

## Moduli of pre-...-modules, preverse sheaves and the Riemann-Hilbert morphism. I.

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## Moduli of pre- $\mathcal{D}$ -modules, perverse sheaves and the Riemann-Hilbert morphism – I

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### 1 Introduction

This paper is devoted to the moduli problem for regular holonomic  $\mathcal{D}$ -modules and perverse sheaves on a complex projective variety  $X$ . It treats the case where the singular locus of the  $\mathcal{D}$ -module is a smooth divisor  $S$  and the characteristic variety is contained in the union of the zero section  $T_X^*X$  of the cotangent bundle of  $X$  and the conormal bundle  $N_{S^*,X}$  of  $S$  in  $X$  (also denoted  $T_S^*X$ ). The sequel (part II) will treat the general case of arbitrary singularities.

A moduli space for  $\mathcal{C}$ -coherent  $\mathcal{D}$ -modules on a smooth projective variety was constructed by Simpson [S]. These are vector bundles with integrable connections, and they are the simplest case of  $\mathcal{D}$ -modules. In this moduli construction, the requirement of semistability is automatically fulfilled by all the objects.

Next in order of complexity are the so called ‘regular meromorphic connections’. These  $\mathcal{D}$ -modules can be generated by vector bundles with connections which have logarithmic singularities on divisors with normal crossing. These  $\mathcal{D}$ -modules are not  $\mathcal{O}$ -coherent, but are torsion free as  $\mathcal{O}$ -modules. A moduli scheme does not exist for these  $\mathcal{D}$ -modules themselves (see Sect. 1 of [N]), but it is possible to define a notion of stability and construct a moduli for vector bundles with logarithmic connections. This was done in [N]. Though many logarithmic connections give rise to the same meromorphic connection, the choice of a logarithmic connection is infinitesimally rigid if its residual eigenvalues do not differ by nonzero integers (see Sect. 5 of [N]).

In the present paper and its sequel, we deal with general regular holonomic  $\mathcal{D}$ -modules. Such modules are in general neither  $\mathcal{O}$ -coherent, nor  $\mathcal{O}$ -torsion free or pure dimensional. We define objects called pre- $\mathcal{D}$ -modules, which play the same role for regular holonomic  $\mathcal{D}$ -modules that logarithmic connections played for regular meromorphic connections. We define a notion of (semi-)stability, and construct a moduli scheme for (semi-)stable pre- $\mathcal{D}$ -modules with prescribed singularity stratification and other numerical data. We also construct a moduli scheme for perverse sheaves with prescribed singularity stratification and other numerical data on a nonsingular variety, and show that the Riemann-Hilbert correspondence defines an analytic morphism between (an open set of) the moduli of pre- $\mathcal{D}$ -modules and the moduli of perverse sheaves.

The contents of this paper are as follows. Let  $X$  be a smooth projective variety, and let  $S$  be a smooth hypersurface on  $X$ . In Sect. 2, we define pre- $\mathcal{D}$ -modules on  $(X, S)$  which may be regarded as  $\mathcal{O}_X$ -coherent descriptions of those regular holonomic  $\mathcal{L}_X$ -modules whose characteristic variety is contained in  $T_X^*X \cup T_S^*X$ . The pre- $\mathcal{L}$ -modules form an algebraic stack in the sense of Artin, which is a property that does not hold for the corresponding  $\mathcal{D}$ -modules.

In Sect. 3, we define a functor from the pre- $\mathcal{D}$ -modules to  $\mathcal{D}$ -modules (in fact we mainly use the presentation of holonomic  $\mathcal{D}$ -modules given by Malgrange [Mal], that we call Malgrange objects). This is a surjective functor, and though not injective, it has an infinitesimal rigidity property (see Proposition 3.9) which generalizes the corresponding result for meromorphic connections.

In Sect. 4, we introduce a notion of (semi-)stability for pre- $\mathcal{D}$ -modules, and show that semistable pre- $\mathcal{D}$ -modules with fixed numerical data form a moduli scheme.

Next, we consider perverse sheaves on  $X$  which are constructible with respect to the stratification  $(X - S) \cup S$ . These have finite descriptions through the work Verdier, recalled in Sect. 5.

We observe that these finite descriptions are objects which naturally form an Artin algebraic stack. Moreover, we show in Sect. 6 that S-equivalence classes (Jordan-Hölder classes) of finite descriptions with given numerical data form a coarse moduli space which is an affine scheme. Here, no hypothesis about stability is necessary.

In Sect. 7, we consider the Riemann-Hilbert correspondence. When a pre- $\mathcal{D}$ -module has an underlying logarithmic connection for which residual eigenvalues

do not differ by nonzero integers, we functorially associate to it a finite description, which is the finite description of the perverse sheaf associated to the corresponding  $\mathcal{D}$ -module by the Riemann-Hilbert correspondence from regular holonomic  $\mathcal{D}$ -modules to perverse sheaves. We show that this gives an analytic morphism of stacks from the analytic open subset of the stack (or moduli) of pre- $\mathcal{D}$ -modules on  $(X, S)$  where the ‘residual eigenvalues are good’, to the stack (or moduli) of finite descriptions on  $(X, S)$ .

In Sect. 8, we show that the above morphism of analytic stacks is in fact a spread (surjective local isomorphism) in the analytic category. We also show that it has removable singularities in codimension 1, that is, it can be defined outside codimension two on any parameter space which is smooth in codimension 1.

## 2 Pre- $\mathcal{D}$ -modules

Let  $X$  be a nonsingular variety and let  $S \subset X$  be a smooth divisor (reduced). Let  $\mathcal{I}_S \subset \mathcal{O}_X$  be the ideal sheaf of  $S$ , and let  $T_X[\log S] \subset T_X$  be the sheaf of all tangent vector fields on  $X$  which preserve  $\mathcal{I}_S$ . Let  $\mathcal{D}_X[\log S] \subset \mathcal{D}_X$  be the algebra of all partial differential operators which preserve  $\mathcal{I}_S$ ; it is generated as an  $\mathcal{O}_X$  algebra by  $T_X[\log S]$ .

The  $\mathcal{I}_S$ -adic filtration on  $\mathcal{O}_X$  gives rise to a (decreasing) filtration of  $\mathcal{D}_X$  as follows: for  $k \in \mathbb{Z}$  define  $V^k \mathcal{D}_X$  as the subsheaf of  $\mathcal{D}_X$  whose local sections consist of operators  $P$  which satisfy  $P \cdot \mathcal{I}_S^j \subset \mathcal{I}_S^{k+j}$  for all  $j$ . By construction, one has  $\mathcal{D}_X[\log S] = V^0 \mathcal{D}_X$  and every  $V^k(\mathcal{D}_X)$  is a coherent  $\mathcal{D}_X[\log S]$ -module.

Let  $p : N_{S,X} \rightarrow S$  denote the normal bundle of  $S$  in  $X$ . The graded ring  $\mathrm{gr}_V \mathcal{D}_X$  is naturally identified with  $p_* \mathcal{D}_{N_{S,X}}$ . Its  $V$ -filtration (corresponding to the inclusion of  $S$  in  $N_{S,X}$  as the zero section) is then split.

There exists a canonical section  $\theta$  of the quotient ring  $\mathcal{D}_X[\log S] / \mathcal{I}_S \mathcal{D}_X[\log S] = \mathrm{gr}_V^0 \mathcal{D}_X$ , which is locally induced by  $x \partial_x$ , where  $x$  is a local equation for  $S$ . It is a central element in  $\mathrm{gr}_V^0 \mathcal{D}_X$ . This ring contains  $\mathcal{O}_S$  as a subring and  $\mathcal{D}_S$  as a quotient (one has  $\mathcal{D}_S = \mathrm{gr}_V^0 \mathcal{D}_X / \theta \mathrm{gr}_V^0 \mathcal{D}_X$ ). One can identify locally on  $S$  the ring  $\mathrm{gr}_V^0 \mathcal{D}_X$  with  $\mathcal{D}_S[\theta]$ .

A coherent  $\mathrm{gr}_V^0 \mathcal{D}_X$ -module on which  $\theta$  acts by 0 is a coherent  $\mathcal{D}_S$ -module. The locally free rank one  $\mathcal{O}_S$ -module  $\mathcal{N}_{S,X} = \mathcal{O}_X(S) / \mathcal{O}_X$  is a  $\mathrm{gr}_V^0 \mathcal{D}_X$ -module on which  $\theta$  acts by  $-1$ .

**Definition 2.1** A **logarithmic module** on  $(X, S)$  will mean a sheaf of  $\mathcal{D}_X[\log S]$ -modules, which is coherent as an  $\mathcal{O}_X$ -module. A **logarithmic connection** on  $(X, S)$  will mean a logarithmic module which is coherent and torsion-free as an  $\mathcal{O}_X$ -module.

*Remark 2.2* It is known that when  $S$  is nonsingular, any logarithmic connection on  $(X, S)$  is locally free as an  $\mathcal{O}_X$ -module.

**Definition 2.3 (Family of logarithmic modules)** Let  $f : Z \rightarrow T$  be a smooth morphism of schemes. Let  $Y \subset Z$  be a divisor such that  $Y \rightarrow T$  is smooth. Let

$T_{Z/T}[\log Y] \subset T_{X/Y}$  be the sheaf of germs of vertical vector fields which preserve the ideal sheaf of  $Y$  in  $\mathcal{O}_Z$ . This generates the algebra  $\mathcal{D}_{Z/T}[\log Y]$ . A family of logarithmic modules on  $Z/T$  is a  $\mathcal{D}_{Z/T}[\log Y]$ -module which is coherent as an  $\mathcal{O}_Z$ -module, and is flat over  $\mathcal{O}_T$ . When  $f : Z \rightarrow T$  is the projection  $X \times T \rightarrow T$ , and  $Y = S \times T$ , we get a family of logarithmic modules on  $(X, S)$  parametrized by  $T$ .

*Remark 2.4* The restriction to  $S$  of a logarithmic module is acted on by  $\theta$ : for a logarithmic connection, this is the action of the residue of the connection, which is an  $\mathcal{O}_S$ -linear morphism.

*Remark 2.5* There is an equivalence (restriction to  $S$ ) between logarithmic modules supported on the reduced scheme  $S$  and  $\mathrm{gr}_V^0 \mathcal{D}_X$ -modules which are  $\mathcal{O}_S$ -coherent, (hence locally free  $\mathcal{O}_S$ -modules, since they are locally  $\mathcal{D}_S$ -modules). In the following, we shall not make any difference between the corresponding objects.

We give two definitions of pre- $\mathcal{D}$ -modules. The two definitions are ‘equivalent’ in the sense that they give not only equivalent objects, but also equivalent families, or more precisely, the two definitions give rise to isomorphic algebraic stacks. To give a familiar example of such an equivalence, this is the way how vector bundles and locally free sheaves are ‘equivalent’. Note also that mere equivalence of objects is not enough to give equivalence of families — for example, the category of flat vector bundles is equivalent to the category of  $\pi_1$  representations, but an algebraic family of flat bundles gives in general only a holomorphic (not algebraic) family of  $\pi_1$  representations.

In their first version, pre- $\mathcal{D}$ -modules are objects that live on  $X$ , and the functor from pre- $\mathcal{D}$ -modules to  $\mathcal{D}$ -modules has a direct description in their terms. The second version of pre- $\mathcal{D}$ -modules is more closely related to the Malgrange description of  $\mathcal{D}$ -modules and the Verdier description of perverse sheaves, and the Riemann-Hilbert morphism to the stack of perverse sheaves has direct description in its terms.

**Definition 2.6 (Pre- $\mathcal{D}$ -module of first kind on  $(X, S)$ )** Let  $X$  be a nonsingular variety, and  $S \subset X$  a smooth divisor. A pre- $\mathcal{D}$ -module  $\mathbf{E} = (E, F, t, s)$  on  $(X, S)$  consists of the following data

- (1)  $E$  is a logarithmic connection on  $(X, S)$ .
- (2)  $F$  is a logarithmic module on  $(X, S)$  supported on the reduced scheme  $S$  (hence a flat connection on  $S$ ).
- (3)  $t : (E|_S) \rightarrow F$  and  $s : F \rightarrow (E|_S)$  are  $\mathcal{D}_X[\log S]$  linear maps, which satisfies the following conditions:
- (4) On  $E|_S$ , we have  $st = R$  where  $R \in \mathrm{End}(E|_S)$  is the residue of  $E$ .
- (5) On  $F$ , we have  $ts = \theta_F$  where  $\theta_F : F \rightarrow F$  is the  $\mathcal{D}_X[\log S]$  linear endomorphism induced by any Eulerian vector field  $x\partial/\partial x$ .

