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Vanishing Cycles and Hermitian Duality[1]

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We show the compatibility between the moderate nearby or vanishing cycle functor for regular holonomic $\mathcal{D}$-modules, as defined by Beilinson, Kashiwara, and Malgrange, and the Hermitian duality functor, as defined by Kashiwara.

INTRODUCTION

The Hermitian dual of a $\mathcal{D}$-module was introduced by M. Kashiwara in [9], who showed that the Hermitian dual of a regular holonomic $\mathcal{D}$-module is also regular holonomic (hence, coherent). In this paper, we show a compatibility result between this functor and the nearby or vanishing cycle functor relative to a holomorphic function for such modules. The latter can be defined using the V-filtration (introduced by Beilinson, Kashiwara, and Malgrange).

Moreover, we make the link with asymptotic expansions of integrals along fibers of the function. This gives a generalization of the previous work of D. Barlet on Hermitian duality for the local Gauss–Manin system of an analytic function. In particular, this gives a simpler approach to the "tangling phenomenon" described by D. Barlet in [3].

1. HERMITIAN DUALITY

1.a. Notation. Let $(X,\mathcal{O}_X)$ be a complex analytic manifold of dimension n, $(X_{\mathbb{R}},\mathcal{A}_{X_{\mathbb{R}}})$ be the underlying real analytic manifold, and $(\overline{X},\mathcal{O}_{\overline{X}}=\overline{\mathcal{O}}_X)$ be the complex conjugate manifold. Denote by $\mathcal{D}_X$ (respectively, $\mathcal{D}_{\overline{X}}$) the sheaf of holomorphic linear differential operators on X (respectively, $\overline{X}$).

Denote by $\overline{}: f\mapsto \overline{f}$ the $\mathbb{R}$-isomorphism $\mathcal{O}_X\to\mathcal{O}_{\overline{X}}$ and $\mathcal{D}_X\to\mathcal{D}_{\overline{X}}$. It induces a trivial conjugation functor, sending $\mathcal{D}_X$-modules to $\mathcal{D}_{\overline{X}}$-modules; if $\mathcal{M}$ is a $\mathcal{D}_X$-module, we denote by $\overline{\mathcal{M}}$ the sheaf of $\mathbb{R}$-vector spaces $\mathcal{M}$ equipped with the action of $\mathcal{D}_{\overline{X}}$ defined as follows: denote by $\overline{m}$ the local section m of $\mathcal{M}$ viewed as a local section of $\overline{M}$; then, $\overline{P}\cdot\overline{m}=Pm$.

Let $\mathfrak{Db}_{X_{\mathbb{R}}}$ (also denoted by $\mathfrak{Db}_X$ for short) be the sheaf of distributions on $X_{\mathbb{R}}$. It acts on the sheaf of C^∞-forms φ with compact support of maximal degree, which is a right $\mathcal{D}_X$- and $\mathcal{D}_{\overline{X}}$-module. Then, $\mathfrak{Db}_X$ is a left $\mathcal{D}_X$- and $\mathcal{D}_{\overline{X}}$-module by the formula $(P\overline{Q}\mu)(\varphi)=\mu(\varphi\cdot P\overline{Q})$. The sheaf $\mathfrak{C}_{X_{\mathbb{R}}}=\mathfrak{Db}_X^{(n,n)}$ of currents of maximal degree is a right $\mathcal{D}_X$- and $\mathcal{D}_{\overline{X}}$-module obtained from $\mathfrak{Db}_X$ by "going from left to right."

It will be convenient in the following to denote by $\mathcal{O}_{X,\overline{X}}$ (respectively, $\mathcal{D}_{X,\overline{X}}$) the sheaf $\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}$ (respectively, $\mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_{\overline{X}}$) and to view $\mathfrak{Db}_X$ (respectively, $\mathfrak{Db}_X^{(n,n)}$) as a left (respectively, right) $\mathcal{D}_{X,\overline{X}}$-module.

Let Z be a reduced divisor in X and $\mathcal{O}_X[*Z]$ be the sheaf of meromorphic functions on X with poles along Z. There is an exact sequence of left $\mathcal{D}_{X,\overline{X}}$-modules

$$0\longrightarrow\mathfrak{Db}_{X,Z}\longrightarrow\mathfrak{Db}_X\longrightarrow\mathfrak{Db}_X^{\mathrm{mod}\,Z}\longrightarrow 0,$$

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where $\mathfrak{Db}_{X,Z}$ denotes the sheaf of distributions supported on $Z_{\mathbb{R}}$ and (see, e.g., [10, Ch. VII])

$$\mathfrak{Db}_X^{\mathrm{mod}\, Z} = \mathcal{O}_X[*Z] \underset{\mathcal{O}_X}{\otimes} \mathfrak{Db}_X = \mathrm{image}[\mathfrak{Db}_X \to j_*\mathfrak{Db}_{X-Z}]$$

denotes the subsheaf of $j_*\mathfrak{Db}_{X-Z}$ (where $j\colon X-Z \hookrightarrow X$ denotes the open inclusion) of distributions on $X-Z$ with *moderate growth* along Z.

1.b. The Hermitian duality functor [9]**.** Denote by C_X the Hermitian duality functor[3]. Recall that C_X is a contravariant functor from the derived category $D^-(\mathcal{D}_X)$ to the category $D^+(\mathcal{D}_{\overline{X}})$ defined as

$$C_X(\mathcal{M}^\bullet) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathfrak{Db}_X).$$

It can be restricted as a functor from the full subcategory $D^b_{hr}(\mathcal{D}_X)$ of bounded complexes with regular holonomic cohomology to $D^b_{hr}(\mathcal{D}_{\overline{X}})$ and is equal to the functor $\mathcal{H}om_{\mathcal{D}_X}(\bullet, \mathfrak{Db}_X)$ on the category of regular holonomic $\mathcal{D}_X$-modules (see [9], see also [5, Ch. VII]), defining there an anti-equivalence of categories between $\mathrm{Mod}_{hr}(\mathcal{D}_X)$ and $\mathrm{Mod}_{hr}(\mathcal{D}_{\overline{X}})$ and between $D^b_{hr}(\mathcal{D}_X)$ and $D^b_{hr}(\mathcal{D}_{\overline{X}})$, $C_{\overline{X}}$ being a quasi-inverse functor. On $D^b_{hr}(\mathcal{D}_X)$, we have

$$\mathcal{H}^k C_X \mathcal{M}^\bullet = C_X \mathcal{H}^k \mathcal{M}^\bullet.$$

Finally, recall (see [9]) that the *conjugate* of a regular holonomic right $\mathcal{D}_X$-module $\mathcal{M}$ is the right $\mathcal{D}_X$-module that is defined as

$$\mathcal{M}^c = \mathcal{T}or_n^{\mathcal{D}_{\overline{X}}}(\overline{\mathcal{M}}, \mathfrak{Db}_X^{(n,0)})$$

and satisfies $\mathrm{DR}\,\mathcal{M}^c = \overline{\mathrm{DR}\,\mathcal{M}}$. The conjugate of a left module is then obtained in the usual way.

For Z as above, we will denote by $C_X^{\mathrm{mod}\, Z}$ the functor defined as

$$C_X^{\mathrm{mod}\, Z}(\mathcal{M}) = C_X(\mathcal{M})[*\overline{Z}] = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X^{\mathrm{mod}\, Z}).$$

We will call $\overline{C_X(\mathcal{M})}$ the *Hermitian dual* of $\mathcal{M}$.

We can also define the Hermitian dual of a right regular holonomic $\mathcal{D}$-module by using the sheaf $\mathfrak{Db}_X^{(n,n)}$ of currents instead of the sheaf $\mathfrak{Db}_X$ of distributions.

Remark 1.1 (extension to the holonomic case)**.** Kashiwara conjectured (see [9, Remark 3.5]) that the previous results remain true for holonomic modules. This is proved in [13] when the support of $\mathcal{M}$ has dimension 1 and, in some cases, when it has dimension 2. If this conjecture is true, Theorem 3.2 also applies to holonomic modules. It would then be interesting to extend Theorem 2.1 below to the nonregular holonomic case in order to get a holonomic analogue of Theorem 4.13.

1.c. Sesquilinear forms on $\mathcal{D}_X$-modules. Let $\mathcal{M}'$ and $\mathcal{M}''$ be two left $\mathcal{D}_X$-modules. A sesquilinear form will be a $\mathcal{D}_{X,\overline{X}}$-linear morphism

$$S\colon \mathcal{M}' \underset{\mathbb{C}}{\otimes} \overline{\mathcal{M}''} \longrightarrow \mathfrak{Db}_X.$$

The datum of S is equivalent to the datum of a $\mathcal{D}_{\overline{X}}$-linear morphism

$$L_S\colon \overline{\mathcal{M}''} \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}', \mathfrak{Db}_X).$$

We say that S is nondegenerate if this morphism is an isomorphism.

When $\mathcal{M}' = \mathcal{M}'' = \mathcal{M}$ is regular holonomic, this is equivalent to saying that $L_S\colon \overline{\mathcal{M}} \to C_X\mathcal{M}$ is injective (or surjective), because and $C_X\mathcal{M}$ have the same characteristic variety (as their de Rham complexes are Verdier dual to each other). We say that S is $\pm$-Hermitian if $\overline{C_X(L_S)} = \pm L_S$, in other words, if $S(m, \overline{\mu}) = \pm S(\mu, \overline{m})$ in $\mathfrak{Db}_X$.

[3] It is called improperly the "conjugation functor" in [5].

1.d. Direct and inverse image by a closed immersion of codimension one and Hermitian duality. Let Z be a reduced divisor in X and $i\colon Z \hookrightarrow X$ (respectively, $\overline{\imath}\colon \overline{Z} \hookrightarrow \overline{X}$) denote the inclusion. Let j_+j^+ be the localization functor along Z, and denote by $j_\dagger j^+$ its adjoint by duality, i.e., $j_\dagger j^+ = Dj_+j^+D$, where D denotes the duality functor on holonomic $\mathcal{D}_X$-modules given by $D\mathcal{M} = \mathcal{H}om_{\mathcal{D}_X}(\Omega_X^n, \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))$ with $n = \dim X$.

We also consider two functors i_+i^+ and $i_+i^\dagger$. Recall that, for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have the following two dual exact sequences:

$$0 \longrightarrow \mathcal{H}^{-1}(i_+i^+\mathcal{M}) \longrightarrow \mathcal{M} \xrightarrow{\text{loc}} j_+j^+\mathcal{M} \longrightarrow \mathcal{H}^0(i_+i^+\mathcal{M}) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{H}^0(i_+i^\dagger\mathcal{M}) \longrightarrow j_\dagger j^+\mathcal{M} \xrightarrow[\text{coloc}]{} \mathcal{M} \longrightarrow \mathcal{H}^1(i_+i^\dagger\mathcal{M}) \longrightarrow 0.$$

Proposition 1.2. *There is a natural isomorphism of contravariant functors from* $\mathrm{Mod}_{hr}(\mathcal{D}_X)$ *to* $\mathrm{Mod}_{hr}(\mathcal{D}_{\overline{X}})$

$$C_X^{\mathrm{mod}\, Z} \simeq C_X \circ j_\dagger j^+$$

under which $C_X(\mathrm{coloc}_{\mathcal{M}})$ *corresponds to* $\mathrm{loc}_{C_X\mathcal{M}}$.

