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HODGE THEORY, SINGULARITIES AND $\mathscr{D}$-MODULES

LECTURE NOTES (CIRM, LUMINY, MARCH 2007)

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Abstract. These notes, which consist of five lectures, intend to explain the notion of a polarized Hodge $\mathscr{D}$-module, after the work of M. Saito, and give some applications.

– In the first lecture, we recall classical results of Hodge theory on smooth complex projective varieties and we introduce the notion of a (polarized) Hodge structure.

– In the second lecture, we introduce the notion of a variation of Hodge structure parametrized by a Riemann surface. Analyzing the problems which arise when the Riemann surface is a punctured disc, we introduce the notion of a Hodge $\mathscr{D}$-module on a disc.

– In the third lecture, we consider global questions on a compact Riemann surface, and show how L^2-cohomology enters in the story.

– In the fourth lecture, we introduce the notion a a Hodge $\mathscr{D}$-module in any dimension and explain the Hodge theorem in this context.

– In the last lecture, we give examples of applications of the theory, and we try to show how these tools can be used in some explicit problems.

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INTRODUCTION

Hodge theory decomposes the cohomology of any smooth complex projective variety X (more generally, any compact Kähler complex manifold):

$$\forall\, k \in \mathbb{N}, \quad H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ stands for $H^q(X, \Omega_X^p)$ or, equivalently, for the Dolbeault cohomology space $H_{d''}^{p,q}(X)$. This result is classically [(1)] proved by methods of analysis (Hodge theory for the Laplace operator) and is of a global nature: the cohomology of a non-compact Kähler manifold, or of a non-projective smooth quasi-projective variety, does not usually satisfy this decomposition property.

Singularities of the variety also prevent from such a decomposition. As we will mainly work with projective varieties, the singularities can be introduced by considering other coefficients than the constant sheaf $\mathbb{C}_X$ on a smooth variety X. Therefore, in the following, X will denote a smooth complex projective variety (for instance the projective space). A system of coefficients will be a complex $\mathscr{F}^\bullet$ of sheaves on X of $\mathbb{C}$-vector spaces (for instance the constant sheaf $\mathbb{C}_X$). We will be interested in the hypercohomology $\mathbb{H}^*(X, \mathscr{F}^\bullet)$ of X with coefficients in $\mathscr{F}^\bullet$. Let us note that the support Z of the complex $\mathscr{F}^\bullet$ is possibly smaller than X and could have singularities.

The first reasonable assumption to be made on $\mathscr{F}^\bullet$, in order that the hypercohomology is finite dimensional, is that $\mathscr{F}^\bullet$ is a *bounded complex with constructible cohomology*. Its support Z is then a projective subvariety of X. A classical result (Verdier) then gives the finiteness of the hypercohomology spaces.

The second reasonable assumption to be made on $\mathscr{F}^\bullet$, in order that the hypercohomology satisfies Poincaré duality is that $\mathscr{F}^\bullet$ is a self-dual *perverse complex* (Goresky-MacPherson). Typically, $\mathscr{F}^\bullet$ is the intersection complex with coefficients in a local system defined on the smooth part of an irreducible projective variety, and the local system is assumed to be isomorphic to its dual local system.

1. See, however, Remark 1.2.6.

Thirdly, in order that the hypercohomology spaces to be defined over $\mathbb{R}$, so that it is meaningful to write $H^{q,p} = \overline{H^{p,q}}$, the complex itself should be defined over $\mathbb{R}$ or, better, over $\mathbb{Q}$: there should exist a complex of $\mathbb{Q}$-vector spaces $\mathscr{F}_{\mathbb{Q}}^{\bullet}$ such that $\mathscr{F}^{\bullet} = \mathbb{C} \otimes_{\mathbb{Q}} \mathscr{F}_{\mathbb{Q}}^{\bullet}$. In the example above, the local system should be defined over $\mathbb{Q}$, *i.e.*, should be a local system of $\mathbb{Q}$-vector spaces (so that the all monodromy matrices can simultaneously be chosen with entries in $\mathbb{Q}$).

Let us remark that the first two assumptions above are of a local nature: they can be defined and checked locally on X (for constructibility, this is not completely trivial). However, the third one is global (checking locally the existence of such an $\mathscr{F}_{\mathbb{Q}}^{\bullet}$ is not enough, we need to glue the local $\mathbb{Q}$-perverse sheaves into a global one).

It is therefore natural to ask: what kind of local assumptions on $\mathscr{F}^{\bullet}$ are sufficient in order to get a Hodge decomposition of the hypercohomology with coefficients in $\mathscr{F}^{\bullet}$ (assuming the existence of a $\mathbb{Q}$-structure)?

The most general answer to this question has been given by M. Saito [**24**]: if we assume that $\mathscr{F}^{\bullet}$ underlies a *polarizable Hodge $\mathscr{D}_X$-module*, then we get the desired Hodge properties on the hypercohomology of $\mathscr{F}^{\bullet}$. As we will see, in the definition of a polarizable Hodge $\mathscr{D}_X$-module, there are assumptions of a local nature, but the existence of a $\mathbb{Q}$-structure remains global. A Hodge $\mathscr{D}$-module will then consist of a triple $(\mathscr{M}, \mathscr{F}_{\mathbb{Q}}^{\bullet}, \alpha)$ consisting of a (filtered) $\mathscr{D}$-module, a bounded complex with $\mathbb{Q}$-constructible cohomology and an isomorphism $\alpha : \mathrm{DR}\,\mathscr{M} \xrightarrow{\sim} \mathscr{F}_{\mathbb{Q}}^{\bullet} \otimes_{\mathbb{Q}} \mathbb{C}$. This set of data has a local definition, but the existence of such an object on a complex projective variety will lead to a Hodge structure on the hypercohomology of $\mathrm{DR}\,\mathscr{M}$. Finding a sufficient set of conditions on such a set of data in order to imply the previous global result is one of the main achievements of M. Saito. On the other hand, knowing that such a set of data satisfies the local conditions implies strong properties of Hodge-theoretic nature for singularities. Gathering these properties in a global situation leads to rigidity properties: on a given projective variety, the number and position of singularities is constrained by the necessity that the local Hodge properties are compatible with the global ones.

This result was obtained as at the end of a long way. Let us sketch it briefly, as it will serve us as the guide for these notes.

- Deligne introduced the abstract notion of polarized Hodge structure, and remarked that if $\mathscr{F}^{\bullet}$ underlies a *variation of polarized Hodge structure* (in particular, $\mathscr{F}^{\bullet}$ is a locally constant sheaf), then Hodge theory applies to the hypercohomology with coefficients in $\mathscr{F}^{\bullet}$.
- Griffiths and Schmid developed a detailed analysis of such variations on a punctured disc and Schmid introduced the notion of *limit mixed Hodge structure*.
- Zucker has combined the global result of Deligne and the local results of Schmid to obtain Hodge decomposition of the cohomology of a local system on a quasi-projective curve (compact Riemann surface with a finite number of points removed), when this local system underlies a variation of polarized Hodge structure (see § 1.1 for a precise statement).

– Some years before, Steenbrink had revisited the result of Schmid when the variation of Hodge structures is defined by a one-parameter family of smooth projective varieties (Gauss-Manin connection). He was able to reprove one of the theorems of Schmid by a local analysis of the singularities of the special fibre of the family. This analysis was done on the resolution of singularities of this special fibre.

– Later, Varchenko gave, in the case of a family given by a germ of a complex function on $\mathbb{C}^n$ having an isolated singularity, a interpretation of the mixed Hodge structure on the vanishing cycles purely in terms of local properties of the singularity. At this point, there is no more need to descend to the parameter space (the disc). This construction was revisited by Scherk and Steenbrink.

– M. Saito extracted from the previous construction a general method to define Hodge $\mathscr{D}$-modules (pure and mixed) and proved the Hodge decomposition by reducing, with the help of a Lefschetz pencil, to the case of curves, a case treated by Zucker.

In the classical Hodge theory, three kinds of cohomologies are related: the singular cohomology (also called Betti cohomology), the holomorphic de Rham cohomology (through the hypercohomology of the holomorphic de Rham complex) and the Dolbeault cohomology. The first two can be generalized in the framework of Hodge $\mathscr{D}$-modules. However, the third one, of a more C^∞ nature, is not easily defined in this framework. This is the reason why we will insist on the *Hodge filtration* instead of the *Hodge decomposition*. Indeed, the Hodge filtration can be defined at the level of de Rham cohomology.

Good general references are [**10**], [**33**], [**21**]. Details on the analysis in Hodge theory can be found in [**7**].

LECTURE 1

HODGE THEORY: REVIEW OF CLASSICAL RESULTS

1.1. Hodge theory on compact Riemann surfaces

Let X be a compact Riemann surface of genus $g \geqslant 0$. Let us assume for simplicity that it is connected. Then $H^0(X,\mathbb{Z})$ and $H^2(X,\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$ (as X is orientable). The only interesting cohomology group is $H^1(X,\mathbb{Z})$, isomorphic to $\mathbb{Z}^{2g}$.

Poincaré duality induces a skewsymmetric non-degenerate bilinear form

$$\langle\bullet,\bullet\rangle : H^1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X,\mathbb{Z}) \xrightarrow{\bullet\cup\bullet} H^2(X,\mathbb{Z}) \xrightarrow{\int_{[X]}} \mathbb{Z}.$$

One of the main analytic results asserts that the space $H^1(X,\mathscr{O}_X)$ is finite dimensional and has dimension equal to the genus g (see *e.g.*, [**22**, Chap. IX] for a direct approach). Then, *Serre's duality* $H^1(X,\mathscr{O}_X) \xrightarrow{\sim} H^0(X,\Omega^1_X)^\vee$ also gives $\dim H^0(X,\Omega^1_X) = g$. A dimension count implies then the Hodge decomposition

$$H^1(X,\mathbb{C}) = H^{0,1}(X) \oplus H^{1,0}(X), \quad H^{0,1}(X) = H^1(X,\mathscr{O}_X), \quad H^{1,0}(X) = H^0(X,\Omega^1_X).$$

If we regard Serre's duality as the pairing

$$H^{1,0} \otimes_{\mathbb{C}} H^{0,1} \xrightarrow{\bullet\wedge\bullet} H^{1,1} \xrightarrow{\int} \mathbb{C},$$

then Serre's duality is equivalent to the complexified Poincaré duality pairing

$$\langle\bullet,\bullet\rangle_{\mathbb{C}} : H^1(X,\mathbb{C}) \otimes_{\mathbb{C}} H^1(X,\mathbb{C}) \longrightarrow \mathbb{C},$$

as $\langle H^{1,0}, H^{1,0}\rangle = 0$ and $\langle H^{0,1}, H^{0,1}\rangle = 0$.

With respect to the real structure $H^1(X,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} H^1(X,\mathbb{R})$, $H^{1,0}$ is conjugate to $H^{0,1}$, and using Serre's duality (or Poincaré duality) we get a sesquilinear pairing

$$k : H^{1,0} \otimes_{\mathbb{C}} \overline{H^{1,0}} \longrightarrow \mathbb{C}.$$

Then, the Hodge-Riemann bilinear relations assert that $h = ik$ is a *positive definite Hermitian form.*

1.2. Hodge theory of smooth projective varieties

Let X be a smooth complex projective variety of pure complex dimension n (*i.e.*, each of its connected component has dimension n). It will be endowed with the usual topology, which makes it a complex analytic manifold. The classical Hodge theory asserts that each cohomology space $H^k(X,\mathbb{C})$ decomposes as the direct sum

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{1.2.1}$$

where $H^{p,q}(X)$ stands for $H^q(X,\Omega_X^p)$ or, equivalently, for the Dolbeault cohomology space $H^{p,q}_{d''}(X)$. Although this result is classically proved by methods of analysis (Hodge theory for the Laplace operator), it can be expressed in a purely algebraic way, by means of the de Rham complex.

The holomorphic de Rham complex is the complex of sheaves $(\Omega_X^\bullet, d)$, where d is the differential, sending a k-form to a $k+1$-form. Recall (holomorphic Poincaré Lemma) that $(\Omega_X^\bullet, d)$ is a resolution of the constant sheaf. Therefore, the cohomology $H^k(X,\mathbb{C})$ is canonically identified with the hypercohomology $\mathbb{H}^k\big(X,(\Omega_X^\bullet,d)\big)$ of the de Rham complex.

***Exercise 1.2.2* (Algebraic de Rham complex).** Using the *Zariski topology* on X, we get an algebraic variety denoted by X^{alg}. In the algebraic category, it is also possible to define a de Rham complex, called the *algebraic de Rham complex.*

(1) Is the algebraic de Rham complex a resolution of the constant sheaf $\mathbb{C}_{X^{\mathrm{alg}}}$?

(2) Do we have $H^*(X^{\mathrm{alg}},\mathbb{C}) = \mathbb{H}^*\big(X^{\mathrm{alg}},(\Omega^\bullet_{X^{\mathrm{alg}}},d)\big)$?

The de Rham complex can be filtered in a natural way by subcomplexes ("filtration bête" in [**4**]).

Remark 1.2.3. In general, we denote by an upper index a *decreasing filtration* and by a lower index an *increasing filtration*. Filtrations are indexed by $\mathbb{Z}$.

We define the "stupid" (increasing) filtration on $\mathscr{O}_X$ by setting

$$F_p\mathscr{O}_X = \begin{cases} \mathscr{O}_X & \text{if } p \geqslant 0, \\ 0 & \text{if } p \leqslant -1. \end{cases}$$

Observe that, trivially, $d(F_p\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega_X^k) \subset F_{p+1}\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega_X^{k+1}$. Therefore, the de Rham complex can be filtered by

$$F^p(\Omega_X^\bullet, d) = \{0 \longrightarrow F_{-p}\mathscr{O}_X \xrightarrow{\;d\;} F_{-p+1}\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega_X^1 \xrightarrow{\;d\;} \cdots\}. \tag{1.2.4}$$

If $p \leqslant 0$, $F^p(\Omega_X^\bullet, d) = (\Omega_X^\bullet, d)$, although if $p \geqslant 1$,

$$F^p(\Omega_X^\bullet, d) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow \cdots \longrightarrow \Omega_X^{\dim X} \longrightarrow 0\}.$$

Therefore, the p-th graded complex is 0 if $p \leqslant -1$ and, if $p \geqslant 0$, it is given by

$$\mathrm{gr}_F^p(\Omega_X^\bullet, d) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\}.$$

In other words, the graded complex $\mathrm{gr}_F(\Omega_X^\bullet, d) = \bigoplus_p \mathrm{gr}_F^p(\Omega_X^\bullet, d)$, is the complex $(\Omega_X^\bullet, 0)$ (*i.e.*, the same terms as for the de Rham complex, but with differential equal to 0).

From general results on filtered complexes, the filtration of the de Rham complex induces a (decreasing) filtration on the hypercohomology spaces (that is, on the de Rham cohomology of X) and there is a spectral sequence starting from $\mathbb{H}^*\big(X, \mathrm{gr}_F(\Omega_X^\bullet, d)\big)$ and abutting to $\mathrm{gr}_F H^*(X, \mathbb{C})$. Let us note that $\mathbb{H}^*\big(X, \mathrm{gr}_F(\Omega_X^\bullet, d)\big)$ is nothing but $\bigoplus_{p,q} H^q(X, \Omega_X^p)$.

Theorem 1.2.5. *The spectral sequence of the filtered de Rham complex on a smooth projective variety degenerates at E^1, that is,*

$$H^*(X, \mathbb{C}) \simeq H^*_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p,q} H^q(X, \Omega_X^p).$$

Remark 1.2.6. Although the classical proof uses Hodge theory for the Laplace operator which is valid in the general case of compact Kähler manifolds, there is a purely algebraic/arithmetic proof in the projective case, due to Deligne and Illusie [**6**].

For any k, Poincaré duality is a non-degenerate bilinear pairing

$$\langle \bullet, \bullet \rangle_{n-k} : H^{n-k}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^{n+k}(X, \mathbb{Z}) \xrightarrow{\ \bullet \cup \bullet\ } H^{2n}(X, \mathbb{Z}) \xrightarrow{\ \int_{[X]}\ } \mathbb{Z}.$$

In particular (taking $k = 0$), we get a non-degenerate bilinear form on $H^n(X, \mathbb{Z})$. For any $k \in \mathbb{Z}$, we set [(1)] $\varepsilon(k) = (-1)^{k(k-1)/2}$. Let us set $Q_n(\bullet, \bullet) = \varepsilon(n)\langle \bullet, \bullet \rangle_0$. Then Q_n is $(-1)^n$-symmetric and

$$Q_n(H^{p,n-p}, H^{p',n-p'}) = 0 \quad \text{if } p + p' \neq n.$$

As $H^{n-p,p} = \overline{H^{p,n-p}}$, we regard Q_n as a sesquilinear pairing on $H^{p,n-p}$, that we denote $k_{p,n-p}$. In order to obtain a Hermitian form, one has to be careful by choosing the right power of i:

$$h_{p,n-p} := (-1)^p i^{-n} k_{p,n-p} \text{ is a non-degenerate Hermitian form on } H^{p,n-p}.$$

In order to obtain similar results on the spaces $H^{n-k}(X, \mathbb{C})$ for $k \neq 0$, it is necessary to choose an isomorphism between the vector spaces $H^{n-k}(X, \mathbb{C})$ and $H^{n+k}(X, \mathbb{C})$ (we know that they have the same dimension, as Poincaré duality is non-degenerate). A class of good morphisms is given by the *Lefschetz operators* that we define now.

Fix an ample line bundle $\mathscr{L}$ on X (for instance, any embedding of X in a projective space defines a very ample bundle, by restricting the canonical line bundle $\mathscr{O}(1)$ of the projective space to X). The first Chern class $c_1(\mathscr{L}) \in H^2(X, \mathbb{Z})$ defines a Lefschetz operator

$$\mathrm{L}_{\mathscr{L}} := c_1(\mathscr{L}) \cup \bullet : H^k(X, \mathbb{Z}) \longrightarrow H^{k+2}(X, \mathbb{Z}).$$

---

1. The presence of $\varepsilon(n)$ and of powers of i basically comes from the following relation on $\mathbb{C}^n$: if we take complex coordinates $z_1, \dots, z_n$ and set $z_j = x_j + y_j$, then the volume form (giving the orientation) is $2^n dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$, and can also be written as $(-1)^{n(n-1)/2} i^n (dz_1 \wedge d\overline{z}_1) \wedge \dots \wedge (dz_n \wedge d\overline{z}_n)$.

(Note that, wedging on the left or on the right amounts to the same, as c_1 has degree 2.)

The *Hard Lefschetz theorem*, usually proved together with the previous results of Hodge theory, asserts that, *for any smooth complex projective variety X, any ample line bundle $\mathscr{L}$, and any $k \geqslant 1$, the k-th power $\mathrm{L}_{\mathscr{L}}^k : H^{n-k}(X, \mathbb{Q}) \to H^{n+k}(X, \mathbb{Q})$ is an isomorphism.* (2)

If we fix such a Lefschetz operator, we can identify $H^{n+k}(X, \mathbb{Q})$ to $H^{n-k}(X, \mathbb{Q})$ with $\mathrm{L}_{\mathscr{L}}^k$ and get a bilinear form Q_{n-k} on $H^{n-k}(X, \mathbb{Q})$ by setting

$$Q_{n-k}(u, v) := \varepsilon(n-k)\langle u, \mathrm{L}_{\mathscr{L}}^k v\rangle_k$$

In such a way, one obtains a $(-1)^{n-k}$-symmetric non-degenerate bilinear form which satisfies, for any p, q and p', q' with $p + q = p' + q' = n - k$:

$$Q_{n-k}(H^{p,q}, H^{p',q'}) = 0 \quad \text{if } p + p' \neq n - k.$$

Therefore, for any p, q with $p+q = n-k$, one is left with a *Hermitian form* as above:

$$h_{p,q} := i^{p-q} k_{p,q} = (-1)^p i^{-(n-k)} k_{p,q} \quad \text{on } H^{p,q}.$$

Let us note however that this Hermitian form is possibly not positive definite (in general). In order *to get positivity*, we have to restrict it to the *primitive part* defined as follows. One note that the Lefschetz operator has type $(1, 1)$ with respect to the Hodge decomposition, hence sends $H^{p,q}$ to $H^{p+1,q+1}$. Therefore, for any $k \geqslant 0$, $\mathrm{L}_{\mathscr{L}}^k$ induces and isomorphism $H^{p,q} \to H^{p+k,q+k}$ for any p, q with $p + q = n - k$. The primitive part $P^{p,q}$ is by definition the kernel of $\mathrm{L}_{\mathscr{L}}^{k+1} : H^{p,q} \to H^{p+k+1,q+k+1}$.