If  $(E, F, t, s)$  and  $(E', F', t', s')$  are two pre- $\mathcal{D}$ -modules, a morphism between them consists of  $\mathcal{D}_X[\log S]$  linear morphisms  $f_0 : E \rightarrow E'$  and  $f_1 : F \rightarrow F'$  which commute with  $t, t'$  and with  $s, s'$ .

*Remark 2.7* It follows from the definition of a pre- $\mathcal{D}$ -module  $(E, F, t, s)$  that  $E$  and  $F$  are locally free on  $X$  and  $S$  respectively, and the vector bundle morphisms  $R, s$  and  $t$  all have constant ranks on irreducible components of  $S$ .

*Example* Let  $E$  be a logarithmic connection on  $(X, S)$ . We can associate functorially to  $E$  the following three pre- $\mathcal{D}$ -modules. Take  $F_1$  to be the restriction of  $E$  to  $S$  as an  $\mathcal{O}$ -module. Let  $t_1 = R$  (the residue) and  $s_1 = 1_F$ . Then  $\mathbf{E}_1 = (E, F_1, t_1, s_1)$  is a pre- $\mathcal{D}$ -module, which under the functor from pre- $\mathcal{D}$ -modules to  $\mathcal{D}$ -modules defined later will give rise to the meromorphic connection corresponding to  $E$ . For another choice, take  $F_2 = E|_S$ ,  $t_2 = 1_F$  and  $s_2 = R$ . This gives a pre- $\mathcal{D}$ -module  $\mathbf{E}_2 = (E, F_2, t_2, s_2)$  which will give rise to a  $\mathcal{D}$ -module which has nonzero torsion part when  $R$  is not invertible. For the third choice (which is in some precise sense the minimal choice), take  $F_3$  to be the image vector bundle of  $R$ . Let  $t_3 = R : (E|_S) \rightarrow F_3$ , and let  $s_3 : F_3 \hookrightarrow (E|_S)$ . This gives a pre- $\mathcal{D}$ -module  $\mathbf{E}_3 = (E, F_3, t_3, s_3)$ . We have functorial morphisms  $\mathbf{E}_3 \rightarrow \mathbf{E}_2 \rightarrow \mathbf{E}_1$  of pre- $\mathcal{D}$ -modules.

**Definition 2.8 (Families of pre- $\mathcal{D}$ -modules)** *Let  $T$  be a complex scheme. A family  $\mathbf{E}_T$  of pre- $\mathcal{D}$ -modules on  $(X, S)$  parametrized by the scheme  $T$ , a morphism between two such families, and pullback of a family under a base change  $T' \rightarrow T$  have obvious definitions (starting from definition of families of  $\mathcal{D}_X[\log S]$ -modules), which we leave to the reader. This gives us a fibered category  $PD$  of pre- $\mathcal{D}$ -modules over the base category of  $\mathbb{C}$  schemes. Let  $\mathcal{PD}$  be the largest (nonfull) subcategory of  $PD$  in which all morphisms are isomorphisms. This is a groupoid over  $\mathbb{C}$  schemes.*

**Proposition 2.9** *The groupoid  $\mathcal{PD}$  is an algebraic stack in the sense of Artin.*

*Proof* It can be directly checked that  $\mathcal{PD}$  is a sheaf, that is, descent and effective descent are valid for faithfully flat morphisms of parameter schemes of families of pre- $\mathcal{D}$ -modules. Let  $Bun_X$  be the stack of vector bundles on  $X$ , and let  $Bun_S$  be the stack of vector bundles on  $S$ . Then  $\mathcal{PD}$  has a forgetful morphism to the product stack  $Bun_X \times_{\mathbb{C}} Bun_S$ . The latter stack is algebraic and the forgetful morphism is representable, hence the desired conclusion follows.

Before giving the definition of pre- $\mathcal{D}$ -modules of the second kind, we observe the following.

*Remark 2.10* Let  $N$  be any line bundle on a smooth variety  $S$ , and let  $\bar{N} = P(N \oplus \mathcal{O}_S)$  be its projective completion, with projection  $\pi : \bar{N} \rightarrow S$ . Let  $S^\infty = P(N)$  be the divisor at infinity. For any logarithmic connection  $E$  on  $(\bar{N}, S \cup S^\infty)$ , the restriction  $E|_S$  is of course a  $\mathcal{D}_{\bar{N}}[\log S \cup S^\infty]$ -module. But conversely, for any  $\mathcal{O}$ -coherent  $\mathcal{D}_{\bar{N}}[\log S \cup S^\infty]$ -module  $F$  scheme theoretically supported on  $S$ , there is a natural structure of a logarithmic connection on  $(\bar{N}, S \cup S^\infty)$  on its pullup  $\pi^*(F)$  to  $\bar{N}$ . The above correspondence is well behaved in families, giving an isomorphism between the algebraic stack of  $\mathcal{D}_{\bar{N}}[\log S \cup S^\infty]$ -modules  $F$  supported on  $S$  and the algebraic stack of logarithmic connections  $E$  on  $(\bar{N}, S \cup S^\infty)$ .

$S^\infty$ ) such that the vector bundle  $E$  is trivial on the fibers of  $\pi : \bar{N} \rightarrow S$ . The functors  $\pi^*(-)$  and  $(-)|_S$  are fully faithful.

*Remark 2.11* Let  $S \subset X$  be a smooth divisor, and let  $N = N_{S,X}$  be its normal bundle. Then the following are equivalent in the sense that we have fully faithful functors between the corresponding categories, which give naturally isomorphic stacks.

- (1)  $\mathcal{D}_X[\log S]$ -modules which are scheme theoretically supported on  $S$ .
- (2)  $\mathcal{D}_N[\log S]$ -modules which are scheme theoretically supported on  $S$ .
- (3)  $\mathcal{D}_{\bar{N}}[\log S \cup S^\infty]$ -modules which are scheme theoretically supported on  $S$ .

The equivalence between (2) and (3) is obvious, while the equivalence between (1) and (2) is obtained as follows. The Poincaré residue map  $res : \Omega_X^1[\log S] \rightarrow \mathcal{O}_S$  gives the following short exact sequence of  $\mathcal{O}_S$ -modules.

$$0 \rightarrow \Omega_S^1 \rightarrow \Omega_X^1[\log S]|_S \rightarrow \mathcal{O}_S \rightarrow 0$$

By taking duals, this gives

$$0 \rightarrow \mathcal{O}_S \rightarrow T_X[\log S]|_S \rightarrow T_S \rightarrow 0.$$

It can be shown that there exists a canonical isomorphism  $T_X[\log S]|_S \rightarrow T_N[\log S]|_S$  which makes the following diagram commute, where the rows are exact.

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_S & \rightarrow & T_N[\log S]|_S & \rightarrow & T_S \rightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \rightarrow & \mathcal{O}_S & \rightarrow & T_X[\log S]|_S & \rightarrow & T_S \rightarrow 0 \end{array}$$

*Remarks 2.12* (1) Observe that the element  $\theta$  is just the image of 1 under the map  $\mathcal{O}_S \rightarrow T_X[\log S]|_S$ .

(2) Using the notations of the beginning of this section, one can identify the ring  $\pi_* \mathcal{D}_{\bar{N}}[\log S \cup S^\infty]$  with  $\mathrm{gr}_V^0 \mathcal{D}_X$ . Hence  $\theta$  is a global section of  $\mathcal{D}_{\bar{N}}[\log S \cup S^\infty]$ .

We now make the following important definition.

**Definition 2.13 (Specialization of a logarithmic module)** *Let  $E$  be a logarithmic module on  $(X, S)$ . Then the specialization  $\mathrm{sp}_S E$  will mean the logarithmic connection  $\pi^*(E|_S)$  on  $(\bar{N}_{S,X}, S \cup S^\infty)$ .*

Now we are ready to define the second version of pre- $\mathcal{D}$ -modules.

**Definition 2.14 (Pre- $\mathcal{D}$ -modules of the second kind on  $(X, S)$ )** *A pre- $\mathcal{D}$ -module (of the second kind)  $\mathbf{E} = (E_0, E_1, c, v)$  on  $(X, S)$  consists of the following data*

- (1)  $E_0$  is a logarithmic connection on  $(X, S)$ ,
- (2)  $E_1$  is a logarithmic connection on  $(\bar{N}_{S,X}, S \cup S^\infty)$ ,
- (3)  $c : \mathrm{sp}_S E_0 \rightarrow E_1$  and  $v : E_1 \rightarrow \mathrm{sp}_S E_0$  are  $\mathcal{D}_{\bar{N}_{S,X}}[\log S \cup S^\infty]$ -linear maps,

which satisfies the following conditions:

(4) on  $\mathrm{sp}_S E_0$ , we have  $cv = \theta_{\mathrm{sp}_S E_0}$ ,

(5) on  $E_1$ , we have  $vc = \theta_{E_1}$ ,

(6) the vector bundle underlying  $E_1$  is trivial in the fibers of  $\pi : \overline{N_{S,X}} \rightarrow S$  (that is,  $E_1$  is locally over  $S$  isomorphic to  $\pi^*(E_1|S)$ ).

If  $(E_0, E_1, c, v)$  and  $(E'_0, E'_1, c', v')$  are two pre- $\mathcal{D}$ -modules, a morphism between them consists of  $\mathcal{L}_X[\log S]$  linear morphisms  $f_0 : E_0 \rightarrow E'_0$  and  $f_1 : E_1 \rightarrow E'_1$  such that  $\mathrm{sp}_S f_0$  and  $f_1$  commute with  $v, v'$  and with  $c, c'$ .

**Definition 2.15 (Families of pre- $\mathcal{D}$ -modules of the second kind)** Let  $T$  be a complex scheme. A family  $\mathbf{E}_T$  of pre- $\mathcal{D}$ -modules on  $(X, S)$  parametrized by the scheme  $T$ , a morphism between two such families, and pullback of a family under a base change  $T' \rightarrow T$  have obvious definitions which we leave to the reader. This gives us a fibered category  $PM$  of pre- $\mathcal{D}$ -modules of second kind over the base category of  $\mathbb{C}$  schemes.

**Proposition 2.16** The functor which associates to each family of pre- $\mathcal{D}$ -module  $(E_0, E_1, c, v)$  of the second kind parametrized by  $T$  the family of pre- $\mathcal{D}$ -module of the first kind  $(E_0, E_1|S, c|S, v|S)$  is an equivalence of fibered categories.

*Proof* This follows from Remarks 2.10 and 2.11 above.

### 3 From pre- $\mathcal{D}$ -modules to $\mathcal{D}$ -modules

In this section we first recall the description of regular holonomic  $\mathcal{D}$ -modules due to Malgrange [Mal] and we associate a ‘Malgrange object’ to a pre- $\mathcal{D}$ -module of the second kind (Proposition 3.6), which has good residual eigenvalues (definition 3.4), each component of  $S$  do not differ by positive integers. Having such a direct description of the Malgrange object enables us to prove that every regular holonomic  $\mathcal{D}$ -module with characteristic variety contained in  $T_X^*X \cup T_S^*X$  arises from a pre- $\mathcal{D}$ -module (Corollary 3.8), and also helps us to prove an infinitesimal rigidity property for the pre- $\mathcal{D}$ -modules over a given  $\mathcal{D}$ -module (Proposition 3.9).

#### Malgrange objects

Regular holonomic  $\mathcal{D}$ -modules on  $X$  whose characteristic variety is contained in  $T_X^*X \cup T_S^*X$  have an equivalent presentation due to Malgrange and Verdier, which we now describe.