Proof. The first part is proved in [13, Proposition II.3.2.2]. We now want to prove that the following diagram commutes:

$$\begin{array}{ccccc} C_X(\mathcal{M}) & \xrightarrow{C_X(\mathrm{coloc}_{\mathcal{M}})} & C_X(j_\dagger j^+\mathcal{M}) & \xrightarrow{\sim} & C_X^{\mathrm{mod}\, Z}(j_\dagger j^+\mathcal{M}) \\ & \searrow^{\mathrm{loc}_{C_X\mathcal{M}}} & & & \wr\uparrow C_X^{\mathrm{mod}\, Z}(\mathrm{coloc}_{\mathcal{M}}) \\ & & & & C_X^{\mathrm{mod}\, Z}(\mathcal{M}) \end{array}$$

Remark first that it clearly commutes on $X - Z$. Put $\mathcal{N} = \overline{C_X(\mathcal{M})}$. The upper part of the diagram gives a morphism $\varphi\colon \mathcal{N} \to \mathcal{N}[*Z]$ that induces the identity on $X - Z$. It thus factorizes uniquely through $\mathrm{loc}\colon \mathcal{N} \to \mathcal{N}[*Z]$ to give a morphism $\psi\colon \mathcal{N}[*Z] \to \mathcal{N}[*Z]$ equal to Id on $X - Z$. It follows that $\psi = \mathrm{Id}$ (indeed, ψ is injective because $\mathcal{N}[*Z]$ has no torsion supported on Z, and, therefore, is onto since $\mathcal{N}[*Z]$ is holonomic). □

Corollary 1.3. *The nondegenerate pairing*

$$j_\dagger j^+\mathcal{M} \underset{\mathbb{C}}{\otimes} C_X^{\mathrm{mod}\, Z}(\mathcal{M}) \longrightarrow \mathfrak{Db}_X$$

induces a nondegenerate pairing

$$\mathcal{H}^{-k}(\overline{\imath}_+\overline{\imath}^+C_X(\mathcal{M})) \underset{\mathbb{C}}{\otimes} \mathcal{H}^k(i_+i^\dagger\mathcal{M}) \longrightarrow \mathfrak{Db}_X$$

and, hence, an isomorphism

$$\mathcal{H}^{-k}(\overline{\imath}_+\overline{\imath}^+C_X(\mathcal{M})) \xrightarrow{\sim} C_X\mathcal{H}^k(i_+i^\dagger\mathcal{M})$$

for $k = 0, 1$. □

Corollary 1.4. *Assume that* Z *is smooth. Then, there is a natural isomorphism of functors* $(k = 0, 1)$

$$C_Z \circ \mathcal{H}^k(i^\dagger) \simeq \mathcal{H}^{-k}(\overline{\imath}^+) \circ C_X.$$

Proof. Remark first that there is a natural isomorphism of functors

$$C_X \circ i_+ \simeq \overline{\imath}_+ \circ C_Z.$$

Indeed, denoting by i_{++} the direct image of $\mathcal{D}_{Z,\overline{Z}}$-modules, recall that one has $\mathfrak{Db}_{X,Z} = i_{++}\mathfrak{Db}_Z$: indeed, put $\mathcal{D}_{Z\to X} \otimes_{\mathbb{C}} \mathcal{D}_{\overline{Z}\to\overline{X}} = \mathcal{D}_{Z\to X,\overline{Z}\to\overline{X}}$ and consider the natural morphism of right $\mathcal{D}_{X,\overline{X}}$-modules

$$\mathfrak{C}_Z \underset{\mathcal{D}_{Z,\overline{Z}}}{\otimes} (\mathcal{D}_{Z\to X,\overline{Z}\to\overline{X}}) \longrightarrow \mathfrak{C}_X$$

such that, for any local section μ of $\mathfrak{C}_Z$, the image of $\mu\otimes 1$ evaluated on any function $\varphi \in \mathcal{C}_c^\infty(X)$ is equal to $\mu(\varphi_{|Z})$; this morphism is an isomorphism, as can be seen from a local computation; going from right to left, one gets the assertion.

It follows that

$$\begin{aligned}
\mathcal{H}om_{\mathcal{D}_X}(i_+\mathcal{M}, \mathfrak{Db}_X) &= \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(i_+\mathcal{M}, \mathfrak{Db}_{X,Z}) \\
&= \boldsymbol{R}i_*\boldsymbol{R}\mathcal{H}om_{i^{-1}\mathcal{D}_X}\Big(\mathcal{D}_{X\leftarrow Z} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_Z}{\otimes}} \mathcal{M}, \mathcal{D}_{X\leftarrow Z,\overline{X}\leftarrow\overline{Z}} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_{Z,\overline{Z}}}{\otimes}} \mathfrak{Db}_Z\Big) \\
&= \boldsymbol{R}i_*\mathcal{D}_{\overline{X}\leftarrow\overline{Z}} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_{\overline{Z}}}{\otimes}} \boldsymbol{R}\mathcal{H}om_{i^{-1}\mathcal{D}_X}\Big(\mathcal{D}_{X\leftarrow Z} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_Z}{\otimes}} \mathcal{M}, \mathcal{D}_{X\leftarrow Z} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_Z}{\otimes}} \mathfrak{Db}_Z\Big) \\
&= \boldsymbol{R}i_*\mathcal{D}_{\overline{X}\leftarrow\overline{Z}} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_{\overline{Z}}}{\otimes}} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{M}, \mathfrak{Db}_Z) \qquad \text{(Kashiwara's equivalence)} \\
&= \overline{\imath}_+ C_Z\mathcal{M}.
\end{aligned}$$

Since i_+ and $\overline{\imath}_+$ are exact functors, from Corollary 1.3 we obtain an isomorphism

$$\overline{\imath}_+ C_Z \circ \mathcal{H}^k(i^\dagger) \simeq \overline{\imath}_+\mathcal{H}^{-k}(\overline{\imath}^+) \circ C_X,$$

and, thus, the result, because $\overline{\imath}_+$ is an equivalence. □

2. REGULAR HOLONOMIC DISTRIBUTIONS

2.a. Regular holonomic distributions [9; 5, Ch. VII]. Let Ω be an open set in X. A distribution $u \in \mathfrak{Db}(\Omega)$ is *regular holonomic* if the sub-$\mathcal{D}_\Omega$-module $\mathcal{D}_\Omega \cdot u$ (or, equivalently, the sub-$\mathcal{D}_{\overline{\Omega}}$-module $\mathcal{D}_{\overline{\Omega}} \cdot u$, see [9, Proposition 4; 5, Proposition 7.4.2]) of $\mathfrak{Db}_\Omega$ is regular holonomic. The notion is local; i.e., there exists a sheaf $\mathrm{RH}\mathfrak{Db}_X$ such that the set of regular holonomic distributions on Ω is $\Gamma(\Omega, \mathrm{RH}\mathfrak{Db}_X)$.

Notice that $\mathrm{RH}\mathfrak{Db}_X$ is a left $\mathcal{D}_X$- and $\mathcal{D}_{\overline{X}}$-module. It will be convenient to consider the subsheaf $\mathcal{C}_X^\infty \cdot \mathrm{RH}\mathfrak{Db}_X$ of $\mathfrak{Db}_X$ whose local sections are finite combinations of regular holonomic distributions with C^∞ coefficients.

Analogous results hold for $\mathrm{RH}\mathfrak{Db}_X^{\mathrm{mod}\,Z}$. Notice that we have

$$\mathrm{RH}\mathfrak{Db}_X^{\mathrm{mod}\,Z} = \mathcal{O}_X[*Z] \underset{\mathcal{O}_X}{\otimes} \mathrm{RH}\mathfrak{Db}_X = \mathrm{image}[\mathrm{RH}\mathfrak{Db}_X \to \mathfrak{Db}_{X-Z}].$$

The following is a slight generalization of [1] and [4, Theorem 11].

Theorem 2.1. *Let $X = Z \times \mathbb{C}$ have dimension $n+1$ and let u be a regular holonomic distribution on the open set $\Omega \times D$ of X. Let $\varphi \in \mathcal{D}^{n,n}(\Omega)$ be a C^∞ (n,n)-form with compact support. Then, $\langle u, \varphi\rangle \in \mathfrak{Db}(D)$ is in $\Gamma(D, \mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C}})$.*

Proof. According to Remark 2.2 and Proposition 2.3 below, it is identical to the one of loc. cit., using the existence of a good Bernstein relation (see, for instance, [5, Theorem 8.8.16] and the references given there) in order to prove the existence of a good operator as in [4, Proposition 8]. □

2.b. Regular holonomic distributions in dimension 1. Assume now that $X=\mathbb{C}$ and $Z=\{0\}$. Let t be a coordinate on $\mathbb{C}$. For $\alpha\in\mathbb{C}$ such that $-1\leq\operatorname{Re}\alpha<0$ and $p\in\mathbb{Z}$, put

$$u_{\alpha,p}=\begin{cases}|t|^{-2(\alpha+1)}\dfrac{(\log|t|^2)^p}{p!}\in L^1_{\mathrm{loc}}(\mathbb{C}) & \text{if } p\geq 0,\\ 0 & \text{if } p<0.\end{cases}$$

Then, it is easy to show that the family $u_{\alpha,p}$ satisfies

$$(\partial_t t+\alpha)u_{\alpha,p}=(\partial_{\overline{t}}\overline{t}+\alpha)u_{\alpha,p}=u_{\alpha,p-1}. \tag{2.1}$$

This implies, in particular, that $u_{\alpha,p}\in\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$.

Let $\mathrm{RH}\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}$ denote the image of $\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ in $\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}=\mathfrak{Db}_{\mathbb{C},0}[1/t]=\mathfrak{Db}_{\mathbb{C},0}[1/\overline{t}\,]$. We then have

$$\mathrm{RH}\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}=\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}[1/t]=\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}[1/\overline{t}\,].$$

It is known (see [5, Ch. VII, §7]) that

$$\mathrm{RH}\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}=\sum_{-1\leq\operatorname{Re}\alpha<0}\sum_p\mathbb{C}\{t\}[t^{-1}]\cdot\mathbb{C}\{\overline{t}\}[\overline{t}^{-1}]\cdot\widetilde{u}_{\alpha,p}, \tag{2.2}$$

where $\widetilde{u}$ denotes the image of the distribution germ u in $\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}$.

Thus, (2.1) implies that

$$\mathrm{RH}\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}=\sum_{-1\leq\operatorname{Re}\alpha<0}\sum_p\mathcal{D}_{\mathbb{C},0}\cdot\mathcal{D}_{\overline{\mathbb{C}},0}\cdot\widetilde{u}_{\alpha,p},$$

from which we deduce that

$$\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}=\sum_{-1\leq\operatorname{Re}\alpha<0}\sum_p\mathcal{D}_{\mathbb{C},0}\cdot\mathcal{D}_{\overline{\mathbb{C}},0}\cdot u_{\alpha,p}+\mathbb{C}[\partial_t,\partial_{\overline{t}}]\cdot\delta=\sum_{-1\leq\operatorname{Re}\alpha<0}\sum_p\mathcal{D}_{\mathbb{C},0}\cdot\mathcal{D}_{\overline{\mathbb{C}},0}\cdot u_{\alpha,p} \tag{2.3}$$

because the Dirac distribution δ can be written as

$$-2i\pi\delta=\partial_t\partial_{\overline{t}}\log|t|^2.$$

Remarks 2.2. (1) Let $u\in\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ and put $\mathcal{M}=\mathcal{D}_{\mathbb{C},0}u\subset\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. Denote by $\widetilde{u}$ the image of u in $\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}$. Then, the regular holonomic module $\mathcal{M}[t^{-1}]$ is naturally embedded in $\mathrm{RH}\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}$, the image of u in $\mathcal{M}[t^{-1}]$ is $\widetilde{u}$ via this embedding, and $\mathcal{M}[t^{-1}]=\mathcal{D}_{\mathbb{C},0}[t^{-1}]\widetilde{u}$, so that $\mathcal{D}_{\mathbb{C},0}\widetilde{u}$ is identified with the quotient of $\mathcal{M}$ by its torsion supported at the origin.

(2) Using Borel's lemma, one can show that $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}$ is equal to the subspace of germs at 0 of C^∞ functions on $\mathbb{C}^*$ having an infinitely termwise differentiable asymptotic expansion at 0, in the sense of [4], the exponents of which belong to a finite union of lattices in $\mathbb{C}$.

Mellin transform. We also have a characterization of $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ in terms of the Mellin transform [4]. Let $u\in\mathfrak{Db}_{\mathbb{C},0}$ be a germ of distribution, and denote also by u a representative of this germ in $\Gamma(D,\mathfrak{Db}_{\mathbb{C}})$, where D is a small disc centered at the origin. Let $\chi\in\mathcal{C}^\infty_c(D)$ such that $\chi\equiv 1$ near 0 and has a sufficiently small support. Then, for any $k',k''\in\mathbb{Z}$, define

$$\mathcal{J}_u^{(k',k'')}(s)=\langle\chi u,t^{k'}\overline{t}^{k''}|t|^{2s}dt\wedge d\overline{t}\rangle,$$

which are holomorphic on $\operatorname{Re}s\gg 0$. These functions depend on χ up to the addition of an entire function. Thus, the classes of $\mathcal{J}_u^{(k',k'')}$ modulo $\mathcal{O}(\mathbb{C})$ only depend on the germ u. Moreover, these functions can be recovered from the functions $\mathcal{J}_u^{(k,0)}$ and $\mathcal{J}_u^{(0,k)}$ for $k\in\mathbb{N}$, because if, for instance, $k'\geq k''$, we clearly have $\mathcal{J}_u^{(k',k'')}(s)=\mathcal{J}_u^{(k'-k'',0)}(s+k'')$. Moreover, $\mathcal{J}_u^{(k',k'')}$ only depends on the image of u in $\mathfrak{Db}^{\bmod 0}_{\mathbb{C},0}$.

Proposition 2.3 [4, Theorem 4]. *Let $u \in \mathfrak{Db}_{\mathbb{C},0}$. Then, $u \in \mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ if and only if there exists a finite set $\mathcal{R} \subset \mathbb{C}$ such that, for all $k \in \mathbb{N}$, the functions $\mathcal{J}_u^{(k,0)}$ and $\mathcal{J}_u^{(0,k)}$, which are holomorphic on $\operatorname{Re} s \gg 0$, extend to meromorphic functions on $\mathbb{C}$ with poles at most in $\mathcal{R}+\mathbb{Z}$ and satisfy*

$$(\exists R > 0),\ (\forall N > 0),\ (\forall \ell > 0),\ (\forall \ell' > 0),$$
$$|s+k/2|^{\ell}|k|^{\ell'} \sup(|\mathcal{J}_u^{(k,0)}(s)|, |\mathcal{J}_u^{(0,k)}(s)|) \le C(u,N,\ell,\ell')R^{\operatorname{Re}(s+k/2)}$$

for $\operatorname{Re}(s+k/2+N) \ge -1$ and $|s+k/2| \gg 0$.

