widetilde{\mathcal{L}}_X$ is the inductive limit of its coherent $\widetilde{\mathcal{O}}_X$-submodules this morphism is an isomorphism.

(2) Reduce the statement to the case where $\widetilde{\mathcal{L}}_X$ is $\widetilde{\mathcal{O}}_X$-coherent. [*Hint*: Proper pushforward commutes with inductive limits.]

(3) Consider first the case of $\mathcal{O}_X$-modules. Use Grauert's theorem (see e.g. [**BS76**, Th. 4.12]) to prove the result.

(4) For $\widetilde{\mathcal{O}}_X$-modules, apply the previous result to each graded piece and conclude.

Exercise 8.74 **(Proof of Lemma 8.8.42).** Let $\widetilde{\mathcal{M}}_i$ $(i = 1, 2)$ be left $\widetilde{\mathcal{D}}_X$-modules (consider right modules as left modules on $\widetilde{\mathcal{D}}_X^{\mathrm{op}}$).

(1) Show that

$$\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}) \simeq \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1}}(\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{D}}_{X_1}) \boxtimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_2, \widetilde{\mathcal{D}}_{X_2})$$

[*Hint*: let $I_i^\bullet$ (resp. $I^\bullet$) be a $\widetilde{\mathcal{D}}_{X_i} \otimes \widetilde{\mathcal{D}}_{X_i}^{\mathrm{op}}$ (resp. $(\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}) \otimes (\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2})^{\mathrm{op}}$)-injective resolution of $\widetilde{\mathcal{D}}_{X_i}$ (resp. $I_1^\bullet \boxtimes_{\widetilde{\mathbb{C}}} I_2^\bullet$). Show the existence of a $(\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2})^{\mathrm{op}}$-linear morphism

$$\begin{aligned}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1}}(\widetilde{\mathcal{M}}_1, I_1^\bullet) \boxtimes_{\widetilde{\mathbb{C}}} \mathcal{H}om_{\widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_2, I_2^\bullet) = \mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, I_1^\bullet \boxtimes_{\widetilde{\mathbb{C}}} I_2^\bullet)\\ \longrightarrow \mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, I^\bullet)\end{aligned}$$

That it is an isomorphism is a local question, and as each $\widetilde{\mathcal{M}}_i$ is coherent, by taking a local free resolution of $\widetilde{\mathcal{M}}_i$, it is enough to check this for $\widetilde{\mathcal{D}}_{X_i}$, for which the assertion is easy.]

(2) Show that there exists a natural morphism (in $\mathsf{D}^+(\widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}})$):

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}) \otimes_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}} \widetilde{\mathcal{D}}_{X_1 \times X_2}\\ \longrightarrow \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{D}}_{X_1 \times X_2}),\end{aligned}$$

where $\widetilde{\mathcal{D}}_{X_1 \times X_2}$ is regarded as a $\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}$-module and as a $\widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}}$-module. [*Hint*: consider an injective resolution $J^\bullet$ of $\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}$ as a $(\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}) \otimes (\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2})^{\mathrm{op}}$-module. Deduce a natural morphism of $\widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}}$-modules:

$$\begin{aligned}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, J^\bullet) \otimes_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}} \widetilde{\mathcal{D}}_{X_1 \times X_2}\\ \longrightarrow \mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, J^\bullet \otimes_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}} \widetilde{\mathcal{D}}_{X_1 \times X_2}).\end{aligned}$$

Then choose an injective resolution $K^\bullet$ of $J^\bullet \otimes_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}} \widetilde{\mathcal{D}}_{X_1 \times X_2}$ as a $(\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}) \otimes \widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}}$-module, and obtain the desired morphism.]

Show moreover that, if $\widetilde{\mathcal{M}}_i$ are $\widetilde{\mathcal{D}}_{X_i}$-coherent, then this morphism is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}})$. [*Hint*: the assertion is local, so by taking a local free resolution of $\widetilde{\mathcal{M}}_i$, reduce to the case where $\widetilde{\mathcal{M}}_i = \widetilde{\mathcal{D}}_{X_i}$and conclude.]