Let us recall the definition of the **specialization**  $\mathrm{sp}_S(M)$  of a regular holonomic  $\mathcal{L}_X$ -module  $M$ : fix a section  $\sigma$  of the projection  $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  and denote  $A$  its image; every such module admits a unique (decreasing) filtration  $V^k M$  ( $k \in \mathbb{Z}$ ) by  $\mathcal{L}_X[\log S]$ -submodules which is good with respect to  $V\mathcal{L}_X$  and satisfies the following property: on  $\mathrm{gr}_V^k M$ , the action of  $\theta$  admits a minimal polynomial all of whose roots are in  $A + k$ . Then by definition one puts



$\mathrm{sp}_S M = \bigoplus_{k \in \mathbf{Z}} \mathrm{gr}_V^k M$ . One has  $(\mathrm{sp}_S M)[*S] = \mathrm{sp}_S(M[*S]) = (\mathrm{gr}_V^{\geq k} M)[*S]$  for all  $k \geq 1$ , if we put  $\mathrm{gr}_V^{\geq k} M = \bigoplus_{\ell \geq k} \mathrm{gr}_V^\ell M$ . The  $p_*\mathcal{D}_{N_S, X}$ -module  $\mathrm{sp}_S M$  does not depend on the choice of  $\sigma$  (if one forgets its gradation).

If  $\theta$  acts in a locally finite way on a  $\mathrm{gr}_V^0 \mathcal{D}_X$  or a  $p_*\mathcal{D}_{N_S, X}$ -module, we denote  $\Theta$  the action of  $\exp(-2i\pi\theta)$ .

Given a regular holonomic  $\mathcal{D}_X$ -module, we can functorially associate to it the following modules:

(1)  $M[*S] = \mathcal{C}_X[*S] \otimes_{\mathcal{C}_X} M$  is the  $S$ -localized  $\mathcal{D}_X$ -module; it is also regular holonomic;

(2)  $\mathrm{sp}_S M$  is the specialized module; this is a regular holonomic  $p_*\mathcal{D}_{N_S, X}$ -module, which is also **monodromic**, i.e. the action of  $\theta$  on each local section is locally (on  $S$ ) finite.

The particular case that we shall use of the result proved in [Mal] is then the following:

**Theorem 3.1** *There is an equivalence between the category of regular holonomic  $\mathcal{D}_X$ -modules and the category which objects are triples  $(\mathcal{M}, \bar{M}, \alpha)$ , where  $\mathcal{M}$  is a  $S$ -localized regular holonomic  $\mathcal{D}_X$ -module,  $\bar{M}$  is a monodromic regular holonomic  $p_*\mathcal{D}_{N_S, X}$ -module and  $\alpha$  is an isomorphism (of localized  $p_*\mathcal{D}_{N_S, X}$ -modules) between  $\mathrm{sp}_S \mathcal{M}[*S]$  and  $\bar{M}[*S]$ .*

In fact, the result of [Mal] does mention neither holonomicity nor regularity. Nevertheless, using standard facts of the theory, one obtains the previous proposition. Regularity includes here regularity at infinity, i.e. along  $S^\infty$ . This statement can be simplified when restricted to the category of regular holonomic  $\mathcal{D}$ -modules which characteristic variety is contained in the union  $T_X^*X \cup T_S^*X$ .

**Definition 3.2** A **Malgrange object** on  $(X, S)$  is a tuple  $(M_0, M_1, C, V)$  where

(1)  $M_0$  is an  $S$ -localized regular holonomic  $\mathcal{D}_X$ -module which is a regular meromorphic connection on  $X$  with poles on  $S$ ,

(2)  $M_1$  is a  $S$ -localized monodromic regular holonomic  $p_*\mathcal{D}_{N_S, X}$ -module which is a regular meromorphic connection on  $N_{S, X}$  (or  $\overline{N_{S, X}}$ ) with poles on  $S$  (or on  $S \cup S^\infty$ ),

(3)  $C, V$  are morphisms (of  $p_*\mathcal{D}_{N_S, X}$ -modules) between  $\mathrm{sp}_S M_0$  and  $M_1$  satisfying  $VC = \Theta - \mathrm{id}$  on  $\mathrm{sp}_S M_0$  and  $CV = \Theta - \mathrm{id}$  on  $M_1$ .

The morphisms between two Malgrange objects are defined in an obvious way, making them an abelian category.

The previous result can be translated in the following way, using [Ve]:

**Corollary 3.3** *There is an equivalence between the category of regular holonomic  $\mathcal{D}$ -modules which characteristic variety is contained in  $T_X^*X \cup T_S^*X$  and the category of Malgrange objects on  $(X, S)$ .*

*From pre- $\mathcal{L}$ -modules to Malgrange objects*

**Definition 3.4** (1) We say that a logarithmic connection  $F$  on  $(X, S)$  has **good residual eigenvalues** if for each connected component  $S_a$  of the divisor  $S$ , the residual eigenvalues  $(\lambda_{a,k})$  of  $F$  along  $S_a$  do not include a pair  $\lambda_{a,i}, \lambda_{a,j}$  such that  $\lambda_{a,i} - \lambda_{a,j}$  is a nonzero integer.

(2) We say that a pre- $\mathcal{L}$ -module  $\mathbf{E} = (E_0, E_1, s, t)$  has **good residual eigenvalues** if the logarithmic connection  $E_0$  has good residual eigenvalues as defined above.

We now functorially associate a Malgrange object  $\mathbf{M} = \eta(\mathbf{E}) = (M_0, M_1, C, V)$  to each pre- $\mathcal{L}$ -module  $\mathbf{E} = (E_0, E_1, c, v)$  on  $(X, S)$  with  $E_0$  having good residual eigenvalues.

*Remark 3.5* By definition of a pre- $\mathcal{L}$ -module it follows that the nonzero eigenvalues of  $\theta_a$  on  $E_0|_{S_a}$  (the residue along  $S_a$ ) are the same as the nonzero eigenvalues of  $\theta_a$  on  $E_{1,a}$ .

**Proposition 3.6 (The Malgrange object associated to a pre- $\mathcal{L}$ -module with good residual eigenvalues)** Let  $\mathbf{E} = (E_0, E_1, c, v)$  be a pre- $\mathcal{L}$ -module on  $(X, S)$  of the second kind (definition 2.14), such that  $E_0$  has good residual eigenvalues. Let  $\eta(\mathbf{E}) = (M_0, M_1, C, V)$  where

- (1)  $M_0 = E_0[*S]$ ,
- (2)  $M_1 = E_1[*S]$ ,
- (3)  $C = c \circ \frac{e^{-2i\pi\theta_{E_0}} - 1}{\theta_{E_0}}$ .
- (4)  $V = v$

Then  $\eta(\mathbf{E})$  is a Malgrange object, and  $\eta$  is functorial in an obvious way.

*Proof* Because  $E_0$  has good residual eigenvalues, one can use the filtration  $V^k E_0[*S] = I_S^k E_0 \subset E_0[*S]$  in order to compute  $\mathrm{sp}_S E_0[*S]$ . It follows that the specialization of  $E_0[*S]$  when restricted to  $N_{S,X} - S$  is canonically isomorphic to the restriction of  $\mathrm{sp}_S E_0 = \pi^*(E_0|_S)$  to  $N_{X,S} - S$ .

*Essential surjectivity*

**Proposition 3.7** Every Malgrange object  $(M_0, M_1, C, V)$  on  $(X, S)$  can be obtained in this way from a pre- $\mathcal{L}$ -module.

*Proof* This follows from [Ve]: one chooses Deligne lattices in  $M_0$  and  $M_1$ . One uses the fact that every  $\mathcal{D}$ -linear map between holonomic  $\mathcal{D}$ -modules is compatible with the  $V$ -filtration, so sends the specialized Deligne lattice of  $M_0$  to the one of  $M_1$ . Moreover, the map  $v$  can be obtained from  $V$  because the only integral eigenvalue of  $\theta$  on the Deligne lattice is 0, so  $\frac{e^{-2i\pi\theta} - 1}{\theta}$  is invertible on it.

The previous two propositions give the following.

**Corollary 3.8** *The functor from pre- $\mathcal{L}$ -modules on  $(X, S)$  to regular holonomic  $\mathcal{D}$ -modules on  $X$  with characteristic variety contained in  $T_X^*X \cup T_S^*X$  is essentially surjective.*

### *Infinitesimal rigidity*

For a regular holonomic  $\mathcal{D}$ -module  $\mathbf{M}$  with characteristic variety  $T_X^*X \cup T_S^*X$ , there exist several nonisomorphic pre- $\mathcal{D}$ -modules  $\mathbf{E}$  which give rise to the Malgrange object associated to  $\mathbf{M}$ . However, we have the following infinitesimal rigidity result, which generalizes the corresponding results in [N].

**Proposition 3.9 (Infinitesimal rigidity)** *Let  $T = \text{Spec } \frac{\mathbb{C}[\epsilon]}{(\epsilon^2)}$ . Let  $\mathbf{E}_T$  be a family of pre- $\mathcal{D}$ -modules on  $(X, S)$  parametrized by  $T$ . Let the associated family  $\mathbf{M}_T$  of  $\mathcal{D}$ -modules on  $X$  be constant (pulled back from  $X$ ). Let  $\mathbf{E}$  (which is the specialization at  $\epsilon = 0$ ) be of the form  $\mathbf{E} = (E, F, s, t)$  where along any component of  $S$ , no two distinct eigenvalues of the residue of the logarithmic connection  $E$  differ by an integer. Then the family  $\mathbf{E}_T$  is also constant.*

*Proof* By [N], the family  $E_T$  is constant, as well as the specialization  $\text{sp}_S E_T$ . As a consequence, the residue  $\theta_{E_T}$  is constant. Let us now prove that the family  $F_T$  is constant.

Let  $S_a$  be a component of  $S$  along which the only possible integral eigenvalue of  $\theta_E$  is 0. Then it is also the only possible integral eigenvalue of  $\theta_F$  along  $S_a$  because the generalized eigenspaces of  $\theta_E$  and  $\theta_F$  corresponding to a nonzero eigenvalue are isomorphic by  $s$  and  $t$  (see remark 3.5). We also deduce from [N] that  $F_T$  is constant as a logarithmic module along this component.

Assume now that 0 is not an eigenvalue of  $\theta_E$  along  $S_a$  but is an eigenvalue of  $\theta_F$  along this component. Then  $\theta_F$  may have two distinct integral eigenvalues, one of which is 0. Note that, in this case,  $\theta_E$  is an isomorphism (along  $S_a$ ), as well as  $\theta_{E_T}$  which is obtained by pullback from  $\theta_E$ . It follows that on  $S_a$  we have an isomorphism  $F_T \simeq E_T|_{S_a} \oplus \text{Ker } \theta_{F_T}$ . Consequently  $\text{Ker } \theta_{F_T}$  is itself a family. It is enough to show that this family is constant. But the corresponding meromorphic connection on  $N_{S, X} - S$  is constant, being the cokernel of the constant map  $C_T : M_{0T} \rightarrow M_{1T}$ . We can then apply Proposition 5.3 of [N] because the only eigenvalue on  $\text{Ker } \theta_F$  is 0.

The maps  $s_T$  and  $t_T$  are constant if and only if for each component  $S_a$  of  $S$  and for some point  $x_a \in S_a$  their restriction to  $F_T|_{x_a} \times T$  and  $E_T|_{x_a} \times T$  are constant. This fact is a consequence of the following lemma.

**Lemma 3.10** *Let  $E$  and  $F$  be finite dimensional complex vector spaces, and let  $\theta_E \in \text{End}(E)$  and  $\theta_F \in \text{End}(F)$  be given. Let  $V \subset W = \text{Hom}(F, E) \times \text{Hom}(E, F)$  be the closed subscheme consisting of  $(s, t)$  with  $st = \theta_E$  and  $ts = \theta_F$ . Let  $\phi : W \rightarrow W$  be the holomorphic map defined by*

$$\phi(s, t) = \left( s, t \frac{e^{st} - 1}{st} \right).$$

Then the differential  $d\phi$  is injective on the Zariski tangent space to  $V$  at any closed point  $(s, t)$ .