Proof. First, notice that $u \in \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ if and only if its image in $\mathfrak{Db}_{\mathbb{C},0}^{\mathrm{mod}\,0}$ belongs to $\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}^{\mathrm{mod}\,0}$. Moreover, we can fix a representative for u and consider χu to define $\mathcal{J}_u^{(k,0)}(s)$ or $\mathcal{J}_u^{(0,k)}$. The condition in the proposition is easily seen to be independent of these choices. The result is then a direct consequence of [4, Theorem 4]. □

3. HERMITIAN DUALITY AND MODERATE NEARBY/VANISHING CYCLES

We will show in this section the compatibility between these functors. We will first recall briefly the construction of moderate and vanishing cycles for holonomic $\mathcal{D}$-modules in order to be able to give a detailed account of the compatibility.

3.a. Notation. We fix a total ordering on $\mathbb{C}$, denoted by $\le$, that is assumed to satisfy (a), (b), and (c) below:

(a) it induces the usual ordering on $\mathbb{R}$,

(b) for $a \in \mathbb{R}$, $\{z \in \mathbb{C} \mid z < a\} = \{z \in \mathbb{C} \mid \operatorname{Re} z < a\}$,

(c) for $a \in \mathbb{R}$ and $z, z' \in \mathbb{C}$, $z \le z' \ \Leftrightarrow\ z + a \le z' + a$.

In the following, we will choose the ordering on $\mathbb{C}$ induced by lexicographic ordering of the triples $(\operatorname{Re} a, |\operatorname{Im} a|, \operatorname{Im} a)$. With such an ordering, we have

$$\{\alpha \in \mathbb{C} \mid -1 \le \alpha < 0\} = \{\alpha \in \mathbb{C} \mid -1 \le \operatorname{Re}\alpha < 0\}.$$

For a complex number γ, denote by $[\gamma]$ the largest integer less than or equal to γ, using the fixed total ordering on $\mathbb{C}$.

3.b. Review on the Malgrange–Kashiwara filtration. Let Z be a complex analytic manifold of dimension n, put $X = Z \times \mathbb{C}$, let t denote the coordinate on $\mathbb{C}$ or the projection $X \to \mathbb{C}$, and consider the inclusion $Z = Z \times \{0\} \hookrightarrow X$.

For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, let $V_\bullet(\mathcal{M})$ be the Malgrange–Kashiwara filtration on $\mathcal{M}$ relative to $Z \times \{0\}$ (see, e.g., [12]): this is a filtration indexed by the union of a (locally on Z) finite number of lattices $\sigma + \mathbb{Z} \subset \mathbb{C}$ ($\sigma \in \mathcal{S}$, and we can choose the finite set $\mathcal{S} \subset \mathbb{C}$ contained in $\operatorname{Re}\sigma \in [0,1[$), using the ordering specified above. For any $\alpha \in \mathbb{C}$, the graded module $\operatorname{gr}_\alpha^V \mathcal{M} \stackrel{\mathrm{def}}{=} V_\alpha\mathcal{M}/V_{<\alpha}\mathcal{M}$ is $\mathcal{D}_Z$-holonomic (and, moreover, regular when $\mathcal{M}$ is) and comes equipped with a nilpotent endomorphism N induced by the action of $-(\partial_t t + \alpha)$.

We have isomorphisms

$$t\colon V_\alpha\mathcal{M} \xrightarrow{\sim} V_{\alpha-1}\mathcal{M} \qquad (\alpha < 0) \tag{3.1}$$

and

$$\partial_t\colon \operatorname{gr}_\alpha^V \mathcal{M} \xrightarrow{\sim} \operatorname{gr}_{\alpha+1}^V \mathcal{M} \qquad (\alpha > -1). \tag{3.2}$$

The complex $i^+\mathcal{M}$ is quasi-isomorphic to the complex

$$\operatorname{gr}_0^V \mathcal{M} \xrightarrow{t} \operatorname{gr}_{-1}^V \mathcal{M}$$

(where the right term has degree 0), and, if $\mathcal{M} = j_\dagger j^+\mathcal{M}$, it is also isomorphic to the complex

$$\mathrm{gr}^V_{-1}\mathcal{M} \xrightarrow{t\partial_t} \mathrm{gr}^V_{-1}\mathcal{M}.$$

Similarly, the complex $i^\dagger\mathcal{M}$ is quasi-isomorphic to the complex

$$\mathrm{gr}^V_{-1}\mathcal{M} \xrightarrow{\partial_t} \mathrm{gr}^V_0\mathcal{M}$$

(where the left term has degree 0), and, if $\mathcal{M} = j_+j^+\mathcal{M}$, it is also isomorphic to the complex

$$\mathrm{gr}^V_{-1}\mathcal{M} \xrightarrow{t\partial_t} \mathrm{gr}^V_{-1}\mathcal{M}.$$

In particular, if $\mathcal{M} = j_+j^+\mathcal{M}$, we will identify

$$\mathcal{H}^0(i^\dagger\mathcal{M}) \qquad \text{with} \qquad \mathrm{Ker}[t\partial_t\colon \mathrm{gr}^V_{-1}\mathcal{M} \to \mathrm{gr}^V_{-1}\mathcal{M}] \tag{3.3}$$

and, if $\mathcal{M} = j_\dagger j^+\mathcal{M}$,

$$\mathcal{H}^0(i^+\mathcal{M}) \qquad \text{with} \qquad \mathrm{Coker}[t\partial_t\colon \mathrm{gr}^V_{-1}\mathcal{M} \to \mathrm{gr}^V_{-1}\mathcal{M}]. \tag{3.4}$$

Analogous results hold for holonomic $\mathcal{D}_{\overline{X}}$-modules. We still denote by $V_\bullet$ the Malgrange–Kashiwara filtration and by N the nilpotent endomorphism induced by $-(\partial_{\bar t}\bar t + \alpha)$ on gr^V_α.

3.c. Review on moderate nearby and vanishing cycles (see, e.g., [12, 14]). Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module (specializable would be enough; see, e.g., [12]). Let α be such that $-1 \le \alpha < 0$ and put $\lambda = \exp(2i\pi\alpha)$. For $p \in \mathbb{N}$, put $\mathcal{M}_{\alpha,p} = (\mathcal{M}[t^{-1}])^{p+1} = \bigoplus_{k=0}^p \mathcal{M}[t^{-1}] \otimes e_{\alpha,k}$. The $\mathcal{D}_{X/\mathbb{C}}$-structure on $\mathcal{M}_{\alpha,p}$ is the direct sum of the $\mathcal{D}_{X/\mathbb{C}}$-structures on each term $\mathcal{M}[t^{-1}]$, and the $\mathcal{D}_X$-structure is given by the relation

$$t\partial_t(m \otimes e_{\alpha,k}) = [(\partial_t t + \alpha)m] \otimes e_{\alpha,k} + m \otimes e_{\alpha,k-1},$$

with the convention that $e_{\alpha,k} = 0$ for $k < 0$. Remark that $\mathcal{M}[t^{-1}]$ is a direct summand of $\mathcal{M}_{-1,p}$ for any $p \ge 0$ (we can assume that $e_{\alpha,k}$ plays the role of the multivalued function $t^{\alpha+1}(\log t)^k/k!$).

We have natural morphisms of $\mathcal{D}_X$-modules

$$\begin{aligned} \mathcal{M}_{\alpha,p} &\xhookrightarrow{a_{p,p+1}} \mathcal{M}_{\alpha,p+1},\\ \sum_{k=0}^p m_{\alpha,k} \otimes e_{\alpha,k} &\longmapsto \sum_{k=0}^p m_{\alpha,k} \otimes e_{\alpha,k} \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{\alpha,p+1} &\xrightarrow{b_{p+1,p}} \mathcal{M}_{\alpha,p},\\ \sum_{k=0}^{p+1} m_{\alpha,k} \otimes e_{\alpha,k} &\longmapsto \sum_{k=0}^p m_{\alpha,k+1} \otimes e_{\alpha,k}. \end{aligned}$$

We will denote by N (without index p) any of the endomorphisms

$$N = a_{p-1,p} \circ b_{p,p-1}\colon \mathcal{M}_{\alpha,p} \longrightarrow \mathcal{M}_{\alpha,p}$$

sending $m \otimes e_{\alpha,k}$ to $m \otimes e_{\alpha,k-1}$. The inductive (respectively, projective) system $\mathcal{H}^0(i^\dagger\mathcal{M}_{\alpha,p})$ (respectively, $\mathcal{H}^0(i^+j_\dagger j^+\mathcal{M}_{\alpha,p})$) where the maps are induced by $a_{p,p+1}$ (respectively, $b_{p+1,p}$) is stationary locally on X, and both systems have a common limit isomorphic to $\mathrm{gr}^V_\alpha\mathcal{M}$: we can identify

$\mathrm{gr}_{-1}^V \mathcal{M}_{\alpha,p}$ with $\bigoplus_{k=0}^p \mathrm{gr}_\alpha^V \mathcal{M} \otimes e_{\alpha,k}$; the natural mappings

$$\begin{aligned} \mathrm{gr}_\alpha^V \mathcal{M} &\longrightarrow \mathrm{gr}_{-1}^V \mathcal{M}_{\alpha,p}, \\ m_0 &\longmapsto \bigoplus_{k=0}^p [-(\partial_t t+\alpha)]^k m_0 \otimes e_{\alpha,k} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \mathrm{gr}_{-1}^V \mathcal{M}_{\alpha,p} &\longrightarrow \mathrm{gr}_\alpha^V \mathcal{M}, \\ \bigoplus_{k=0}^p m_k \otimes e_{\alpha,k} &\longmapsto \bigoplus_{k=0}^p [-(\partial_t t+\alpha)]^k m_{p-k} \end{aligned} \tag{3.6}$$

induce, for p large enough, an isomorphism from $\mathrm{gr}_\alpha^V \mathcal{M}$ to $\operatorname{Ker} t\partial_t \simeq \mathcal{H}^0(i^\dagger \mathcal{M}_{\alpha,p})$ and from $\operatorname{Coker} t\partial_t \simeq \mathcal{H}^0(i^+ j_\dagger j^+ \mathcal{M}_{\alpha,p})$ to $\mathrm{gr}_\alpha^V \mathcal{M}$.

We denote this limit by $\psi_{t,\lambda}^{\mathrm{mod}} \mathcal{M}$ and call it the moderate nearby cycle module associated with $\mathcal{M}$, with eigenvalue λ. We also denote by N the endomorphism induced by the previous N. It corresponds naturally to $-(\partial_t t+\alpha)$ via both isomorphisms with $\mathrm{gr}_\alpha^V \mathcal{M}$. Notice also that the inductive system of $\mathcal{H}^1$ (respectively, the projective system of $\mathcal{H}^{-1}$) has limit 0.

The construction of the moderate vanishing cycle module $\phi_{t,1}^{\mathrm{mod}}(\mathcal{M})$ is achieved by considering the inductive system of complexes $\mathcal{M} \to \mathcal{M}_{-1,p}$ (where the right term has degree 0 and the map is the composition of $\mathrm{loc}\colon \mathcal{M} \to \mathcal{M}[t^{-1}]$ with $a_{0,p}\colon \mathcal{M}[t^{-1}] \to \mathcal{M}_{-1,p}$) instead of the single module $\mathcal{M}_{\alpha,p}$. The only possible nonvanishing limit is also obtained for $\mathcal{H}^0 i^\dagger$. It can also be achieved by considering the projective system of complexes $j_\dagger j^+ \mathcal{M}_{-1,p} \to \mathcal{M}$ (where the left term has degree 0 and the map is the composition of $j_\dagger j^+ b_{p,0}$ and $\mathrm{coloc}\colon j_\dagger j^+ \mathcal{M} \to \mathcal{M}$) and the projective limit of $\mathcal{H}^0 i^+$. Let us give a precise description. The complex $i^\dagger(\mathcal{M} \to \mathcal{M}_{-1,p})$ is the single complex associated to the double complex

$$\begin{array}{ccc} j_\dagger j^+ \mathcal{M} & \longrightarrow & j_\dagger j^+ \mathcal{M}_{-1,p} \\ \mathrm{coloc}\downarrow & & \downarrow\mathrm{coloc} \\ \mathcal{M} & \longrightarrow & \mathcal{M}_{-1,p} \end{array} \quad \simeq \quad \begin{array}{ccc} \mathrm{gr}_{-1}^V \mathcal{M} & \longrightarrow & \mathrm{gr}_{-1}^V(\mathcal{M}_{-1,p}) \\ \partial_t\downarrow & & \downarrow t\partial_t \\ \mathrm{gr}_0^V \mathcal{M} & \xrightarrow[t]{} & \mathrm{gr}_{-1}^V(\mathcal{M}_{-1,p}) \end{array}$$

which is isomorphic to the complex

$$\mathrm{gr}_{-1}^V \mathcal{M} \longrightarrow \mathrm{gr}_0^V \mathcal{M} \oplus \mathrm{gr}_{-1}^V(\mathcal{M}_{-1,p}) \longrightarrow \mathrm{gr}_{-1}^V(\mathcal{M}_{-1,p}),$$

where the middle term has degree 0. The kernel of the second morphism can be identified with $\mathrm{gr}_{-1}^V \mathcal{M} \oplus \mathrm{gr}_0^V \mathcal{M}$ via

$$m_0 \oplus n_0 \longmapsto n_0 \oplus (m_0 \otimes e_{-1,0}) \oplus \left[\bigoplus_{k=1}^p (-t\partial_t)^{k-1}(-t\partial_t m_0 + t n_0) \otimes e_{-1,k}\right],$$

and the $\mathcal{H}^0$ of this complex is identified with $\mathrm{gr}_0^V \mathcal{M}$ via

$$\mathrm{gr}_0^V \mathcal{M} \xrightarrow{0\oplus \mathrm{Id}} \mathrm{gr}_{-1}^V \mathcal{M} \oplus \mathrm{gr}_0^V \mathcal{M}.$$

The action of $0 \oplus N$ on $\mathrm{gr}_0^V \mathcal{M} \oplus \mathrm{gr}_{-1}^V(\mathcal{M}_{-1,p})$ induces, via these isomorphisms, an action of $-\partial_t t$ on $\mathrm{gr}_0^V \mathcal{M}$.