For instance, when $k = 0$, we get positivity on $\operatorname{Ker} \mathrm{L}_{\mathscr{L}} : H^{p,n-p} \to H^{p+1,n-p+1}$. In the case of curves ($n = 1$), such a restriction is empty as, whatever the choice of $\mathscr{L}$ is, we have $\mathrm{L}_{\mathscr{L}} = 0$ on $H^1(X, \mathbb{Q})$ (as it takes values in $H^3 = 0$).

1.3. Polarized Hodge structures

The previous properties of the cohomology of a projective variety can be put in an axiomatic form. This will happen to be useful as a first step to Hodge $\mathscr{D}$-modules.

1.3.a. Hodge structures. This is, in some sense, a category looking like that of finite dimensional complex vector spaces. In particular, it is *abelian*, that is, the kernel and cokernel of a morphism exist in this category. This category is very useful as an intermediate category for building that of mixed Hodge structures, but the main results in Hodge theory use a supplementary property, namely the existence of a polarization (*cf.* § 1.3.b)

***Definition 1.3.1* (Hodge structures).** Given a finite dimensional $\mathbb{C}$-vector space H and an integer $w \in \mathbb{Z}$, a *pure Hodge structure of weight w* on H consists of

2. It is known that the same statement is not true in general if one replaces the coefficients $\mathbb{Q}$ with $\mathbb{Z}$.

(1) a decomposition $H = \bigoplus_{p\in\mathbb{Z}} H^p$, that it will be more convenient to denote as $H = \bigoplus_{p+q=w} H^{p,q}$ in order to keep in mind the weight w,

(2) a $\mathbb{Q}$-structure, that is, a $\mathbb{Q}$-vector subspace $H_{\mathbb{Q}} \subset H$ such that $H = \mathbb{C} \otimes_{\mathbb{Q}} H_{\mathbb{Q}}$, giving therefore also a real structure on H and a conjugation operator.

These data are subject to the relation

$$\forall\, p, q \text{ with } p+q=w, \quad H^{q,p} = \overline{H^{p,q}} \quad \text{in } H.$$

It will be useful, when considering families, to introduce the *Hodge filtration*, which is the decreasing filtration of H defined by

$$F^pH = \bigoplus_{p'\geqslant p} H^{p',w-p'}.$$

***Exercise 1.3.2* (Properties of the Hodge filtration).** Given any decreasing filtration $F^\bullet H$ of H by vector subspaces, show that the following are equivalent:

(1) the filtration $F^\bullet H$ and its complex conjugate $\overline{F^\bullet H}$ are w-opposite, that is, for any p, $F^pH \cap \overline{F^{w-p+1}H} = 0$;

(2) setting $H^{p,w-p} = F^pH \cap \overline{F^{w-p}H}$, then $H^{w-p,p} = \overline{H^{p,w-p}}$ and $H = \bigoplus_p H^{p,w-p}$.

Let us introduce a new variable z and let us consider in the free $\mathbb{C}[z,z^{-1}]$-module $\mathscr{H} := \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}} H$ the object $\mathscr{F} := \bigoplus_p F^pHz^{-p}$; show that $\mathscr{F}$ is a $\mathbb{C}[z]$-submodule of $\mathscr{H}$ which generates $\mathscr{H}$, that is, $\mathscr{H} = \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z]} \mathscr{F}$. Similarly, denote by $\overline{\mathscr{F}}$ the object [3] $\bigoplus_q \overline{F^qH} z^q$; show that $\overline{\mathscr{F}}$ is a $\mathbb{C}[z^{-1}]$-submodule of $\mathscr{H}$ which generates $\mathscr{H}$, that is, $\mathscr{H} = \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z^{-1}]} \overline{\mathscr{F}}$. Using the gluing

$$\mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z]} \mathscr{F} \xrightarrow{\ \sim\ } \mathscr{H} \xleftarrow{\ \sim\ } \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z^{-1}]} \overline{\mathscr{F}}$$

the pair $(\mathscr{F}, \overline{\mathscr{F}})$ defines an algebraic vector bundle on $\mathbb{P}^1$ of rank $\dim H$. Show that the properties (1) and (2) are also equivalent to

(3) The vector bundle determined by $(\mathscr{F}, \overline{\mathscr{F}})$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(w)^{\dim H}$.

***Exercise 1.3.3* (The category of Hodge structures).** The category of Hodge structures is defined as follows:

(a) the objects are Hodge structures of some weight, as defined above,

(b) the morphisms are the $\mathbb{Q}$-linear morphisms between the underlying $\mathbb{Q}$-vector spaces which are compatible with the Hodge filtration, *i.e.*, sends F^p into F^p for any p.

Prove the following properties:

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3. Notice that the effect of the conjugation on $\mathscr{F}$ is not restricted to F^qH, but it also transforms z into z^{-1}; in fact the right conjugation should transform z to $-z^{-1}$, but this will not matter here.

(1) If one defines the filtration on the tensor product as

$$F^p(H_1 \otimes H_2) = \sum_{p_1+p_2=p} F^{p_1}H_1 \otimes F^{p_2}H_2,$$

the tensor product of Hodge structures of weight w_1, w_2 is a Hodge structure of weight $w_1 + w_2$.

(2) If one defines the filtration on the space of linear morphisms as

$$F^p \operatorname{Hom}(H_1, H_2) = \{f \in \operatorname{Hom}(H_1, H_2) \mid \forall\, k \in \mathbb{Z},\ f(F^k H_1) \subset F^{p+k} H_2\},$$

the Hom of Hodge structures of weight w_1, w_2 is a Hodge structure of weight $w_2 - w_1$. Conclude that the dual of a Hodge structure of weight w is a Hodge structure of weight $-w$.

(3) Show that a morphism of Hodge structures of the same weight preserves the Hodge decomposition as well. Deduce that, if $(F^\bullet H', H'_\mathbb{Q})$ is a Hodge sub-structure of weight w of $(F^\bullet H, H_\mathbb{Q})$, that is, if $H'_\mathbb{Q} \subset H_\mathbb{Q}$ and $F^\bullet H' = H' \cap F^\bullet H$ define a Hodge structure of weight w, then $H'^{p,q} \subset H^{p,q}$.

(4) Show that a morphism between Hodge structures of weights w_1, w_2 induces a morphism between the associated vector bundles on $\mathbb{P}^1$ (*cf.* 1.3.2(3)). Conclude that there is no non-zero morphism if $w_1 > w_2$.

(5) Show that any morphism $f : H_1 \to H_2$ between Hodge structures of weights w_1, w_2 is *strictly* compatible with the Hodge filtration, that is, $f(F^\bullet H_1) = f(H_1) \cap F^\bullet H_2$.

(6) Let $f : H_1 \to H_2$ be a morphism of Hodge structures (*i.e.*, an element of $F^0 \operatorname{Hom}(H_1, H_2) \cap \operatorname{Hom}_\mathbb{Q}(H_{1,\mathbb{Q}}, H_{2,\mathbb{Q}})$). Show that the kernel of f, equipped with the filtration induced by $F^\bullet H_1$, is a Hodge structure of weight w_1 (use 1.3.2(1)). By duality, prove that $\operatorname{Coker} f$ (equipped with the filtration induced by $F^\bullet H_2$) is a Hodge structure of weight w_2. Conclude that the category of Hodge structures is *abelian*.

***Exercise 1.3.4* (The Tate twist).** It is often useful to change the weight of a Hodge structure. We proceed as for homogeneous polynomials, were one can change the degree by multiplying by a monomial. A similar operation can be done on vector bundles on the Riemann sphere, by tensoring with a line bundle $\mathscr{O}(k)$.

The trivial Hodge structure is called $\mathbb{Z}(0)$: this is the complex space $\mathbb{C}$ equipped with the usual $\mathbb{Q}$-structure (and even $\mathbb{Z}$-structure) and Hodge decomposition having only a $(0,0)$-component.

The most basic Hodge structure (defined over $\mathbb{Z}$) is called $\mathbb{Z}(1)$: the underlying vector space is $\mathbb{C}$, but the $\mathbb{Q}$-structure is $2\pi i\mathbb{Q}$ (or $2\pi i\mathbb{Z}$) and it is considered as a Hodge structure of weight -2, *i.e.*, has only a $(-1,-1)$-component.

Using Exercise 1.3.3, prove that

(1) if one defines, for any $k \in \mathbb{Z}$, $\mathbb{Z}(k)$ as $\mathbb{Z}(1)^{\otimes k}$, then $\mathbb{Z}(k)$ has weight $-2k$;

(2) If H is a Hodge structure of weight w, then $H(k) := \mathbb{Z}(k) \otimes H$ has weight $w - 2k$, it has the decomposition $H(k)^{p,q} = H^{p+k,q+k}$ and Hodge filtration $F^p(H(k)) = F^{p+k}H$;

(3) If $f : H_1 \to H_2$ is a morphism of Hodge structures, then it is also a morphism of twisted Hodge structures $H_1(k) \to H_2(k)$.

We say that a morphism $H_1 \to H_2$ is a morphism of type (k,k) of Hodge structures if it is a morphism of Hodge structures $H_1 \to H_2(k)$. Prove that a morphism $H_1 \to H_2$ has type (k,k) if and only if it belongs to $F^k \operatorname{Hom}(H_1, H_2)$.

Exercise 1.3.5 **(The Hodge polynomial).** Let H be a Hodge structure of weight w with Hodge decomposition $H = \bigoplus_{p+q=w} H^{p,q}$. The Hodge polynomial $P_h(H) \in \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ is the two-variable Laurent polynomial defined as $\sum_{p,q\in\mathbb{Z}} h^{p,q} u^p v^q$ with $h^{p,q} = \dim H^{p,q}$. This is a homogeneous Laurent polynomial of degree w. Show the following formulas:

$$\begin{aligned} P_h(H_1 \otimes H_2)(u,v) &= P_h(H_1)(u,v) \cdot P_h(H_2)(u,v), \\ P_h(\operatorname{Hom}(H_1,H_2))(u,v) &= P_h(H_1)(u^{-1},v^{-1}) \cdot P_h(H_2)(u,v), \\ P_h(H^\vee)(u,v) &= P_h(H)(u^{-1},v^{-1}), \\ P_h(H(k))(u,v) &= P_h(H)(u,v) \cdot (uv)^{-k}. \end{aligned}$$

1.3.b. Polarized Hodge structures. In the same way the category of Hodge structures looks like that of complex vector spaces, that of *polarized Hodge structures* looks like that of vector spaces equipped with a positive definite Hermitian form. It will be semisimple, that is, any object can be decomposed into a orthogonal direct sum of irreducible objects. However, although irreducible vector spaces with a positive definite Hermitian form have dimension one (this follows from the classification of positive definite Hermitian form) this does not remain true (fortunately) for polarized Hodge structures. A better analogy would be to consider $\mathbb{Q}$-vector spaces with a positive definite quadratic form: indeed, this is nothing but polarized Hodge structures of type $(0,0)$. The category of polarized Hodge structures will have enough rigidity to be stable under various operations of algebraic geometry.

Definition 1.3.6 **(Polarization).** A *polarization* of a Hodge structure $(F^\bullet H, H_\mathbb{Q})$ of weight w consists of a non-degenerate bilinear pairing Q on $H_\mathbb{Q}$ such that,

(a) Q is $(-1)^w$-symmetric,

(b) $Q(H^{p,w-p}, H^{p',w-p'}) = 0$ if $p' \neq w - p$, and

(c) on each $H^{p,w-p}$, the associated Hermitian form $h_{p,w-p} := (-1)^p i^{-w} k_{p,w-p}$ is positive definite, where $k_{p,w-p}$ is the sesquilinear pairing on $H^{p,w-p}$ induced by Q.

Remark 1.3.7. It may be useful to express Q as a morphism of Hodge structures. According to (b), it defines a morphism $(H, F^\bullet H) \otimes (H, F^\bullet H) \to \mathbb{C}(-w)$ but, when restricted to $H_\mathbb{Q}$, takes values in $\mathbb{Q}$. In order that it takes values in $\mathbb{Q}(-w)$, we should multiply it by $(2\pi i)^{-w}$. Therefore, (a) and (b) are equivalent to giving a non-degenerate morphism of Hodge structures

$$S_{\mathbb{Q}} = (2\pi i)^{-w} Q : (F^\bullet H, H_{\mathbb{Q}}) \otimes (F^\bullet H, H_{\mathbb{Q}}) \longrightarrow \mathbb{Q}(-w),$$

whose sesquilinear complex extension $S : H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ is *Hermitian*. Then (c) means that, when restricted to $H^{p,w-p}$, S is $(-1)^p$-positive definite.

Remark 1.3.8. Although we kept a notation very similar to that of the geometric case of § 1.2, one should be careful, when applying the previous definition to Q_{n-k}, to give H the interpretation of the primitive part (with respect to a chosen ample line bundle) of $H^{n-k}(X, \mathbb{C})$. On the other hand, the previous definition applies to $H^{n-k}(X, \mathbb{C})$, but the definition of the corresponding Q is not Q_{n-k}, but is given by a formula depending on the ample line bundle, that we will not make explicit here. Let us notice that the dependence with respect to the choice of the ample line bundle explains the word 'polarization'. The notion of a Hodge-Lefschetz structure (*i.e.*, Hodge structure with a unipotent automorphism, or nilpotent endomorphism) that we introduce in § 1.4 will give the right analogue for the structure on the cohomology of a complex projective manifold (*cf.* Remark 1.4.15).

Proposition 1.3.9. *The category of polarized Hodge structures of weight w (the morphisms should be moreover compatible with the bilinear forms) is semisimple, i.e., any object can be decomposed as the direct sum of simple objects.*

***Exercise 1.3.10* (Polarization on Hodge sub-structures).** Let $S_{\mathbb{Q}}$ be a polarization of a Hodge structure $(F^\bullet H, H_{\mathbb{Q}})$ of weight w. Let $(F^\bullet H', H'_{\mathbb{Q}})$ be a Hodge sub-structure of weight w of $(F^\bullet H, H_{\mathbb{Q}})$ (*cf.* Exercise 1.3.3(3)).

(1) Show that the restriction $S'_{\mathbb{Q}}$ of $S_{\mathbb{Q}}$ to $H'_{\mathbb{Q}}$ is a polarization of $(F^\bullet H', H'_{\mathbb{Q}})$. (*Hint*: use that the restriction of a positive definite Hermitian form to a subspace remains positive definite.)

(2) Deduce that $(F^\bullet H', H'_{\mathbb{Q}})$ is a direct summand of $(F^\bullet H, H_{\mathbb{Q}})$ in the category of Hodge structures, hence the proof of Proposition 1.3.9

Exercise 1.3.11. Write down the effect of a Tate twist on $S_{\mathbb{Q}}$.

1.4. Polarized Hodge-Lefschetz structures

Let $H_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space equipped with a linear automorphism T. We denote by T_s and T_u the semi-simple and unipotent part of T. Let $F^\bullet H$ be a decreasing filtration of $H = \mathbb{C} \otimes_{\mathbb{Q}} H_{\mathbb{Q}}$. We will define the notion of a (quasi) Hodge-Lefschetz structure of weight $w \in \mathbb{Z}$ relative to the automorphism.

1.4.a. Hodge-Lefschetz structures. Let us first assume that $T_s = \mathrm{Id}$. Let N be a nilpotent endomorphism of $H_{\mathbb{Q}}$ (which could be $T_u - \mathrm{Id}$, but this will not be the final choice).

***Lemma 1.4.1* (Jakobson-Morosov).** *There exists a unique increasing filtration of $H_{\mathbb{Q}}$ indexed by $\mathbb{Z}$, called the* monodromy filtration *relative to* N *and denoted by* $\mathrm{M}_\bullet(\mathrm{N})$, *satisfying the following properties:*

(a) *For any $\ell \in \mathbb{Z}$,* $\mathrm{N}(\mathrm{M}_\ell) \subset \mathrm{M}_{\ell-2}$,

(b) *For any $\ell \geqslant 1$,* N^ℓ *induces an isomorphism* $\mathrm{gr}^{\mathrm{M}}_\ell H_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{M}}_{-\ell} H_{\mathbb{Q}}$.

Example 1.4.2. If N consists only of one lower Jordan block of size $k+1$, one can write the basis as $e_k, e_{k-2}, \dots, e_{-k}$, with $\mathrm{N}e_j = e_{j-2}$. Then M_ℓ is the space generated by the e_j's with $j \leqslant \ell$.

Remark 1.4.3. The proof of the lemma is left as an exercise. One can prove the existence by using the decomposition into Jordan blocks and Example 1.4.2. The uniqueness is interesting to prove. In fact, there is an explicit formula for this filtration in terms of the kernel filtration of N and of its image filtration (*cf.* [**31**]). The choice of a splitting of the filtration (which always exists for a filtration on a finite dimensional vector space) corresponds to the choice of a decomposition of a Jordan block of N. The decomposition (hence the splitting) is not unique, although the filtration is. The filtration exists in the larger context of a nilpotent endomorphism of an object in an abelian category, even though there does not exist a Jordan decomposition. We can apply it to the category of holonomic $\mathscr{D}$-modules for instance.

Exercise 1.4.4. We denote by $\mathrm{grN} : \mathrm{gr}^{\mathrm{M}}_\ell H \to \mathrm{gr}^{\mathrm{M}}_{\ell-2} H$ the morphism induced by N. Show that, for any $\ell \geqslant 1$, $\mathrm{grN} : \mathrm{gr}^{\mathrm{M}}_\ell H \to \mathrm{gr}^{\mathrm{M}}_{\ell-2} H$ is *injective*. Show also that $\mathrm{M}_{-2}H$ is *equal to* (an not only included in) $\mathrm{N}(\mathrm{M}_0 H)$.

As we expect that N will send F^k into F^{k-1} (so is not necessarily compatible with the filtration), we will regard N as a morphism $\big((H, F^\bullet H), H_{\mathbb{Q}}\big) \to \big((H, F^\bullet H), H_{\mathbb{Q}}\big)(-1)$, using the Tate twist notation. The right choice for N, starting from T_u will then be $\mathrm{N} := (2\pi i)^{-1} \log T_u$.

Definition 1.4.5. We say that $\big((H, F^\bullet H), H_{\mathbb{Q}}, T_u\big)$ is a ($\pm$)Hodge-Lefschetz structure of weight w if

(a) $\mathrm{N} := (2\pi i)^{\mp 1} \log T_u$ induces a morphism (compatible with the filtrations and the $\mathbb{Q}$-structure) $\big((H, F^\bullet H), H_{\mathbb{Q}}\big) \to \big((H, F^\bullet H), H_{\mathbb{Q}}\big)(\mp 1)$,

(b) for any $\ell \in \mathbb{Z}$, the object $\big((\mathrm{gr}^{\mathrm{M}}_\ell H, F^\bullet \mathrm{gr}^{\mathrm{M}}_\ell H), \mathrm{gr}^{\mathrm{M}}_\ell H_{\mathbb{Q}}\big)$ is a Hodge structure of weight $w \pm \ell$.

By a Hodge-Lefschetz structure of weight w, we simply mean a (+)Hodge-Lefschetz structure of weight w.

The filtration $F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}H$ is that naturally induced by $F^\bullet H$ on $\mathrm{gr}_\ell^{\mathrm{M}}H$, that is,

$$F^p\mathrm{gr}_\ell^{\mathrm{M}}H := \frac{F^pH\cap \mathrm{M}_\ell H}{F^pH\cap \mathrm{M}_{\ell-1}H}.$$

Let us consider the graded space $\mathrm{gr}^{\mathrm{M}}H=\bigoplus_\ell \mathrm{gr}_\ell^{\mathrm{M}}H$. It is equipped with a (graded) $\mathbb{Q}$-structure and a (graded) decreasing filtration $F^\bullet$. However, this filtration does not induce a pure Hodge structure. The Hodge structure is pure on each graded piece, with a weight depending on the grading index. This graded space is also equipped with a nilpotent endomorphism, that we denote by grN, from $\mathrm{gr}_\ell^{\mathrm{M}}H$ to $\mathrm{gr}_{\ell-2}^{\mathrm{M}}H$, and which is naturally induced by N. From the definition above, it has the following property: for any $\ell\in\mathbb{Z}$,

$$\mathrm{grN}:\big((\mathrm{gr}_\ell^{\mathrm{M}}H,F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}H),\mathrm{gr}_\ell^{\mathrm{M}}H_{\mathbb{Q}}\big)\longrightarrow\big((\mathrm{gr}_{\ell-2}^{\mathrm{M}}H,F^\bullet\mathrm{gr}_{\ell-2}^{\mathrm{M}}H),\mathrm{gr}_{\ell-2}^{\mathrm{M}}H_{\mathbb{Q}}\big)(\mp1)\tag{1.4.6}$$

is a morphism of Hodge structures. In particular, for any $\ell\geqslant 1$,

$$(\mathrm{grN})^\ell:\big((\mathrm{gr}_\ell^{\mathrm{M}}H,F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}H),\mathrm{gr}_\ell^{\mathrm{M}}H_{\mathbb{Q}}\big)\longrightarrow\big((\mathrm{gr}_{-\ell}^{\mathrm{M}}H,F^\bullet\mathrm{gr}_{-\ell}^{\mathrm{M}}H),\mathrm{gr}_{-\ell}^{\mathrm{M}}H_{\mathbb{Q}}\big)(\mp\ell)$$

is an isomorphism and, for any $\ell\geqslant 0$, the primitive subspace $\mathrm{Ker}(\mathrm{grN})^{\ell+1}$ is a Hodge substructure of weight ℓ in $\big((\mathrm{gr}_\ell^{\mathrm{M}}H,F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}H),\mathrm{gr}_\ell^{\mathrm{M}}H_{\mathbb{Q}}\big)$. It will be denoted by $\mathrm{P}\big((\mathrm{gr}_\ell^{\mathrm{M}}H,F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}H),\mathrm{gr}_\ell^{\mathrm{M}}H_{\mathbb{Q}}\big)$.