*Proof* Let  $(a, b)$  be a tangent vector to  $V$  at  $(s, t)$ . Then by definition of  $V$ , we must have  $at + sb = 0$  and  $ta + bs = 0$ . Using  $at + sb = 0$ , we can see that  $d\phi(a, b) = (a, bf(st))$  where  $f$  is the entire function on  $\text{End}(E)$  defined by the power series  $(e^x - 1)/x$ . Suppose  $(a, bf(st)) = 0$ . Then  $a = 0$  and so the condition  $ta + bs = 0$  implies  $bs = 0$ . As the constant term of the power series  $f$  is 1 and as  $bs = 0$ , we have  $bf(st) = b$ . Hence  $b = 0$ , and so  $d\phi$  is injective.

#### 4 Semistability and moduli for pre- $\mathcal{D}$ -modules.

In order to construct a moduli scheme for pre- $\mathcal{D}$ -modules, one needs a notion of semistability. This can be defined in more than one way. What we have chosen below is a particularly simple and canonical definition of semistability. (In an earlier version of this paper, we had employed a definition of semistability in terms of parabolic structures, in which we had to fix the ranks of  $s : E_1 \rightarrow E_0|S$  and  $t : E_0|S \rightarrow E_1$  and a set of parabolic weights.)

Let  $S_a$  be the irreducible components of the smooth divisor  $S \subset X$ . For a pre- $\mathcal{D}$ -module  $\mathbf{E} = (E_0, E_1, s, t)$ , we denote by  $E_a$  the restriction of  $E_1$  to  $S_a$ , and we denote by  $s_a$  and  $t_a$  the restrictions of  $s$  and  $t$ .

##### *Definition of semistability*

We fix an ample line bundle on  $X$ , and denote the resulting Hilbert polynomial of a coherent sheaf  $F$  by  $p(F, n)$ . For constructing a moduli, we fix the Hilbert polynomials of  $E_0$  and  $E_a$ , which we denote by  $p_0(n)$  and  $p_a(n)$ . Recall (see [S]) that an  $\mathcal{C}_X$ -coherent  $\mathcal{D}_X[\log S]$ -module  $F$  is by definition **semistable** if it is pure dimensional, and for each  $\mathcal{C}_X$ -coherent  $\mathcal{D}_X[\log S]$  submodule  $F'$ , we have the inequality  $p(F', n)/\text{rank}(F') \leq p(F, n)/\text{rank}(F)$  for large enough  $n$ . We call  $p(F, n)/\text{rank}(F)$  the **normalized Hilbert polynomial** of  $F$ .

**Definition 4.1** We say that the pre- $\mathcal{D}$ -module  $\mathbf{E}$  is **semistable** if the  $\mathcal{D}_X[\log S]$ -modules  $E_0$  and  $E_a$  are semistable.

*Remarks 4.2* (1) It is easy to prove that the semistability of the  $\mathcal{D}_X[\log S]$ -module  $E_a$  is equivalent to the semistability of the logarithmic connection  $\pi_a^*(E_a)$  on  $P(N_{S_a, X} \oplus 1)$  with respect to a natural choice of polarization.

(2) When  $X$  is a curve, a pre- $\mathcal{D}$ -module  $\mathbf{E}$  is semistable if and only if the logarithmic connection  $E_0$  on  $(X, S)$  is semistable, for then  $E_1$  is always semistable.

(3) Let the dimension of  $X$  be more than one. Then even when a pre- $\mathcal{D}$ -module  $\mathbf{E}$  is a pre meromorphic connection (equivalently, when  $s : E_1 \rightarrow E_0|S$

is an isomorphism), the definition of semistability of  $\mathbf{E}$  does not reduce to the semistability of the underlying logarithmic connection  $E_0$  on  $(X, S)$ . This is to be expected because we do not fix the rank of  $s$  (or  $t$ ) when we consider families of pre- $\mathcal{D}$ -modules. Also note that even in dimension one, meromorphic connections are not a good subcategory of the abelian category of all regular holonomic  $\mathcal{D}$ -modules with characteristic variety contained in  $T_X^*X \cup T_S^*X$ , in the sense that a submodule or a quotient module of a meromorphic connection is not necessarily a meromorphic connection.

### *Boundedness and local universal family*

We let the index  $i$  vary over  $0$  and over the  $a$ .

**Proposition 4.3 (Boundedness)** *Semistable pre- $\mathcal{D}$ -modules with given Hilbert polynomials  $p_i$  form a bounded set, that is, there exists a family of pre- $\mathcal{D}$ -modules parametrized by a noetherian scheme of finite type over  $\mathbb{C}$  in which each semistable pre- $\mathcal{D}$ -module with given Hilbert polynomials occurs.*

*Proof* This is obvious as each  $E_i$  (where  $i = 0, a$ ) being semistable with fixed Hilbert polynomial, is bounded.

Next, we construct a local universal family. By boundedness, there exists a positive integer  $N$  such that for all  $n \geq N$ , the sheaves  $E_0(N)$  and  $E_1(N)$  are generated by global sections and have vanishing higher cohomology. Let  $\Lambda = D_X[\log S]$ . Let  $\mathcal{C}_X = \Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda$  be the increasing filtration of  $\Lambda$  by the order of the differential operators. Note that each  $\Lambda_k$  is an  $\mathcal{C}_X$  bimodule, coherent on each side. Let  $r$  be a positive integer larger than the ranks of the  $E_i$ . Let  $Q_i$  be the quot scheme of quotients  $q_i : \Lambda_r \otimes \mathcal{C}_X(-N)^{p_i(N)} \rightarrow E_i$  where the right  $\mathcal{C}_X$ -module structure on  $\Lambda_r$  is used for making the tensor product. Note that  $G_i = PGL(p_i(N))$  has a natural action on  $Q_i$ . Simpson defines a locally closed subscheme  $C_i \subset Q_i$  which is invariant under  $G_i$ , and a local universal family  $E$  of  $\Lambda$ -modules parametrized by  $C_i$  with the property that for two morphisms  $T \rightarrow C_i$ , the pull back families are isomorphic over an open cover  $T' \rightarrow T$  if and only if the two morphisms define  $T'$  valued points of  $C_i$  which are in a common orbit of  $G_i(T')$ .

On the product  $C_0 \times C_a$ , consider the linear schemes  $A_a$  and  $B_a$  which respectively correspond to  $Hom_\Lambda(E_1, E_0)$  and  $Hom_\Lambda(E_0, E_1)$  (see Lemma 2.7 in [N] for the existence and universal property of such linear schemes). Let  $F_a$  be the fibered product of  $A_a$  and  $B_a$  over  $C_0 \times C_a$ . Let  $H_a$  be the closed subscheme of  $F_a$  where the tuples  $(q_0, q_1, t, s)$  satisfy  $st = \theta$  and  $ts = \theta$ . Finally let  $H$  be the fibered product of the pullbacks of the  $H_a$  to  $C = C_0 \times \prod_a C_a$ . Note that  $H$  parametrizes a tautological family of pre- $\mathcal{D}$ -modules on  $(X, S)$  in which every semistable pre- $\mathcal{D}$ -module with given Hilbert polynomials occurs.

The group

$$\mathcal{G} = G_0 \times \prod_a (G_a \times GL(1))$$

has a natural action on  $H$ , with

$$(q_0, q_a, t_a, s_a) \cdot (g_0, g_a, \lambda_a) = (q_0 g_0, q_a g_a, (1/\lambda_a) t_a, \lambda_a s_a)$$

It is clear from the definitions of  $H$  and this action that two points of  $H$  parametrise isomorphic pre- $\mathcal{L}$ -modules if and only if they lie in the same  $G$  orbit.

The morphism  $H \rightarrow C \times \prod_a C_a$  is an affine morphism which is  $\mathcal{S}$ -equivariant, and by Simpson's construction of moduli for  $\Lambda$ -modules, the action of  $\mathcal{S}$  on  $C \times \prod_a C_a$  admits a good quotient in the sense of geometric invariant theory. Hence a good quotient  $H//\mathcal{S}$  exists by Ramanathan's lemma (see Proposition 3.12 in [Ne]), which by construction and universal properties of good quotients is the coarse moduli scheme of semistable pre- $\mathcal{L}$ -modules with given Hilbert polynomials.

Note that under a good quotient in the sense of geometric invariant theory, two different orbits can in some cases get mapped to the same point (get identified in the quotient). In the rest of this section, we determine what are the closed points of the quotient  $H//\mathcal{S}$ .

*Remark 4.4* For simplicity of notation, we assume in the rest of this section that  $S$  has only one connected component. It will be clear to the reader how to generalize the discussion when  $S$  has more components.

### Reduced modules

Assuming for simplicity that  $S$  has only one connected component, so that  $\mathcal{S} = \mathcal{H} \times GL(1)$  where  $H = G_0 \times G_1$ , we can construct the quotient  $H//\mathcal{S}$  in two steps: first we go modulo the factor  $GL(1)$ , and then take the quotient of  $R = H//GL(1)$  by the remaining factor  $\mathcal{H}$ . The following lemma is obvious.

**Lemma 4.5** *Let  $T$  be a scheme of finite type over  $k$ , and let  $V \rightarrow T$  and  $W \rightarrow T$  be linear schemes over  $T$ . Let  $V \times W$  be their fibered product (direct sum) over  $T$ , and let  $V \otimes W$  be their tensor product. Let  $\phi : V \times W \rightarrow V \otimes W$  be the tensor product morphism. Then its schematic image  $D \subset V \otimes W$  is a closed subscheme which (i) parametrizes all decomposable tensors, and (ii) base changes correctly. Let  $GL(1, k)$  act on  $V \times W$  by the formula  $\lambda \cdot (v, w) = (\lambda v, (1/\lambda)w)$ . Then  $\phi : V \times W \rightarrow D$  is a good quotient for this action.*

*Proof* The statement is local on the base, so we can assume that (i) the base  $T$  is an affine scheme, and (ii) both the linear schemes are closed linear subschemes of trivial vector bundles on the base, that is,  $V \subset A_T^m$  and  $W \subset A_T^n$  are subschemes defined respectively by homogeneous linear equations  $f_p(x_i) = 0$  and  $g_q(y_j) = 0$  in the coordinates  $x_i$  on  $A_T^m$  and  $y_j$  on  $A_T^n$ . Let  $z_{i,j}$  be the coordinates on  $A_T^{mn}$ , so that the map  $\otimes : A_T^m \times_T A_T^n \rightarrow A_T^{mn}$  sends  $(x_i, y_j) \mapsto (z_{i,j})$  where  $z_{i,j} = x_i y_j$ . Its schematic image is the subscheme  $C$  of  $A_T^{mn}$  defined by the equations  $z_{a,b} z_{c,d} - z_{a,d} z_{b,c} = 0$ , that is, the matrix  $(z_{i,j})$  should have rank  $< 2$ . Take

$D$  to be the subscheme of  $C$  defined by the equations  $f_p(z_{1,j}, \dots, z_{m,j}) = 0$  and  $g_q(z_{i,1}, \dots, z_{i,n}) = 0$ . Now the lemma 4.5 follows trivially from this local coordinate description.

The above lemma implies the following. To get the quotient  $H//GL(1)$ , we just have to replace the fibered product  $A \times B$  over  $C_0 \times C_1$  by the closed subscheme  $Z \subset D \subset A \otimes B$ , where  $D$  is the closed subscheme consisting of decomposable tensors  $u$ , and  $Z$  is the closed subscheme of  $D$  defined as follows. Let  $\mu_0$  and  $\mu_1$  be the canonical morphisms from  $A \otimes B$  to the linear schemes representing  $End_\Lambda(E_0|S)$  and  $End_\Lambda(E_1)$  respectively. Then  $Z$  is defined to consist of all  $u$  such that  $\mu_0(u) = \theta \in End_\Lambda(E_0|S)$  and  $\mu_1(u) = \theta \in End_\Lambda(E_1)$ . There is a canonical  $GL(1)$  quotient morphism  $A \times B \rightarrow D$  over  $C_0 \times C_1$ , which sends  $(s, t) \mapsto u = s \otimes t$ . These give the  $GL(1)$  quotient map  $H \rightarrow Z$ . Note that the map  $H \rightarrow C_0 \times C_1$  is  $\mathcal{S}$  equivariant, and the action of  $GL(1)$  on  $C_0 \times C_1$  is trivial, so we get a  $\mathcal{H}$ -equivariant map  $Z \rightarrow C_0 \times C_1$ .