Similarly, the complex $i^+(j_\dagger j^+\mathcal{M}_{-1,p}\to\mathcal{M})$ is isomorphic to the single complex associated with

$$\begin{array}{ccc} \operatorname{gr}^V_{-1}(\mathcal{M}_{-1,p}) & \xrightarrow{\partial_t} & \operatorname{gr}^V_0\mathcal{M} \\ {\scriptstyle t\partial_t}\downarrow & & \downarrow{\scriptstyle t} \\ \operatorname{gr}^V_{-1}(\mathcal{M}_{-1,p}) & \longrightarrow & \operatorname{gr}^V_{-1}\mathcal{M} \end{array}$$

where the middle term has degree 0. Its $\mathcal{H}^0$ is naturally isomorphic to $\operatorname{gr}^V_0\mathcal{M}$, and the action of N on $\mathcal{M}_{-1,p}$ induces that of $-\partial_t t$ on $\operatorname{gr}^V_0\mathcal{M}$.

The morphisms can and Var are defined as

$$\operatorname{gr}^V_{-1}\mathcal{M} \underset{\mathrm{Var}=t}{\overset{\mathrm{can}=-\partial_t}{\rightleftarrows}} \operatorname{gr}^V_0\mathcal{M}$$

and can be obtained, via the previous isomorphisms, from the morphisms of complexes

$$\begin{array}{ccc} 0 & \longrightarrow & \mathcal{M}_{-1,p} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathcal{M} & \longrightarrow & \mathcal{M}_{-1,p} \end{array} \quad\text{or}\quad \begin{array}{ccc} j_\dagger j^+\mathcal{M}_{-1,p} & \longrightarrow & 0 \\ {\scriptstyle N}\downarrow & & \downarrow \\ j_\dagger j^+\mathcal{M}_{-1,p} & \longrightarrow & \mathcal{M} \end{array} \tag{can}$$

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M}_{-1,p} \\ \downarrow & & \downarrow{\scriptstyle N} \\ 0 & \longrightarrow & \mathcal{M}_{-1,p} \end{array} \quad\text{or}\quad \begin{array}{ccc} j_\dagger j^+\mathcal{M}_{-1,p} & \longrightarrow & \mathcal{M} \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow \\ j_\dagger j^+\mathcal{M}_{-1,p} & \longrightarrow & 0 \end{array} \tag{Var}$$

3.d. Compatibility with Hermitian duality. We now assume that $\mathcal{M}$ is regular holonomic. For any α such that $-1\le\alpha<0$, consider the function $u_{-\alpha-2,p}=|t|^{2(\alpha+1)}(\log|t|^2)^k/k!$ analogous to that of Subsection 2.b as a function on X. It has moderate growth along Z together with all its derivatives. Hence, for any moderate distribution $\widetilde{u}$ along Z, the product $u_{-\alpha-2,p}\widetilde{u}$ is well-defined as a moderate distribution along Z.

Lemma 3.1. *The pairing*

$$C_X(\mathcal{M})_{\alpha,p}\underset{\mathbb{C}}{\otimes}\mathcal{M}_{\alpha,p}\longrightarrow \mathfrak{Db}_X^{\operatorname{mod} Z},$$
$$\Big(,\sum_{k=0}^p \mu_{\alpha,k}\otimes\overline{e}_{\alpha,k}\Big)\otimes\Big(,\sum_{\ell=0}^p m_{\alpha,\ell}\otimes e_{\alpha,\ell}\Big)\longmapsto \sum_{k,\ell}\mu_{\alpha,k}(m_{\alpha,\ell})u_{-\alpha,k+\ell-p}$$

is nondegenerate and induces an isomorphism compatible with N and $C_X^{\operatorname{mod} Z}(N)$

$$\eta_{\alpha,p}\colon C_X(\mathcal{M})_{\alpha,p}\xrightarrow{\sim} C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p})$$

such that all diagrams

$$\begin{array}{ccc} C_X(\mathcal{M})_{\alpha,p} & \xrightarrow{\sim} & C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p}) \\ {\scriptstyle a_{p,p+1}}\downarrow & & \downarrow{\scriptstyle C_X^{\operatorname{mod} Z}(b_{p+1,p})} \\ C_X(\mathcal{M})_{\alpha,p+1} & \xrightarrow{\sim} & C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p+1}) \end{array} \quad\textit{and}\quad \begin{array}{ccc} C_X(\mathcal{M})_{\alpha,p} & \xrightarrow{\sim} & C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p}) \\ {\scriptstyle b_{p+1,p}}\uparrow & & \uparrow{\scriptstyle C_X^{\operatorname{mod} Z}(a_{p,p+1})} \\ C_X(\mathcal{M})_{\alpha,p+1} & \xrightarrow{\sim} & C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p+1}) \end{array}$$

commute.

Proof. First, it is easy to see that the morphism $\eta_{\alpha,p}$ induced by the pairing induces commutative diagrams as in the lemma. The compatibility of $\eta_{\alpha,p}$ with N and $C_X^{\operatorname{mod} Z}(N)$ is thus clear. The nondegeneracy of the pairing is then proved by induction on p, the case $p=0$ being easy. □

Theorem 3.2. *There exist natural isomorphisms of functors from* $\mathrm{Mod}_{hr}(\mathcal{D}_X)$ *to* $\mathrm{Mod}_{hr}(\mathcal{D}_{\overline{Z}})$

$$c^{\psi}_{X,\lambda}\colon \psi^{\mathrm{mod}}_{t,\lambda}\circ C_X \longrightarrow C_Z\circ\psi^{\mathrm{mod}}_{t,\lambda},\quad \lambda\in\mathbb{C}^*,\qquad \textit{and}\qquad c^{\phi}_{X,1}\colon \phi^{\mathrm{mod}}_{t,1}\circ C_X \longrightarrow C_Z\circ\phi^{\mathrm{mod}}_{t,1}$$

which satisfy the following properties, where $c_X = c^{\psi}_{X,\lambda}$ *or* $c^{\phi}_{X,1}$:

- $c_X = C_Z\circ c_{\overline{X}}\circ C_X$;
- $c_X\circ N = C_Z(N)\circ c_X$;
- $c^{\phi}_{X,1}\circ\mathrm{can} = C_Z(\mathrm{Var})\circ c^{\psi}_{X,1}$ *and* $c^{\psi}_{X,1}\circ\mathrm{Var} = C_Z(\mathrm{can})\circ c^{\phi}_{X,1}$.

Proof. According to the previous lemma and Corollary 1.3, the inductive system

$$(\mathcal{H}^0\bar{\imath}^{\dagger}C_X(\mathcal{M})_{\alpha,p},\mathcal{H}^0\bar{\imath}^{\dagger}a_{p,p+1})$$

is isomorphic, via $\mathcal{H}^0\bar{\imath}^{\dagger}\eta_{\alpha,p}$, to C_Z of the projective system $(\mathcal{H}^0(i^+j_{\dagger}j^+\mathcal{M}_{\alpha,p}),b_{p+1,p})$. The first part of the theorem then follows from the construction of $\psi^{\mathrm{mod}}_{t,\lambda}$ recalled in Subsection 3.c. The proof for $\phi^{\mathrm{mod}}_{t,1}$ and the other properties also follow from the same arguments. □

3.e. Nearby/vanishing cycles for a sesquilinear form. Let $\mathcal{M}'$ and $\mathcal{M}''$ be two regular holonomic $\mathcal{D}_X$-modules and let $S\colon\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''}\to\mathfrak{Db}_X$ be a sesquilinear pairing.

We will define, for $-1\le\alpha<0$, sesquilinear forms

$$\psi_\lambda S\colon\ \mathrm{gr}^V_\alpha\mathcal{M}'\underset{\mathbb{C}}{\otimes}\mathrm{gr}^V_\alpha\overline{\mathcal{M}''}\longrightarrow\mathfrak{Db}_Z$$

and similarly (for $\alpha=0$) $\phi_1 S$, which satisfy (with obvious notation)

$$\begin{aligned}\psi_\lambda S(N\bullet,\bullet)&=\psi_\lambda S(\bullet,N\bullet),\\ \phi_1 S(N\bullet,\bullet)&=\phi_1 S(\bullet,N\bullet),\\ \psi_1 S(\mathrm{Var}\bullet,\bullet)&=\phi_1 S(\bullet,\mathrm{can}\bullet),\\ \psi_1 S(\bullet,\mathrm{Var}\bullet)&=\phi_1 S(\mathrm{can}\bullet,\bullet).\end{aligned}\tag{3.7}$$

Denote for a while by L_S the $\mathcal{D}_{\overline{X}}$-linear morphism $\overline{\mathcal{M}''}\to C_X\mathcal{M}'$ induced by S. Consider $\psi_\lambda L_S\colon\ \mathrm{gr}^V_\alpha\overline{\mathcal{M}''}\to\mathrm{gr}^V_\alpha C_X\mathcal{M}'$ (and $\phi_1 L_S$ defined similarly). Its composition with $c^{\psi}_{X,\lambda}$ (or $c^{\phi}_{X,1}$) is the linear morphism associated with a sesquilinear form $\psi_\lambda S$ or $\phi_1 S$. Properties (3.7) follow then from the properties of c_X given by Theorem 3.2.

Remark 3.3. Denote by $M_\bullet\,\mathrm{gr}^V_\alpha(\mathcal{M})$ the monodromy filtration associated to the nilpotent endomorphism N, i.e., the increasing filtration such that $NM_k\subset M_{k-2}$ and, for all $\ell\ge 0$,

$$\mathrm{gr}^M_\ell\,\mathrm{gr}^V_\alpha\mathcal{M}\xrightarrow{N^\ell}\mathrm{gr}^M_{-\ell}\,\mathrm{gr}^V_\alpha\mathcal{M}$$

is an isomorphism. Let $P\,\mathrm{gr}^M_\ell\,\mathrm{gr}^V_\alpha\mathcal{M}$ denote the primitive part

$$\mathrm{Ker}\Big[N^{\ell+1}\colon\ \mathrm{gr}^M_\ell\,\mathrm{gr}^V_\alpha\mathcal{M}\longrightarrow\mathrm{gr}^M_{-\ell-2}\,\mathrm{gr}^V_\alpha\mathcal{M}\Big].$$

The pairing $\psi_\lambda S$, being compatible with N, induces for any ℓ a pairing

$$\mathrm{gr}^M_\ell\,\mathrm{gr}^V_\alpha\mathcal{M}'\underset{\mathbb{C}}{\otimes}\mathrm{gr}^M_{-\ell}\,\mathrm{gr}^V_\alpha\overline{\mathcal{M}''}\xrightarrow{\ \psi_{\lambda,\ell}S\ }\mathfrak{Db}_Z$$

and is nondegenerate iff $\psi_{\lambda,\ell}S$ is nondegenerate for any ℓ. This is so iff the pairing induced on the primitive parts

$$P\,\mathrm{gr}^M_\ell\,\mathrm{gr}^V_\alpha\mathcal{M}'\underset{\mathbb{C}}{\otimes}P\,\mathrm{gr}^M_\ell\,\mathrm{gr}^V_\alpha\overline{\mathcal{M}''}\xrightarrow{\ \psi_{\lambda,\ell}S\circ(\mathrm{Id}\otimes N^\ell)\ }\mathfrak{Db}_Z\tag{3.8}$$

is nondegenerate, according to the Lefschetz decomposition. Similar results hold for $\phi_1 S$.