(3) Lastly, show that there exists a natural morphism

$$\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{D}}_{X_1 \times X_2}) \longrightarrow \boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1 \times X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{D}}_{X_1 \times X_2}).$$

[*Hint*: as $\widetilde{\mathcal{D}}_{X_1 \times X_2}$ is $\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}$-flat, an injective $\widetilde{\mathcal{D}}_{X_1 \times X_2}$-module is also an injective $\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}$-module. Take an injective resolution $I^\bullet$ of $\widetilde{\mathcal{D}}_{X_1 \times X_2}$ as a $\widetilde{\mathcal{D}}_{X_1 \times X_2} \otimes \widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}}$-module, show the existence of a morphism of $\widetilde{\mathcal{D}}_{X_1 \times X_2}^{\mathrm{op}}$-complexes

$$\mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1} \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_{X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_2, I^\bullet) \longrightarrow \mathcal{H}om_{\widetilde{\mathcal{D}}_{X_1 \times X_2}}(\widetilde{\mathcal{M}}_1 \boxtimes \widetilde{\mathcal{M}}_2, I^\bullet),$$

and obtain the desired morphism.]

Moreover, show as in (2) that it is an isomorphism if $\widetilde{\mathcal{M}}_i$ are $\widetilde{\mathcal{D}}_{X_i}$-coherent.

By using the usual shifts, deduce

$$\boldsymbol{D}\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \boldsymbol{D}\widetilde{\mathcal{M}}_2 \simeq \boldsymbol{D}(\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_2).$$

[*Hint*: the left-hand term is that considered in (2), after (1), and the right-hand term is the second one in (3).]

8.11.h. Exercises for Section 8.10

Exercise 8.75. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Recall (Exercise 8.29) that $\mathrm{Sp}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is a resolution of $\mathcal{L}$ as an $\mathcal{O}_X$-module. Show that the morphism (8.10.1) is the augmentation morphism $\mathrm{Sp}^0(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \mathcal{L}$.

Exercise 8.76.

(1) Show that any $\mathcal{O}_X$-linear morphism $u : \mathcal{L} \to \mathcal{L}'$ is a differential operator from $\mathcal{L}$ to $\mathcal{L}'$ and that a corresponding v is $u \otimes 1$.

(2) Assume that $\mathcal{L}, \mathcal{L}'$ are right $\mathcal{D}_X$-modules. Let $u : \mathcal{L} \to \mathcal{L}'$ be $\mathcal{D}_X$-linear. Show that the corresponding v is $\mathcal{D}_X$-linear for both structures $(\text{right})_{\text{triv}}$ and $(\text{right})_{\text{tens}}$ (see Exercise 8.19) on $\mathcal{L}^{(\prime)} \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

(3) Show that $\mathcal{H}om_{\mathrm{Diff}}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{D}_X$.

(4) Show that the morphism in Definition 8.10.6 is compatible with composition. Conclude that the composition of differential operators is a differential operator and that it is associative.

Exercise 8.77 **(Integrable connections are differential operators)**

Let $\mathcal{M}$ be an $\mathcal{O}_X$-module and let $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ be an integrable connection on $\mathcal{M}$.

(1) Show that ∇ is a differential morphism, by considering the right $\mathcal{D}_X$-linear morphism

$$v(m \otimes P) := \nabla(m) \otimes P + m \otimes \nabla(P),$$

for any local section m of $\mathcal{M}$ and P of $\mathcal{D}_X$, and where ∇P is defined in Exercise 8.5. Extend this result to connections ${}^{(k)}\nabla$.

(2) Let $\mathcal{M}', \mathcal{M}''$ be $\mathcal{O}_X$-submodules of $\mathcal{M}$ such that ${}^{(k)}\nabla$ induces a $\mathbb{C}$-linear morphism ${}^{(k)}\nabla' : \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{M}' \to \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{M}''$. Show that ${}^{(k)}\nabla'$ is a differential morphism.

Exercise 8.78. Show that $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ is an additive category, i.e.,

- $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$ is a $\mathbb{C}$-vector space and the composition is $\mathbb{C}$-bilinear,
- the 0 $\mathcal{O}_X$-module satisfies $\mathrm{Hom}_{\mathrm{Diff}}(0, 0) = 0$,
- $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}_1 \oplus \mathcal{L}_2, \mathcal{L}') = \mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}_1, \mathcal{L}') \oplus \mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}_2, \mathcal{L}')$ and similarly with $\mathcal{L}'_1, \mathcal{L}'_2$.

Exercise 8.79 **(De Rham and inverse de Rham on induced $\mathcal{D}$-modules)**

(1) Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Show that $H^k\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X)\big) = 0$ for $k \neq 0$ and $H^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X)\big) = \mathcal{L}$. [*Hint*: Use Exercise 8.29.]

(2) Show that $H^0({}^{\mathrm{p}}\mathrm{DR})$ defines a functor $\mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X) \mapsto \mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$, which will be denoted by ${}^{\mathrm{diff}}\mathrm{DR}$.