In order to describe the identifications brought about by the above quotient, we make the following definition.

**Definition 4.6** A **reduced module** is a tuple  $(E_0, E_1, u)$  where  $E_0$  and  $E_1$  are as in a pre- $\mathcal{D}$ -module, and  $u \in Hom_\Lambda(E_1, E_0|S) \otimes Hom_\Lambda(E_0, E_1)$  is a decomposable tensor, such that the canonical maps  $\mu_0 : Hom_\Lambda(E_1, E_0|S) \otimes Hom_\Lambda(E_0, E_1) \rightarrow End_\Lambda(E_0|S)$  and  $\mu_1 : Hom_\Lambda(E_1, E_0|S) \otimes Hom_\Lambda(E_0, E_1) \rightarrow End_\Lambda(E_1)$ , both map  $u$  to the endomorphism  $\theta$  of the appropriate module. In other words, there exist  $s$  and  $t$  such that  $(E_0, E_1, s, t)$  is a pre- $\mathcal{D}$ -module, and  $u = s \otimes t$ . We say that the reduced module  $(E_0, E_1, s \otimes t)$  is the **associated reduced module** of the pre- $\mathcal{D}$ -module  $(E_0, E_1, s, t)$ . Moreover, we say that a reduced module is **semistable** if it is associated to a semistable pre- $\mathcal{D}$ -module.

**Lemma 4.7** Let  $V$  and  $W$  are two vector spaces,  $v, v' \in V$  and  $w, w' \in W$ , then

- (1) If  $v \otimes w = 0$  then  $v = 0$  or  $w = 0$ .
- (2) If  $v \otimes w = v' \otimes w' \neq 0$ , then there exists a scalar  $\lambda \neq 0$  such that  $v = \lambda v'$  and  $w = (1/\lambda)w'$ .

**Remark 4.8** The above lemma shows that if  $\mathbf{E}$  and  $\mathbf{E}'$  are two non-isomorphic pre- $\mathcal{D}$ -modules whose associated reduced modules are isomorphic, then we must have  $s \otimes t = s' \otimes t' = 0$ . In particular,  $\theta$  will act by zero on  $E_0|S$  and also on  $E_1$  for such pre- $\mathcal{D}$ -modules as  $st = 0$  and  $ts = 0$ .

### *S*-equivalence and stability

**Definition 4.9** By a **filtration** of a logarithmic connection  $E$  we shall mean an increasing filtration  $E_p$  indexed by  $\mathbb{Z}$  by subvector bundles which are logarithmic connections. Similarly, a filtration on a  $\mathcal{L}_X[\log S]$ -module  $F$  supported on  $S$  will mean a filtration of the vector bundle  $F|S$  by subbundles  $F_p$  which are  $\mathcal{L}_X[\log S]$ -submodules. A filtration of a pre- $\mathcal{D}$ -module  $(E_0, E_1, s, t)$  is an increasing filtration  $(E_i)_p$  of the logarithmic connection  $E_i$  ( $i = 0, 1$ ) such that  $s$  and  $t$  are filtered

morphisms with respect to the specialized filtration of  $E_0$  and the filtration of  $E_1$ . A filtration of a reduced module  $(E_0, E_1, u)$ , with  $u = s \otimes t$  where we take  $s = 0$  and  $t = 0$  if  $u = 0$ , is a filtration of the pre- $\mathcal{L}$ -module  $(E_0, E_1, s, t)$ . We shall always assume that this filtration is exhaustive, that is,  $(E_i)_p = 0$  for  $p \ll 0$  and  $(E_i)_p = E_i$  for  $p \gg 0$ . A filtration is **nontrivial** if some  $(E_i)_p$  is a proper subbundle of  $E_i$  for  $i = 0$  or  $1$ .

For a filtered pre- $\mathcal{L}$ -module (or reduced module), each step of the filtration as well as the graded object are pre- $\mathcal{L}$ -modules (or reduced modules).

*Remark 4.10* There is a natural family  $(\mathbf{E}_\tau)_{\tau \in A^1}$  of pre- $\mathcal{L}$ -modules or reduced modules parametrized by the affine line  $A^1 = \text{Spec } \mathbb{C}[\tau]$ , which fibre at  $\tau = 0$  is the graded object  $\mathbf{E}'$  and the fibre at  $\tau_0 \neq 0$  is isomorphic to the original pre- $\mathcal{L}$ -module or reduced module  $\mathbf{E}$ : put (for  $i = 0, 1$ )  $\mathcal{E}_i = \bigoplus_{p \in \mathbb{Z}} (E_i)_p \tau^p \subset E_i \otimes \mathbb{C}[\tau, \tau^{-1}]$  and the relative  $\mathcal{L}$  log-structure is the natural one.

**Definition 4.11** A special filtration of a coherent  $\mathcal{O}_X$ -module  $E$  is a filtration for which each  $E_p$  has the same normalized Hilbert polynomial as  $E$ . A **special filtration of a reduced module**  $(E_0, E_1, u)$  is a filtration of this reduced module which is special on  $E_0$  and on  $E_1$ .

The graded reduced module  $\mathbf{E}'$  associated with a special filtration of a semistable reduced module  $\mathbf{E}$  is again semistable.

**Definition 4.12** The equivalence relation on the set of isomorphism classes of all semistable reduced modules generated by this relation (by which  $\mathbf{E}'$  is related to  $\mathbf{E}$ ) will be called *S-equivalence*.

**Definition 4.13** We say that a semistable reduced module is **stable** if it does not admit any nontrivial special filtration.

*Remarks 4.14* (1) Note in particular that if each  $E_0, E_a$  is stable as a  $\Lambda$ -module, then the reduced module  $\mathbf{E}$  is stable. Consequently we have the following. Though the definition of stability depends on the ample line bundle  $L$  on  $X$ , irrespective of the choice of the ample bundle, for any pre- $\mathcal{L}$ -module such that the monodromy representation of  $E_0|(X - S)$  is irreducible, and the monodromy representation of  $\pi_a^* E_a|(N_{S_a, X} - S_a)$  is irreducible for each component  $S_a$ , the corresponding reduced module is stable. The converse is not true – a pre- $\mathcal{L}$ -module, whose reduced module is stable, need not have irreducible monodromies. The example 2.4.1 in [N] gives a logarithmic connection, whose associated pre- $\mathcal{L}$ -module in which  $s$  is identity and  $t$  is the residue, gives a stable reduced module, but the monodromies are not irreducible.

(2) If  $u = 0$ , the reduced module is stable if and only  $E_0$  and each  $E_a$  is stable.

(3) When  $X$  is a curve, a reduced module with  $u \neq 0$  is stable if and only if the logarithmic connection  $E_0$  is stable. If  $u = 0$ , each  $E_a$  must moreover have length at most one as an  $\mathcal{O}_X$ -module. Hence over curves, there is a plentiful supply of stable reduced modules.



**Lemma 4.15** *Let  $(E_0, E_1, u)$  be a reduced module and let  $(E_i)_p$  be filtrations of  $E_i$  ( $i = 0, 1$ ). Then  $s$  and  $t$  are filtered morphisms with respect to the specialized filtrations if and only if there exists some point  $P \in S$  such that the restrictions of  $s$  and  $t$  to the fibre  $E_{i,p}$  at  $P$  are filtered with respect to the restricted filtrations.*

*Proof* This follows from the fact that if a section  $\sigma$  of a vector bundle with a flat connection has a value  $\sigma(P)$  in the fibre at  $P$  of a sub flat connection, then it is a section of this subbundle: we apply this to  $s$  (resp.  $t$ ) as a section of  $\text{Hom}((E_0)_p|_S, (E_1)|_S)$  (resp.  $\text{Hom}((E_1)_p|_S, (E_0)|_S)$ ).

#### A criterion for stability

Let  $\mathbf{E} = (E_0, E_1, u = s \otimes t)$  be a reduced module. Assume that we are given filtrations  $0 = F_0(E_i) \subset F_1(E_i) \subset \dots \subset F_{\ell_i}(E_i) = E_i$  of  $E_i$  ( $i = 0, 1$ ) by vector subbundles which are  $\mathcal{L}_X[\log S]$ -submodules.

For  $j = 0, \dots, \ell_0$  let  $k(j)$  be the smallest  $k$  such that  $s(\text{sp}_S F_j(E_0)) \subset F_k(E_1)$  and let  $J(s)$  be the graph of the map  $j \rightarrow k(j)$ . A *jump point* is a point  $(j, k(j))$  on this graph such that  $k(j-1) < k(j)$ . Consider also the set  $G_s$  made by points under the graph:  $G_s = \{(j, k) \mid k \leq k(j)\}$ . For  $t$  there is an equivalent construction: we have a map  $k \rightarrow j(k)$  and a set  $G_t$  on the left of the graph  $I(t)$ :  $G_t = \{(j, k) \mid j \leq j(k)\}$ .

**Definition 4.16**  $u = s \otimes t$  is **compatible with the filtrations** if the two sets  $G_s$  and  $G_t$  intersect at most at (common) jump points (where if  $u = 0$ , take  $s = 0$  and  $t = 0$ ).

**Proposition 4.17** *Let  $\mathbf{E} = (E_0, E_1, u)$  be a semistable reduced module. The following conditions are equivalent:*

- (1)  $\mathbf{E}$  is not stable,
- (2) there exists a nontrivial special filtration  $F_j(E_i)$  ( $j = 0, \dots, \ell_i$ ) of each  $E_i$  where all inclusions are proper and  $u$  is compatible with these filtrations.

*Proof* (1)  $\Rightarrow$  (2): If  $\mathbf{E}$  is not stable, we can find two nontrivial special filtrations  $(E_0)_p$  and  $(E_1)_q$  such that  $s$  and  $t$  are filtered morphisms. Let  $p_j$  ( $j = 1, \dots, \ell_0$ ) be the set of jumping indices for  $(E_0)_p$  and  $q_k$  ( $k = 1, \dots, \ell_1$ ) for  $(E_1)_q$ . For each  $j_0$  and  $k_0$  we have  $j(k(j_0)) \leq j_0$  and  $k(j(k_0)) \leq k_0$ . We define  $F_j(E_0) = (E_0)_{p_j}$  and  $F_k(E_1) = (E_1)_{q_k}$ . We get nontrivial filtrations of  $E_0$  and  $E_1$  where all inclusions are proper. Moreover there cannot exist two distinct points  $(j_0, k(j_0))$  and  $(j(k_0), k_0)$  with  $j_0 \leq j(k_0)$  and  $k_0 \leq k(j_0)$  otherwise we would have  $j_0 \leq j(k_0) \leq j(k(j_0)) \leq j_0$  and the same for  $k_0$  so the two points would be the equal. Consequently  $u$  is compatible with these filtrations.

(2)  $\Rightarrow$  (1): We shall construct a special filtration  $((E_0)_p, (E_1)_q)$  of the reduced module from the filtrations  $F_j(E_i)$  of each  $E_i$ . Choose a polygonal line with only positive slopes, going through each jump point of  $G_s$  and for which each jump point of  $G_t$  is on or above this line (see Fig. 1). Choose increasing functions  $p(j)$  and  $q(k)$  such that  $p(j) - q(k)$  is identically 0 on this polygonal line, is

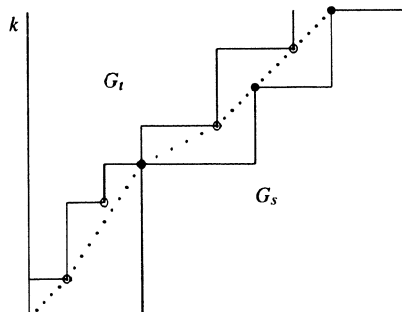


Fig. 1.  $\bullet$  = jump points of  $s$ ,  $\circ$  = jump points of  $t$

$< 0$  above it and  $> 0$  below it (for instance, on each segment  $[(j_0, k_0), (j_1, k_1)]$  of this polygonal line, parametrised by  $j = j_0 + m\varepsilon_1$ ,  $k = k_0 + m\varepsilon_2$ , put  $p(j) = p(j_0) + \varepsilon_2(j - j_0)$  and  $q(k) = q(k_0) + \varepsilon_1(k - k_0)$ , and  $p(0) = q(0) = 0$ ). For  $p(j) \leq p < p(j + 1)$  put  $(E_0)_p = F_j(E_0)$  and for  $q(k) \leq q < q(k + 1)$  put  $(E_1)_q = F_k(E_1)$ . The filtration  $((E_0)_p, (E_1)_q, u)$  is then a nontrivial special filtration of the reduced module  $\mathbf{E}$ .