For $\ell \geq 0$, we will set

$$P\psi_{\lambda,\ell}S \overset{\text{def}}{=} \psi_{\lambda,\ell}S \circ (\mathrm{Id} \otimes N^\ell) \qquad \text{and} \qquad P\phi_{1,\ell}S \overset{\text{def}}{=} \phi_{1,\ell}S \circ (\mathrm{Id} \otimes N^\ell).$$

From Theorem 3.2, we deduce

Corollary 3.4. *The sesquilinear form S is nondegenerate in a neighborhood of Z if and only if all sesquilinear forms $P\psi_{\lambda,\ell}S$, $\lambda \in \mathbb{C}^*$, $\ell \geq 0$, and $P\phi_{1,\ell}S$, $\ell \geq 0$, are nondegenerate.*

Proof. According to Remark 3.3, it is enough to show that S is nondegenerate iff all $\psi_\lambda S$ and $\phi_1 S$ are so. Now, L_S is an isomorphism in a neighborhood of Z if and only if all $\psi_\lambda L_S$ and $\phi_1 L_S$ are isomorphisms: this follows from the fact that a regular holonomic module $\mathcal{M}$ is equal to zero near Z if and only if all its moderate nearby or vanishing cycles vanish on Z. The result is then a consequence of the definition of $\psi_\lambda S$ and $\phi_1 S$ and of Theorem 3.2. □

4. HERMITIAN DUALITY AND ASYMPTOTIC EXPANSIONS

We will give in this section a more explicit description of the compatibility morphisms given in Theorem 3.2, using asymptotic expansions (in the sense of Remark 2.2(2)). The main goal will be to give a more precise version of Theorem 2.1 taking into account the order with respect to the Malgrange–Kashiwara filtration.

We begin with some easy results in dimension 1.

4.a. Dimension 1. *Regular holonomic distributions and Malgrange–Kashiwara filtration.* If u is the germ at $0 \in \mathbb{C}$ of a regular holonomic distribution defined on some open disc D centered at the origin, we denote by $\alpha'(u)$ the order of u with respect to the Malgrange–Kashiwara filtration of the regular holonomic module $\mathcal{D}_D \cdot u \subset \mathfrak{Db}_D$ and by $\alpha''(u)$ its $V_\bullet$-order in $\mathcal{D}_{\overline{D}}u$. Notice that, according to the strictness property of any morphism between holonomic modules with respect to the Malgrange–Kashiwara filtration, if $v \in \mathcal{D}_D \cdot u$, then $\alpha'(v)$ is equal to the V-order of v when viewed as an element of $\mathcal{D}_D \cdot u$.

We obtain in this way increasing filtrations

$$V'_{\alpha'}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = \{u \in \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0} \mid \alpha'(u) \leq \alpha'\},$$
$$V''_{\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = \{u \in \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0} \mid \alpha''(u) \leq \alpha''\}$$

(where $\leq$ is the fixed total ordering on $\mathbb{C}$) and thus a doubly indexed filtration

$$V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = V'_{\alpha'}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) \cap V''_{\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}).$$

We then put

$$\mathrm{gr}^V_{\alpha',\alpha''} \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0} \overset{\text{def}}{=} V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})/(V_{<\alpha',\alpha''} + V_{\alpha',<\alpha''})(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}).$$

For $\lambda \in \mathbb{C}^*$, choose $\alpha \in \mathbb{C}$ with $-1 \leq \alpha < 0$ such that $\lambda = \exp(2i\pi\alpha)$; put, as in Subsection 2.b, $u_{\alpha,p} = |t|^{-2(\alpha+1)}(\log|t|^2)^p/p!$ and

$$\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}(\lambda) = \sum_{p\geq 0} \mathcal{D}_{\mathbb{C},0}\mathcal{D}_{\overline{\mathbb{C}},0} \cdot u_{\alpha,p}.$$

We then have

$$\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0} = \bigoplus_{\lambda\in\mathbb{C}^*} \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}(\lambda).$$

Proposition 4.1. *The filtration* $V_{\bullet,\bullet}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$ *satisfies the following properties.*

(1) $tV_{\alpha',\alpha''} \subset V_{\alpha'-1,\alpha''}$, *respectively* $\bar{t}V_{\alpha',\alpha''} \subset V_{\alpha',\alpha''-1}$, *with equality if* $\alpha' < 0$, *respectively* $\alpha'' < 0$.

(2) $\partial_t V_{\alpha',\alpha''} \subset V_{\alpha'+1,\alpha''}$ *and* $\partial_{\bar{t}} V_{\alpha',\alpha''} \subset V_{\alpha',\alpha''+1}$.

(3) *Let* $u \in \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. *Then,* $u \in V_{\alpha',\alpha''}$ *iff there exist* $k', k'' \in \mathbb{N}$ *with*

$$(\partial_t t + \alpha')^{k'} u \in V_{<\alpha',\alpha''} \qquad \text{and} \qquad (\partial_{\bar{t}}\bar{t} + \alpha'')^{k''} u \in V_{\alpha',<\alpha''}.$$

(4) *We have*

$$V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = \sum_{-1\le\alpha<0} \sum_{\substack{k',k''\in\mathbb{Z}\\ \alpha+k'\le\alpha'\\ \alpha+k''\le\alpha''}} \sum_{p\ge0} V_{k'}(\mathcal{D}_{\mathbb{C},0}) \cdot V_{k''}(\mathcal{D}_{\overline{\mathbb{C}},0}) \cdot u_{\alpha,p}$$

and $\mathrm{gr}^V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = 0$ *if* $\alpha' - \alpha'' \notin \mathbb{Z}$.

(5) *For any* $\alpha \in \mathbb{C}$, $(\partial_t t - \partial_{\bar{t}}\bar{t})V_{\alpha,\alpha} \subset V_{<\alpha,\alpha} + V_{\alpha,<\alpha}$.

Proof. Assertion (2) and the first part of (1) follow immediately from the properties of the Malgrange–Kashiwara filtration on holonomic modules.

Let us prove the second part of (1). Let $u \in V_{\alpha'-1,\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$ with $\alpha' < 0$. According to (3.1), there exists then $v \in \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ with $\alpha'(v) \le \alpha'$ such that $u = tv$. Let $b(\partial_{\bar{t}}\bar{t})$ be the minimal polynomial satisfying $b(\partial_{\bar{t}}\bar{t})u = \bar{t}P(\bar{t}, \partial_{\bar{t}}\bar{t})u$ with $P \in V_0\mathcal{D}_{\overline{D},0}$. Then, $w \overset{\text{def}}{=} [b(\partial_{\bar{t}}\bar{t}) - \bar{t}P(\bar{t}, \partial_{\bar{t}}\bar{t})]v$ is supported at the origin and satisfies $\alpha'(w) \le \alpha'(v) \le \alpha' < 0$. Therefore, by (3.1), we have $w = 0$, and v satisfies $\alpha''(v) \le \alpha''(u)$.

For (3), notice that there exist β' and β'' such that $u \in V_{\beta',\beta''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$. We may assume that $\beta' \ge \alpha'$ and $\beta'' \ge \alpha''$. There exist polynomials $B'(-s)$ and $B''(-s)$ with roots in $]\alpha', \beta']$ and $]\alpha'', \beta'']$, respectively, such that $B'(\partial_t t)B''(\partial_{\bar{t}}\bar{t})u$ belongs to $V_{\alpha',\alpha''}$. Applying the Bézout theorem and the condition in (3), we conclude that u belongs to $V_{\alpha',\alpha''}$.

Let us now prove (4) and (5). We will first need the following lemma.

Lemma 4.2. (a) *We have* $\mathrm{gr}^V_{\beta',\beta''}\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}(\lambda) = 0$ *if* $\beta' \notin \alpha + \mathbb{Z}$ *or* $\beta'' \notin \alpha + \mathbb{Z}$.

(b) *For all* $k', k'' \in \mathbb{Z}$, *we have*

$$V_{k'+\alpha,k''+\alpha}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}(\lambda)) = \sum_{p\ge0} V_{k'}(\mathcal{D}_{\mathbb{C},0})V_{k''}(\mathcal{D}_{\overline{\mathbb{C}},0}) \cdot u_{\alpha,p}.$$

(c) *For* $-1 \le \alpha < 0$, *the classes of* $u_{\alpha,p}$, $p \ge 0$, *form a basis of the* $\mathbb{C}$-*vector space* $\mathrm{gr}^V_{\alpha,\alpha}\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$.

(d) *The classes of* $\partial_t\partial_{\bar{t}}u_{-1,p}$, $p \ge 1$, *form a basis of* $\mathrm{gr}^V_{0,0}\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$.

Proof. According to (2.1), the distribution $u_{\alpha,p}$, $-1 \le \alpha < 0$ and $p \in \mathbb{N}$, satisfies

$$(\partial_t t + \alpha)^{p+1}u_{\alpha,p} = (\partial_{\bar{t}}\bar{t} + \alpha)^{p+1}u_{\alpha,p} = 0. \tag{4.1}$$

It is then in $V_{\alpha,\alpha}$.

It follows that, for any $\overline{P} \in \mathcal{D}_{\overline{\mathbb{C}},0}$, the correspondence $1 \mapsto \overline{P}u_{\alpha,p}$ induces a surjective $\mathcal{D}_{\mathbb{C},0}$-linear morphism

$$\mathcal{D}_{\mathbb{C},0}/\mathcal{D}_{\mathbb{C},0}(\partial_t t + \alpha)^{p+1} \longrightarrow \mathcal{D}_{\mathbb{C},0} \cdot \overline{P}u_{\alpha,p}.$$

This implies that, for any $k \in \mathbb{Z}$, we have

$$V'_{k+\alpha}(\mathcal{D}_{\mathbb{C},0} \cdot \overline{P}u_{\alpha,p}) = V_k(\mathcal{D}_{\mathbb{C},0}) \cdot \overline{P}u_{\alpha,p} \tag{4.2}$$

because a similar property is easily seen to be true for $\mathcal{D}_{\mathbb{C},0}/\mathcal{D}_{\mathbb{C},0}(\partial_t t + \alpha)^{p+1}$, and any morphism of holonomic $\mathcal{D}$-modules is strict with respect to the Malgrange–Kashiwara filtration.

By the same argument, we also get that, for $-1 \le \alpha < 0$,

$$V_{k'+\alpha,k''+\alpha}(\mathcal{D}_{\mathbb{C},0}\mathcal{D}_{\overline{\mathbb{C}},0}u_{\alpha,p}) = V_{k'}(\mathcal{D}_{\mathbb{C},0})V_{k''}(\mathcal{D}_{\overline{\mathbb{C}},0})u_{\alpha,p}.$$

As $\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}(\lambda) = \varinjlim_p \mathcal{D}_{\mathbb{C},0}\mathcal{D}_{\overline{\mathbb{C}},0}u_{\alpha,p}$, statements (a) and (b) are clear.

Part (b) shows that the elements given in part (c) or (d) generate the corresponding bigraded object. If we had, for $-1 \le \alpha < 0$, a linear relation between the classes of $u_{\alpha,p}$, $p \ge 0$, in $\mathrm{gr}^V_{\alpha,\alpha}\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$, then, by applying a suitable power of $\partial_t t + \alpha$ and using relation (2.1), we would have a relation

$$|t|^{-2(\alpha+1)} \in t\sum_p \mathcal{O}_{\mathbb{C},0}\mathcal{O}_{\overline{\mathbb{C}},0}\cdot u_{\alpha,p} + \overline{t}\sum_p \mathcal{O}_{\mathbb{C},0}\mathcal{O}_{\overline{\mathbb{C}},0}\cdot u_{\alpha,p},$$

which is clearly impossible in view of the valuation at 0. Similarly, a linear relation between the classes $\partial_t\partial_{\overline{t}}u_{-1,p}$, $p \ge 1$, would imply that $\delta \in (V_{-1,0} + V_{0,-1})\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. Notice now that

$$t\colon V_{-1,0}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) \longrightarrow V_{-2,0}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$$

is bijective: part (b) shows that it is onto; it is injective because $t\colon V'_{-1} \to V'_{-2}$ is so, as follows from (3.1).

Thus, if $\delta = u^{(-1,0)} + u^{(0,-1)}$, we have $tu^{(-1,0)} \in V_{-2,0}\cap V_{-1,-1} = V_{-2,-1}$; hence, $u^{(-1,0)} \in V_{-1,-1}$ and, similarly, $u^{(0,-1)} \in V_{-1,-1}$; therefore, $\delta \in V_{-1,-1}$, which is impossible because t acting on $V_{-1,-1}$ is injective. □

Statement (4) of the proposition follows from (b) in the lemma.