Definition 1.4.7. We say that the $(\pm)$Hodge-Lefschetz structure is *graded* if it is isomorphic to its graded structure with respect to the monodromy filtration.

***Remark 1.4.8* (Mixed Hodge structures).** The symmetry between $(+)$ and $(-)$ Hodge-Lefschetz structures is only apparent. Although the $(+)$ ones are examples of *mixed Hodge structures*, with (increasing) weight filtration $W_\bullet$ defined by $W_kH_{\mathbb{Q}}=\mathrm{M}_{w+k}H_{\mathbb{Q}}$, the $(-)$ ones are not necessarily mixed Hodge structures. They are so in the graded case, as an increasing weight filtration can easily been constructed from the grading in such a case. In fact, we will only encounter *graded* $(-)$Hodge-Lefschetz structures. On the other hand, we will encounter (non graded) $(+)$Hodge-Lefschetz structures in the theory of vanishing cycles, *cf.* § 2.3.

Example 1.4.9. The cohomology $H^*(X,\mathbb{Q})$ of a smooth complex projective variety, equipped with the nilpotent endomorphism $\mathrm{N}=(2\pi i)\mathrm{L}_{\mathscr{L}}$ (or the unipotent automorphism $T_u=\exp\mathrm{L}_{\mathscr{L}}$), is naturally graded. We define the filtration $F^\bullet H^*(X,\mathbb{C})$ as being the direct sum of the Hodge filtration on each term. Then, the graded component of degree $n+\ell$ ($\ell\in\mathbb{Z}$) equipped with its filtration is a Hodge structure of weight $n+\ell$. The cohomology $H^*(X,\mathbb{Q})$ is thus a graded $(-)$Hodge-Lefschetz structure of weight n.

***Exercise 1.4.10* (Tate twist).** We define the Tate twist of a set of data $((H,F^\bullet H),H_{\mathbb{Q}},T_u)$ by Tate-twisting the first to set of data and leaving T_u unchanged. Show that $((H,F^\bullet H),H_{\mathbb{Q}},T_u)(k)$ is a $(\pm)$Hodge-Lefschetz structure of weight $w\mp 2k$.

Exercise 1.4.11. The category of Hodge-Lefschetz structures is defined in such a way that morphisms should be compatible with the filtration, with the $\mathbb{Q}$-structure and with the automorphism T_u.

(1) Show that this category is *abelian* and that any morphism is strict with respect to the filtration $F^\bullet$.

(2) Show that $\mathrm{N} : \big((H, F^\bullet H), H_\mathbb{Q}, T_u\big) \to \big((H, F^\bullet H), H_\mathbb{Q}, T_u\big)(-1)$ is a morphism in this category.

(3) Compute the filtration $\mathrm{M}_\bullet$ on $\mathrm{Im}\,\mathrm{N}$ in terms of $\mathrm{M}_\bullet(H_\mathbb{Q})$.

(4) Conclude that the image (taken in the category) by N of a Hodge-Lefschetz structure of weight w is a Hodge-Lefschetz structure of weight $w+2$ with unipotent automorphism $T_{u|\,\mathrm{Im}\,\mathrm{N}}$.

Make precise the filtration $F^\bullet$ on $\mathrm{N}(H)$ and on the graded pieces by the monodromy filtration.

(5) Show similar results for graded $(-)$Hodge-Lefschetz structures.

1.4.b. Quasi-Hodge-Lefschetz structures. We now assume that the eigenvalues λ of T are roots of unity, *i.e.*, that T is quasi-unipotent (in what follows, it would be enough to assume that they have a modulus equal to one). We will write $\lambda = \exp 2\pi i b$ with $b \in \,]-1, 0] \cap \mathbb{Q}$. The decomposition of H into eigenspaces of T_s is not defined over $\mathbb{Q}$ in general. We will *not* assume that the filtration $F^\bullet H$ is compatible with this decomposition. In order to induce a filtration on each of these eigenspaces, we proceed as follows: let us consider the decreasing filtration $(H^b)_{b\in]-1,0]}$ such that $\mathrm{gr}^b H := H^b/H^{>b}$ is the $\exp(2\pi i b)$-eigenspace of T_s. We define the filtration $F^\bullet \mathrm{gr}^b H$ by setting

$$F^p \mathrm{gr}^b H := \frac{F^p H \cap H^b}{F^p H \cap H^{>b}}.$$

We therefore get a filtration on $\mathrm{gr} H = \bigoplus_{b\in]-1,0]} \mathrm{gr}^b H$. Of course, $\mathrm{gr} H$ and H are canonically identified as $\mathbb{C}$-vector space, but not as *filtered* $\mathbb{C}$-vector spaces.

Definition 1.4.12. We say that $\big((H, F^\bullet H), H_\mathbb{Q}, T\big)$ is a quasi-Hodge-Lefschetz structure of weight w if $\big((\mathrm{gr} H, F^\bullet \mathrm{gr} H), H_\mathbb{Q}, T_u\big)$ is a Hodge-Lefschetz structure of weight w as in Definition 1.4.12.

Remark 1.4.13. Notice that, a priori, the F-filtration depends on the ordering of the logarithms b. For instance, with the choice we made, the filtration $F^\bullet \mathrm{gr}^0 H$ is a sub-filtration of $F^\bullet H$, as $H^{>0} = 0$. If we had chosen $b \in [0, 1[$, it would have been a quotient filtration.

1.4.c. Polarization. We will only consider the case of a unipotent automorphism $T = T_u$, and thus of a $(\pm)$Hodge-Lefschetz structure (graded, in the $(-)$ case). Let $S_\mathbb{Q} : \big((H, F^\bullet H), H_\mathbb{Q}\big) \otimes \big((H, F^\bullet H), H_\mathbb{Q}\big) \to \mathbb{Q}(-w)$ be a pairing. Assume that N is an infinitesimal automorphism of $S_{\mathbb{Q}}$, that is, $S_{\mathbb{Q}}(\mathrm{N}\bullet,\bullet)+S_{\mathbb{Q}}(\bullet,\mathrm{N}\bullet)=0$. Then, for any $\ell\geqslant 0$, $S_{\mathbb{Q}}$ induces a pairing

$$S_{\mathbb{Q}}^{(\ell,-\ell)}:\mathrm{gr}_{\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)\otimes\mathrm{gr}_{-\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)\longrightarrow\mathbb{Q}(-w).$$

If we set $S_{\mathbb{Q}}^{(\ell,\ell)}(x,y):=S_{\mathbb{Q}}^{(\ell,-\ell)}(x,(\mathrm{grN})^{\ell}y)$, we define in this way a pairing

$$S_{\mathbb{Q}}^{(\ell,\ell)}:\mathrm{gr}_{\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)\otimes\mathrm{gr}_{\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)\longrightarrow\mathbb{Q}(-w\mp\ell)$$

and we can restrict this pairing to the primitive part. The polarization condition is that, for any $\ell\geqslant 0$, the pairing

$$S_{\mathbb{Q}}^{(\ell,\ell)}:\mathrm{Pgr}_{\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)\otimes\mathrm{Pgr}_{\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)\longrightarrow\mathbb{Q}(-w\mp\ell)$$

is a polarization of the Hodge structure.

Remark 1.4.14. In fact, $\mathrm{gr}_{\ell}^{\mathrm{M}}\big((H,F^{\bullet}H),H_{\mathbb{Q}}\big)$ is also a polarized Hodge structure, but the polarization form is not equal to $S_{\mathbb{Q}}^{(\ell,\ell)}$. It is however obtained by an explicit formula, *cf.* [**11**].

Remark 1.4.15. This definition of polarized graded $(-)$Hodge-Lefschetz structure is the right analogue of the geometric Hodge structure of § 1.2 (*cf.* Remark 1.3.8). We have there a canonical splitting of the monodromy filtration of $\mathrm{N}=(2\pi i)\mathrm{L}_{\mathscr{L}}$. If we denote by $H_{\mathbb{Q}}$ the *total cohomology* $H^{*}(X,\mathbb{Q})$, then $H_{\mathbb{Q}}^{n-\ell}=\mathrm{gr}_{\ell}^{\mathrm{M}}H_{\mathbb{Q}}$, $w=n=\dim_{\mathbb{C}}X$ and $S_{\mathbb{Q}}^{(\ell,-\ell)}=Q_{n-\ell}=\varepsilon(n-\ell)\langle\bullet,\bullet\rangle_{\ell}$.

Exercise 1.4.16. Define the notion of a polarization when the automorphism is quasi-unipotent.

LECTURE 2

HODGE $\mathscr{D}$-MODULES ON CURVES
LOCAL PROPERTIES

2.0. Introduction

A Hodge structure, as explained in the previous lecture, can be considered as a Hodge structure on a vector bundle supported by a point, that is, a vector space. The question we address in this lecture and the next one is the definition and properties of Hodge structures on a vector bundle on a curve (Riemann surface).

The case "without singularity" is called a *variation of Hodge structure*. We explain this notion in § 2.1 from a local point of view. The global properties will be considered in the next lecture.

In order to analyze singularities, we restrict ourselves to a local setting, where the base manifold is a disc D centered at the origin in $\mathbb{C}$ (or simply the germ of D at the origin). We denote by t a coordinate on the disc, by $\mathbb{C}\{t\}$ the ring of convergent power series in the variable t and by $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$ the ring of germs of holomorphic differential operators. There is a natural increasing filtration $F_\bullet\mathscr{D}$ indexed by $\mathbb{Z}$ defined by

$$F_k\mathscr{D} = \begin{cases} 0 & \text{if } k \leqslant -1, \\ \sum_{j=0}^{k} \mathbb{C}\{t\}\cdot\partial_t^j & \text{if } k \geqslant 0. \end{cases}$$

This filtration is compatible with the ring structure (*i.e.*, $F_k \cdot F_\ell \subset F_{k+\ell}$ for any $k, \ell \in \mathbb{Z}$. The graded ring $\mathrm{gr}^F\mathscr{D} := \bigoplus_k \mathrm{gr}_k^F\mathscr{D} = \bigoplus_k F_k/F_{k-1}$ is isomorphic to the polynomial ring $\mathbb{C}\{t\}[\tau]$ (graded with respect to the degree in τ).

We also denote by $\mathscr{D}_D$ the *sheaf* of differential operators with holomorphic coefficients on D. This is a coherent sheaf, similarly equipped with an increasing filtration $F_\bullet\mathscr{D}_D$ by free $\mathscr{O}_D$-modules of finite rank. The graded sheaf $\mathrm{gr}^F\mathscr{D}_D$ is identified with the sheaf on D of functions on the cotangent bundle T^*D which are polynomial in the fibres of the fibration $T^*D \to D$.

Like a Hodge structure on a vector space, a "Hodge structure" on a holonomic $\mathscr{D}$-module consists of the data of a filtered holonomic $\mathscr{D}_D$-module $(\mathscr{M}, F_\bullet)$ (analogue of a filtered vector space), of a constructible sheaf $\mathscr{F}_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces on the disc (analogue of a $\mathbb{Q}$-vector space), and of an isomorphism $\alpha : \mathrm{DR}\,\mathscr{M} \xrightarrow{\sim} \mathbb{C}\otimes_\mathbb{Q}\mathscr{F}_\mathbb{Q}$.

In such a presentation, the *Hodge decomposition* is lacking, as it involves *complex conjugation*, which is not defined on $\mathscr{M}$. The idea of M. Saito is to use complex conjugation *only after restricting* the $\mathscr{D}$-module to a point of the disc D. If this does not cause any trouble at points of $D^* := D \setminus \{0\}$, this leads to problems at the origin for two reasons:

– the stalk of the sheaf $\mathscr{F}_{\mathbb{Q}}$ at the origin gives few information on the sheaf in the neighbourhood of the origin,

– the restriction of $\mathscr{M}$ consists usually of two vector spaces.

The right way to introduce the restriction consists in introducing *nearby and vanishing cycles*. Therefore, the compatibility of the data with the nearby and vanishing cycles functors will be the main tool in the theory of Hodge $\mathscr{D}$-modules.

2.1. Variation of Hodge structure on a Riemann surface

The definitions below are modelled on the behaviour of the cohomology of a family of smooth projective varieties parametrized by an algebraic curve, that is, a smooth projective morphism $f : Y \to X$, that we call below the "geometric setting".

Let X be a connected (possibly non compact) Riemann surface. In such a setting, the generalization of a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ is a *locally constant sheaf of* $\mathbb{Q}$*-vector spaces* $\mathscr{H}_{\mathbb{Q}}$ *on* X. Let us choose a universal covering $\widetilde{X} \to X$ of X and let us denote by G its group of deck-transformations, which is isomorphic to $\pi_1(X, \star)$ for any base-point $\star \in X$. Let us denote by $\widetilde{H}_{\mathbb{Q}}$ the space of global sections of the pull-back $\widetilde{\mathscr{H}_{\mathbb{Q}}}$ of $\mathscr{H}_{\mathbb{Q}}$ to $\widetilde{X}$. Then, giving $\mathscr{H}_{\mathbb{Q}}$ is equivalent to giving the *monodromy representation* $G \to \mathrm{GL}(\widetilde{H}_{\mathbb{Q}})$.

The analogue of a complex vector space H could be a locally constant sheaf $\mathscr{H}$ of finite dimensional $\mathbb{C}$-vector spaces, so that the isomorphism $H \simeq \mathbb{C} \otimes_{\mathbb{Q}} H_{\mathbb{Q}}$ would be easily translated at the level of locally constant sheaves. However, it is known that, in the geometric setting, the Hodge decomposition in each fibre of the family does not give rise to a locally constant sheaf, but to C^∞-bundles.

In order to avoid the use of C^∞-bundles and remain in the holomorphic framework, we consider the Hodge filtration, which is known to give rise to holomorphic bundles.

Therefore, a better analogue of the complex vector space H is a holomorphic vector bundle V equipped with a holomorphic connection $\nabla : V \to \Omega^1_X \otimes_{\mathscr{O}_X} V$, so that the locally constant sheaf $\mathscr{H} = \operatorname{Ker} \nabla$ is the desired local system. A filtration is then a finite (exhaustive) decreasing filtration by subbundles $F^\bullet V$. The main property, known as *Griffiths transversality property* is that the filtration should satisfy $\nabla(F^p V) \subset \Omega^1_X \otimes_{\mathscr{O}_X} F^{p-1} V$ for any $p \in \mathbb{Z}$.

***Definition 2.1.1* (Variation of Hodge structure).** A variation of Hodge structure of weight w is a tuple $\big((V, \nabla, F^\bullet V), \mathscr{H}_{\mathbb{Q}}, \alpha\big)$ such that

(a) $(V, \nabla, F^\bullet V)$ is a vector bundle with connection and filtration satisfying Griffiths transversality,

(b) $\mathscr{H}_{\mathbb{Q}}$ is a locally constant sheaf of $\mathbb{Q}$-vector spaces,

(c) α is an isomorphism $\mathscr{H} = \operatorname{Ker}\nabla \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} \mathscr{H}_{\mathbb{Q}}$,

such that the restriction to any point x of X gives rise to a Hodge structure ($\mathscr{H}_x = V_x, F^{\bullet}\mathscr{H}_x, \mathscr{H}_{\mathbb{Q},x}, \alpha_x$).

Let us remark that we did not introduce the complex conjugation at the level of the family, as this would lead to consider C^{∞}-bundles (or, at least, real analytic bundles).

A *polarization* is a non-degenerate bilinear form $\mathscr{S}_{\mathbb{Q}} : \mathscr{H}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathscr{H}_{\mathbb{Q}} \to \mathbb{Q}_X(-w)$ inducing a polarization in each fibre (*cf.* Remark 1.3.7).

In the next lecture we will give an idea of the proof of the Hodge theorem:

***Theorem 2.1.2* (Hodge-Deligne theorem on a compact Riemann surface)**
Let $\big((V,\nabla,F^{\bullet}V),\mathscr{H}_{\mathbb{Q}},\alpha,\mathscr{S}_Q\big)$ be a polarized variation of Hodge structure of weight w on a compact Riemann surface. Then the cohomology $H^k_{\mathbb{Q}} := H^k(X,\mathscr{H}_{\mathbb{Q}})$ is naturally equipped with a polarized Hodge structure of weight $w+k$.

2.2. Variation of Hodge structure on a punctured disc

We now consider the behaviour of a variation of Hodge structure near a singular point. From now on, we will work on the disc D, as indicated in the introduction of this lecture and we will denote by D^* the punctured disc $D \smallsetminus \{0\}$. Assume that $\big((V,\nabla,F^{\bullet}V),\mathscr{H}_{\mathbb{Q}},\alpha\big)$ is a variation of Hodge structure on D^*. Our goal is to define a suitable restriction of these data to the origin. As for the case of a generic point, the underlying vector space of the restricted object should have a dimension equal to the generic rank of the bundle.

2.2.a. The locally constant sheaf. Let $\mathscr{H}_{\mathbb{Q}}$ be a locally constant sheaf on D^*. We wish to define a "restriction" of $\mathscr{H}_{\mathbb{Q}}$ at the origin. This should be a $\mathbb{Q}$-vector space of the same dimension as the generic dimension of the stalks of $\mathscr{H}_{\mathbb{Q}}$. This condition eliminates the natural candidate, namely the space of sections of $\mathscr{H}_{\mathbb{Q}}$ over D^*.

On the other hand, let us choose a universal covering $\widetilde{D}^* \to D^*$ (for instance, one can use the exponential map). As $\pi_1(D^*,\star) \simeq \mathbb{Z}$, giving $\mathscr{H}_{\mathbb{Q}}$ is equivalent, as we have indicated in § 2.1, to giving an element $T \in \mathrm{GL}(\widetilde{H}_{\mathbb{Q}})$ called *monodromy*. Our "restriction" of $\mathscr{H}_{\mathbb{Q}}$ at the origin will be the pair $(\widetilde{H}_{\mathbb{Q}}, T)$. This is not only a $\mathbb{Q}$-vector space, but it is a vector space with an automorphism. We will use the notation

$$\Psi_t(\mathscr{H}_{\mathbb{Q}}) := \widetilde{H}_{\mathbb{Q}}.$$

Moreover, for any $\lambda \in \mathbb{C}^*$, the generalized eigenspace $\Psi_t^{\lambda}(\mathscr{H}_{\mathbb{Q}}) = \operatorname{Ker}(T-\lambda\,\mathrm{Id})^N$ ($N \gg 0$) is a subspace of $\Psi_t(\mathscr{H}) = \mathbb{C} \otimes_{\mathbb{Q}} \Psi_t(\mathscr{H}_{\mathbb{Q}})$. We have a decomposition $\Psi_t(\mathscr{H}) = \bigoplus_{\lambda} \Psi_t^{\lambda}(\mathscr{H})$. Over $\mathbb{Q}$, we only keep the decomposition as by considering only $\Psi_t^1(\mathscr{H}_{\mathbb{Q}}) \oplus \Psi_t^{\neq 1}(\mathscr{H}_{\mathbb{Q}})$, with $\Psi_t^{\neq 1}(\mathscr{H}_{\mathbb{Q}}) := \bigoplus_{\lambda\neq 1} \Psi_t^{\lambda}(\mathscr{H}_{\mathbb{Q}})$ (we can also distinguish $\lambda = -1$ if we wish).

The space of sections of $\mathscr{H}_{\mathbb{Q}}$ on D^* is identified with the subspace $\mathrm{Ker}(T-\mathrm{Id})$ of the space $\Psi_t(\mathscr{H}_{\mathbb{Q}})$ of *multivalued* sections of $\mathscr{H}_{\mathbb{Q}}$ on D^*. More precisely, it is contained in $\Psi_t^1(\mathscr{H}_{\mathbb{Q}})$.