(3) Show that ${}^{\mathrm{diff}}\mathrm{DR}^{-1} : \mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X)$ is an equivalence of categories, a quasi-inverse functor being ${}^{\mathrm{diff}}\mathrm{DR} : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \mapsto \mathcal{L}$, ${}^{\mathrm{diff}}\mathrm{DR}(v) = u$.

(4) Show that the composed functor $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X) \mapsto \mathsf{Mod}(\mathcal{D}_X)$, still denoted by ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$, is *fully faithful*, i.e., it induces a bijective morphism

$$\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

(One may think that we "embed" the additive (non abelian) category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ in the abelian category $\mathsf{Mod}(\mathcal{D}_X)$; we will use this "embedding" to define below acyclic objects).

Exercise 8.80 **(The de Rham functor ${}^{\mathrm{diff}}\mathrm{DR}$).**

(1) Show that the de Rham complex of a left $\mathcal{D}_X$-module $\mathcal{M}$ is a complex in $\mathsf{C}^{\mathrm{b}}(\mathcal{O}_X, \mathrm{Diff}_X)$. [*Hint*: Use Exercise 8.76(1).]

(2) By using Exercise 8.26(1), show that the de Rham complex of a right $\mathcal{D}_X$-module $\mathcal{M}$ is a complex in $\mathsf{C}^{\mathrm{b}}(\mathcal{O}_X, \mathrm{Diff}_X)$

(3) Show that the de Rham complex of a $\star$-bounded complex of right $\mathcal{D}_X$-modules has its associated single complex in $\mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$. [*Hint*: Use Exercise 8.24.]

(4) Conclude that ${}^{\mathrm{p}}\mathrm{DR}$ induces a functor ${}^{\mathrm{diff}}\mathrm{DR} : \mathsf{C}^\star(\mathcal{D}_X) \mapsto \mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$.

(5) Extend this functor as a functor of triangulated categories $\mathsf{K}^\star(\mathcal{D}_X) \to \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$.

Exercise 8.81. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Show that $\mathrm{God}^\bullet\, {}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}$ is a differential complex. [*Hint*: Identify this complex with ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^\bullet\,\mathcal{M}$.]

Exercise 8.82. Show that the family N of Diff-acyclic objects forms a *null system* in $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, i.e.,

- the object 0 belongs to N,
- an object $\mathcal{L}^\bullet$ belongs to N iff $\mathcal{L}^\bullet[1]$ does so,
- if $\mathcal{L}^\bullet \to \mathcal{L}'^\bullet \to \mathcal{L}''^\bullet \to \mathcal{L}^\bullet[1]$ is a distinguished triangle of $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, and if $\mathcal{L}^\bullet, \mathcal{L}'^\bullet$ are objects in N, then so is $\mathcal{L}''^\bullet$.

[*Hint*: Use that the extension of ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ to the categories $\mathsf{K}^\star$ is a functor of triangulated categories.]

Exercise 8.83 **(The functor $\mathsf{D}^\star(\mathcal{O}_X) \mapsto \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$).** Using Exercise 8.76(1), define a functor $\mathsf{C}^\star(\mathcal{O}_X) \mapsto \mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ and $\mathsf{K}^\star(\mathcal{O}_X) \mapsto \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$. By using that $\mathcal{D}_X$ is $\mathcal{O}_X$-flat, show that if $\mathcal{L}^\bullet$ is acyclic in $\mathsf{K}^\star(\mathcal{O}_X)$, then $\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is acyclic in $\mathsf{K}^\star(\mathcal{D}_X)$. Conclude that the previous functor extends as a functor $\mathsf{D}^\star(\mathcal{O}_X) \mapsto \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$.

Exercise 8.84. Show that the following diagram commutes:

$$\mathsf{D}^\star(\mathcal{D}_X) \xrightarrow[{}^{\mathrm{diff}}\mathrm{DR}]{} \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \xrightarrow[\mathsf{Forget}]{} \mathsf{D}^\star(\mathbb{C}_X), \qquad \mathsf{D}^\star(\mathcal{D}_X) \xrightarrow{{}^{\mathrm{p}}\mathrm{DR}} \mathsf{D}^\star(\mathbb{C}_X)$$

Exercise 8.85. Assume that $\mathcal{L}^\bullet$ is Diff-acyclic. Show that $\mathsf{Forget}\,\mathcal{L}^\bullet$ is acyclic. [*Hint*: By definition, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^\bullet)$ is acyclic; then ${}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^\bullet)$ is also acyclic and quasi-isomorphic to $\mathsf{Forget}\,\mathcal{L}^\bullet$.]