Relation (2.1) clearly implies that, for all $k,\ell \in \mathbb{N}$, we have $(\partial_t t - \partial_{\overline{t}}\overline{t})u = 0$ if $u = t^k\overline{t}^k\partial_t^\ell\partial_{\overline{t}}^\ell u_{\alpha,p}$, for $-1 \le \alpha < 0$. Then, (5) follows immediately. □

The Malgrange–Kashiwara filtration for $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. In order to apply similar considerations to an asymptotic expansion, we will introduce the Malgrange–Kashiwara filtration on $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. Put

$$V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) \stackrel{\mathrm{def}}{=} \mathcal{C}^\infty\cdot V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}).$$

We clearly have

$$V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) + (V_{<\alpha',\alpha''} + V_{\alpha',<\alpha''})(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) \tag{4.3}$$

and, hence, a surjective morphism $\mathrm{gr}^V_{\alpha',\alpha''}\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0} \to \mathrm{gr}^V_{\alpha',\alpha''}\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$.

Proposition 4.3. *The results of Proposition* 4.1 *apply to* $V_{\bullet,\bullet}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$ *as well and, moreover,*

$$\mathrm{gr}^V_{\alpha',\alpha''}\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0} = \mathrm{gr}^V_{\alpha',\alpha''}\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}.$$

Remark 4.4. It follows from Propositions 4.1(5) and 4.3 that the nilpotent endomorphisms induced by $\partial_t t + \alpha$ or $\partial_{\overline{t}}\overline{t} + \alpha$ on $\mathrm{gr}^V_{\alpha,\alpha}\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ coincide for $-1 \le \alpha \le 0$.

Proof of Proposition 4.3. (1), (2), (3), and (5) in Proposition 4.1 immediately extend to $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. Moreover, (4) clearly gives

$$V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}) = \sum_{-1\le\alpha<0}\ \sum_{\substack{k',k''\in\mathbb{Z}\\ \alpha+k'\le\alpha'\\ \alpha+k''\le\alpha''}}\ \sum_{p\ge 0}\mathcal{C}^\infty\cdot V_{k'}(\mathcal{D}_{\mathbb{C},0})\cdot V_{k''}(\mathcal{D}_{\overline{\mathbb{C}},0})\cdot u_{\alpha,p}. \tag{4.4}$$

An argument similar to that of (c) and (d) in Lemma 4.2 for $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ gives the last assertion of Proposition 4.3. □

Localization and Mellin transform. We may give similar definitions and similar arguments for the germ $\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}$. We say that a germ $\widetilde{u}\in\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}$ has order less than α' if it belongs to $V_{\alpha'}(\mathcal{D}_{\mathbb{C},0}\widetilde{u})$, etc. In particular, we get

$$\begin{aligned} V'_{\alpha'}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}) &= \sum_{-1\le\alpha<0}\sum_p t^{-[\alpha'-\alpha]}\mathcal{O}_{\mathbb{C},0}\mathcal{O}_{\overline{\mathbb{C}},0}[\overline{t}^{-1}]\cdot u_{\alpha,p},\\ V''_{\alpha''}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}) &= \sum_{-1\le\alpha<0}\sum_p \mathcal{O}_{\mathbb{C},0}[t^{-1}]\overline{t}^{-[\alpha''-\alpha]}\mathcal{O}_{\overline{\mathbb{C}},0}\cdot u_{\alpha,p},\\ V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}) &\overset{\mathrm{def}}{=} V'_{\alpha'}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})\cap V''_{\alpha''}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})\\ &= \sum_{-1\le\alpha<0}\sum_p t^{-[\alpha'-\alpha]}\mathcal{O}_{\mathbb{C},0}\cdot\overline{t}^{-[\alpha''-\alpha]}\mathcal{O}_{\overline{\mathbb{C}},0}\cdot u_{\alpha,p}. \end{aligned}\tag{4.5}$$

Put

$$V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})\overset{\mathrm{def}}{=}\mathcal{C}^\infty V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})=\sum_{-1\le\alpha<0}\sum_p t^{-[\alpha'-\alpha]}\overline{t}^{-[\alpha''-\alpha]}\mathcal{C}^\infty_{\mathbb{C},0}\cdot u_{\alpha,p}.\tag{4.6}$$

From [4, Theorem 4] we obtain

Proposition 4.5. *Let $u\in\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$, and denote by $\widetilde{u}$ its image in $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}$. Let $\alpha',\alpha''\in\mathbb{C}$. Then, $\widetilde{u}\in V_{\alpha',\alpha''}$ if and only if, for any $k',k''\in\mathbb{Z}$, the poles of the meromorphic function $\mathcal{J}_u^{(k',k'')}(s)$ are $\le\min(\alpha'-k',\alpha''-k'')$.* □

Note that it is enough to verify the previous criterion, for instance, for $k''=0$ and $k'\in\mathbb{Z}$.

Put $V'_{\alpha'}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})=\bigcup_{\beta''}V_{\alpha',\beta''}$ and define $V''_{\alpha''}$ similarly.

Corollary 4.6. *For any $\alpha',\alpha''\in\mathbb{C}$ with $\alpha'-\alpha''\in\mathbb{Z}$, we have*

$$V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})=V'_{\alpha'}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})\cap V''_{\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}).\quad\square$$

It also follows from Lemma 4.2(b) and formula (4.5) above that

$$V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})=\mathrm{image}[V_{\alpha',\alpha''}(\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})\longrightarrow\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}].\tag{4.7}$$

Thus, we have a similar result for $\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0}$.

Corollary 4.7. *For any $\alpha',\alpha''\in\mathbb{C}$, we have*

$$V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})=V'_{\alpha'}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})\cap V''_{\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}).$$

Proof. Let $u\in V'_{\alpha'}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})\cap V''_{\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$. Then, $\widetilde{u}\in V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}^{\mathrm{mod}\,0}_{\mathbb{C},0})$; hence, by (4.7), there exists $v\in V_{\alpha',\alpha''}(\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$ such that $u-v$ is supported at 0, i.e., belongs to

$$V'_{\alpha'}(\mathbb{C}[\partial_t,\partial_{\overline{t}}]\cdot\delta)\cap V''_{\alpha''}(\mathbb{C}[\partial_t,\partial_{\overline{t}}]\cdot\delta),$$

which is easily seen to be equal to $V_{\alpha',\alpha''}(\mathbb{C}[\partial_t,\partial_{\overline{t}}]\cdot\delta)$. □

Remark 4.8. For $u\in\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$, we have $\alpha'(\widetilde{u})\le\alpha'(u)$ with equality if $\alpha'(u)<0$, and a similar result for α''.

The morphisms L_α. According to (c) and (d) in Lemma 4.2, we can give the following definition:

Definition 4.9. For $-1\le\alpha\le0$, the linear morphism

$$L_\alpha\colon\ \mathrm{gr}^V_{\alpha,\alpha}\mathcal{C}^\infty\mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}\longrightarrow\mathbb{C}$$

is obtained by taking the coefficient of the class of $\frac{u_{\alpha,0}}{2i\pi}$ if $-1\le\alpha<0$ and the coefficient of the class of $\delta=-\frac{1}{2i\pi}\partial_t\partial_{\overline{t}}\log|t|^2$ if $\alpha=0$.

It will be convenient to denote also by L_α the map composed with the previous L_α and the projection $V_{\alpha,\alpha} \to \mathrm{gr}^V_{\alpha,\alpha}$, so that $L_\alpha(u) = 0$ if $\alpha'(u) < \alpha$ or $\alpha''(u) < \alpha$.

Proposition 4.10. *Let $v \in \mathrm{gr}^V_{-1,0}\, \mathcal{C}^\infty \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ and $w \in \mathrm{gr}^V_{0,-1}\, \mathcal{C}^\infty \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$. We then have*

$$L_0(-\partial_t v) = L_{-1}(\bar{t}v) \qquad \textit{and} \qquad L_0(-\partial_{\bar{t}} w) = L_{-1}(tw).$$

Proof. Any such v can be written as $\partial_{\bar{t}}(\sum_{p\geq 1} v_p \frac{u_{-1,p}}{2i\pi})$. Then, we have $L_{-1}(\bar{t}v) = v_1$ since $\bar{t}\partial_{\bar{t}} \log|t|^2 = 1$, and

$$\partial_t v = \partial_t \partial_{\bar{t}} \Bigg(\sum_{p\geq 1} v_p \frac{u_{-1,p}}{2i\pi} \Bigg);$$

hence, $L_0(-\partial_t v) = v_1$. □

Proposition 4.11. *For $-1 \leq \alpha < 0$ and $u \in V_{\alpha,\alpha}(\mathcal{C}^\infty \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0})$, we have*

$$L_\alpha(u) = \star \operatorname{Res}_{s=\alpha} \mathcal{J}_u^{(0,0)} \qquad \textit{with} \quad \star \neq 0.$$

Proof. This follows from the computation in the proof of Theorem 4 in [4]. □

Remark 4.12. For $u \in V_{0,0}\mathcal{C}^\infty \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$, we can also compute $L_0(u)$ as the residue of the Mellin transform of the *localized Fourier transform* of the germ u:

$$L_0(u) = \star \operatorname{Res}_{s=-1} \mathcal{J}^{(0,0)}_{\mathcal{F}_{\mathrm{loc}} u} \qquad \text{with} \quad \star \neq 0.$$

4.b. The morphism $\mathrm{gr}^V_\alpha C_X \mathcal{M} \to C_Z \mathrm{gr}^V_\alpha \mathcal{M}$ defined using asymptotic expansions. Keep the notation of Subsection 3.b. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. In order to define morphisms $\mathrm{gr}^V_\alpha C_X \mathcal{M} \to C_Z \mathrm{gr}^V_\alpha \mathcal{M}$, we will show the following

Assertion. *For any open set $\Omega \subset Z$, any disc $D \subset \mathbb{C}$ centered at 0, and for $-1 \leq \alpha \leq 0$, the mapping*

$$\begin{aligned} \Gamma(\Omega \times D, V_\alpha(\mathcal{M})) \underset{\mathbb{C}}{\otimes} \Gamma(\Omega \times D, V_\alpha(C_X\mathcal{M})) &\longrightarrow \Gamma(\Omega, \mathfrak{Db}_Z), \\ (m, \mu) &\longmapsto [\varphi \mapsto L_\alpha(\langle \mu(m), \varphi \rangle)] \end{aligned} \tag{4.8}$$

is well defined (that it takes values in $\mathfrak{Db}_Z$ can be seen as in [2, lemme 1]*) and induces 0 on $V_\alpha(\mathcal{M}) \otimes_{\mathbb{C}} V_{<\alpha}(C_X\mathcal{M})$ and $V_{<\alpha}(\mathcal{M}) \otimes_{\mathbb{C}} V_\alpha(C_X\mathcal{M})$.*

Therefore, (4.8) well defines a $\mathcal{D}_{Z,\overline{Z}}$-linear map

$$\Gamma(\Omega, \mathrm{gr}^V_\alpha \mathcal{M}) \underset{\mathbb{C}}{\otimes} \Gamma(\Omega, \mathrm{gr}^V_\alpha (C_X\mathcal{M})) \xrightarrow{\langle \bullet, \bullet \rangle_\alpha} \mathfrak{Db}_Z(\Omega). \tag{4.9}$$

Moreover, denoting by N the action of $\partial_t t + \alpha$ on the left as well as the action of $\partial_{\bar{t}} \bar{t} + \alpha$ on the right, according to Propositions 4.1(5) and 4.3, we have

$$\langle Nm, \mu \rangle_\alpha = \langle m, N\mu \rangle_\alpha.$$

Therefore, $\langle \bullet, \bullet \rangle_\alpha$ defines, for $-1 \leq \alpha \leq 0$, a $\mathcal{D}_{\overline{Z}}$-linear morphism

$$c_{X,\alpha}(\mathcal{M})\colon\ \mathrm{gr}^V_\alpha C_X \mathcal{M} \longrightarrow C_Z \mathrm{gr}^V_\alpha \mathcal{M}$$

changing N to $C_Z(N)$; i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathrm{gr}^V_\alpha C_X \mathcal{M} & \xrightarrow{c_{X,\alpha}(\mathcal{M})} & C_Z \mathrm{gr}^V_\alpha \mathcal{M} \\ {\scriptstyle N}\big\downarrow & & \big\downarrow{\scriptstyle C_Z(N)} \\ \mathrm{gr}^V_\alpha C_X \mathcal{M} & \xrightarrow{c_{X,\alpha}(\mathcal{M})} & C_Z \mathrm{gr}^V_\alpha \mathcal{M} \end{array} \tag{4.10}$$

More precisely, $c_{X,\alpha}$ is a functorial morphism between the functors $\mathrm{gr}_\alpha^V C_X$ and $C_Z \mathrm{gr}_\alpha^V$. We can also consider $c_{\overline{X},\alpha}$ defined in a similar way. We then have

$$c_{X,\alpha} = C_Z \circ c_{\overline{X},\alpha} \circ C_X. \tag{4.11}$$

According to Proposition 4.10, the following diagram and its Hermitian dual analogue commute:

$$\begin{array}{ccc} \mathrm{gr}_{-1}^V C_X\mathcal{M} & \xrightarrow{c_{X,-1}(\mathcal{M})} & C_Z \mathrm{gr}_{-1}^V \mathcal{M} \\ {\scriptstyle -\partial_{\overline{t}}}\big\downarrow & & \big\downarrow{\scriptstyle C_Z(t)} \\ \mathrm{gr}_0^V C_X\mathcal{M} & \xrightarrow{c_{X,0}(\mathcal{M})} & C_Z \mathrm{gr}_0^V \mathcal{M} \end{array} \tag{4.12}$$

Theorem 4.13. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Then, the morphism $c_{X,\alpha}$, $-1 \le \alpha < 0$, coincides with $c^\psi_{X,\lambda}$, $\lambda = \exp(2i\pi\alpha)$, and $c_{X,0}$ coincides with $c^\phi_{X,1}$.*

Before proving Theorem 4.13, we will justify the construction of the morphisms $c_{X,\alpha}(\mathcal{M})$ by proving the assertion.