Assume now that we have a non-degenerate bilinear form $\mathscr{S}_{\mathbb{Q}} : \mathscr{H}_{\mathbb{Q}} \otimes \mathscr{H}_{\mathbb{Q}} \to \mathbb{Q}_{D^*}(-w)$ on the locally constant sheaf $\mathscr{H}_{\mathbb{Q}}$ on D^* (here, the Tate twist only means multiplication by $(2\pi i)^{-w}$). It gives rise to a non-degenerate bilinear form $\Psi_t\mathscr{S}_{\mathbb{Q}}$ on the space of multivalued sections $\widetilde{H}_{\mathbb{Q}}$, and the monodromy is an automorphism of this bilinear form, that is, $\Psi_t\mathscr{S}_{\mathbb{Q}}(T\bullet, T\bullet) = \Psi_t\mathscr{S}_{\mathbb{Q}}(\bullet,\bullet)$. In particular, the decomposition $\Psi_t(\mathscr{H}_{\mathbb{Q}}) = \Psi_t^1(\mathscr{H}_{\mathbb{Q}}) \oplus \Psi_t^{\neq 1}(\mathscr{H}_{\mathbb{Q}})$ is orthogonal with respect to $\Psi_t\mathscr{S}_{\mathbb{Q}}$. Using the decomposition into $\Psi_t^\lambda(\mathscr{H})$'s, we find $\Psi_t\mathscr{S}(\Psi_t^\lambda(\mathscr{H}), \Psi_t^\mu(\mathscr{H})) = 0$ unless $\mu = \lambda^{-1}$ (which also reads as $\mu = \overline{\lambda}$ if T is quasi-unipotent).

2.2.b. The vector bundle with connection. If we are given (V, ∇) on D^*, there exists a unique meromorphic extension, called *Deligne meromorphic extension*, of the bundle V to a meromorphic bundle $\widetilde{V}$ (that is, a free sheaf of $\mathscr{O}_D[1/t]$-modules) equipped with a connection. It consists of all local sections of j_*V (where $j : D^* \hookrightarrow V$ is the inclusion) whose coefficients in some (or any) basis of multivalued horizontal sections have moderate growth in any sector with bounded arguments. Equivalently, it is characterized by the property that the coefficients of any multivalued horizontal section expressed in some basis of $\widetilde{V}$ are multivalued functions on D^* with moderate growth in any sector with bounded arguments.

Remark 2.2.1. This statement is a form of the Riemann-Hilbert correspondence on the disc: there exists a unique regular holonomic $\mathscr{D}_D$-module having $\boldsymbol{R}j_*\mathscr{H}$ as its de Rham complex. This $\mathscr{D}_D$-module is $\widetilde{V}$.

Similarly, there exists a $\mathscr{O}_D$-submodule $\widetilde{V}^0$ of $\widetilde{V}$, called the *Deligne canonical lattice*, consisting of *all* local sections of j_*V whose coefficients in any basis of horizontal sections on any bounded sector are holomorphic functions on this sector with at most logarithmic growth. On this bundle, the connection $\widetilde{\nabla}$ has a logarithmic pole. The residue R of the connection on $\widetilde{V}^0$ is an endomorphism of the vector space $\widetilde{V}^0/t\widetilde{V}^0$, that we denote by $\psi_t\widetilde{V}$. The real part of its eigenvalues belong to $[0,1[$. The latter two properties also characterize $\widetilde{V}^0$. There is a natural isomorphism

$$\psi_t\widetilde{V} \xrightarrow{\sim} \Psi_t\mathscr{H} \tag{2.2.2}$$

(recall that $\mathscr{H} = \mathrm{Ker}\,\nabla$), under which the monodromy T is expressed as $\exp(-2\pi i\mathrm{R})$.

Let us note that the connection $\widetilde{\nabla}$ makes $\widetilde{V}$ a left $\mathscr{D}_D$-module by setting $\partial_t\widetilde{v} := \widetilde{\nabla}_{\partial_t}\widetilde{v}$ (*cf.* also Exercise 2.4.9).

We can more generally consider a whole family of Deligne canonical lattices: for any $b \in \mathbb{R}$, we denote by $\widetilde{V}^b$ the lattice defined by the property that the eigenvalues of the residue of the connection have their real part in $[b, b+1[$. If we set $\widetilde{V}^{>b} = \bigcup_{b'>b}\widetilde{V}^{b'}$, then $\widetilde{V}^{>b}$ is the Deligne canonical lattice for which the eigenvalues of the residue of the connection belong to $]b, b+1]$.

2.2.c. The filtration. We would like to extend the filtration $F^\bullet V$ as a filtration $F^\bullet \widetilde{V}$ by subbundles satisfying the Griffiths transversality property with respect to the meromorphic connection $\widetilde{\nabla}$. Here the first delicate problem shows up. A first natural choice would be to set

$$F^p\widetilde{V} := j_*F^pV \cap \widetilde{V},$$

where $j : D^* \hookrightarrow D$ denotes the inclusion. This choice can lead to a non-coherent $\mathscr{O}_D$-module: for instance, if $p \ll 0$, we have $F^pV = V$ and we get $F^p\widetilde{V} = \widetilde{V}$, which is not $\mathscr{O}_D$-coherent. Being more clever, one first defines

$$F^p\widetilde{V}^{>-1} := j_*F^pV \cap \widetilde{V}^{>-1}. \tag{2.2.3}$$

If this sheaf is $\mathscr{O}_D$-coherent, it will then be natural to define, for any p, in order to obtain Griffiths transversality,

$$F^p\widetilde{V} = \sum_{j\geqslant 0}(\widetilde{\nabla}_{\partial_t})^j F^{p+j}\widetilde{V}^{>-1}. \tag{2.2.4}$$

Indeed, with this definition, the relation $\widetilde{\nabla}_{\partial_t}F^p\widetilde{V} \subset F^{p-1}\widetilde{V}$ is clearly satisfied.

***Exercise 2.2.5* (Extension of the filtration).** Show that

(1) For any $b > -1$, we have $F^p\widetilde{V} \cap \widetilde{V}^b = j_*F^pV \cap \widetilde{V}^b$ and for any $b \geqslant -1$, $F^p\widetilde{V} \cap \widetilde{V}^{>b} = j_*F^pV \cap \widetilde{V}^{>b}$;

(2) if $F^p\widetilde{V}^{>-1}$ is $\mathscr{O}_D$-coherent, it is $\mathscr{O}_D$-locally free, hence free (use that $F^p\widetilde{V}^{>-1} \subset \widetilde{V}^{>-1}$);

(3) $F^p\widetilde{V}$ is a $\mathscr{O}_D$-module;

(4) under the assumption in (2), $F^p\widetilde{V}$ is $\mathscr{O}_D$-coherent, and thus $\mathscr{O}_D$-free;

(5) the sheaf $\bigcup_p F^p\widetilde{V}$ is a coherent $\mathscr{D}_D$-module; it is equal to the $\mathscr{D}_D$-submodule of $\widetilde{V}$ generated by $\widetilde{V}^{>-1}$.

(*Hint:* Recall that there exists an integer $p_0 \gg 0$ such that $F^{p_0}V = 0$ and $F^{-p_0}V = V$.)

This discussion shows that, assuming that each $F^p\widetilde{V}$ is $\mathscr{O}_D$-coherent, the $\mathscr{D}_D$-module $\widetilde{V}$ contains an irrelevant part for our purpose: if we denote by $\widetilde{V}_{\min}$ the $\mathscr{D}_D$-submodule of $\widetilde{V}$ generated by the Deligne lattice $\widetilde{V}^{>-1}$, then $\widetilde{V}/\widetilde{V}_{\min}$ is not used in the construction.

Definition 2.2.6. We call $\widetilde{V}_{\min}$ the *minimal extension*[1] of (V, ∇) across the origin (whereas $\widetilde{V}$ should be called the maximal one).

Exercise 2.2.7. Assuming that each $F^p\widetilde{V}$ is $\mathscr{O}_D$-coherent and setting $F_k\widetilde{V}_{\min} = F^{-k}\widetilde{V}$, show that $F_\bullet\widetilde{V}_{\min}$ is a good filtration of $\widetilde{V}_{\min}$ (*cf.* § 2.4.a).

***Exercise 2.2.8* (*cf.* [24, Prop. 3.2.2]).** Assume that each $F^p\widetilde{V}$ is $\mathscr{O}_D$-coherent and that the eigenvalues of the residue R of $\widetilde{\nabla}$ on $\widetilde{V}^0$ are *real*. One can then consider the filtration $\widetilde{V}^\bullet_{\min}$ indexed by $\mathbb{R}$ (*cf.* § 2.4.e). Prove that the filtration $F^\bullet\widetilde{V}$ satisfies the following properties:

1. It is also called *middle extension* or *intermediate extension*; this is justified by Exercise 2.4.25.

(1) for any $b > -1$, $t(F^p\widetilde{V} \cap \widetilde{V}^b_{\min}) = F^p\widetilde{V} \cap \widetilde{V}^{b+1}_{\min}$;

(2) for any $b < 0$, $\partial_t F^p \mathrm{gr}^b(\widetilde{V}_{\min}) = F^{p-1}\mathrm{gr}^{b-1}(\widetilde{V}_{\min})$.

[The inclusions $\subset$ are easy, *cf.* Exercise 2.4.17; the remarkable property is the existence of inclusions $\supset$; we will call this property *strict specializability*, *cf.* § 4.3.b.]

Conversely, prove that if $F_\bullet\widetilde{V}_{\min}$ is a good filtration satisfying (1) and (2), it can be obtained by the formulas (2.2.3) and (2.2.4).

2.3. Hodge $\mathscr{D}$-modules on a Riemann surface

2.3.a. Gathering the properties. We can now give a name to the object we want to single out. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_D$-module which is a minimal extension of its restriction to D^*, which we assume to be a vector bundle with connection. We will assume that

(a) there exists a logarithmic lattice in $\mathscr{M}$, that is to say a free $\mathscr{O}_D$-submodule on which the connection is logarithmic and which generates $\mathscr{M}$ as a $\mathscr{D}_D$-module, [(2)]

(b) the residue of the connection on some (or any) logarithmic lattice has *real* eigenvalues.

Let $F_\bullet\mathscr{M}$ be a good F-filtration of $\mathscr{M}$.

***Definition 2.3.1* (Strict specializability).** We say that the filtered $\mathscr{D}_D$-module $(\mathscr{M}, F_\bullet)$ is *strictly specializable* at the origin if the properties 2.2.8(1) and (2) are satisfied.

***Definition 2.3.2* (Hodge $\mathscr{D}_D$-module with strict support the disc)**

A *Hodge $\mathscr{D}_D$-module* of weight w and having the germ of disc D as its strict support consists of the data $\big((\widetilde{V}_{\min}, F^\bullet), j_*\mathscr{H}_\mathbb{Q}, j_*\alpha\big)$ such that

(a) The residue of the connection on some (or any) logarithmic lattice of $\widetilde{V}_{\min}$ has rational eigenvalues (equivalently, the monodromy of $\mathscr{H}_\mathbb{Q}$ is quasi-unipotent),

(b) it restricts to a variation of Hodge structure $\big((V, F^\bullet), \mathscr{H}_\mathbb{Q}, \alpha\big)$ of weight w on D^*,

(c) $(\widetilde{V}_{\min}, F^\bullet)$ is strictly specializable at the origin,

(d) the object $\big((\psi_t\widetilde{V}_{\min}, F^\bullet\psi_t\widetilde{V}_{\min}), \Psi_t\mathscr{H}_\mathbb{Q}, T, \psi_t\alpha\big)$ is a quasi-Hodge-Lefschetz structure of weight w.

Recall that α is an isomorphism $\mathrm{Ker}\,\nabla \xrightarrow{\sim} \mathbb{C}\otimes_\mathbb{Q}\mathscr{H}_\mathbb{Q}$. It defines an isomorphism $\Psi_t\alpha : \Psi_t\,\mathrm{Ker}\,\nabla \xrightarrow{\sim} \mathbb{C}\otimes_\mathbb{Q}\Psi_t\mathscr{H}_\mathbb{Q}$ compatible with the monodromy. We denote by $\psi_t\alpha$ the morphism composed of $\Psi_t\alpha$ and of the isomorphism $\psi_t\widetilde{V} \xrightarrow{\sim} \Psi_t\,\mathrm{Ker}\,\nabla$ whose existence is indicated in § 2.2.b.

***Definition 2.3.3* (Hodge $\mathscr{D}_D$-module).** A *Hodge $\mathscr{D}$-module* on the disc with singularity at the origin at most will then consist of the direct sum of

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2. We then say that $\mathscr{M}$ has a regular singularity at the origin.

– a *Hodge $\mathscr{D}$-module* with strict support equal to the disc and with singularity at the origin at most,
– a *Hodge $\mathscr{D}$-module* with strict support equal to the origin: this is mainly a Hodge structures on a finite dimensional $\mathbb{Q}$-vector space, extended to the disc by the direct image functor (i_+, i_*) induced by the inclusion $i : \{0\} \hookrightarrow D$.

***Exercise 2.3.4* (Abelianity).** Prove that the category of Hodge $\mathscr{D}$-modules (where the morphisms are pairs of morphisms in the categories of $\mathscr{D}$-modules and sheaves respectively, compatible with α) is *abelian*.

***Definition 2.3.5* (Polarized Hodge $\mathscr{D}_D$-module with strict support the disc)**
Let $\big((\widetilde{V}_{\min}, F^\bullet), j_*\mathscr{H}_{\mathbb{Q}}, j_*\alpha\big)$ be a Hodge $\mathscr{D}$-module of weight w. A *polarization* is a polarization $\mathscr{S}_{\mathbb{Q}}$ of the restriction to D^* such that

(f) $\Psi_t\mathscr{S}_{\mathbb{Q}}$ induces a polarization of the quasi Hodge-Lefschetz structure of weight w $\big((\psi_t\widetilde{V}_{\min}, F^\bullet\psi_t\widetilde{V}_{\min}), \Psi_t\mathscr{H}_{\mathbb{Q}}, T, \psi_t\alpha\big)$.

The definition of a polarized Hodge $\mathscr{D}$-module is obtained in a way similar to that of 2.3.3.

Remark 2.3.6. In § 4.4, we will use the convention of M. Saito: the F-filtration $F^\bullet_{\mathrm{Saito}}$ on V is $F^\bullet[-1] := F^{\bullet-1}V$ and, correspondingly, the F-filtration $F^\bullet_{\mathrm{Saito}}\psi_t\widetilde{V}_{\min}$ is the filtration induced by $F^\bullet_{\mathrm{Saito}}[1]\widetilde{V}_{\min}$, that is, the filtration induced by $F^\bullet\widetilde{V}_{\min}$. Correspondingly, one should be change the constants in the polarization on ψ_t. We postpone the adjustment of multiplicative constants in the Saito convention until § 4.4.

***Exercise 2.3.7* (Semi-simplicity).** Prove that the category of polarized Hodge $\mathscr{D}$-modules is *semi-simple*.

2.3.b. So what? After all these definitions, one is entitled to ask: *Does there exist any Hodge $\mathscr{D}$-module on D other than the variations of Hodge structure on D?*

This question has various aspects:

(a) Existence,

(b) openness of the notion,

(c) closedness of the notion.

In the first direction, we start from a quasi-Hodge-Lefschetz structure of weight w on some given $\mathbb{Q}$-vector space $\widetilde{H}_{\mathbb{Q}}$. It is then possible to construct $\big((\widetilde{V}, \widetilde{\nabla}), \mathscr{H}_{\mathbb{Q}}, \alpha\big)$ giving rise to $(\widetilde{H}, \widetilde{H}_{\mathbb{Q}}, \psi_t\alpha, T)$ by specialization. The question is to define a filtration $F^\bullet\widetilde{V}$ specializing to the given filtration $F^\bullet\widetilde{H}$ and such that, on D^*, we get a variation of Hodge structure of weight w.

For (b), assume that we are given a well-filtered $\mathscr{D}_D$-module with regular singularity, and which is the minimal extension of its restriction to D^*, that is, it takes the form $(\widetilde{V}_{\min}, F_\bullet\widetilde{V}_{\min})$. Assume that we are also given $\mathscr{H}_{\mathbb{Q}}$ on D^* with quasi-unipotent monodromy and an isomorphism $\alpha : \mathscr{H} = \operatorname{Ker}\nabla_{|D^*} \xrightarrow{\sim} \mathbb{C}\otimes_{\mathbb{Q}}\mathscr{H}_Q$. Lastly, assume that the 'restriction' to the origin of these data give rise to a quasi-Hodge-Lefschetz structure of weight w. The question is to know whether the restriction of these data to any point in D^* is also a Hodge structure of weight w. In other words, *is the previous construction open?*

For (c), the question is if, starting from a variation of Hodge structure on D^*, the previous construction does give rise to coherent $\mathscr{O}_D$-modules $F^\bullet \widetilde{V}$ and, if so, if we get a Hodge structure of weight w with a quasi-unipotent automorphism on the special fibre, by the construction ψ_t.

The fundamental theorems of W. Schmid **[29]** assert that the answer to each of these questions is positive, *provided that we moreover assume the existence of a polarization.* More precisely, they give a positive answer to the questions when we add in the assumption *and in the conclusion* the existence of a polarization.

For the existence, the result is known as the *Nilpotent orbit theorem*, at least when the automorphism we start with is unipotent.

That (b) is true can be related, at least when the weight is zero, to the rigidity of trivial vector bundles on the Riemann sphere (according to Exercise 1.3.2(3)).

In the proof of (c) enters the easy fact that, if we have a continuous one-parameter family of Hermitian forms on a vector space such that the special Hermitian form is non-degenerate and the general one is positive definite, then the special one is also positive definite.

However, a lot of analysis has to be developed in order to use these simple facts.

2.4. Appendix: Basics on holonomic $\mathscr{D}$-modules

We keep the notation given in the introductory part of this lecture.

2.4.a. Good F-filtrations, holonomic modules. Let M be a finitely generated $\mathscr{D}$-module. By an F-filtration of M we mean increasing filtration $F_\bullet M$ by $\mathscr{O} = \mathbb{C}\{t\}$-submodules, indexed by $\mathbb{Z}$, such that, for any $k, \ell \in \mathbb{Z}$, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$. Such a filtration is said to be *good* if it satisfies the following properties:

(1) $F_kM = 0$ for $k \ll 0$,

(2) each F_kM is finitely generated over $\mathscr{O}$,

(3) for any $k, \ell \in \mathbb{Z}$, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$,

(4) there exists $\ell_0 \in \mathbb{Z}$ such that, for any $k \geqslant 0$ and any $\ell \geqslant \ell_0$, $F_k\mathscr{D} \cdot F_\ell M = F_{k+\ell}M$.

***Remark 2.4.1* (Increasing or decreasing?)** In Hodge theory, one usually uses decreasing filtrations. The trick to go from increasing to decreasing filtrations is to set [3], for

3. However, when one considers the V-filtration below, the rule for going from an increasing to a decreasing filtration on a holonomic $\mathscr{D}$-module (but not on $\mathscr{D}$ itself) will also include a translation:

$$V^bM = V_aM \quad \text{with } a = -b-1.$$

This can be useful to keep in mind when referring to the literature.

any $p \in \mathbb{Z}$,

$$F^p M := F_{-p} M.$$

***Exercise 2.4.2* (The Rees module).** The previous properties can be expressed in a simpler way by adding a dummy variable. Let M be a left $\mathscr{D}$-module and let $F_\bullet M$ be an F-filtration of M. Let z be such a variable and let us set $R_F\mathscr{D} = \bigoplus_{k\in\mathbb{Z}} F_k\mathscr{D} \cdot z^k$ and $R_F M = \bigoplus_{k\in\mathbb{Z}} F_k M \cdot z^k$.

(1) Prove that $R_F\mathscr{D}$ is a Noetherian ring.

(2) Prove that $R_F M$ has no $\mathbb{C}[z]$-torsion.

(3) Prove that the F-filtration condition is equivalent to: *$R_F M$ is a left $R_F\mathscr{D}$-module.*

(4) Prove that $R_F M/zR_F M = \mathrm{gr}^F M$ and $R_F M/(z-1)R_F M = M$.

(5) Prove that the goodness of $F_\bullet M$ is equivalent to: *$R_F M$ is a finitely generated left $R_F\mathscr{D}$-module.*

(6) Prove that M has a good F-filtration if and only if it is finitely generated.

Definition 2.4.3. We say that M is *holonomic* if it is finitely generated and any element of M is annihilated by some $P \in \mathscr{D}$.

One can prove that any holonomic $\mathscr{D}$-module can be generated by one element (*i.e.*, it is cyclic), hence of the form $\mathscr{D}/I$ where I is a left ideal in $\mathscr{D}$, and that this ideal can be generated by two elements (*cf.* [**1**]).