**Proposition 4.18** *Semistability and stability are Zariski open conditions on the parameter scheme of any family of reduced modules.*

*Proof* As semistability is an open condition on  $\mathcal{D}_X[\log S]$ -modules, it follows it is an open condition on reduced modules. Now, for any family of semistable reduced modules parametrised by a scheme  $T$ , all possible special filtrations of the form given by 4.17 on the specializations of the family are parametrised by a scheme  $U$  which is projective over  $T$ . The image of  $U$  in  $T$  is the set of non stable points in  $T$ , hence its complement is open.

#### Points of the moduli

We are now ready to prove the following theorem.

**Theorem 4.19** *Let  $X$  be a projective variety together with an ample line bundle, and let  $S \subset X$  be a smooth divisor.*

(1) *There exists a coarse moduli scheme  $\mathcal{P}$  for semistable pre- $\mathcal{D}$ -modules on  $(X, S)$  with given Hilbert polynomials  $p_i$ . The scheme  $\mathcal{P}$  is quasiprojective, in particular, separated and of finite type over  $\mathbb{C}$ .*

(2) *The points of  $\mathcal{P}$  correspond to  $S$ -equivalence classes of semistable pre- $\mathcal{D}$ -modules.*

(3) *The  $S$ -equivalence class of a semistable reduced module  $\mathbf{E}$  equals its isomorphism class if and only if  $\mathbf{E}$  is stable.*

(4)  *$\mathcal{P}$  has an open subscheme  $\mathcal{P}^s$  whose points are the isomorphism classes of all stable reduced modules. This is a coarse moduli for (isomorphism classes of) stable reduced modules.*

*Proof* Let  $\mathcal{P} = H // \mathcal{G}$ . Then (1) follows by the construction of  $\mathcal{P}$ . To prove (2), first note that by the existence of the deformation  $\mathbf{E}_t$  (see 4.10) of any reduced module  $\mathbf{E}$  corresponding to a weighted special filtration, and by the separatedness of  $\mathcal{P}$ , the reduced module  $\mathbf{E}$  and its limit  $\mathbf{E}'$  go to the same point of  $\mathcal{P}$ . Hence an S-equivalence class goes to a common point of  $\mathcal{P}$ . For the converse, first recall that  $\mathcal{G} = \mathcal{H} \times GL(1)$ , and the quotient  $\mathcal{P}$  can be constructed in two steps:  $\mathcal{P} = R // \mathcal{H}$  where  $R = H / \mathcal{G}$ . The scheme  $R$  parametrizes a canonical family of reduced modules. Let the  $\mathcal{H}$  orbit of a point  $x$  of  $R$  corresponding to the reduced module  $\mathbf{E}$  not be closed in  $R$ . Let  $x_0$  be any of its limit points. Then there exists a 1-parameter subgroup  $\lambda$  of  $\mathcal{H}$  such that  $x_0 = \lim_{t \rightarrow 0} \lambda(t)x$ . This defines a map from the affine line  $A^1$  to  $R$ , which sends  $t \mapsto \lambda(t)x$ . Let  $\mathbf{E}_t$  be the pull back of the tautological family of reduced modules parametrized by  $R$ . Then from the description of the limits of the actions of 1-parameter subgroups on a quot scheme given in Sect. 1 of Simpson [S], it follows that  $\mathbf{E}$  has a special filtration such that the family  $\mathbf{E}_t$  is isomorphic to a deformation of the type constructed in 4.10 above. Hence the reduced modules parametrized by  $x$  and  $x_0$  are S-equivalent. This proves (2).

If the orbit of  $x$  is not closed, then it has a limit  $x_0$  outside it under a 1-parameter subgroup, which by above represents a reduced module  $\mathbf{E}'$  which is the limit of  $\mathbf{E}$  under a special filtration. As by assumption  $\mathbf{E}'$  is not isomorphic to  $\mathbf{E}$ , the special filtration must be nontrivial. Hence  $\mathbf{E}$  is not stable. Hence stable points have closed orbits in  $R$ . If  $x$  represents a stable reduced module, then  $x$  cannot be the limit point of any other orbit. For, if  $x$  is a limit point of the orbit of  $y$ , then by openness of stability (see 4.18),  $y$  should again represent a stable reduced module. But then by above, the orbit of  $y$  is closed. This proves (3).

Let  $H^s \subset H$  be the open subscheme where the corresponding pre- $\mathcal{D}$ -module is stable. By (2) and (3) above,  $H^s$  is saturated under the quotient map  $H \rightarrow \mathcal{P}$ , hence by properties of a good quotient, its image  $\mathcal{P}^s$  is open in  $\mathcal{P}$ . Moreover by (2) and (3) above,  $H^s$  is the inverse image of  $\mathcal{P}^s$ . Hence  $H^s \rightarrow \mathcal{P}^s$  is a good quotient, which again by (2) and (3) is an orbit space. Hence points of  $\mathcal{P}^s$  are exactly the isomorphism classes of stable reduced modules, which proves (4).

## 5 Perverse sheaves, Verdier objects and finite descriptions

Let  $X$  be a nonsingular projective variety and let  $S$  be a smooth divisor. The abelian category of perverse sheaves constructible with respect to the stratification  $(X - S, S)$  of  $X$  is equivalent to the category of ‘Verdier objects’ on  $(X, S)$ . Before defining this category, let us recall the notion of specialization along  $S$ .

Let  $\mathcal{E}$  be a local system (of finite dimensional vector spaces) on  $X - S$ . The *specialization*  $\text{sp}_S \mathcal{E}$  is a local system (of the same rank) on  $N_{S, X} - S$  equipped with an endomorphism  $\tau_{\mathcal{E}}$ . It is constructed using the nearby cycle functor  $\psi$  defined by Deligne applied to the morphism which describes the canonical deformation from  $X$  to the normal bundle  $N_{S, X}$ .

A local system  $\mathcal{F}$  on  $N_{S,X} - S$  equipped with an endomorphism  $\tau_{\mathcal{F}}$  is said to be **monodromic** if  $\tau_{\mathcal{F}}$  is equal to the monodromy of  $\mathcal{F}$  around  $S$ . Then  $\text{sp}_S \mathcal{E}$  is monodromic.

**Definition 5.1** A **Verdier object** on  $(X, S)$  is a tuple  $\mathbf{V} = (\mathcal{E}, \mathcal{F}, C, V)$  where

- (1)  $\mathcal{E}$  is a local system on  $X - S$ ,
- (2)  $\mathcal{F}$  is a monodromic local system on  $N_{S,X} - S$ ,
- (3)  $C : \text{sp}_S \mathcal{E} \rightarrow \mathcal{F}$  and  $V : \mathcal{F} \rightarrow \text{sp}_S \mathcal{E}$  are morphisms of (monodromic) local systems on  $N_{S,X} - S$  satisfying
- (4)  $CV = \tau_{\mathcal{F}} - \text{id}$  and  $VC = \tau_{\mathcal{E}} - \text{id}$ .

*Remark 5.2* The morphisms between Verdier objects on  $(X, S)$  are defined in an obvious way, and the category of Verdier objects is an abelian category in which each object has finite length. Hence the following definition makes sense.

**Definition 5.3** We say that two Verdier objects are **S-equivalent** if they admit Jordan-Hölder filtrations such that the corresponding graded objects are isomorphic.

*Remark 5.4* Let  $B$  be a tubular neighbourhood of  $S$  in  $X$ , diffeomorphic to a tubular neighbourhood of  $S$  in  $N_{S,X}$ . Put  $B^* = B - S$ . The specialized local system  $\text{sp}_S \mathcal{E}$  can be realized as the restriction of  $\mathcal{E}$  to  $B^*$ , its monodromy  $\tau_{\mathcal{E}}$  at some point  $x \in B^*$  being the monodromy along the circle normal to  $S$  going through  $x$ . Hence a Verdier object can also be described as a tuple  $\mathbf{V}$  where  $\mathcal{F}$  is a local system on  $B^*$  and  $C, V$  are morphisms between  $\mathcal{E}|_{B^*}$  and  $\mathcal{F}$  subject to the same condition (4).

The notion of a family of perverse sheaves is not straightforward. We can however define the notion of a family of Verdier objects. Let us define first a family of local systems on  $X - S$  (or on  $N_{S,X} - S$ ) parametrized by a scheme  $T$ . This is a locally free  $p^{-1}\mathcal{O}_T$ -module of finite rank, where  $p$  denotes the projection  $X - S \times T \rightarrow T$ . Morphisms between such objects are  $p^{-1}\mathcal{O}_T$ -linear. The notion of a family of Verdier objects is then straightforward.

In order to make a moduli space for Verdier objects, we shall introduce the category of ‘finite descriptions’ on  $(X, S)$ . Let us fix the following data (D):

- (D1) finitely generated groups  $G$  and  $G_a$  for each component  $S_a$  of  $S$ ,
- (D2) for each  $a$  an element  $\tau_a$  which lies in the center of  $G_a$  and a group homomorphism  $\phi_a : G_a \rightarrow G$ .

**Definition 5.5** A **finite description D** (with respect to the data (D)) is a tuple  $(E, \rho, F_a, \rho_a, C_a, V_a)$  where

- (1)  $\rho : G \rightarrow GL(E)$  is a finite dimensional complex representation of the group  $G$ ; for each  $a$  we will regard  $E$  as a representation of  $G_a$  via the homomorphism  $\phi_a : G_a \rightarrow G$ ;
- (2) for each  $a$ ,  $\rho_a : G_a \rightarrow GL(F_a)$  is a finite dimensional complex representation of the group  $G$ ;
- (3) for each  $a$ ,  $C_a : E \rightarrow F_a$  and  $V_a : F_a \rightarrow E$  are  $G_a$ -equivariant linear maps such that  $V_a C_a = \rho(\tau_a) - \text{id}$  in  $GL(E)$  and  $C_a V_a = \rho_a(\tau_a) - \text{id}$  in  $GL(F_a)$ .

A morphism between two finite descriptions has an obvious definition.

*Remark 5.6* Let  $P_0 \in X - S$  and let  $P_a$  be a point in the component  $B^*_{a}$  of  $B^*$ . Choose paths  $\sigma_a : [0, 1] \rightarrow X - S$  with  $\sigma_a(0) = P_0$  and  $\sigma_a(1) = P_a$ . Let  $G$  be the fundamental group  $\pi_1(X - S, P_0)$ , and let  $G_a = \pi_1(B^*_{a}, P_a)$ . Let  $\tau_a \in G_a$  be the positive loop based at  $P_a$  in the fiber of  $B^*_{a} \rightarrow S_a$ . Finally, let  $\phi_a : G_a \rightarrow G$  be induced by the inclusion  $B^*_{a} \hookrightarrow X - S$  by using the path  $\sigma_a$  to change base points. Then, under the equivalence between representations of fundamental group and local system, the category of finite description with respect to the previous data is equivalent to the category of Verdier objects on  $(X, S)$ .

*Remark 5.7* The category of finite descriptions is an abelian category in which each object has finite length. Therefore the notion of S-equivalence as in definition 5.3 above makes sense for finite descriptions.

**Definition 5.8** A family of finite descriptions parametrized by a scheme  $T$  is a tuple  $(E_T, \rho_T, F_{T,a}, \rho_{T,a}, C_{T,a}, V_{T,a})$  where  $E_T$  and the  $F_{T,a}$  are locally free sheaves on  $T$ ,  $\rho$  and  $\rho_{T,a}$  are families of representations into these, and the  $C_{T,a}$  and  $V_{T,a}$  are  $\mathcal{C}_T$ -homomorphisms of sheaves satisfying the analogues of condition 5.5.3 over  $T$ . The pullback of a family under a morphism  $T' \rightarrow T$  is defined in an obvious way, giving a fibered category. Let  $PS$  denote the corresponding groupoid.