Lemma 4.14. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Then, for any $\alpha', \alpha'' \in \mathbb{C}$, $k \in \mathbb{Z}$, any sections $m \in \Gamma(\Omega \times D, V_{\alpha'}\mathcal{M})$ and $\mu \in \Gamma(\Omega \times D, V_{\alpha''}C_X\mathcal{M})$, and any $\varphi \in \mathcal{D}^{(n,n)}(\Omega)$, the meromorphic functions $\mathcal{J}^{(k,0)}_{u_\varphi}(s)$ have poles $\le \min(\alpha' - k, \alpha'')$.*

Proof. Denote by K the support of φ and by $p \ge 0$ the order of the distribution $\mu(m)$ on $K \times D$. For $q \in \mathbb{Z}$, the functions $(t,s) \mapsto |t|^{2s} t^{p+q}$ and $(t,s) \mapsto |t|^{2s}\overline{t}^{p+q}$ are C^p on $\{\mathrm{Re}\, s > -q\} \times \mathbb{C}$ and depend holomorphically on s. Consequently, if $\chi \in \mathcal{C}^\infty_c(D)$, the function $s \mapsto \langle u_\varphi, \chi(t)|t|^{2s}t^{p+q} dt \wedge d\overline{t}\rangle$ is holomorphic for $\mathrm{Re}\, s > -q$.

Let b_m be the Bernstein polynomial for m on $K \times D$: there exists $P \in \Gamma(K \times D, V_0\mathcal{D}_{X\times\mathbb{C}})$ such that $b_m(\partial_t t)m = tP(x,t,\partial_x,\partial_t t)m$. By assumption, the roots of $b_m(-s)$ are $\le \alpha'$. Denote by b_μ the Bernstein polynomial for μ; the roots of $b_\mu(-s)$ are $\le \alpha''$. Fix $k \in \mathbb{Z}$, choose $r \ge 0$ so large that $p - r < \alpha'$ and consider the polynomial

$$B'(\partial_t t) = \prod_{j=-k-r}^{-k} b_m(\partial_t t + j).$$

We have $B'(\partial_t t)t^k \mu(m) \in t^{k+r}(V_0\mathcal{D}_{\Omega\times D}) \cdot \mu(m)$. Hence,

$$B'(-s)\mathcal{J}^{(k,0)}_{u_\varphi}(s) \equiv \mathcal{J}^{(k+r,0)}_{u_\psi}(s) \qquad \mathrm{mod}\ \mathcal{O}(\mathbb{C})$$

for some $\psi \in \mathcal{C}^\infty_K(\Omega)$. But $\mathcal{J}^{(k+r,0)}_{u_\psi}(s)$ is holomorphic for $\mathrm{Re}\, s > p-k-r$ and, hence, for $\mathrm{Re}\, s > \alpha' - k$. Since the zeros of $B'(-s)$ are $\le \alpha' - k$, we conclude that the poles of $\mathcal{J}^{(k,0)}_{u_\varphi}(s)$ are $\le \alpha' - k$.

A similar argument for b_μ shows that the poles are $\le \alpha''$. □

Proof of the assertion. If α' and α'' are < 0, the desired assertion follows from Proposition 4.5 and Remark 4.8.

In general, one uses Proposition 4.1(3) together with Proposition 4.3 to show the assertion for any pair (α', α''). □

4.c. Proof of Theorem 4.13. *First step.* Assume that $\mathcal{M}$ is supported on Z. We then have

$$i^+\mathcal{M} = V_0\mathcal{M} = \mathrm{Ker}[t\colon \mathcal{M} \to \mathcal{M}],$$

$$\overline{\imath}^+ C_X\mathcal{M} = V_0 C_X\mathcal{M} = \mathrm{Ker}[\overline{t}\colon C_X\mathcal{M} \to C_X\mathcal{M}].$$

On the other hand, we have

$$\mathrm{Ker}[t\colon \mathfrak{Db}_{X,Z} \to \mathfrak{Db}_{X,Z}] = \mathrm{Ker}[\overline{t}\colon \mathfrak{Db}_{X,Z} \to \mathfrak{Db}_{X,Z}]$$

and can identify this sheaf with $\mathfrak{Db}_Z$ by defining, for any $\mu \in \operatorname{Ker} t$ and $\varphi \in \mathcal{C}_c^{n,n}(Z)$ (with $n = \dim Z$), $\langle \mu, \varphi \rangle = \mu(\psi dt \wedge d\overline{t})$, where ψ is any C_c^∞ (n,n) form on X such that $\psi_{|Z} = \varphi$. The pairing $C_X\mathcal{M} \otimes_{\mathbb{C}} \mathcal{M} \to \mathfrak{Db}_{X,Z}$ induces a pairing

$$V_0 C_X\mathcal{M} \underset{\mathbb{C}}{\otimes} V_0\mathcal{M} \longrightarrow \operatorname{Ker} t \simeq \mathfrak{Db}_Z.$$

This is the pairing constructed in Corollary 1.3. It coincides with the pairing defined with the help of L_0 in Definition 4.9.

Since the theorem is true for the modules supported on Z, it is enough to prove it for the modules satisfying $\mathcal{M} = j_+ j^+ \mathcal{M} = \mathcal{M}[t^{-1}]$. Moreover, since $(c^\psi_{X,1}, c^\phi_{X,1})$ and $(c_{X,-1}, c_{X,0})$ are both compatible with can and Var, it is enough to prove the theorem for $-1 \le \alpha < 0$.

Second step. Assume now that $\mathcal{M} = j_+ j^+ \mathcal{M} = \mathcal{M}[t^{-1}]$. We will show that the nondegenerate pairing

$$\mathcal{H}^0(\overline{\imath}^+ C_X^{\operatorname{mod} Z}\mathcal{M}) \underset{\mathbb{C}}{\otimes} \mathcal{H}^0(i^+ j_\dagger j^+ \mathcal{M}) \longrightarrow \mathfrak{Db}_Z$$

given in Corollary 1.3 coincides with that defined with the help of $c_{X,0}$. Notice that $C_X\mathcal{M} = \overline{\jmath}_\dagger \overline{\jmath}^+ C_X\mathcal{M}$, so that

$$\mathcal{H}^0(\overline{\imath}_+ \overline{\imath}^+ C_X^{\operatorname{mod} Z}\mathcal{M}) = \operatorname{Ker}[\operatorname{loc}\colon C_X\mathcal{M} \to C_X^{\operatorname{mod} Z}\mathcal{M}],$$
$$\mathcal{H}^0(i_+ i^+ j_\dagger j^+ \mathcal{M}) = \operatorname{Coker}[\operatorname{coloc}\colon j_\dagger j^+\mathcal{M} \to \mathcal{M}].$$

Identify

$$\mathcal{H}^0(\overline{\imath}^+ C_X^{\operatorname{mod} Z}\mathcal{M}) \qquad \text{with} \qquad V_0\mathcal{H}^0(\overline{\imath}_+ \overline{\imath}^+ C_X^{\operatorname{mod} Z}\mathcal{M}) \subset V_0 C_X\mathcal{M}$$

and

$$\mathcal{H}^0(i^+ j_\dagger j^+ \mathcal{M}) \qquad \text{with} \qquad V_0\mathcal{H}^0(i_+ i^+ j_\dagger j^+ \mathcal{M}) \quad \text{(a quotient of } V_0\mathcal{M}\text{)}.$$

Let μ be a local section of $V_0 C_X\mathcal{M}$ and m be a local section of $V_0\mathcal{M}$. Then, if μ is in $\mathcal{H}^0(\overline{\imath}^+ C_X^{\operatorname{mod} Z}\mathcal{M})$, the distribution $\mu(m)$ is supported on Z and is in $\operatorname{Ker} \overline{t}$. Thus, we can apply the first step to get the result.

Third step: proof for $-1 \le \alpha < 0$. We may assume that $\mathcal{M} = j_+ j^+ \mathcal{M}$. We will show that $c_{X,\alpha}$ can be computed using $\mathcal{M}_{\alpha,p}$, as we did for $c^\psi_{X,\lambda}$. The second step will then give $c_{X,\alpha} = c^\psi_{X,\lambda}$.

We have the isomorphisms

$$\begin{aligned}
\operatorname{gr}^V_\alpha C_X(\mathcal{M}) &\xrightarrow[(3.5)]{\sim} \operatorname{Ker} \overline{t}\partial_{\overline{t}} (\subset \operatorname{gr}^V_{-1} C_X(\mathcal{M})_{\alpha,p}) \xrightarrow[\text{Lemma 3.1}]{\sim} \operatorname{Ker} \overline{t}\partial_{\overline{t}} (\subset \operatorname{gr}^V_{-1} C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p}))\\
&\xrightarrow{\sim} \operatorname{Ker} \overline{t}\partial_{\overline{t}} (\subset \operatorname{gr}^V_{-1} C_X(\mathcal{M}_{\alpha,p})) \xrightarrow[c_{X,-1}(\mathcal{M}_{\alpha,p})]{\sim} C_Z(\operatorname{Coker} t\partial_t)(\subset C_Z(\operatorname{gr}^V_{-1} \mathcal{M}_{\alpha,p}))\\
&\xrightarrow[(3.6)]{\sim} C_Z(\operatorname{gr}^V_\alpha \mathcal{M}).
\end{aligned}$$

We will identify the composed isomorphism

$$\operatorname{gr}^V_\alpha C_X(\mathcal{M}) \xrightarrow{\sim} C_Z(\operatorname{gr}^V_\alpha \mathcal{M}) \tag{4.13}$$

with $c_{X,\alpha}(\mathcal{M})$.

Let μ be a local section of $V_\alpha C_X\mathcal{M}$ and put

$$\widetilde{\mu}_{\alpha,p} = \sum_{k=0}^{p} [-(\partial_{\overline{t}}\overline{t} + \alpha)]^k \mu \otimes \overline{e}_{\alpha,k} \in C_X(\mathcal{M})_{\alpha,p}.$$

According to Lemma 3.1, we can view $\widetilde{\mu}_{\alpha,p}$ as a local section of $C_X^{\operatorname{mod} Z}(\mathcal{M}_{\alpha,p})$ by putting, for $\sum_{\ell=0}^p m_\ell \otimes e_{\alpha,\ell} \in \mathcal{M}_{\alpha,p}$,

$$\Big\langle \widetilde{\mu}_{\alpha,p}, \sum_{\ell=0}^{p} m_\ell \otimes e_{\alpha,\ell} \Big\rangle = \sum_{k,\ell=0}^{p} [-(\partial_{\overline{t}}\overline{t} + \alpha)]^k \mu(m_\ell) \cdot u_{-\alpha-2,k+\ell-p}.$$

To understand the image of (the class of) $\widetilde{\mu}_{\alpha,p}$ by the morphism $c_{X,-1}(\mathcal{M}_{\alpha,p})$, we fix a local form φ of maximal degree and with compact support on Z and consider, under the condition that all m_ℓ are in $V_\alpha\mathcal{M}$, the coefficient of $\frac{u_{-1,0}}{2i\pi}$ in

$$\sum_{k,\ell=0}^{p} \langle [-(\partial_{\bar t}\bar t+\alpha)]^k \mu(m_\ell),\varphi\rangle \cdot u_{-\alpha-2,k+\ell-p}. \tag{4.14}$$

The only terms contributing to it are those for which $k+\ell=p$. Put

$$\langle \mu(m_\ell),\varphi\rangle = \sum_{j\geq 0} v_{\ell,j}\frac{u_{\alpha,j}}{2i\pi} \qquad \text{with} \quad v_{\ell,j}\in\mathbb{C}.$$

The coefficient of $u_{-1,0}$ in (4.14) is $\sum_{k=0}^{p}(-1)^k v_{p-k,k}$.