2.4.b. The V-filtration. In order to analyze the behaviour of a holonomic module near the origin, we will use another kind of filtration, called the Kashiwara-Malgrange filtration. It is an extension to holonomic modules of the notion of Deligne lattice for meromorphic bundle with connection.

We first define the decreasing filtration $V^\bullet\mathscr{D}$ indexed by $\mathbb{Z}$, by giving to any monomial $t^{a_1}\partial_t^{b_1}\cdots t^{a_n}\partial_t^{b_n}$ the V-degree $\sum_i a_i - \sum_i b_i$, and by defining the V-order of an operator $P \in \mathscr{D}$ as the smallest V-degree of its monomials.

Exercise 2.4.4

(1) Check that the V-order of P does not depend on the way we write a monomial.

(2) Check that each $V^k\mathscr{D}$ is a $\mathscr{O}$-module, and that, for $k \geqslant 0$, $V^k\mathscr{D} = t^k V^0\mathscr{D}$.

(3) Check that the filtration by the V-order is compatible with the product, and more precisely that

$$V^k\mathscr{D}\cdot V^\ell\mathscr{D} \begin{cases} \subset V^{k+\ell}\mathscr{D} & \text{for any } k,\ell \in \mathbb{Z},\\ = V^{k+\ell}\mathscr{D} & \text{if } k,\ell \leqslant 0 \text{ or if } k,\ell \geqslant 0.\end{cases}$$

Conclude that $V^0\mathscr{D}$ is a ring and that each $V^k\mathscr{D}$ is a left $V^0\mathscr{D}$-module.

(4) Check that the Rees object $R_V\mathscr{D} := \bigoplus_{k\in\mathbb{Z}} V^k\mathscr{D}\cdot z^{-k}$ is a Noetherian ring.

(5) Show that $\mathrm{gr}_V^0\mathscr{D}$ can be identified with the polynomial ring $\mathbb{C}[E]$, where E is the class of $t\partial_t$ in $\mathrm{gr}_V^0\mathscr{D}$.

(6) Show that E does not depend on the choice of the coordinate t on the disc.

Definition 2.4.5. Let M be a left $\mathscr{D}$-module. By a V-filtration we mean an decreasing filtration $U^\bullet M$ of M, indexed by $\mathbb{Z}$, which satisfies $V^k\mathscr{D}\cdot U^\ell M\subset U^{k+\ell}M$ for any $k,\ell\in\mathbb{Z}$. We say that $U^\bullet M$ is *good* if there exists $\ell_0\in\mathbb{N}$ such that the previous inclusion is an equality for any $k\leqslant 0$ and $\ell\leqslant -\ell_0$, and for any $k\geqslant 0$ and $\ell\geqslant\ell_0$.

Exercise 2.4.6

(1) Show that a filtration $U^\bullet M$ is a V-filtration if and only if the Rees object $R_UM:=\bigoplus_{k\in\mathbb{Z}}U^kMz^{-k}$ is naturally a left $R_V\mathscr{D}$-module.

(2) Show that, for any V-filtration $U^\bullet M$ on M, $R_UM/zR_UM=\mathrm{gr}^UM$ and $R_UM/(z-1)R_UM=M$.

(3) Show that any finitely generated $\mathscr{D}$-module has a good V-filtration.

(4) Show that a V-filtration is good if and only if the Rees module R_UM is finitely generated over $R_V\mathscr{D}$.

(5) Show that, if M is holonomic, then for any good V-filtration the graded spaces gr_U^kM are finite dimensional $\mathbb{C}$-vector spaces equipped with a linear action of E.

(6) Show that, if $U^\bullet M$ is a V-filtration of M, then the left multiplication by t induces for any $k\in\mathbb{Z}$ a $\mathbb{C}$-linear homomorphism $\mathrm{gr}_U^kM\to\mathrm{gr}_U^{k+1}M$ and that the action of ∂_t induces $\mathrm{gr}_U^kM\to\mathrm{gr}_U^{k-1}M$. How does E commute with these morphisms?

(7) Show that if a V-filtration is good, then $t:U^kM\to U^{k+1}M$ is an isomorphism for any $k\gg 0$ and $\partial_t:\mathrm{gr}_U^kM\to\mathrm{gr}_U^{k-1}M$ is so for any $k\ll 0$.

***Theorem 2.4.7* (The Kashiwara-Malgrange filtration).** *Let M be a holonomic $\mathscr{D}$-module. Then there exists a unique good V-filtration denoted by $V^\bullet M$ and called* the Kashiwara-Malgrange filtration of M, *such that the eigenvalues of E acting on the finite dimensional vector space gr_V^0M have their real part in $[0,1[$.* □

Exercise 2.4.8. Show that the Kashiwara-Malgrange filtration satisfies the following properties:

(1) for any $k\geqslant 0$, the morphism $V^kM\to V^{k+1}M$ induced by t is an isomorphism;

(2) for any $k\geqslant 0$, the morphism $\mathrm{gr}_V^{-1-k}M\to\mathrm{gr}_V^{-2-k}M$ induced by ∂_t is an isomorphism.

Exercise 2.4.9. Show that, for any holonomic module M, the module $\mathscr{O}[t^{-1}]\otimes_{\mathscr{O}}M$ is still holonomic and is a finite dimensional vector space over the field of Laurent series $\mathscr{O}[t^{-1}]$, equipped with a connection. Conversely, prove that any finite dimensional vector space over the field of Laurent series $\mathscr{O}[t^{-1}]$ equipped with a connection is a holonomic $\mathscr{D}$-module. Conclude that the germ at the origin of $(\widetilde{V},\widetilde{\nabla})$ considered in § 2.2.b is a holonomic $\mathscr{D}$-module.

2.4.c. Nearby and vanishing cycles. For simplicity, in the following we always assume that M is holonomic.

Definition 2.4.10 **(Nearby and vanishing cycles).** Let M be a holonomic $\mathscr{D}$-module.

- For any $\beta \in \mathbb{C}$ with real part $b \in [0,1[$, we denote by $\psi_t^\beta M \subset \mathrm{gr}_V^0 M$ the generalized eigenspace of $E : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^0 M$ with respect to the eigenvalue β,
- we denote by $\psi_t^{-1} M \subset \mathrm{gr}_V^{-1} M$ the generalized eigenspace of $E : \mathrm{gr}_V^{-1} M \to \mathrm{gr}_V^{-1} M$ with respect to the eigenvalue 1.
- Using Exercise 2.4.8, we define $\psi_t^\beta M$ for any $\beta \in \mathbb{C}$. We note that there exists a finite set $B \in \mathbb{C}$ such that $\psi_t^\beta M = 0$ for any $\beta \notin B + \mathbb{Z}$.

We denote by N the nilpotent part of the endomorphism induced by $-E$ on $\psi_t^\beta M$ ($\beta \in \mathbb{C}$). Moreover, can : $\psi_t^0 M \to \psi_t^{-1} M$ is the homomorphism induced by $-\partial_t$ and var : $\psi_t^{-1} M \to \psi_t^0 M$ is that induced by t, so that $\mathrm{var} \circ \mathrm{can} = \mathrm{N}$ and $\mathrm{can} \circ \mathrm{var} = \mathrm{N}$. We also denote by $\mathrm{M}_\bullet \psi_t^\beta M$ the monodromy filtration defined by the nilpotent endomorphism N on $\psi_t^\beta M$ (*cf.* § 1.4.a).

Exercise 2.4.11. Let M be a holonomic $\mathscr{D}$-module. Prove that

(1) the construction of ψ_t^β ($\beta \in \mathbb{C}$), can, var, N, is functorial with respect to M and ψ_t^β are compatible with short exact sequences;

(2) can is onto iff M has no quotient supported at the origin (*i.e.*, there is no surjective morphism $M \to N$ where each element of N is annihilated by some power of t);

(3) var is injective if and only if M has no submodule supported at the origin (*i.e.*, whose elements are annihilated by some power of t);

(4) $\psi_t^{-1} = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$ if and only if $M = M' \oplus M''$, where M'' is supported at the origin and M' has neither a quotient nor a submodule supported at the origin (in such a case, we say that M is *S(upport)-decomposable*).

Definition 2.4.12 **(Middle extension).** We say that a holonomic M is a middle (or minimal, or intermediate) extension of $\mathscr{O}[t^{-1}] \otimes_{\mathscr{O}} M$ if can is onto and var is injective, that is, if M has neither a quotient nor a submodule supported at the origin.

Examples 2.4.13

(1) If 0 is not a singular point of M, then M is $\mathscr{O}$-free of finite rank and $\psi_t^\beta M = 0$ unless $\beta \in -\mathbb{N}^*$. Then can $= 0$, var $= 0$ and $\mathrm{N} = 0$.

(2) If M is *purely irregular*, *e.g.*, $M = (\mathscr{O}, \nabla)$ with $\nabla = d + dt/t^2$, then $\psi_t^\beta M = 0$ for any β. In such a case, the ψ-functor does not bring any information on M.

Exercise 2.4.14. Show that M is a middle extension if and only if M is equal to the $\mathscr{D}$-submodule generated by $V^0 M$.

Definition 2.4.15 **(Regular singularity).** We say that M has a regular singularity (or is *regular*) at the origin if $V^0 M$ has finite type over $\mathscr{O}$.

Exercise 2.4.16. Prove that the Deligne meromorphic extension of § 2.2.b has regular singularity at the origin.

2.4.d. The monodromy filtration. Each vector space $\psi_t^\beta M$ is equipped with a nilpotent endomorphism N. As we indicated above, it acquires then a *monodromy filtration* $\mathrm{M}_\bullet(\mathrm{N})$ (*cf.* § 1.4.a).

Let us assume that all β's such that $\psi_t^\beta M \neq 0$ are *real*. We denote them by the letter b to recall they are real. It is possible to extend the V-filtration $V^\bullet M$ to a filtration indexed by (a discrete subset of) $\mathbb{R}$, in such a way that 2.4.5 holds when $\ell \in \mathbb{R}$ and that, for any $b \in \mathbb{R}$, we have $\psi_t^b M = \mathrm{gr}_V^b M := V^b M/V^{>b}M$. Then we can lift the monodromy filtration, by defining $\mathrm{M}_\ell V^b M$ to be the pull-back of $\mathrm{M}_\ell \psi_t^b M$ by the projection $V^b M \to V^b M/V^{>b}M = \psi_t^b M$.

Exercise 2.4.17

(1) Show the analogue of Exercise 2.4.8, that is,
- $t : V^b M \to V^{b+1} M$ is an isomorphism for any $b > -1$,
- $\partial_t : \mathrm{gr}_V^b M \to \mathrm{gr}_V^{b-1} M$ is an isomorphism for any $b < 0$.

(2) Show that, if M has a regular singularity, $V^b M$ has finite type over $\mathscr{O}$ for any $b \in \mathbb{R}$ and is $\mathscr{O}$-free of finite rank for any $b > -1$.

(3) With the same assumption, show that $\mathrm{M}_\ell V^b M$ has finite type over $\mathscr{O}$ for any $\ell \in \mathbb{Z}$ and $b \in \mathbb{R}$ and is stable by $V_0\mathscr{D}$. Show that, for any $b > -1$, it is a free $\mathscr{O}$-module of finite rank with a logarithmic connection (*i.e.*, stable by $V_0\mathscr{D}$).

2.4.e. F-filtration on nearby and vanishing cycles. Let M be holonomic and equipped with a good F-filtration $F_\bullet M$. There is a natural way to induce a filtration on each vector space $\mathrm{gr}_V^\ell M$ by setting

$$F_p \mathrm{gr}_V^\ell M := \frac{F_p M \cap V^\ell M}{F_p M \cap V^{\ell+1} M}.$$

Note that, unless $\psi_t^\beta M = 0$ for any $\beta \notin \mathbb{Z}$, there is no natural way to induce a filtration on each $\psi_t^\beta M$. There is nevertheless a canonical way, as we assumed that all β's are real, hence naturally totally ordered. In such a case, using the notation of § 2.4.d, we can set

$$F_p \psi_t^b M := \frac{F_p M \cap V^b M}{F_p M \cap V^{>b} M}. \tag{2.4.18}$$

Exercise 2.4.19. Show that $\mathrm{N} \cdot F_p \psi_t^b M \subset F_{p+1} \psi_t^b M$ for any $b \in \mathbb{C}$ and that

$$\begin{aligned} \mathrm{can}(F_p \psi_t^0 M) &\subset F_{p+1} \psi_t^{-1} M, \\ \mathrm{var}(F_p \psi_t^{-1} M) &\subset F_p \psi_t^0 M. \end{aligned}$$

2.4.f. The de Rham complex

The germic version of the de Rham complex. Let us first consider the de Rham complex of M. The holomorphic de Rham complex $\mathrm{DR}\, M$ is defined as the complex

$$\mathrm{DR}\, M = \{0 \longrightarrow M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} M \longrightarrow 0\}.$$

The de Rham complex can be V-filtered, by setting

$$V^k \mathrm{DR}\, M = \{0 \longrightarrow V^k M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} V^{k-1} M \longrightarrow 0\},$$

for any $k \in \mathbb{Z}$, and if moreover the eigenvalues of E acting on $\mathrm{gr}_V^0 \mathscr{M}$ are real, then we can extend this filtration to indices $b \in \mathbb{R}$. The terms of this complex are finite type $\mathscr{O}$-modules. As the morphism $\mathrm{gr}_V^b M \to \mathrm{gr}_V^{b-1} M$ induced by ∂_t is an isomorphism for any $b < 0$, it follows that the inclusion of complexes

$$V^0 \mathrm{DR}\, M \hookrightarrow \mathrm{DR}\, M \tag{2.4.20}$$

is a quasi-isomorphism.

Example 2.4.21 **(The de Rham complex of a minimal extension)**

Let us assume that M is a minimal extension, that is, that can is onto and var is injective. Then $V^{-1}M = \partial_t V^0 M + V^{>-1} M$ and $t : V^{-1}M \to tV^{-1}M$ is an isomorphism. Therefore, $V^0 \mathrm{DR}\, M$ is quasi-isomorphic to

$$\{0 \longrightarrow V^0 M \xrightarrow{-t\partial_t} \Omega^1 \otimes_{\mathscr{O}} (t\partial_t V^0 M + V^{>0} M) \longrightarrow 0\}. \tag{2.4.22}$$

We can refine the presentation of this complex by using the lifted monodromy filtration $\mathrm{M}_\bullet V^\bullet M$.

Lemma 2.4.23. *If M is a minimal extension, then* $\mathrm{DR}\, M$ *is quasi-isomorphic to*

$$\{0 \longrightarrow \mathrm{M}_0 V^0 M \xrightarrow{-t\partial_t} \Omega^1 \otimes_{\mathscr{O}} \mathrm{M}_{-2} V^0 M \longrightarrow 0\}.$$

Proof. Clearly, the complex in the lemma is a subcomplex of (2.4.22). Let us consider the quotient complex. This is

$$0 \longrightarrow (\psi_t^0 M / \mathrm{M}_0 \psi_t^0 M) \xrightarrow{\mathrm{N}} (\mathrm{image}\, \mathrm{N} / \mathrm{M}_{-2} \psi_t^0 M) \longrightarrow 0. \tag{2.4.24}$$

Applying Exercise 1.4.4, we find that this complex is quasi-isomorphic to 0 (*i.e.*, the middle morphism is an isomorphism). □

The sheaf version of the de Rham complex. We now sheafify the previous constructions and consider a $\mathscr{D}_D$-module $\mathscr{M}$. We assume it is holonomic, that is, its germ at any point of D is holonomic in the previous sense. Then the $\mathscr{D}_D$-module $\mathscr{M}$ is a $\mathscr{O}_D$-module and is equipped with a connection. Moreover, away from the origin (and if the disc D is chosen sufficiently small), it is locally $\mathscr{O}_{D^*}$-free of finite rank.

The holomorphic de Rham complex $\mathrm{DR}\, \mathscr{M}$ is defined as the complex

$$\mathrm{DR}\, \mathscr{M} = \{0 \longrightarrow \mathscr{M} \xrightarrow{\nabla} \Omega_D^1 \otimes_{\mathscr{O}_D} \mathscr{M} \longrightarrow 0\}.$$

Away from the origin, the de Rham complex has cohomology in degree -1 only, and $\operatorname{Ker}\nabla$ is a local system of finite dimensional $\mathbb{C}$-vector spaces on D^*. In general, $\mathrm{DR}\,\mathscr{M}$ is a constructible complex on D, that is, its cohomology spaces at the origin are finite dimensional $\mathbb{C}$-vector spaces.

Exercise 2.4.25. Show that the holomorphic de Rham complex $\mathrm{DR}\,\widetilde{V}_{\min}$ has cohomology in degree 0 only and that this cohomology sheaf is $j_*\mathscr{H}$, if $j : D^* \hookrightarrow D$ denotes the inclusion.

LECTURE 3

HODGE $\mathscr{D}$-MODULES ON CURVES GLOBAL PROPERTIES

In Section 3.1, we state various generalizations of the Hodge theorem on Riemann surfaces. The case of constant coefficients has been considered in § 1.1. The next case is that of a unitary representation of the fundamental group of a compact Riemann surface. This is a variation of polarized Hodge structure of weight $(0, 0)$. We then consider the case of a general variation of polarized Hodge structure on a compact Riemann surface.

The next step consists in introducing singularities, that is, in considering a punctured compact Riemann surface. We first consider a unitary representation of the fundamental group, and then the general case of a variation of polarized Hodge structure. The general result is due to Zucker [**34**]. Using the results of the previous lecture, we interpret this theorem as a theorem on direct images of a Hodge $\mathscr{D}$-module on a compact Riemann surface (direct image with respect to the constant map).

Given a local system $\mathscr{L}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces on a punctured compact Riemann surface $X^* \overset{j}{\hookrightarrow} X$ which underlies a variation of Hodge structure, the cohomology which is expected to underlie a Hodge structure is the cohomology $H^k(X, j_*\mathscr{L}_{\mathbb{Q}})$. The Hodge properties will be a consequence of an identification of this cohomology (tensored with $\mathbb{C}$) with a L^2-cohomology, which has harmonic representatives. The theorems of Schmid in § 3.2 justify this identification. It remains then to do Hodge theory on this L^2-cohomology: this is the content of § 3.3.

3.1. The Hodge theorem

3.1.a. The Hodge theorem for unitary representations. We will extend the Hodge theorem (Theorem 1.2.5 and the results indicated after its statement concerning the polarization) to the case of the cohomology with coefficients in a unitary representation. It is not needed to assume here that the underlying manifold is a Riemann surface.

Let us start with a holomorphic vector bundle V of rank d on a complex projective manifold X equipped with a flat holomorphic connection ∇ (flatness is tautological on a Riemann surface). The local system $\mathscr{H}=\operatorname{Ker}\nabla$ corresponds to a representation $\pi_1(X,\star)\to \mathrm{GL}_d(\mathbb{C})$, up to conjugation. The unitarity assumption means that we can conjugate the given representation in such a way that it takes values in the unitary group.

In other words, there exists a Hermitian metric h on the associated C^∞-bundle $H=\mathscr{C}^\infty\otimes_{\mathscr{O}_X}V$ such that, if we denote by D the connection on H defined by $D(\varphi\otimes v)=d\varphi\otimes v+\varphi\otimes\nabla v$ (this is a flat connection which decomposes as $D=D'+D''$ and $D''=d''\otimes\mathrm{Id}$), the connection D is compatible with a metric (*i.e.*, is the Chern connection of the metric).

Exercise 3.1.1. Show that a connection D is flat if and only if it satisfies

$$D'^2=0,\quad D''^2=0,\quad D'D''+D''D'=0.$$

That D is a connection *compatible with the metric* implies that its formal adjoint (with respect to the metric) is obtained with a Hodge $\star$ operator by the formula $D^*=-\star D\star$. This leads to the decomposition of the space of C^∞ k-forms on X with coefficients in H (resp. (p,q)-forms) as the orthogonal sum of the kernel of the Laplace operator with respect to D (resp. D' or D''), that is, harmonic section, and its image.

As the connection D is *flat*, there is a C^∞ de Rham *complex* $(\mathscr{E}_X^\bullet\otimes H,D)$, and standard arguments give

$$H^k(X,\mathscr{H})=\mathbb{H}^k(X,\mathrm{DR}(V,\nabla))=H^k\big(\Gamma(X,(\mathscr{E}_X^\bullet\otimes H,D))\big).$$

One can also define the Dolbeault cohomology groups by decomposing $\mathscr{E}^\bullet$ into $\mathscr{E}^{p,q}$'s and by decomposing D as $D'+D''$. Then $H^{p,q}_{D''}(X,H)=H^q(X,\Omega^p_X\otimes V)$.

As the projective manifold X is *Kähler*, we obtain the Kähler identities for the various Laplace operators: $\Delta_D=2\Delta_{D'}=2\Delta_{D''}$.