*Remark 5.9* It can be checked (we omit the details) that the groupoid  $PS$  is an Artin algebraic stack.

## 6 Moduli for perverse sheaves

Let us fix data (D) as above.

**Theorem 6.1** *There exists an affine scheme of finite type over  $\mathbb{C}$ , which is a coarse moduli scheme for finite descriptions  $\mathbf{D} = (E, \rho, F_a, \rho_a, C_a, V_a)$  relative to (D) with fixed numerical data  $n = \dim E$  and  $n_a = \dim F_a$ . The closed points of this moduli scheme are the S-equivalence classes of finite descriptions with given numerical data  $(n, n_a)$ .*

Using remark 5.6 we get

**Corollary 6.2** *There exists an affine scheme of finite type over  $\mathbb{C}$ , which is a coarse moduli scheme for Verdier objects  $\mathbf{V} = (\mathcal{E}, \mathcal{F}, C, V)$  (or perverse sheaves on  $(X, S)$ ) with fixed numerical data  $n = \text{rank } \mathcal{E}$  and  $n_a = \text{rank } \mathcal{F}|_{B^*_a}$ . The closed points of this moduli scheme are the S-equivalence classes of Verdier objects with given numerical data  $(n, n_a)$ .*

The above corollary and its proof does not need  $X$  to be a complex projective variety, and the algebraic structure of  $X$  does not matter. All that is needed is that the fundamental group of  $X - S$  and that of each  $S_a$  is finitely generated.

The rest of this section contains the proof of the above theorem.

**Proposition 6.3** (1) Let  $\mathbf{D}$  be a finite description, and let  $\text{gr}(\mathbf{D})$  be its semisimplification. Then there exists a family  $\mathbf{D}_T$  of finite descriptions parametrized by the affine line  $T = A^1$  such that the specialization  $\mathbf{D}_0$  at the origin  $0 \in T$  is isomorphic to  $\text{gr}(\mathbf{D})$ , while  $\mathbf{D}_t$  is isomorphic to  $\mathbf{D}$  at any  $t \neq 0$ .

(2) In any family of finite descriptions parametrized by a scheme  $T$ , each S-equivalence class (Jordan-Hölder class) is Zariski closed in  $T$ .

*Proof* The statement (1) has a proof by standard arguments which we omit. To prove (2), first note that if  $\mathbf{D}_T$  is any family and  $\mathbf{D}'$  a simple finite description, then the condition that  $\mathbf{D}' \times \{t\}$  is a quotient of  $\mathbf{D}_t$  defines a closed subscheme of  $T$ . From this, (2) follows easily.

*Construction of Moduli* Let  $E$  and  $F_a$  be vector spaces with  $\dim(E) = n$  and  $\dim(F_a) = n_a$ . Let  $\mathcal{R}$  be the affine scheme of all representations  $\rho$  of  $G$  in  $E$ , made as follows. Let  $h_1, \dots, h_r$  be generators of  $G$ . Then  $\mathcal{R}$  is the closed subscheme of the product  $GL(E)^r$  defined by the relations between the generators. Similarly, choose generators for each  $G_a$ , and let  $\mathcal{R}_a$  be the corresponding affine scheme of all representations  $\rho_a$  of  $G_a$  in  $F_a$ .

Let

$$A \subset \mathcal{R} \times \prod_a (\mathcal{R}_a \times \text{Hom}(E, F_a) \times \text{Hom}(F_a, E))$$

be the closed subscheme defined by condition 5.5.3 above. Its closed points are tuples  $(\rho, \rho_a, C_a, V_a)$  where the linear maps  $C_a : E \rightarrow F_a$  and  $V_a : F_a \rightarrow E$  are  $G_a$ -equivariant under the representations  $\rho\phi_a : G_a \rightarrow GL(E)$  and  $\rho_a : G_a \rightarrow GL(F_a)$ , and satisfy  $V_a C_a = \rho(\tau_a) - 1$  in  $GL(E)$ , and  $C_a V_a = \rho_a(\tau_a) - 1$  in  $GL(F_a)$  for each  $a$ .

The product group  $\mathcal{S} = GL(E) \times (\prod_a GL(F_a))$  acts on the affine scheme  $A$  by the formula

$$(\rho, \rho_a, C_a, V_a) \cdot (g, g_a) = (g^{-1}\rho g, g_a^{-1}\rho_a g_a, g_a^{-1}C_a g, g^{-1}V_a g_a).$$

The orbits under this action are exactly the isomorphism classes of finite descriptions. The moduli of finite descriptions is the good quotient  $\mathcal{F} = A // \mathcal{S}$ , which exists as  $A$  is affine and  $\mathcal{S}$  is reductive. It is an affine scheme of finite type over  $\mathbb{C}$ . It follows from 6.3.1 and 6.3.2 and properties of a good quotient that the Zariski closures of two orbits intersect if and only if the two finite descriptions are S-equivalent. Hence closed points of  $\mathcal{F}$  are S-equivalence classes (Jordan-Hölder classes) of finite descriptions.

## 7 Riemann-Hilbert morphism

To any Malgrange object  $\mathbf{M}$ , there is an obvious associated Verdier object  $\mathbf{V}(\mathbf{M})$  obtained by applying the de Rham functor to each component of  $\mathbf{M}$ . This defines a functor, which is in fact an equivalence of categories from Malgrange objects to Verdier objects. We have already defined a functor  $\eta$  from pre- $\mathcal{S}$ -modules with good residual eigenvalues to Malgrange objects. Composing, we get an exact

functor from pre- $\mathcal{D}$ -modules with good residual eigenvalues to Verdier objects. Choosing base points in  $X$  and paths as in remark 5.6 we get an exact functor  $\mathcal{R}\mathcal{H}$  from pre- $\mathcal{D}$ -modules to finite descriptions. This construction works equally well for families of pre- $\mathcal{D}$ -modules, giving us a holomorphic family  $\mathcal{R}\mathcal{H}(\mathbf{E}_T)$  of Verdier objects (or finite descriptions) starting from a holomorphic family  $\mathbf{E}_T$  of pre- $\mathcal{D}$ -modules with good residual eigenvalues.

*Remark 7.1* Even if  $\mathbf{E}_T$  is an algebraic family of pre- $\mathcal{D}$ -modules with good residual eigenvalues, the associated family  $\mathcal{R}\mathcal{H}(\mathbf{E}_T)$  of Verdier objects may not be algebraic.

*Remark 7.2* If a semistable pre- $\mathcal{D}$ -module has good residual eigenvalues, then any other semistable pre- $\mathcal{D}$ -module in its S-equivalence class has (the same) good residual eigenvalues. Hence the analytic open subset  $T_g$  of the parameter space  $T$  of any analytic family of semistable pre- $\mathcal{D}$ -modules defined by the condition that residual eigenvalues are good is saturated under S-equivalence.

**Lemma 7.3** *If two semistable pre- $\mathcal{D}$ -modules with good residual eigenvalues are S-equivalent (in the sense of definition 4.13 above), then the associated finite descriptions are S-equivalent (that is, Jordan-Hölder equivalent).*

*Proof* Let  $\mathbf{E} = (E_0, E_1, s, t)$  be a pre- $\mathcal{D}$ -module with good residual eigenvalues (that is, the logarithmic connection  $E_0$  has good residual eigenvalues on each component of  $S$ ) such that  $s \otimes t = 0$ . Then one can easily construct a family of pre- $\mathcal{D}$ -modules parametrized by the affine line  $A^1$  which is the constant family  $\mathbf{E}$  outside some point  $P \in A^1$ , and specializes at  $P$  to  $\mathbf{E}' = (E_0, E_1, 0, 0)$ . Let  $\phi : A^1 \rightarrow \mathcal{F}$  be the resulting morphism to the moduli  $\mathcal{F}$  of finite descriptions. By construction,  $\phi$  is constant on  $A^1 - P$ , and so as  $\mathcal{F}$  is separated,  $\phi$  is constant. As the points of  $\mathcal{F}$  are the S-equivalence classes of finite descriptions, it follows that the finite descriptions corresponding to  $\mathbf{E}$  and  $\mathbf{E}'$  are S-equivalent. Hence the S-equivalence class of the finite description associated to a pre- $\mathcal{D}$ -module depends only on the reduced module made from the pre- $\mathcal{D}$ -module. Now we must show that any two S-equivalent (in the sense of 4.13) reduced semistable modules have associated finite descriptions which are again S-equivalent (Jordan-Hölder equivalent). This follows from the deformation given in 4.10 by using the separatedness of  $\mathcal{F}$  as above.

Now consider the moduli  $\mathcal{P} = H // \mathcal{G}$  of semistable pre- $\mathcal{D}$ -modules. Let  $H_g$  be the analytic open subspace of  $H$  where the family parametrized by  $H$  has good residual eigenvalues. By the above remark,  $H_g$  is saturated under  $H \rightarrow \mathcal{P}$ . Hence its image  $\mathcal{P}_g \subset \mathcal{P}$  is analytic open. Let  $\phi : H_g \rightarrow \mathcal{F}$  be the classifying map to the moduli  $\mathcal{F}$  of finite descriptions for the tautological family of pre- $\mathcal{D}$ -modules parametrized by  $H$ , which is defined because of the above lemma. By the analytic universal property of GIT quotients (see Proposition 5.5 of Simpson [S] and the remark below),  $\phi$  factors through an analytic map  $\mathcal{R}\mathcal{H} : \mathcal{P}_g \rightarrow \mathcal{F}$ , which we call as the *Riemann-Hilbert morphism*.

*Remark 7.4* In order to apply Proposition 5.5 of [S], note that a  $\mathcal{S}$ -linear ample line bundle can be given on  $H$  such that all points of  $H$  are semistable. Moreover, though the proposition 5.5 in [S] is stated for semisimple groups, its proof works for reductive groups.

*Remark 7.5* The Riemann-Hilbert morphism can also be thought of as a morphism from the analytic stack of pre- $\mathcal{D}$ -modules with good residual eigenvalues to the analytic stack of perverse sheaves.

## 8 Some properties of the Riemann-Hilbert morphism

In this section we prove some basic properties of the morphism  $\mathcal{RH}$ , which can be interpreted either at stack or at moduli level.

**Lemma 8.1 (Relative Deligne construction)** (1) *Let  $T$  be the spectrum of an Artin local algebra of finite type over  $\mathbb{C}$ , and let  $\rho_T$  be a family of representations of  $G$  (the fundamental group of  $X - S$  at base point  $P_0$ ) parametrized by  $T$ . Let  $E$  be a logarithmic connection with eigenvalue not differing by nonzero integers, such that the monodromy of  $E$  equals  $\rho$ , the specialization of  $\rho_T$ . Then there exists a family  $E_T$  of logarithmic connections parametrized by  $T$  such that  $E_0 = E$  and  $E_T$  has monodromy  $\rho_T$ .*

(2) *A similar statement is true for analytic germs of  $G$ -representations.*

*Proof* For each  $a$ , choose a fundamental domain  $\Omega_a$  for the exponential map ( $z \mapsto \exp(2\pi\sqrt{-1}z)$ ) such that the eigenvalues of the residue  $R_a(E)$  of  $E$  along  $S_a$  are in the interior of the set  $\Omega_a$ . As the differential of the exponential map  $M(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$  is an isomorphism at all those points of  $M(n, \mathbb{C})$  where the eigenvalues do not differ by nonzero integers, using the fundamental domains  $\Omega_a$  we can carry out the Deligne construction locally to define a family  $E_T$  of logarithmic connections on  $(X, S)$  with  $E_0 = E$ , which has the given family of monodromies.

Note that for the above to work, we needed the inverse function theorem, which is valid for Artin local algebras.

*Remark 8.2* If in the above, the family  $\rho_T$  of monodromies is a constant family (that is, pulled back under  $T \rightarrow \text{Spec}(\mathbb{C})$ ), then  $E_T$  is also a constant family as follows from Proposition 5.3 of [N].