On the other hand, we have

$$L_\alpha\Biggl(\Biggl\langle \mu\Biggl(\sum_{\ell=0}^{p}[-(\partial_t t+\alpha)]^\ell m_{p-\ell}\Biggr),\varphi\Biggr\rangle\Biggr) = L_\alpha\Biggl(\sum_{\ell=0}^{p}(-1)^\ell\sum_{j\geq 0} v_{p-\ell,j}\frac{u_{\alpha,j-\ell}}{2i\pi}\Biggr) = \sum_{\ell=0}^{p}(-1)^\ell v_{p-\ell,\ell}.$$

Consequently, (4.13) coincides with $c_{X,\alpha}(\mathcal{M})$. This completes the proof of Theorem 4.13. □

4.d. Relation to some results of D. Barlet. We will show that Theorems 3.2 and 4.13 give a generalization to regular holonomic modules of some results of D. Barlet concerning the effective contribution of monodromy to poles of $\int|f|^{2s}$ for a holomorphic function $f\colon Z\to\mathbb{C}$ on a smooth manifold Z (see [2]). Remark that the assumption on monodromy made by D. Barlet concerns monodromy on the cohomology of the Milnor fiber of f; here, however, the assumption concerns monodromy on the complex of nearby or vanishing cycles and may give better results (see, e.g., [11]).

Let $\psi_f\mathbb{C}_Z$ and $\phi_f\mathbb{C}_Z$ denote the complexes of nearby and vanishing cycles (see [7]) and, for $\lambda\in\mathbb{C}^*$, denote by $\psi_{f,\lambda}\mathbb{C}_Z$ and (for $\lambda=1$) $\phi_{f,1}\mathbb{C}_Z$ the complexes corresponding to the eigenvalue λ (and 1) of the monodromy (see the construction in [6] or [14]). These complexes are perverse up to a shift and are equipped with a nilpotent endomorphism (the nilpotent part of monodromy). Let $M_\bullet$ denote the monodromy filtration in the perverse category (see, e.g., [8, §1.6] or [14, §1.3.9]).

Corollary 4.15. *Let $x^\circ\in f^{-1}(0)$ and assume that x° belongs to the support of $\mathrm{gr}_\ell^M\psi_{f,\lambda}\mathbb{C}_Z$ for some $\lambda=\exp(2i\pi\alpha)\in\mathbb{C}^*$, $-1\leq\alpha<0$, and $\ell\in\mathbb{N}$. Then, for any sufficiently small neighborhood V of x°, there exists $\varphi\in\mathcal{D}^{(n,n)}(V)$, $n=\dim Z$, such that the function*

$$I_\varphi\colon s\longmapsto\int_Z|f|^{2s}\varphi$$

has a pole of order at least $\ell+1$ at some $\alpha-k$ with $k\in\mathbb{N}$. Similarly, if x° belongs to $\mathrm{supp}\,\mathrm{gr}_\ell^M\phi_{f,1}\mathbb{C}_Z$, then, for each V, there exists φ such that the pole order is at least $\ell+2$ at some negative integer.

Remarks. (1) To say that x° belongs to the support of $\mathrm{gr}_\ell^M\psi_{f,\lambda}\mathbb{C}_Z$ means that the monodromy on the *perverse complex* $\psi_f\mathbb{C}_Z$ has a Jordan block of size at least $\ell+1$ for the eigenvalue λ, near x°.

(2) If ℓ_0 is the maximal integer ℓ such that x° belongs to $\mathrm{supp}\,\mathrm{gr}_\ell^M\psi_{f,\lambda}\mathbb{C}_Z$ (or belongs to $\mathrm{supp}\,\mathrm{gr}_\ell^M\psi_{f,1}\mathbb{C}_Z\cup\mathrm{supp}\,\mathrm{gr}_\ell^M\phi_{f,1}\mathbb{C}_Z$ if $\lambda=1$), then the pole of any function $I_\varphi(s)$ at points $\alpha-k$, for φ supported in a small neighborhood of x°, has order $\leq\ell_0+1$.

(3) If I_φ has a pole of order $\ell+1$ at some $\alpha-p$ for some φ, then, for any $p'\geq p$, there exists ψ such that I_ψ has a pole of order $\ell+1$ at $\alpha-p'$: put $\psi=|f|^{2(p'-p)}\varphi$.

Proof of Corollary 4.15. Denote by $i_f\colon Z\hookrightarrow X=Z\times\mathbb{C}$ the graph inclusion. Put $\mathcal{M}=i_{f+}\mathcal{O}_Z$. Since $C_Z\mathcal{O}_Z=\overline{\mathcal{O}}_Z$ (the Dolbeault lemma), we have $C_X\mathcal{M}=\overline{\mathcal{M}}$. We then get a sesquilinear pairing

$$S\colon\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\longrightarrow\mathfrak{Db}_X.$$

Let us consider first the case of nearby cycles ($-1 \leq \alpha < 0$). By assumption and the Riemann–Hilbert correspondence for nearby cycles, there is a local section m of $\mathcal{M}$ such that $m \in V_\alpha(\mathcal{M})$, the class of m in $\operatorname{gr}_\alpha^V \mathcal{M}$ belongs to $M_\ell \operatorname{gr}_\alpha^V \mathcal{M}$, and its class in $\operatorname{gr}_\ell^M \operatorname{gr}_\alpha^V \mathcal{M}$ is nonzero at x°. Moreover, we can assume that the class of m is *primitive*; otherwise, there would exist a primitive class in $\operatorname{gr}_{\ell+2k}^M \operatorname{gr}_\alpha^V \mathcal{M}$ for some $k \geq 1$, and we would apply the reasoning to this class.

As the pairing (3.8) is nondegenerate, there exists μ in $V_\alpha \mathcal{M}$ such that $S([m], N^\ell[\overline{\mu}]) \neq 0$ in $\mathfrak{Db}_Z$. This means that there exists $\psi \in \mathcal{D}^{(n,n)}(Z)$ such that, if we put

$$m = \sum_{i \geq 0} m_i \partial_t^i \delta(t-f), \qquad \mu = \sum_{j \geq 0} \mu_j \partial_t^j \delta(t-f),$$

where m_i and μ_j are holomorphic functions on a neighborhood of x° in Z, the germ

$$\sum_{i,j} \partial_t^i \partial_{\overline{t}}^j (\partial_{\overline{t}} \overline{t} + \alpha)^\ell \left[\int_{f=t} m_i \overline{\mu_j} \psi \right]$$

in $\mathcal{C}^\infty \mathrm{RH}\mathfrak{Db}_{\mathbb{C},0}$ has a nonzero coefficient on $u_{\alpha,0}$. Hence, there exist i and j such that $\partial_t^i \partial_{\overline{t}}^j \int_{f=t} m_i \overline{\mu_j} \psi$ has a nonzero coefficient on $u_{\alpha,\ell}$. From the computation of the Mellin transform [4, Theorem 4] it follows that

$$\left\langle \partial_t^i \partial_{\overline{t}}^j \int_{f=t} m_i \overline{\mu_j} \psi, |t|^{2s}\, dt \wedge d\overline{t} \right\rangle$$

has a pole of order at least $\ell + 1$ at $s = \alpha$. Therefore,

$$s(s-1)\ldots(s-i+1) \cdot s(s-1)\ldots(s-j+1) \int |f|^{2(s-(i+j))} f^j \overline{f}^i m_i \overline{\mu_j}\, \psi dt \wedge d\overline{t}$$

has a pole of order at least $\ell + 1$ at $s = \alpha$. Put then $\varphi = f^j \overline{f}^i m_i \overline{\mu_j}\, \psi dt \wedge d\overline{t}$ and $k = i + j$ to get the assertion for ψ_f.

The assertion for ϕ_f follows from

$$\operatorname{gr}_\ell^M (\phi_1 \mathcal{M})_{x^\circ} \neq 0 \quad \Rightarrow \quad \operatorname{gr}_{\ell+1}^M (\psi_1 \mathcal{M})_{x^\circ} \neq 0, \tag{4.15}$$

for which we briefly recall the proof. Since $\mathcal{O}_Z$ is a simple $\mathcal{D}_Z$-module, $\mathcal{M}$ is a simple $\mathcal{D}_X$-module according to Kashiwara's equivalence theorem. In particular, it has neither submodule nor quotient module supported by Z. This implies (see [14, lemme 5.1.4] forgetting the filtration F) that $\mathrm{can}\colon \psi_1 \to \phi_1$ is onto and $\mathrm{Var}\colon \phi_1 \to \psi_1$ is injective. From [14, lemme 5.1.12] we deduce that, for any ℓ, we have

$$\mathrm{can}(M_\ell \psi_1) \subset M_{\ell-1} \phi_1, \qquad \mathrm{Var}(M_\ell \phi_1) \subset M_{\ell-1} \psi_1$$

and that the induced morphisms

$$\mathrm{can}\colon \operatorname{gr}_\ell^M \psi_1 \to \operatorname{gr}_{\ell-1}^M \phi_1, \qquad \mathrm{Var}\colon \operatorname{gr}_\ell^M \phi_1 \to \operatorname{gr}_{\ell-1}^M \psi_1$$

are onto and injective, respectively. □

Remark 4.16. In [3], D. Barlet introduces the topological notion of "tangling of strata" and shows how this tangling can be detected by inspection of the order of poles of the functions $I_\varphi(s)$. This notion has the following interpretation. Assume as in loc. cit. that, for some eigenvalue $\lambda \neq 1$, the support of $\psi_{f,\lambda} \mathbb{C}_Z$ is a curve Σ near x° and assume furthermore for simplicity that the germ (Σ, x°) is irreducible (one can easily extend what follows to the reducible case). The complex $\psi_{f,\lambda} \mathbb{C}_Z$ is perverse up to a shift by $\dim Z - 1$. Let z be a local coordinate on the normalization of Σ. Consider the corresponding diagram of vector spaces:

$$\bigoplus_\mu \psi_{z,\mu} \psi_{f,\lambda} \mathbb{C}_Z \underset{v}{\overset{c}{\rightleftarrows}} \bigoplus_\mu \phi_{z,\mu} \psi_{f,\lambda} \mathbb{C}_Z.$$

The left-hand term corresponds to the generic fiber of the local system $\psi_{f,\lambda}$ on $\Sigma - \{x^\circ\}$ and $N' = v \circ c$ is the nilpotent part of the monodromy relative to z of this local system. Moreover, $\operatorname{Coker} c$ (respectively, $\operatorname{Ker} v$ and $\operatorname{Ker} c$) is isomorphic to the generalized eigenspace with eigenvalue λ of the cohomology of the Milnor fiber F_{x° of f at x° in maximal degree $\dim Z - 1$ (respectively, the cohomology with compact support and the cohomology in degree $\dim Z - 2$). As usual, c and v are compatible with the direct-sum decomposition indexed by μ, and their μ-components are isomorphisms if $\mu \neq 1$. Moreover, c and v commute with the nilpotent part N of the monodromy of f.

The tangling phenomenon (for the eigenvalue λ) appears when the nilpotency indices of N on the *cohomology sheaves* of $\psi_{f,\lambda}\mathbb{C}_Z$ are strictly smaller than the nilpotency index of N on the *complex* $\psi_{f,\lambda}\mathbb{C}_Z$. The latter can be read from the pole order of functions $I_\varphi(s)$ (Corollary 4.15).

This also means that the nilpotency indices of N on the spaces $\psi_{z,1}\psi_{f,\lambda}\mathbb{C}_Z$ and $\operatorname{Coker} c = H^{\dim Z-1}(F_{x^\circ})_\lambda$ are strictly smaller than the nilpotency index of N on the space $\phi_{z,1}\psi_{f,\lambda}\mathbb{C}_Z$.

This would not happen if c were *strict* relative to the monodromy filtration $M(N)$. In such a case, still denoting by $M(N)$ the monodromy filtration on $\operatorname{Coker} c$, we would have

$$\operatorname{gr}_\ell^M \operatorname{Coker} c = \operatorname{Coker} \operatorname{gr}_\ell^M c,$$

and $\operatorname{gr}_\ell^M \phi_{z,1}\psi_{f,\lambda}\mathbb{C}_Z$ would vanish as soon as $\operatorname{gr}_\ell^M \psi_{z,1}\psi_{f,\lambda}\mathbb{C}_Z$ and $\operatorname{gr}_\ell^M H^{\dim Z-1}(F_{x^\circ})_\lambda$ do so.

More generally, as $\operatorname{Im} c$ and $\operatorname{Ker} v$ are stable by N, the tangling phenomenon would not happen if $\phi_{z,1}\psi_{f,\lambda}\mathbb{C}_Z$ could be decomposed as $\operatorname{Im} c \oplus \operatorname{Ker} v$, which is equivalent to the property that the canonical morphism $H_c^{\dim Z-1}(F_{x^\circ})_\lambda \to H^{\dim Z-1}(F_{x^\circ})_\lambda$ (i.e., $\operatorname{Ker} v \to \operatorname{Coker} c$) is an isomorphism (or injective, or onto, as $\dim \operatorname{Ker} v = \dim \operatorname{Coker} c$ by the duality and self-conjugation of $\psi_f\mathbb{C}_Z$). When such an isomorphism occurs, there is no “topological tangling” in the sense of Barlet [3].

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