Then, exactly as in Theorem 1.2.5, we get:

Theorem 3.1.2. *Under these conditions, one has a canonical decomposition*

$$\mathbb{H}^k(X,\mathrm{DR}(V,\nabla))=\bigoplus_{p+q=k}H^{p,q}(X,H)$$

and $H^{q,p}(X,H)$ is identified with $\overline{H^{p,q}(X,H^\vee)}$, where $H^\vee$ is the dual bundle. (1)

The Hard Lefschetz theorem also holds in this context.

3.1.b. Variation of polarized Hodge structure on a compact Kähler manifold: the Hodge-Deligne theorem. Let us keep notation of § 3.1.a. We do not assume anymore that $\mathscr{H}$ is unitary. We only assume that it underlies a variation of polarized Hodge structure of some weight w. In such a situation, we have flat connection D on the C^∞-bundle H associated to V, with $D=D'+d''$, and we also

1. When we work with a variation of polarized Hodge structure, the polarization $\mathscr{S}_{\mathbb{Q}}$ identifies (H,D) and $(H^\vee,D^\vee)$ and we recover the usual conjugation relation between $H^{q,p}$ and $H^{p,q}$.

have a Hermitian metric h on H, but D is possibly not compatible with the metric. The argument using the Hodge $\star$ operator is not valid anymore.

Exercise 3.1.3. Let $(V, \nabla, F^\bullet V, \mathscr{H}_\mathbb{Q}, \mathscr{S}_\mathbb{Q})$ be a variation of Hodge structure of weight w on X (*cf.* Definition 2.1.1). Let H be the C^∞-bundle associated to V, let h be the Hermitian metric deduced from $\mathscr{S}_\mathbb{Q}$ and let $D = D' + D''$ with $D'' = d'' \otimes \mathrm{Id}_V$ be the flat C^∞ connection deduced from ∇. Let $H = \bigoplus_{p+q=w} H^{p,q}$ be the Hodge decomposition (which is h-orthogonal by construction).

Show that

(1) the Griffiths transversality relation $\nabla F^p V \subset \Omega^1_X \otimes F^{p-1}V$ can be translated as

$$D' H^{p,q} \subset \Omega^1_X \otimes (H^{p,q} \oplus H^{p-1,q+1}),$$
$$D'' H^{p,q} \subset \overline{\Omega^1_X} \otimes (H^{p,q} \oplus H^{p+1,q-1});$$

(2) the composition of D' (resp. D'') with the projection on the first summand defines a $(1,0)$ (resp. $(0,1)$)-connection D'_E (resp. D''_E), and that the projection to the second summand defines a C^∞-linear morphism θ'_E (resp. θ''_E).

(3) Show that $D_E := D'_E + D''_E$ is compatible with the metric h, but is possibly not flat.

(4) Show that the connection $\mathcal{D}'' := D''_E + \theta'_E$ has square zero, as well as the connection $\mathcal{D}' := D'_E + \theta''_E$.

The decomposition $D = D' + D''$ is replaced with the decomposition $D = \mathcal{D}' + \mathcal{D}''$. The disadvantage is that we loose the decomposition into types $(1,0)$ and $(0,1)$, but we keep the flatness property. On the other hand, as D_E is compatible with the metric, its formal adjoint is computed with a Hodge $\star$ operator. Using the *Kähler* metric, one shows that θ_E satisfies the right relations in order to ensure the equality of Laplace operators $\Delta_D = 2\Delta_{\mathcal{D}'} = 2\Delta_{\mathcal{D}''}$.

We remark we did not really loose the decomposition into type if we are more careful: the operator $\mathcal{D}''$ sends a section of $H^{p,q}$ to a section of $\Omega^1_X \otimes H^{p-1,q+1} + \overline{\Omega^1_X} \otimes H^{p,q}$. Counting the total type, we find $(p, q+1)$ for both terms. In other word, taking into account the Hodge type of a section, the operator $\mathcal{D}''$ is indeed of type $(0,1)$. A similar argument applies to $\mathcal{D}'$.

This being understood, the arguments of Hodge theory apply to this situation as in the case considered in § 3.1.a, to get the Hodge-Deligne theorem 2.1.2. The polarization is obtained from $\mathscr{S}_\mathbb{Q}$ and Poincaré duality as we did for Q_n in § 1.2, still using the sign ε, and from it we cook up the form $S_\mathbb{Q}$.

3.1.c. Unitary representation on a Riemann surface with a complete metric. The compactness assumption in Hodge theory is not mandatory. One can relax it, provided that the metric remains *complete* (*cf. e.g.*, [**7**, § 12]). Let us indicate the new phenomena that occur in the setting of § 3.1.a.

One works with C^∞ sections v of $\mathscr{E}^\bullet_X \otimes H$ which are globally L^2 with respect to the metric h and to the complete metric on X, and whose differential Dv is L^2. The analysis of the Laplace operator is now similar to that of the compact case. One uses a L^2 de Rham complex and a L^2 Dolbeault complex (*i.e.*, one puts a L^2 condition on sections and their derivatives).

One missing point, however, is the finite dimensionality of the L^2-cohomologies involved. In the compact case, it is ensured, for instance, by the finiteness of the Betti cohomology $H^k(X, \mathscr{H}_{\mathbb{Q}})$. So the theorem is stated as

Theorem 3.1.4. *Let (X, ω) be a complete Kähler manifold and (V, ∇) be a holomorphic bundle with a flat connection ∇ corresponding to an irreducible representation* $\operatorname{Ker}\nabla$ *of $\pi_1(X, \star)$. Then, with the assumption that all the terms involved are finite dimensional, one has a canonical isomorphisms*

$$H^k_{L^2}(X, H, D) \simeq \bigoplus_{p+q=k} H^{p,q}_{L^2}(X, H, D''), \qquad H^{q,p}_{L^2}(X, H, D) \simeq \overline{H^{p,q}_{L^2}(X, H^\vee, D^\vee)}.$$

It remains to relate the L^2 de Rham cohomology with topology. If we are lucky, then this will not only provide a relation with Betti cohomology, but the Betti cohomology will be finite dimensional and this will also provide the finiteness assumption needed for the L^2 de Rham cohomology.

There will also be a need for the finiteness of the L^2 Dolbeault cohomology. In the case, that will occupy us later, where X is a punctured compact Riemann surface, this will be done by relating L^2 Dolbeault cohomology with the cohomology of a coherent sheaf on the compact Riemann surface.

We will indicate in § 3.3 the way to solve these two problems, through the L^2 Poincaré Lemma and the L^2 Dolbeault Lemma.

3.1.d. Variation of polarized Hodge structure on a punctured compact Riemann surface: the Hodge-Zucker theorem. We now mix the setting of §§ 3.1.b and 3.1.c, that is we consider a variation of polarized Hodge structure of weight w on a punctured compact Riemann surface $X^* \overset{j}{\hookrightarrow} X$. We will prove:

***Theorem 3.1.5* (Zucker [34]).** *In such a case, the cohomology $H^k(X, j_*\mathscr{H})$ $(k = 0, 1, 2)$ carries a natural polarized Hodge structure of weight $w + k$.*

The way of using L^2 cohomology is the exactly the same as in § 3.1.c, provided that we replace D' and D'' with $\mathcal{D}'$ and $\mathcal{D}''$. Then we are left with the corresponding L^2 Poincaré and Dolbeault Lemmas, that we will consider in § 3.3.

3.1.e. Hodge $\mathscr{D}$-modules on a compact Riemann surface: the Hodge-Saito theorem. Let us now start with a polarized Hodge $\mathscr{D}_X$-module on a compact Riemann surface X (*cf.* Definition 2.3.5). We denote it by $\big((\widetilde{V}_{\min}, F^\bullet), j_*\mathscr{H}_{\mathbb{Q}}, j_*\alpha, \mathscr{S}_{\mathbb{Q}}\big)$. Away from a finite set $\Sigma \overset{i}{\hookrightarrow} X$, it corresponds to a variation of Hodge structure of weight w. The de Rham complex $\mathrm{DR}\,\widetilde{V}_{\min}$ is naturally filtered (*cf.* Formula 4.1.2 below) and we get in a natural way a filtration on its hypercohomology. Recall that (*cf.* Exercise 2.4.25), this de Rham complex is a resolution of $j_*\mathscr{H}$, where we denote of course by $X^* = X \setminus \Sigma \overset{j}{\hookrightarrow} X$ the inclusion.

The Hodge-Saito theorem asserts that the k-th cohomology of $j_*\mathscr{H}_\mathbb{Q}$ gets in this way a polarized Hodge structure of weight $w+k$.

What is the difference with the Hodge-Zucker theorem? A priori, it seems that the Hodge-Saito theorem is implied by the Hodge-Zucker theorem, and such is the case, indeed, as we will see in a moment. However, the object with start with is defined *on* X, and for instance is algebraic by GAGA, although in the theorem of Hodge-Zucker it is only defined on X^*. Part of the theorem of Hodge-Zucker consists in extending the object to X (at least at the level of de Rham or Dolbeault complexes). Then the main part in proving the Hodge-Saito theorem will be checking that the extended object defined by Zucker and the a priori object defined by M. Saito correspond. In other words, a polarized Hodge $\mathscr{D}_X$-module is nothing but the exact algebraic expression, before taking 'de Rham' of the extended object constructed by Zucker (through the theorems of Schmid, *cf.* § 3.2.b).

The main point will then be Theorem 3.3.4 together with Theorem 3.3.5, putting together Schmid's theorem 3.2.7 and Lemma 2.4.23 (this called the L^2 Poincaré lemma) and a similar statement for the filtered object (this is called the L^2 Dolbeault lemma).

3.2. Metric characterization of the minimal extension

Before trying to prove the Hodge theorem in the singular case, we will have to come back to the local setting of § 2.2. Let us consider a variation of polarized Hodge structure $\big((V,\nabla,F^\bullet V),\mathscr{H}_\mathbb{Q},\alpha,\mathscr{S}_\mathbb{Q}\big)$ of weight w on the punctured disc. The polarization $\mathscr{S}_\mathbb{Q}$ gives rise to a positive definite Hermitian metric h on the C^∞-bundle associated to V.

3.2.a. Reminder on Hermitian bundles on the punctured disc. Let V be a holomorphic vector bundle on D^* and let h be a Hermitian metric on the associated C^∞-bundle $\mathscr{C}^\infty_{D^*}\otimes_{\mathscr{O}_{D^*}} V$.

Let V be a holomorphic bundle on D^*, equipped with a Hermitian metric h. We denote by ${}^h\widetilde{V}$ the subsheaf of j_*V consisting of local sections, the h-norm of which has moderate growth in the neighbourhood of the origin. This is a $\mathscr{O}_D[1/t]$-module, which coincides with V when restricted to D^*.

The *parabolic filtration* ${}^h\widetilde{V}^\bullet$ is the decreasing filtration, indexed by $\mathbb{R}$, consisting of local sections such that: for any compact neighbourhood K of the origin, in the punctured neighbourhood of which the local section is defined, and for any $\varepsilon>0$, there exists $C=C(K,\varepsilon)>0$ such that the h-norm on $K^* := K\setminus\{0\}$ of the local section is locally bounded by $C|t|^{\bullet-\varepsilon}$. By definition, we have ${}^h\widetilde{V}^b=\bigcap_{b'<b}{}^h\widetilde{V}^{b'}$.

Clearly, each ${}^h\widetilde{V}^b$ is a $\mathscr{O}_D$-submodule of ${}^h\widetilde{V}$, which coincides with V when restricted to D^*, and we have

$$ {}^h\widetilde{V}=\bigcup_b {}^h\widetilde{V}^b, \qquad \text{and} \qquad \forall\, k\in\mathbb{Z},\quad t^k\,{}^h\widetilde{V}^\bullet={}^h\widetilde{V}^{\bullet+k}. $$

A *jump* (or, more correctly, jumping index) of the parabolic filtration is a real number b such that the quotient ${}^h\widetilde{V}^b/{}^h\widetilde{V}^{>b} \neq 0$, where ${}^h\widetilde{V}^{>b} := \bigcup_{b'>b} {}^h\widetilde{V}^{b'}$. Clearly, if b is a jump, then $b+k$ is a jump for any $k \in \mathbb{Z}$. We denote by $J(b)$ the set of jumping indices which belong to $[b, b+1[$. We have $J(b+k) = J(b)$ for any $k \in \mathbb{Z}$.

Definition 3.2.1. We say that the metric is *moderate* if ${}^h\widetilde{V}$ is $\mathscr{O}_D[1/t]$-locally free and each ${}^h\widetilde{V}^b$ is $\mathscr{O}_D$-locally free.

When the metric is moderate, $J(b)$ is finite for any $b \in \mathbb{R}$ and we have

$$ {}^h\widetilde{V}^b/t\,{}^h\widetilde{V}^b = \bigoplus_{b' \in J(b)} \mathrm{gr}^{b'}({}^h\widetilde{V}). $$

3.2.b. The theorems of Schmid. Let us consider a variation of Hodge structure on D^*, as in the beginning of this section, with associated Hermitian metric h.

***Theorem 3.2.2* (Schmid).** *The metric h on $\mathscr{C}^\infty_{D^*} \otimes_{\mathscr{O}_{D^*}} V$ is moderate and the meromorphic extension ${}^h\widetilde{V}$ of V with respect to the metric h is equal to the canonical Deligne meromorphic extension $\widetilde{V}$ of (V, ∇).*

***Example 3.2.3* (The unitary case).** Let us consider the simple case where the connection is compatible with the Hermitian metric h. This corresponds to a variation of Hodge structure of pure type $(0,0)$. Then the norm of any horizontal section of H is constant, hence bounded. By definition of Deligne meromorphic extension, the norm of any section of $\widetilde{V}$ has thus moderate growth. Hence $\widetilde{V} \subset {}^h\widetilde{V}$.

In fact, both extensions are then equal, and therefore the metric is moderate, as asserted in the general case by Schmid's theorem. Indeed, given any section v of V, we express it on a unitary frame of multivalued horizontal sections, and the norm of the section has moderate growth if and only if the coefficients are multivalued functions with moderate growth in any bounded angular sector. Similarly, we can express a $\mathscr{O}_D[t^{-1}]$-basis of $\widetilde{V}$ on this unitary frame, and the coefficients have moderate growth. Expressing now v in the chosen $\mathscr{O}_D[t^{-1}]$-basis of $\widetilde{V}$, we find univalued coefficients with moderate growth, that is, meromorphic functions. In other words, ${}^h\widetilde{V} \subset \widetilde{V}$.

From now on, we will not distinguish between ${}^h\widetilde{V}$ and $\widetilde{V}$. One can go further, and analyze the parabolic filtration. But first, we need a result, due to Borel (*cf.* [**29**, Lemma 4.5]), which asserts:

Lemma 3.2.4. *For such a variation, the monodromy is quasi-unipotent.*

With such a result, we can extend the Kashiwara-Malgrange filtration to a filtration indexed by $\mathbb{R}$ (in fact $\mathbb{Q}$). The next result is:

***Theorem 3.2.5* (Schmid).** *The parabolic filtration ${}^h\widetilde{V}^\bullet$ on $\widetilde{V}$ induced by the metric h is equal to the (extended) Kashiwara-Malgrange filtration $\widetilde{V}^\bullet$.*

Exercise 3.2.6. Prove the result in the unitary case of Example 3.2.3.

This result characterizes sections of $\widetilde{V}^b$ in terms of growth of their norm with respect to real powers of t. In order to analyze the L^2 behaviour of the norm, we will need to refine this result by using a logarithmic scale. Recall that we can lift the monodromy filtration $\mathrm{M}_\bullet\psi_t^b\widetilde{V}$ to $\mathrm{M}_\bullet\widetilde{V}^b$.

***Theorem 3.2.7* (Schmid).** *A local section of $\mathrm{M}_\ell\widetilde{V}^b$ has a non-zero image in $\mathrm{gr}_\ell^{\mathrm{M}}\psi_t^b\widetilde{V}$ if and only if its norm has the same order of growth as $|t|^b\mathrm{L}(t)^{\ell/2}$.*

Remark 3.2.8. In § 2.2.b, when extending the vector bundle V with holomorphic connection ∇ from D^* to D, we chose Deligne's meromorphic extension, that is, we chose an extension with *regular singularities*. Such a choice, while being canonical and, in some sense, as simple as possible, was not the only one. We could have chosen other kinds of extensions, with irregular singularities. A posteriori, when considering variations of *polarized* Hodge structures, the theorems of Schmid strongly justify the previous choice.

3.3. Proof of the Hodge-Zucker theorem

3.3.a. Hermitian bundle and volume form. If we fix a metric on the punctured disc, with volume element vol, we can define the L^2-norm of a section v of V on an open set $U \subset D^*$ by the formula

$$\|v\|_2^2 = \int_U h(v,v)\,d\,\mathrm{vol}\,.$$

In order to be able to apply the techniques of § 3.1.c, we choose a metric which is complete in the neighbourhood of the puncture. We will assume that, near the puncture, it takes the form

$$d\,\mathrm{vol} = \frac{dx^2+dy^2}{|t|^2\mathrm{L}(t)^2}, \quad \text{with } x=\mathrm{Re}\,t,\ y=\mathrm{Im}\,t,\ \mathrm{L}(t):=|\log|t|^2| = -\log t\bar{t}. \tag{3.3.1}$$

3.3.b. L^2 computation of $j_*\,\mathrm{Ker}\,\nabla$. The goal of this section is to explain the proof, by S. Zucker [**34**], of the following theorem:

Theorem 3.3.2. *The cohomology $H^*(X, j_*\,\mathrm{Ker}\,\nabla)$ is equal to the L^2 cohomology of the C^∞-bundle with flat connection (H,D) associated with the holomorphic bundle (V,∇), the L^2 condition being taken with respect to the Hodge metric h on H and a complete metric on X^*, locally equivalent near each puncture to the Poincaré metric.*

Instead of directly proving this result, one proves a local result: one introduces a L^2 complex, whose global cohomology computes the desired L^2 cohomology, and one shows a quasi-isomorphism of this complex with $j_*\,\mathrm{Ker}\,\nabla$. Therefore, we will start a local analysis on a punctured disc D^*. We keep notation of the previous section.

Let us be more explicit concerning the Poincaré metric. Working in polar coordinates $t = re^{i\theta}$, we find a characterization of the L^2 behaviour of forms near the puncture:

(0) $f \in L^2(d\,\mathrm{vol}) \Leftrightarrow |\log r|^{-1} f \in L^2(d\theta\, dr/r)$;

(1) $\omega = f\, dr/r + g\, d\theta \in L^2(d\,\mathrm{vol}) \Leftrightarrow f$ and $g \in L^2(d\theta\, dr/r)$;

(2) $\eta = h\, d\theta\, dr/r \in L^2(d\,\mathrm{vol}) \Leftrightarrow |\log r|\, h \in L^2(d\theta\, dr/r)$.

On the other hand, for any integer ℓ, we have $|\log r|^{\ell/2} \in L^2(d\theta\, dr/r)$ if and only if $\ell \leqslant -2$.

3.3.c. The holomorphic L^2 de Rham complex. Let us consider the holomorphic de Rham complex of $\widetilde{V}$:

$$\mathrm{DR}\,\widetilde{V} = \{0 \longrightarrow \widetilde{V} \xrightarrow{\nabla} \Omega^1_D \otimes \widetilde{V} \longrightarrow 0\}.$$

We will consider the holomorphic L^2 de Rham complex

$$(\mathrm{DR}\,\widetilde{V})_{(2)} = \{0 \longrightarrow \widetilde{V}_{(2)} \xrightarrow{\nabla} (\Omega^1_D \otimes \widetilde{V})_{(2)} \longrightarrow 0\},$$

defined in the following way:

– $(\Omega^1_D \otimes \widetilde{V})_{(2)}$ is the subsheaf of $\Omega^1_D \otimes \widetilde{V}$ consisting of L^2-sections (with respect to the metric h on $\widetilde{V}$ and the volume $d\,\mathrm{vol}$ on D^*),

– $\widetilde{V}_{(2)}$ is the subsheaf of $\widetilde{V}$ consisting of sections v which are L^2, and such that ∇v belongs to $(\Omega^1_D \otimes \widetilde{V})_{(2)}$ defined above.

Let us note that, by the very construction, we get a complex.

Exercise 3.3.3

(1) Let $r_0 \in\,]0,1[$, let $b \in \mathbb{R}$ and $\ell \in \mathbb{Z}$. Show that the integral

$$\int_0^{r_0} r^{2b+1} \mathrm{L}(r)^\ell dr$$

is finite iff $b > -1$ or $b = -1$ and $\ell \leqslant -2$ (recall that $\mathrm{L}(r) = |\log r| = -\log r$).

(2) Deduce from Schmid's theorem 3.2.7 and the characterization of $L^2(d\,\mathrm{vol})$ given above for 1-forms that $(\Omega^1_D \otimes \widetilde{V})_{(2)} = \mathrm{M}_{-2}\widetilde{V}^{-1}$.