**Proposition 8.3 ('Injectivity' of  $\mathcal{RH}$ )** *Let  $\mathbf{E} = (E, F, t, s)$  and  $\mathbf{E}' = (E', F', t', s')$  be pre- $\mathcal{D}$ -modules having good residual eigenvalues, such that for each  $a$ , the eigenvalues of the residues of  $E$  and  $E'$  over  $S_a$  belong a common fundamental domain  $\Omega_a$  for the exponential map  $\exp : \mathbb{C} \rightarrow \mathbb{C}^* : z \mapsto \exp(2\pi\sqrt{-1}z)$ . Then  $\mathbf{E}$  and  $\mathbf{E}'$  are isomorphic if and only if the finite descriptions  $\mathcal{RH}(\mathbf{E})$  and  $\mathcal{RH}(\mathbf{E}')$  are isomorphic.*

*Proof* It is enough to prove that if the Malgrange objects  $\mathbf{M}$  and  $\mathbf{M}'$  are isomorphic, then so are the pre- $\mathcal{D}$ -modules  $\mathbf{E}$  and  $\mathbf{E}'$ . First use the fact that, in a given



meromorphic connection  $M$  on  $X - S$  (or on  $N_{S,X} - S$ ), there exists one and only one logarithmic connection having its residue along  $S_a$  in  $\Omega_a$  for each  $a$ , to conclude that  $E$  and  $E'$  (resp.  $F$  and  $F'$ ) are isomorphic logarithmic modules. To obtain the identification between  $s$  and  $s'$  (resp.  $t$  and  $t'$ ), use the fact that these maps are determined by their value at a point in each connected component  $N_{S_a,X} - S_a$  of  $N_{S,X} - S$  and this value is determined by the corresponding  $C_a$  or  $C'_a$  (resp.  $V_a$  or  $V'_a$ ).

**Proposition 8.4 (Surjectivity of  $\mathcal{R}\mathcal{H}$ )** *Let  $\mathbf{D}$  be a finite description, and let  $\sigma_a : \mathbb{C}^* \rightarrow \mathbb{C}$  be set theoretic sections of  $z \mapsto \exp(2\pi\sqrt{-1}z)$ . Then there exists a pre- $\mathcal{L}$ -module  $\mathbf{E}$  whose eigenvalues of residue over  $S_a$  are in image( $\sigma_a$ ), for which  $\mathcal{R}\mathcal{H}(\mathbf{E})$  is isomorphic to  $\mathbf{D}$ .*

*Proof* This follows from Proposition 3.7.

**Remark 8.5** The Propositions 8.3 and 8.4 together say that the set theoretic fiber of  $\mathcal{R}\mathcal{H}$  over a given finite description is in bijection with the choices of ‘good’ logarithms for the local monodromies of the finite description (here ‘good’ means eigenvalues do not differ by nonzero integers).

**Proposition 8.6 (Tangent level injectivity for  $\mathcal{R}\mathcal{H}$ )** *Let  $(E, F, t, s)_T$  be a family of pre- $\mathcal{L}$ -modules having good residual eigenvalues parametrized by the spectrum  $T$  of an Artinian local algebra. Let the family  $\mathcal{R}\mathcal{H}(E, F, t, s)_T$  of finite descriptions parametrized by  $T$  be constant (pulled back under  $T \rightarrow \text{Spec } \mathbb{C}$ ). Then the family  $(E, F, t, s)_T$  is also constant.*

*Proof* This is just the rigidity result of Proposition 3.9.

**Proposition 8.7 (Infinitesimal surjectivity for  $\mathcal{R}\mathcal{H}$ )** *Let  $T$  be the spectrum of an Artin local algebra of finite type over  $\mathbb{C}$ , and let  $\mathbf{D}$  be a family of finite descriptions parametrized by  $T$ . Let  $\mathbf{E}$  be a pre- $\mathcal{L}$ -module having good residual eigenvalues such that  $\mathcal{R}\mathcal{H}(\mathbf{E}) = \mathbf{D}_\xi$ , the restriction of  $\mathbf{D}$  over the closed point  $\xi$  of  $T$ . Then there exists a family  $\mathbf{E}'_T$  of pre- $\mathcal{L}$ -modules having good residual eigenvalues with  $\mathbf{E}'_\xi = \mathbf{E}$  and  $\mathcal{R}\mathcal{H}(\mathbf{E}'_T) = \mathbf{D}_T$ .*

*Proof* This follows from Lemma 8.1 and the proof of Proposition 3.7 which works for families over Artin local algebras.

**Theorem 8.8** *The analytic open substack of the stack (or analytic open subset of the moduli) of pre- $\mathcal{L}$ -modules on  $(X, S)$ , where  $\mathbf{E}$  has good residual eigenvalues, is an analytic spread over the stack (or moduli) of perverse sheaves on  $(X, S)$  under the Riemann-Hilbert morphism.*

*Proof* This follows from Propositions 8.4, 8.6 and 8.7 above.

Note that we have not defined  $\mathcal{R}\mathcal{H}$  on the closed analytic subset  $T_o$  of the parameter space of a family where  $\mathbf{E}$  does not have good residual eigenvalues. Note that  $T_o$  is defined by a ‘codimension one’ analytic condition, that is, if  $T$  is nonsingular, and if  $T_o$  is a nonempty and proper subset of  $T$ , then  $T_o$

has codimension 1 in  $T$ . However, it follows from Proposition 8.9 below that the morphism  $\mathcal{R}\mathcal{H}$  on  $T - T_0$  can be extended to an open subset of  $T$  of complementary codimension at least two. However, on the extra points to which it gets extended, it may not represent the de Rham functor.

**Proposition 8.9 (Removable singularities for  $\mathcal{R}\mathcal{H}$ )** *Let  $T$  be an open disk in  $\mathbb{C}$  centered at 0. Let  $\mathbf{E}_T = (E, F, t, s)_T$  be a holomorphic family of pre- $\mathcal{L}$ -modules parametrized by  $T$ . Let the restriction  $E_z$  have good residual eigenvalues for all  $z \in T - \{0\}$ . Then there exists a holomorphic family  $\mathbf{D}_U$  of finite descriptions parametrized by a neighbourhood  $U$  of  $0 \in T$  such that on  $U - \{0\}$ , the families  $\mathcal{R}\mathcal{H}(\mathbf{E}_U|U - \{0\})$  and  $\mathbf{D}_{U - \{0\}}$  are isomorphic.*

If at  $z = 0$  the logarithmic connection  $E$  does not have good residual eigenvalues, it is possible to change it to obtain a new logarithmic connection having good residual eigenvalues. This is done by the classical ‘shearing transformation’ that we adapt below (**inferior and superior modifications** for pre- $\mathcal{L}$ -modules). This can be done in family and has no effect on the Malgrange object at least locally.

**Definition 8.10** *If  $E$  is a vector bundle on  $X$ , and  $V$  a subbundle of the restriction  $E|_S$ , then the **inferior modification**  ${}_{\vee}E$  is the sheaf of all sections of  $E$  which lie in  $V$  at points of  $S$ . This is a locally free subsheaf of  $E$  (but not generally a subbundle). The **superior modification**  ${}^{\vee}E$  is the vector bundle  $\mathcal{C}_X(S) \otimes {}_{\vee}E$ .*

**Remark 8.11** If  $E|_S = V \oplus V'$ , then we have a canonical isomorphism

$${}_{\vee}E|_S \rightarrow V \oplus (\mathcal{C}_S^* \otimes V')$$

and hence also a canonical isomorphism

$${}^{\vee}E|_S \rightarrow (\mathcal{C}_S \otimes V) \oplus V'$$

**Remark 8.12** If  $(E, \nabla)$  is a logarithmic connection on  $(X, S)$  and  $V$  is invariant under the residue, then it can be seen that  ${}_{\vee}E$  is invariant under  $\nabla$ , so is again a logarithmic connection. We call it the inferior modification of the logarithmic connection  $E$  along the residue invariant subbundle  $V \subset E|_S$ . It has the effect that the residual eigenvalues along  $V$  get increased by 1 when going from  $E$  to  ${}_{\vee}E$ . As  $\mathcal{C}_X(S)$  is canonically a logarithmic connection, the superior modification  ${}^{\vee}E$  is also a logarithmic connection, with the residual eigenvalues along  $V$  getting decreased by 1.

Let  $(E, F, t, s)$  be pre- $\mathcal{L}$ -module on  $(X, S)$  such that  $E$  has good residual eigenvalues. Let us for simplicity of writing assume that  $S$  is connected. Let  $E|_S = \bigoplus_{\alpha} E^{\alpha}$  and  $F = \bigoplus_{\alpha} F^{\alpha}$  be the respective direct sum decompositions into generalized eigen subbundles for the action of  $\theta$ . Then (see also Remark 3.5) as  $\theta$  commutes with  $s$  and  $t$ , it follows that  $t(E^{\alpha}) \subset F^{\alpha}$  and  $s(F^{\alpha}) \subset E^{\alpha}$ . Moreover, when  $\alpha \neq 0$ , the maps  $s$  and  $t$  are isomorphisms between  $E^{\alpha}$  and  $F^{\alpha}$ .

Now let  $\alpha \neq 0$ . Let  $V = E^\alpha$  and  $V' = \bigoplus_{\beta \neq \alpha} E^\beta$ . Let  $F'' = \bigoplus_{\beta \neq \alpha} F^\beta$ . Let  $F' = F^\alpha \oplus \int_S^* \otimes F''$ . Let  $E' = \nu E$ . Then using 8.11 and the above, we get maps  $t' : E'|S \rightarrow F'$  and  $s' : F' \rightarrow E'|S$  such that  $(E', F', s', t')$  is a pre- $\mathcal{L}$ -module.

**Definition 8.13** We call the pre- $\mathcal{L}$ -module  $(E', F', s', t')$  constructed above as the inferior modification of  $(E, F, s, t)$  along the generalized eigenvalue  $\alpha \neq 0$ .

Similarly, we can define the superior modification along a generalized eigenvalue  $\alpha \neq 0$  by tensoring with  $\mathcal{C}_X(S)$ .

*Remark 8.14* The construction of inferior or superior modification of pre- $\mathcal{L}$ -modules can be carried out over a parameter space  $T$  (that is, for families) provided the subbundles  $V$  and  $V'$  form vector subbundles over the parameter space  $T$  (their ranks are constant).

*Proof of 8.9* If the restriction  $E = E_{T|z=0}$  has good residual eigenvalues, then  $\mathcal{A}\mathcal{H}\mathcal{E}_T$  has the desired property. So suppose  $E$  does not have good residual eigenvalues.

We first assume for simplicity of writing that  $E$  fails to have good residual eigenvalues because its residue  $R_a$  on  $S_a$  has exactly one pair  $(\alpha, \alpha - 1)$  of distinct eigenvalues which differ by a positive integer, with  $\alpha - 1 \neq 0$ . Let  $f_T$  be the characteristic polynomial of  $R_{a,T}$ . Then  $f_0$  has a factorization  $f_0 = gh$  such that the polynomials  $g$  and  $h$  are coprime,  $g(\alpha) = 0$  and  $h(\alpha - 1) = 0$ . On a neighbourhood  $U$  of 0 in  $T$  we get a unique factorization  $f_T|U = g_U h_U$  where  $g_U$  specializes to  $g$  and  $h_U$  specializes to  $h$  at 0. By taking  $U$  small enough, we may assume that  $g_U$  and  $h_U$  have coprime specializations at all points of  $U$ . Let  $V_U$  be the kernel of the endomorphism  $g_U(R_{a,U})$  of the bundle  $E_{a,U}$ . If  $U$  is small enough then  $F_U$  is a subbundle. Now take the inferior modification  $\mathbf{E}' = (\nu E_U, F'_U, t'_U, s'_U)$  of the family  $(E, F, t, s)_U$  as given by construction 8.13. Then  $\nu E_U$  is a family of logarithmic connections having good residual eigenvalues, so by definition  $\mathbf{E}'$  has good residual eigenvalues.

If  $(0, 1)$  are the eigenvalues, then use superior modification along the eigenvalue 1.

If  $R_a$  has eigenvalues  $(\alpha, \alpha - k)$  for some integer  $k \geq 1$ , then repeat the above inferior (or superior) modification  $k$  times (whether to choose an inferior or superior modification is governed by the following restriction : the multiplicity of the generalized eigenvalue 0 should not decrease at any step). By construction, we arrive at the desired family  $(E', F', s', t')$ .

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