Conclude that Forget induces a functor $\mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{D}^\star(\mathbb{C}_X)$, and that we have an isomorphism of functors

$${}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1} \xrightarrow{\sim} \mathsf{Forget} : \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \longmapsto \mathsf{D}^\star(\mathbb{C}_X).$$

Compare with Exercise 8.29.

Exercise 8.86. Let $\mathcal{L},\mathcal{L}'$ be two $\mathcal{O}_X$-modules and

$$v:\mathcal{M}=\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X\longrightarrow\mathcal{M}'=\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X$$

a $\mathcal{D}_X$-linear morphism. It defines a $f^{-1}\mathcal{D}_Y$-linear morphism

$$v\otimes\mathbf{1}:\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}\longrightarrow\mathcal{M}'\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y},$$

where $\mathbf{1}$ is the section introduced in Exercise 8.54(1). This is therefore a morphism

$$\widetilde{v}:\mathcal{L}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y\longrightarrow\mathcal{L}'\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y.$$

Show that ${}^{\mathrm{diff}}\mathrm{DR}_Y(\widetilde{v})={}^{\mathrm{diff}}\mathrm{DR}_X(v)$.

[*Hint*: Since the problem is local, argue with coordinates on X and Y and write $f=(f_1,\dots,f_m)$. Let ℓ be a local section of $\mathcal{L}$, and let $\mathbf{1}_X$ be the unit of $\mathcal{D}_X$. Set $v(\ell\otimes\mathbf{1}_X)=w(\ell)=\sum_\alpha w(\ell)_\alpha\otimes\partial_x^\alpha$ and $\widetilde{v}(\ell\otimes\mathbf{1}_X)=v(\ell\otimes\mathbf{1}_X)\otimes\mathbf{1}_{X\to Y}$. Show that, if $\alpha_i\neq0$,

$$\partial_{x_i}^{\alpha_i}\otimes\mathbf{1}_{X\to Y}=\partial_{x_i}^{\alpha_i-1}\sum_j\frac{\partial f_j}{\partial x_i}\otimes\partial_{y_j}.$$

Deduce that the image of $\widetilde{v}(\ell\otimes\mathbf{1}_X)$ by the map $\mathcal{L}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y\to\mathcal{L}$ is equal to the image of $w(\ell)_0$, which is nothing but $u(\ell)$ by definition of $u:=H^0\,\mathrm{DR}_X(v)$.]

Exercise 8.87.

(1) Show that the Leray filtration is a decreasing finite filtration and that it is compatible with the differential.

(2) Show that, locally, being in Ler^p means having at least p factors dy_j in any summand.

***Exercise 8.88* (The connecting morphism).** Let $0\to C_1^\bullet\to C_2^\bullet\to C_3^\bullet\to0$ be an exact sequence of complexes. Let $[\mu]\in H^kC_3^\bullet$ and choose a representative in C_3^k with $d\mu=0$. Lift μ as $\widetilde{\mu}\in C_2^k$.

(1) Show that $d\widetilde{\mu}\in C_1^{k+1}$ and that its differential is zero, so that the class $[d\widetilde{\mu}]\in H^{k+1}C_1^\bullet$ is well-defined.

(2) Show that $\delta:[\mu]\mapsto[d\widetilde{\mu}]$ is a well-defined morphism $H^kC_3^\bullet\to H^{k+1}C_1^\bullet$.

(3) Deduce the existence of the cohomology long exact sequence, having δ as its connecting morphism.

8.12. Comments

Most of the results in this chapter are now classical and explained in various reference books (e.g. [**MS93a, MS93b**], [**Bjö93**], [**MN04**], [**Kas03**], [**HTT08**]). We have emphasized their adaptation to the case of filtered $\mathcal{D}$-modules, or more precisely to the case of $\widetilde{\mathcal{D}}$-modules, in a way similar to what is done in [**Sab05**] with the analytification $\mathcal{R}$ with respect to the variable z of the sheaf $\widetilde{\mathcal{D}}$, and [**Moc07, Moc11a, Moc15**].

The notion of induced $\mathcal{D}$-module and the idea of inverting the de Rham functor is due to M. Saito [**Sai89a**]. The comparison of the notion of pushforward of a

$\mathcal{D}$-module with the Katz-Oda construction of the Gauss-Manin connection is taken from [**DMSS00**].

The pushforward of a holonomic $\mathcal{D}$-module $\mathcal{M}$ by a finite morphism (or finite on the support of $\mathcal{M}$) is worth considering in detail. This is done in [**Käl18**] in the algebraic setting. In particular, the decomposition theorem holds without any Hodge assumption for such morphisms.