(3) Similarly, show that the holomorphic sections of V which are L^2 near the origin are the sections of $\mathrm{M}_0\widetilde{V}^0$.

(4) Conclude that $\widetilde{V}_{(2)} = \mathrm{M}_0\widetilde{V}^0$ (use that $t\mathrm{M}_{-2}\widetilde{V}^{-1} = \mathrm{M}_{-2}\widetilde{V}^0$ and that $t\partial_t(\mathrm{M}_0\widetilde{V}^0) \subset \mathrm{M}_{-2}\widetilde{V}^0$).

According to Lemma 2.4.23, we get

***Theorem 3.3.4* (Zucker).** *We have* $(\mathrm{DR}\,\widetilde{V})_{(2)} \simeq \mathrm{DR}\,\widetilde{V}_{\min} = j_*\,\mathrm{Ker}\,\nabla$.

This theorem is the first step toward a L^2 computation of $j_*\,\mathrm{Ker}\,\nabla$.

3.3.d. The L^2 de Rham complex. We now work with the associated C^∞ bundle $H = \mathscr{C}^\infty_{D^*} \otimes_{\mathscr{O}_{D^*}} V$. It is equipped with a flat C^∞ connection $D = D' + D''$, with $D'' = d'' \otimes \mathrm{Id}$ and D' induced by ∇. We can similarly define the L^2 de Rham complex

$$0 \longrightarrow \mathscr{L}^0_{(2)}(H) \xrightarrow{D} \mathscr{L}^1_{(2)}(H) \xrightarrow{D} \mathscr{L}^2_{(2)}(H) \longrightarrow 0,$$

where the upper index refers to the degree of forms. One should give a precise definition of each term. Let us only say that we consider sections of H having as coefficients currents of degree k ($k = 0, 1, 2$, currents of degree 0 are distributions) and the norm of these sections should be locally L^2 on the disc D. Moreover, we have to ensure that the differential of theses sections are also L^2, in order to get a complex.

Let us note that, with such a definition, it is not clear that we can decompose the complex with respect to type, that is, it is not clear that, if a section v is such that Dv is L^2, then $D'v$ and $D''v$ are also L^2.

Theorem 3.3.5 **(L^2 Poincaré Lemma, Zucker).** *The natural inclusion of complexes* $(\mathrm{DR}\,\widetilde{V})_{(2)} \hookrightarrow \mathscr{L}^\bullet_{(2)}(H, D)$ *is a quasi-isomorphism.*

According to Theorem 3.3.4, taking hypercohomology gives Theorem 3.3.2.

Indication for the proof of Theorem 3.3.5. The proof is purely local near the origin of the disc. The main observation is

Lemma 3.3.6 **(*cf.* [34, Prop. 6.4]).** *Let L be a holomorphic line bundle on D^* (equipped with the complete metric* (3.3.1)*) with Hermitian metric h and having a frame v such that $\|v\|_h \sim \mathrm{L}(t)^k$ for $k \in \mathbb{Z}$. Then, if $k \neq 1$, any germ $\eta = f d\bar{t} \otimes v$ of section of $\mathscr{L}^{(0,1)}_{(2)}(L, h)$ at the origin is equal to $\overline{\partial}\psi \otimes v$ for some local section $\psi \otimes v$ of $\mathscr{L}^0_{(2)}(L, h)$.*

This is a $\overline{\partial}$ equation with logarithmically twisted L^2 conditions. It is proved using the decomposition in Fourier series and Hardy inequalities.

Once this lemma is proved, the proof of Theorem 3.3.5 when the monodromy is unipotent follows (this is not completely straightforward) from Schmid's theorem 3.2.7. The quasi-unipotent case can be deduced by considering a suitable ramified covering of the disc. □

3.3.e. The L^2 Dolbeault lemma. One of the important points in order to prove the E_1-degeneracy of the Hodge-to-de Rham spectral sequence in the context of the Hodge-Zucker theorem is the Dolbeault lemma, making the bridge between the holomorphic world and the L^2 world of harmonic sections. We will briefly give indications on its proof to end this lecture.

Recall that the Dolbeault lemma, on a compact complex manifold X, says that $H^q(X, \Omega^p_X) \simeq H^{p,q}_{d''}(X) = H^q\big(\Gamma(X, \mathscr{E}^{p,\bullet}_X), d''\big)$.

If we now consider a variation of polarized Hodge structure on X, as in § 3.1.b, the complex $\mathscr{E}^\bullet_X(V)$ is filtered by taking into account the holomorphic degree of the form and the Hodge degree of the section. Moreover, this filtration splits as direct sum of terms $\mathscr{E}_X^{i,j}\otimes H^{k,\ell}$, and each of this term is a summand in the p,q term of the decomposition if $p=i+k$ and $q=j+\ell$. The Dolbeault lemma then says that

$$H^q\big(X,\mathrm{gr}_F^p\mathrm{DR}(V,\nabla)\big)=H^q\big(\Gamma(X,\mathrm{gr}_F^p\mathscr{E}_X^\bullet(V))\big).$$

Let us note that the differential in the complex $\mathrm{gr}_F^p\mathscr{E}_X^\bullet(V))$ is $\mathcal{D}''$ introduced in Exercise 3.1.3(4).

Let us now come back to the context of the Hodge-Zucker theorem. The first point to be settled is the freeness of each step of the F-filtration of $\widetilde{V}_{\min}$. Recall that it is defined with (2.2.3) and (2.2.4).

Let us first consider $F^p\widetilde{V}_{(2)}:=j_*F^pV\cap\widetilde{V}_{(2)}$. According to Exercise 3.3.3(4), this is also $j_*F^pV\cap\mathrm{M}_0\widetilde{V}^0$. If we show its coherence, then $F^p\widetilde{V}^0$ will be coherent as well, as $\widetilde{V}^0/\mathrm{M}_0\widetilde{V}^0$ is finite dimensional. In the same way, $F^p\widetilde{V}^{>-1}$ will be coherent. It will then also be locally free of rank equal to $\mathrm{rk}\,F^pV$, and $F^p\widetilde{V}_{\min}$ defined by (2.2.4) will be a good filtration of $\widetilde{V}_{\min}$ satisfying $F^p\widetilde{V}_{\min}\cap\widetilde{V}_{(2)}=F^p\widetilde{V}_{(2)}$.

The coherence of $F^p\widetilde{V}_{(2)}$ is shown in [**34**, Prop. 5.2], using results of Schmid [**29**]. Moreover, one can give the following interpretation of these results:

Theorem 3.3.7 **(Schmid).** $\big((\widetilde{V}_{\min},F^\bullet\widetilde{V}_{\min}),j_*\mathscr{H}_{\mathbb{Q}},\alpha,\mathscr{S}_{\mathbb{Q}}\big)$ *is a polarized Hodge $\mathscr{D}$-module on the disc D, in the sense of Definitions* 2.3.3 *and* 2.3.5.

It is not difficult to filter the complex $\mathrm{DR}\,\widetilde{V}_{\min}$ by the usual procedure (*cf.* (4.1.2) below) from the filtration of $\widetilde{V}_{\min}$. On the other hand, according to Theorem 3.3.4, the inclusion $(\mathrm{DR}\,\widetilde{V})_{(2)}\hookrightarrow\mathrm{DR}\,\widetilde{V}_{\min}$ is a quasi-isomorphism. Is it a *filtered* quasi-isomorphism?

Firstly, we have to define the filtration $F^\bullet(\mathrm{DR}\,\widetilde{V})_{(2)}$. Using the interpretation of Exercise 3.3.3, we are reduced to defining the filtration on $\mathrm{M}_0\widetilde{V}^0$ and $\mathrm{M}_{-2}\widetilde{V}^{-1}$. The natural choice is simply to induce the filtration $F^\bullet\widetilde{V}_{\min}$ on these submodules. Therefore, answering the question above amounts to answering the following ones:

(1) Is (2.4.20) a filtered isomorphism, when the terms are equipped with the induced filtration?

(2) Is (2.4.24) a filtered isomorphism, when the terms are equipped with the induced filtration?

The answer to both questions is *yes*. For the first question, we have to show that, for any $b<0$ and any p, the complex

$$0\longrightarrow F^p\psi_t^b\widetilde{V}_{\min}\xrightarrow{\;\partial_t\;}F^{p-1}\psi_t^{b-1}\widetilde{V}_{\min}\longrightarrow 0$$

is an isomorphism. This is Exercise 2.2.8(2). Using now Exercise 2.2.8(1), we can replace the filtered complex $F^\bullet V^0\,\mathrm{DR}\,\widetilde{V}_{\min}$ with the filtered complex corresponding to (2.4.22).

For the second question, we have to prove the filtered analogue of Lemma 2.4.23. This is done by an argument of strictness: both terms in (2.4.24) are shown to be mixed Hodge modules, and the morphism between them (if one Tate-twists the right-hand term by -1) a morphism of mixed Hodge modules, hence is strictly compatible with the Hodge filtration.

Let us come back to the Dolbeault lemma. On the holomorphic side, we have $H^q(X, \mathrm{gr}_F^p \mathrm{DR}\, \widetilde{V}_{\min})$, that we now can write as $H^q(X, \mathrm{gr}_F^p(\mathrm{DR}\, \widetilde{V}_{\min})_{(2)})$, a form which will help us to compare with the L^2 side.

The L^2 Dolbeault complex has to be taken with respect to the differential $\mathcal{D}''$ and the L^2 condition on a section $\eta \otimes v$ concerns the derivative $\mathcal{D}''(\eta \otimes v)$.

By using Lemma 3.3.6, one gets

Theorem 3.3.8 **(L^2 Dolbeault Lemma, Zucker).** *The natural morphism (induced by the inclusion of complexes) $H^q(X, \mathrm{gr}_F^p(\mathrm{DR}\, \widetilde{V}_{\min})_{(2)}) \to H^q(X, \mathrm{gr}_F^p \mathscr{L}_{(2)}(V, \mathcal{D}''))$ is an isomorphism.*

LECTURE 4

HODGE $\mathscr{D}$-MODULES: AN INTRODUCTION

In this lecture, we introduce the general notion of a Hodge $\mathscr{D}$-module and state the Hodge-Saito theorem. The analytic part of the proof is contained in the Hodge-Zucker theorem of § 3.1.d. The remaining part is a consequence of the properties of the nearby and vanishing cycles. The functor of nearby cycles is instrumental in defining the notion of a Hodge module. It has to be defined at various levels:

– for perverse complexes,
– for holonomic $\mathscr{D}$-modules,
– for filtered holonomic $\mathscr{D}$-modules.

We will extend to holonomic $\mathscr{D}$-modules in arbitrary dimension the results of § 2.4 which are given for dimension one.

4.1. Good filtrations on $\mathscr{D}_X$-modules

To any complex manifold X is associated the sheaf of holomorphic functions $\mathscr{O}_X$ and the sheaf of holomorphic differential operators $\mathscr{D}_X$. In local coordinates $x_1, \dots, x_n$, a differential operator can be written as $\sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\partial_x^\alpha$. The degree of a local section of $\mathscr{D}_X$ is $\max\{|\alpha| \mid a_\alpha(x) \not\equiv 0\}$. The sheaf $\mathscr{D}_X$ comes equipped with an increasing filtration $F_\bullet\mathscr{D}_X$ by the degree. Each $F_k\mathscr{D}_X$ is a locally free $\mathscr{O}_X$-module of finite rank.

Exercise 4.1.1. Prove that, in an intrinsic way, the graded sheaf $\operatorname{gr}^F\mathscr{D}_X := \bigoplus_k \operatorname{gr}_k^F\mathscr{D}_X$ is identified with the sheaf $\mathscr{O}_X[TX]$ of holomorphic functions on the cotangent bundle which are polynomial in the fibers of $T^*X \to X$.

Given a $\mathscr{O}_X$-module $\mathscr{M}$, the following are equivalent:

(1) a flat connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$, *i.e.*, a $\mathbb{C}$-linear operator satisfying the Leibniz rule $\nabla(fm) = df \otimes m + f\nabla m$, and with vanishing curvature $\nabla\circ\nabla : \mathscr{M} \to \Omega^2_X\mathscr{M}$,

(2) a structure of left $\mathscr{D}_X$-module on $\mathscr{M}$.

Therefore, to any left $\mathscr{D}_X$-module $\mathscr{M}$ one can attach the de Rham complex

$$\mathrm{DR}\,\mathscr{M} = \{0 \longrightarrow \mathscr{M} \xrightarrow{\nabla} \Omega^1_X \otimes \mathscr{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{\dim X}_X \otimes \mathscr{M} \longrightarrow 0\}.$$

It will be convenient to us the *perverse shift convention*, and to denote by ${}^p\mathrm{DR}\mathscr{M}$ the same complex, but where the last term has degree 0; in other words, ${}^p\mathrm{DR}\mathscr{M} = \mathrm{DR}\,\mathscr{M}[\dim X]$.

Let $F_\bullet\mathscr{M}$ be a good filtration of $\mathscr{M}$. We define a filtration on the de Rham complex by the formula:

$$(4.1.2)\quad F_k{}^p\mathrm{DR}\mathscr{M} = \{0 \longrightarrow F_{k-\dim X}\mathscr{M} \xrightarrow{\nabla} \Omega^1_X \otimes F_{k-\dim X+1}\mathscr{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{\dim X}_X \underset{\bullet}{\otimes} F_k\mathscr{M} \longrightarrow 0\}.$$

Examples 4.1.3

(1) Consider the left $\mathscr{D}_X$-module $(\mathscr{O}_X, d)$ with filtration $F_k\mathscr{O}_X = 0$ if $k \leqslant -1$ and $F_k\mathscr{O}_X = \mathscr{O}_X$ if $k \geqslant 0$. This is a good filtration, and the filtration on the de Rham complex is, up to a shift (and up to changing increasing with decreasing), the same as that defined in (1.2.4).

(2) Let $i : Y \hookrightarrow X$ be the inclusion of a closed submanifold. Then $i_+\mathscr{O}_Y$ is a left $\mathscr{D}_X$-module. It is equipped with a natural F-filtration which can be described in the following way. Let us choose local coordinates (x, y) on X, such that Y is defined by $x = 0$. Then $i_+\mathscr{O}_Y = \bigoplus_{\alpha\in\mathbb{N}^{\operatorname{codim} Y}} \mathscr{O}_Y\partial_x^\alpha$ and the F-filtration is by the total degree in ∂_x.

(3) Let $i : Y \hookrightarrow X$ be the inclusion of a closed analytic hypersurface in X. Let us consider the sheaf $\mathscr{O}_X[*Y]$ of meromorphic functions on X with poles along Y at most. This is a left $\mathscr{D}_X$-module. One can filter it by the order of the pole (so that $\mathscr{O}_X = F_0\mathscr{O}_X[*Y]$.

4.2. Hodge $\mathscr{D}$-modules: a first approach

Hodge $\mathscr{D}$-modules are supposed to play the role of Hodge structures with a multi-dimensional parameter. These objects can acquire singularities. The way each characteristic of a Hodge structure is translated in higher dimension of the parameter space is given by the table below.

dimension 0	dimension $n \geqslant 1$
H a $\mathbb{C}$-vector space	$\mathscr{M}$ a holonomic $\mathscr{D}$-module
$F^\bullet H$ a filtration	$F_\bullet\mathscr{M}$ a good filtration
$H_\mathbb{Q}$ a $\mathbb{Q}$-vector space	$\mathscr{F}_\mathbb{Q}$ a bounded complex with constructible cohomology
$\mathbb{C}\otimes_\mathbb{Q} H_\mathbb{Q} = H$ a $\mathbb{Q}$-structure	$\alpha : {}^p\mathrm{DR}\mathscr{M} \xrightarrow{\sim} \mathbb{C}\otimes_\mathbb{Q}\mathscr{F}_\mathbb{Q}$ an isomorphism

Why choosing holonomic $\mathscr{D}$-modules as analogues of $\mathbb{C}$-vector space? The reason is that the category of holonomic $\mathscr{D}$-modules is artinian, that is to say any holonomic $\mathscr{D}$-module has finite length (locally on the underlying manifold). A related reason is that its de Rham complex has constructible cohomology, making the last line of the table a priori meaningful. On the other hand, the last line implies that $\mathscr{F}_{\mathbb{Q}}$ is a *perverse sheaf*.

In order to define the Hodge properties, we use the same method as in dimension one:

– we only consider holonomic $\mathscr{D}$-modules which are *S-decomposable*, that is, which are direct sum of $\mathscr{D}$-modules having an irreducible *strict support*, in other words supported by an irreducible closed analytic subset of the underlying manifold and having neither sub-module nor quotient module supported in a smaller subset;

– we moreover ask that the F-filtration is compatible with the decomposition by the support, in other words, the filtered $\mathscr{D}$-module is *strictly S-decomposable*;

– similarly, we only consider $\mathbb{Q}$-perverse sheaves which are S-decomposable; their irreducible S-components then take the form $\mathrm{IC}^\bullet \mathscr{L}_{\mathbb{Q}}$, where $\mathscr{L}_{\mathbb{Q}}$ is a locally constant sheaf of $\mathbb{Q}$-vector spaces on a Zariski open dense smooth subset Z^o of an irreducible closed analytic subset Z of the underlying manifold, and where IC means "intersection complex";

– in order to reduce the structure to a point, we use iterated nearby cycles, along a family of functions, the ideal of which define the point; therefore, one has to define the functor of nearby cycles in each of the settings of the table above, and to check the compatibility of it with α.

4.3. The V-filtration and the nearby cycle functor

4.3.a. The Kashiwara-Malgrange filtration. We will repeat here, in higher dimension, some of the constructions of § 2.4.b. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a complex manifold X and let f be a holomorphic function on X. Denote by $i_f : X \hookrightarrow X \times \mathbb{C}$ the map $x \mapsto (x, f(x))$ and let t denote the coordinate on $\mathbb{C}$. We define the filtration $V^\bullet \mathscr{D}_{X\times\mathbb{C}}$ by giving t the weight 1, ∂_t the weight -1 and any operator in $\mathscr{D}_X$ the weight 0. Using the existence of a local Bernstein-Sato polynomial, one deduce a filtration $V^\bullet i_{f,+}\mathscr{M}$ indexed by $\mathbb{Z}$ such that, locally on X, there exists $B \in \mathbb{C}[s] \smallsetminus \{0\}$ with roots having their real part in $[0, 1[$, such that $B(t\partial_t - k)$ vanishes on $\mathrm{gr}^k_V i_{f,+}\mathscr{M}$ for any $k \in \mathbb{Z}$.

For any $\beta \in \mathbb{C}$, we can define a functor ψ_f^β, from holonomic $\mathscr{D}_X$-modules to $\mathscr{D}_X$-modules supported on $f = 0$. It is known that it takes values in *holonomic $\mathscr{D}_X$-modules* supported on $f = 0$. As in dimension one, we define N, can, var and the monodromy filtration $\mathrm{M}_\bullet$ on each ψ_f^β.

We say that $\mathscr{M}$ is *quasi-unipotent along* $f=0$ if the roots of the minimal B belong to $\mathbb{Q}$. In the following, we only consider holonomic $\mathscr{D}_X$-modules which are quasi-unipotent along any germ of hypersurface. We can then extend the V-filtration as a filtration indexed by (a discrete set of) $\mathbb{Q}$, as in § 2.4.d.

4.3.b. Strict specializability. Let $(\mathscr{M}, F_\bullet\mathscr{M})$ be a holonomic $\mathscr{D}_X$-module equipped with a good filtration and let f be a holomorphic function on X. We still assume that $\mathscr{M}$ is quasi-unipotent along $f=0$. We want to define a filtration on each $\psi_f^b\mathscr{M}$.

Firstly, it is easy to define a good filtration $F_\bullet(i_{f,+}\mathscr{M})$, and there is a direct image functor [(1)] $i_{f,+}(\mathscr{M}, F_\bullet\mathscr{M})$.

So the question amounts to defining a filtration on each $\mathrm{gr}_V^b i_{f,+}\mathscr{M}$. A natural candidate is given by a formula similar to (2.4.18).

We say that $(\mathscr{M}, F_\bullet\mathscr{M})$ is *strictly specializable along* $f=0$ if the following properties are satisfied (*cf.* Exercise 2.2.8):

(a) for any $b>-1$, the isomorphism $t: V^b(i_{f,+}\mathscr{M}) \to V^{b+1}(i_{f,+}\mathscr{M})$ is strictly compatible with the filtration induced by $F_\bullet\mathscr{M}$,

(b) for any $b<0$, the isomorphism $\partial_t: \psi_f^b(i_{f,+}\mathscr{M}) \to \psi_f^{b-1}(i_{f,+}\mathscr{M})$ is strictly compatible with the filtration induced by $F_\bullet\mathscr{M}$.

4.3.c. Regularity. The previous definition would be of no use without finiteness properties of the induced F-filtration. Let $\mathscr{M}$ be holonomic and quasi-unipotent along $f=0$, and let $F_\bullet\mathscr{M}$ be a good filtration. Let us assume that $(\mathscr{M}, F_\bullet\mathscr{M})$ is strictly specializable. We then say that $(\mathscr{M}, F_\bullet\mathscr{M})$ is *regular along* $f=0$ if the filtration $F_\bullet\psi_f^b\mathscr{M}$ is good, for any $b\in[-1,0]$ (considering other values of b is not necessary, according to the property of strict specializability).

Exercise 4.3.1. Show that, in dimension one, this definition implies that $\mathscr{M}$ is regular in the sense of Definition 2.4.15.

Remark 4.3.2. In the theory of Hodge $\mathscr{D}$-modules, the notion of regular singularity for holonomic $\mathscr{D}$-modules is not used. What is used is the previous notion, which applies to *filtered* $\mathscr{D}$-modules. One can define the notion of regular singularity for a filtered holonomic $\mathscr{D}$-module by induction on the dimension of the support of the $\mathscr{D}$-module: we ask that $(\mathscr{M}, F_\bullet\mathscr{M})$ is regular along any germ of hypersurface $f=0$ and that, whenever $f=0$ has everywhere codimension one in the support of $\mathscr{M}$, then $(\psi_f^b\mathscr{M}, F_\bullet\psi_f^b\mathscr{M})$ is regular.

Fortunately, it can be shown that, if a filtered $\mathscr{D}$-module is regular in this sense of M. Saito, then the underlying holonomic $\mathscr{D}$-module is regular in the classical sense (Kashiwara, Mebkhout).

1. Let us note however that this is not an equivalence of categories.

As we will see below, the usual way to prove a property for a Hodge $\mathscr{D}$-module is by induction on the dimension of the support. This is due to the inductive way in which the definitions are formulated.

4.3.d. The nearby cycle functor. For a complex $\mathscr{F}_{\mathbb{Q}}$ with $\mathbb{Q}$-constructible cohomology (in particular, for a $\mathbb{Q}$-perverse sheaf) on X, we can define, for any holomorphic function f on X a complex $\Psi_f\mathscr{F}_{\mathbb{Q}}$, which has constructible cohomology, and is equipped with an automorphism T (monodromy). It is known that, if $\mathscr{F}_{\mathbb{Q}}$ is perverse, then $\Psi_f\mathscr{F}_{\mathbb{Q}}$ is perverse up to a shift by -1. We are therefore led to set ${}^p\Psi_f\mathscr{F}_{\mathbb{Q}} := \Psi_f\mathscr{F}_{\mathbb{Q}}[-1]$.

One can decompose ${}^p\Psi_f\mathscr{F}$ with respect to eigenvalues λ of T as a direct sum of perverse sheaves (recall that the category of perverse sheaves is abelian, so it is meaningful to consider the perverse sheaf $\operatorname{Ker}(T-\lambda\operatorname{Id})^N$ for $N\gg 0$): we have

$$ {}^p\Psi_f\mathscr{F} = \bigoplus_{\lambda\in\mathbb{C}^*} {}^p\Psi_f^\lambda\mathscr{F}, \qquad {}^p\Psi_f^\lambda\mathscr{F} := \operatorname{Ker}(T-\lambda\operatorname{Id})^N,\ N\gg 0. $$

On the other hand, let $(\mathscr{M},F_\bullet\mathscr{M})$ be a holonomic $\mathscr{D}_X$-module with a good filtration. We assume that $\mathscr{M}$ is quasi-unipotent along $f=0$, that it is strictly specializable and regular along this hypersurface. Then, for any $b\in\,]-1,0]$, the filtration $F_\bullet\psi_f^b\mathscr{M}$ induced by $F_\bullet i_{f,+}\mathscr{M}$ is good. We set

$$ \psi_f^b(\mathscr{M},F_\bullet\mathscr{M}) := (\psi_f^b\mathscr{M},F_\bullet[1]\psi_f^b\mathscr{M}), $$

where the notation for the shift is $F_\bullet[1]=F_{\bullet-1}$ and, correspondingly, $F^\bullet[1]=F^{\bullet+1}$.

In order to define the nearby cycles for a triple $\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$, it remains to apply the functor to α. Recall that α is a chosen isomorphism ${}^p\mathrm{DR}\mathscr{M}\xrightarrow{\sim}\mathscr{F}=\mathbb{C}\otimes_{\mathbb{Q}}\mathscr{F}_{\mathbb{Q}}$ (*cf.* the table above). Applying the topological functor ${}^p\Psi_f$, we get ${}^p\Psi_f\alpha:{}^p\Psi_f{}^p\mathrm{DR}\mathscr{M}\xrightarrow{\sim}{}^p\Psi_f\mathscr{F}$ compatible with monodromies.

In order to define $\psi_f\alpha$, we need to construct in a canonical and functorial way an isomorphism similar to (2.2.2)

$$ {}^p\mathrm{DR}\psi_f\mathscr{M}\xrightarrow{\sim}{}^p\Psi_f({}^p\mathrm{DR}\mathscr{M}), \tag{4.3.3} $$

such that T corresponds to $\exp(-2\pi i t\partial_t)$. This isomorphism will then decompose as the direct sum of its components

$$ {}^p\mathrm{DR}\psi_f^b\mathscr{M}\xrightarrow{\sim}{}^p\Psi_f^\lambda({}^p\mathrm{DR}\mathscr{M}), \qquad \lambda=\exp-2\pi i b. \tag{4.3.4} $$

Such an isomorphism exists under a regularity assumption along $f=0$. Such a result goes back to Malgrange and Kashiwara. With the assumption of the existence of a good filtration $F_\bullet\mathscr{M}$ satisfying the regularity property 4.3.c, its existence has been proved by M. Saito [**24**, Prop. 3.4.12]. For a good survey, see [**20**] or [**15**].

We now have at our disposal an object $\psi_f\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$, equipped with a quasi-unipotent automorphism. If we compare with the construction of § 1.4.b, we remark that, by its very construction, the filtration $F_\bullet$ respects the gradation $\bigoplus_{b\in]-1,0]}\psi_f^b\mathscr{M}$.

We then regard the operator $\mathrm{N} = (2\pi i)^{-1}\log T_u$ as a morphism

$$\mathrm{N} : \psi_f\big((\mathscr{M}, F_\bullet\mathscr{M}), \mathscr{F}_{\mathbb{Q}}, \alpha\big) \longrightarrow \psi_f\big((\mathscr{M}, F_\bullet\mathscr{M}), \mathscr{F}_{\mathbb{Q}}, \alpha\big)(-1). \tag{4.3.5}$$

4.3.e. What about vanishing cycles? A priori, the results above should be completed by the construction of vanishing cycles (with respect to the eigenvalue 1 of the monodromy): one should have to construct ${}^p\Phi_f^1\mathscr{F}_{\mathbb{Q}}$ with its (unipotent) monodromy and the diagram

$$\begin{array}{c}\text{Can}\\ {}^p\Psi_f^1\mathscr{F}_{\mathbb{Q}} \rightleftarrows {}^p\Phi_f^1\mathscr{F}_{\mathbb{Q}}\ ,\\ \text{Var}\end{array} \qquad \begin{aligned}\mathrm{Var}\circ\mathrm{Can} &= T - \mathrm{Id} : {}^p\Psi_f^1\mathscr{F}_{\mathbb{Q}} \longrightarrow {}^p\Psi_f^1\mathscr{F}_{\mathbb{Q}},\\ \mathrm{Can}\circ\mathrm{Var} &= T - \mathrm{Id} : {}^p\Phi_f^1\mathscr{F}_{\mathbb{Q}} \longrightarrow {}^p\Phi_f^1\mathscr{F}_{\mathbb{Q}}.\end{aligned}$$

One should have a similar diagram

$$\begin{array}{c}\text{can}\\ \psi_f^0\mathscr{M} \rightleftarrows \psi_f^{-1}\mathscr{M}\ ,\\ \text{var}\end{array} \qquad \begin{aligned}\mathrm{var}\circ\mathrm{can} &= \mathrm{N} : \psi_f^0\mathscr{M} \longrightarrow \psi_f^0\mathscr{M},\\ \mathrm{can}\circ\mathrm{var} &= \mathrm{N} : \psi_f^{-1}\mathscr{M} \longrightarrow \psi_f^{-1}\mathscr{M}.\end{aligned}$$

and an isomorphism ${}^p\mathrm{DR}\psi_f^{-1}\mathscr{M} \xrightarrow{\ \sim\ } {}^p\Phi_f^1({}^p\mathrm{DR}\mathscr{M})$ completing (4.3.4), so that the diagrams above suitably correspond.

Although these results are true with the same assumptions as above, we will not have to use them, as we will soon restrict to *strictly S-decomposable* objects. For each strict component of such an object, can (resp. Can) is onto and var (resp. Var) is injective, so the vanishing cycles can be defined as the image of N in (4.3.5). From the point of view of Hodge $\mathscr{D}$-module structure, we also define the vanishing cycles as the image of N in the category of Hodge-Lefschetz $\mathscr{D}$-modules, using the same procedure as in Exercise 1.4.11.

4.4. Polarizable Hodge $\mathscr{D}$-modules: definition

4.4.a. Pure Hodge $\mathscr{D}$-modules. Let X be a complex manifold. One defines by induction on d the category $\mathrm{MH}_{\leqslant d}(X, w)$ of Hodge $\mathscr{D}$-modules of weight w and dimension of support $\leqslant d$.

(MH_0) *A Hodge module supported on a point* $\{x^o\} \overset{i}{\hookrightarrow} X$ *is* i_+ *(in the sense of filtered $\mathscr{D}$-modules and perverse sheaves) of a Hodge structure* $\big((H, F^\bullet H), H_{\mathbb{Q}}\big)$.

An object of $\mathrm{MH}_{\leqslant d}(X, w)$ is a triple $\big((\mathscr{M}, F_\bullet\mathscr{M}), \mathscr{F}_{\mathbb{Q}}, \alpha\big)$ where $(\mathscr{M}, F_\bullet\mathscr{M})$ is a holonomic $\mathscr{D}_X$-module with a good filtration, $\mathscr{F}_{\mathbb{Q}}$ is a $\mathbb{Q}$-perverse sheave and $\alpha : {}^p\mathrm{DR}\mathscr{M} \to \mathbb{C}\otimes_{\mathbb{Q}}\mathscr{F}_{\mathbb{Q}}$ is an isomorphism. This triple is assumed to be *strictly S-decomposable*: it is the direct sum of objects having strict support on an irreducible closed analytic subset of X.

We are now reduced to defining the category $\mathrm{MH}_Z(X, w)$ of Hodge $\mathscr{D}$-modules having strict support the irreducible closed analytic subset Z of X (by induction, we assume that $\dim Z = d$).

The condition is local on Z (or X).

(HS) *For any germ f of holomorphic function on Z, the triple $\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$ is strictly specializable along $f=0$ and the corresponding monodromy is quasi-unipotent.*

We can therefore consider the nearby cycles $\psi_f^b\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$ ($b\in\,]-1,0]$) which are supported on $f=0$. If f is not identically 0 on Z, we get an object supported in dimension $\leqslant d-1$. On the nearby cycle objects, we have the nilpotent endomorphism N and its monodromy filtration $\mathrm{M}_\bullet$.

($\mathrm{MH}_{>0}$) *Each $\mathrm{gr}_\ell^{\mathrm{M}}\psi_f^b\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$ ($b\in\,]-1,0]$, $\ell\in\mathbb{Z}$) is an object of the category $\mathrm{MH}_{\leqslant d-1}(X,w-1+\ell)$.*

Remarks 4.4.1

(1) This definition can look frightening: in order to check that an object $\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$ belongs to $\mathrm{MH}(X,w)$, we have to consider in an inductive way nearby cycles with respect to *all* germs of holomorphic functions.

It should be considered however the other way round. Once we know at least one Hodge $\mathscr{D}$-module, we automatically know an infinity of them, by considering (monodromy-graded) nearby or vanishing cycles with respect to *any* holomorphic function.

(2) It can be proved (by induction on d) that the category of such objects, as well as the category of mixed objects obtained by the ψ functor, are abelian. This is an important property.

(3) If we define vanishing cycles as the image by N of ψ_f^0 (this is justified by the previous remark) we get an object such that the corresponding graded pieces $\mathrm{gr}_\ell^{\mathrm{M}}$ belong to $\mathrm{MH}_{\leqslant d-1}(X,w+\ell)$ (note that the weight is $w+\ell$, not $w-1+\ell$).

(4) It is possible to define the notion of a Hodge-Lefschetz $\mathscr{D}$-module, as we did for Hodge structures in § 1.4. Then $\psi_f\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$ is a quasi-Hodge Lefschetz $\mathscr{D}$-module of weight $w-1$ and $\psi_f^{-1}\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_{\mathbb{Q}},\alpha\big)$ is a Hodge-Lefschetz $\mathscr{D}$-module of weight w.

4.4.b. The polarization. In order to define the notion of a polarization, we need to introduce the notion of a pairing between $\mathbb{Q}$-perverse sheaves (extending the notion of a pairing between $\mathbb{Q}$-vector spaces) and, in order to apply the inductive procedure to reach a polarization on a Hodge structure, we need to determine its behaviour under the topological ${}^p\Psi_f$ functor for any holomorphic function f. This part is purely "topological". One can consider the Poincaré-Verdier duality functor $\mathbb{D}$. A non-degenerate pairing on a constructible complex $\mathscr{F}_{\mathbb{Q}}$ is an isomorphism $\mathscr{Q}:\mathscr{F}_{\mathbb{Q}}\xrightarrow{\sim}\mathbb{D}\mathscr{F}_{\mathbb{Q}}$. In a similar way to what happens to the functor ${}^p\mathrm{DR}$ in (4.3.3), we need to construct a canonical and functorial isomorphism

$$\mathbb{D}({}^p\Psi_f\mathscr{F}_{\mathbb{Q}})\xrightarrow{\sim}{}^p\Psi_f(\mathbb{D}\mathscr{F}_{\mathbb{Q}}). \tag{4.4.2}$$

A polarization of the Hodge $\mathscr{D}$-module $\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_\mathbb{Q},\alpha\big)$ will then be an isomorphism

$$\mathscr{S}_\mathbb{Q}:\mathbb{D}\mathscr{F}_\mathbb{Q}\longrightarrow\mathscr{F}_\mathbb{Q}(-w)$$

inducing, after applying ${}^p\Psi_f$ and the isomorphism (4.4.2) a polarization of the Hodge-Lefschetz $\mathscr{D}$-module $\psi_t\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_\mathbb{Q},\alpha\big)$. At the beginning of the induction, we use the notion of polarization of Remark 1.3.7.

Remark 4.4.3. Strictly speaking, it is necessary to impose a priori neither the non-degeneracy property nor the symmetry property (that I forgot above), as all these properties should be proved to hold by induction on the dimension of the support, starting from the corresponding properties for Hodge structures.

4.5. The Hodge-Saito theorem

The Hodge-Saito theorem is a version with parameter and in higher dimension of the Hodge-Saito theorem of § 3.1.e.

4.5.a. The direct image functor. Let $f:X\to Y$ be a projective morphism between smooth projective varieties over $\mathbb{C}$. Let $\big((\mathscr{M},F_\bullet\mathscr{M}),\mathscr{F}_\mathbb{Q},\alpha\big)$ be a polarizable Hodge $\mathscr{D}$-module on X. The notion of direct image by f is well defined for $\mathscr{D}$-modules, as well as for complexes of sheaves. Moreover, as f is projective (hence proper), $f_+\mathscr{M}$ is a complex of $\mathscr{D}_Y$-modules having $\mathscr{D}_Y$-coherent and holonomic cohomology. Similarly, $\boldsymbol{R}f_*\mathscr{F}_\mathbb{Q}$ is a complex of sheaves having constructible cohomology on Y.

Remark 4.5.1. The direct image $f_+\mathscr{O}_X$ corresponds to the classical construction of the Gauss-Manin connection (*cf.* [**8**], see also [**16**]).

The cohomology modules $\mathscr{H}^kf_+\mathscr{M}$ are $\mathscr{D}_Y$-holonomic. The perverse cohomology sheaves ${}^p\boldsymbol{R}^kf_*\mathscr{F}_\mathbb{Q}$ are perverse sheaves on Y. We wish to cook up, out of α, an isomorphism

$$\alpha_k:{}^p\mathrm{DR}(\mathscr{H}^kf_+\mathscr{M})\xrightarrow{\sim}\mathbb{C}\otimes_\mathbb{Q}{}^p\boldsymbol{R}^kf_*\mathscr{F}_\mathbb{Q}.$$

As for nearby cycles (4.3.3), this amounts to constructing a canonical and functorial isomorphism in the derived category $D(\mathbb{C}_Y)$

$$\text{(4.5.2)}\qquad {}^p\mathrm{DR}f_+\mathscr{M}\xrightarrow{\sim}\boldsymbol{R}f_*{}^p\mathrm{DR}\mathscr{M}.$$

Indeed, taking perverse cohomology of both terms will then give (using that ${}^p\mathrm{DR}\mathscr{M}$ is perverse)

$$\text{(4.5.3)}\qquad {}^p\mathrm{DR}(\mathscr{H}^kf_+\mathscr{M})={}^p\mathscr{H}^k({}^p\mathrm{DR}f_+\mathscr{M})\xrightarrow{\sim}{}^p\boldsymbol{R}^kf_*({}^p\mathrm{DR}\mathscr{M}).$$

We compose this isomorphism with

$${}^p\boldsymbol{R}^kf_*\alpha:{}^p\boldsymbol{R}^kf_*{}^p\mathrm{DR}\mathscr{M}\xrightarrow{\sim}\mathbb{C}\otimes_\mathbb{Q}{}^p\boldsymbol{R}^kf_*\mathscr{F}_\mathbb{Q}$$

to get α_k. That the isomorphism (4.5.2) exists is a very general fact (*cf.* for instance [**19**]).

It remains to define the notion of direct image for a *filtered* $\mathscr{D}_X$-module. This is done by M. Saito **[24]** by using the notion of induced $\mathscr{D}$-module. The approach of induced $\mathscr{D}$-modules is very efficient and gives in particular another proof of the comparison (4.5.2) (*cf.* **[24]**, see also **[16]**).

4.5.b. The Hodge-Saito theorem: statement. Let X be a projective variety and let $\big((\mathscr{M}, F_\bullet\mathscr{M}), \mathscr{F}_\mathbb{Q}, \alpha, \mathscr{S}_\mathbb{Q}\big)$ be a polarized Hodge $\mathscr{D}$-module of weight w on X. The hypercohomology $\mathbb{H}^*(X, {}^p\mathrm{DR}\mathscr{M})$ gets

- a filtration coming from the filtration (4.1.2) on the de Rham complex,
- a $\mathbb{Q}$-structure coming from $\mathbb{H}^k(X, \mathscr{F}_\mathbb{Q})$ through α,
- a morphism $\mathbb{H}^k(X, \mathscr{F}_\mathbb{Q})^\vee \to \mathbb{H}^{-k}(X, \mathscr{F}_\mathbb{Q})(w)$ according to the Poincaré-Verdier duality theorem.

Moreover, if we fix an ample line bundle on X, as in § 1.2, one can define a corresponding Lefschetz operator L on the hypercohomology. The theorem of M. Saito asserts that the properties expressed in § 1.2 hold in such a setting (*i.e.*, we get a polarized graded $(-)$Hodge-Lefschetz structure).

4.5.c. The Hodge-Saito theorem: sketch of proof. It is not known how to prove directly the Hodge-Saito theorem. One does not know how to develop an analysis similar to the one which occurs in the Hodge theorem (or the Hodge-Deligne theorem). The singularities of the support of the Hodge $\mathscr{D}$-module make the direct analysis of the Laplace operator much too hard. On the contrary, the proof uses the machinery of direct images to reduce to dimension one, that is, to the theorem of § 3.1.e.

(1) One first states the Hodge-Saito theorem in a relative form: instead of considering the constant map $f : X \to \mathrm{pt}$, one considers any projective morphism $f : X \to Y$, assuming only that X is quasi-projective, and one rephrases the theorem by replacing 'k-th hypercohomology' by 'k-th direct image' as in § 4.5.a.

(2) The proof of the relative Hodge-Saito theorem is done by induction on the pair $(d_\mathscr{M}, d_{f_+(\mathscr{M})})$, where $d_\mathscr{N}$ is the dimension of the support of $\mathscr{N}$ by using the following steps:

(a) the case $\dim X = 1$ and $\dim Y = 0$ is nothing but the Hodge-Saito theorem of § 3.1.e;

(b) If the theorem is true for a pair (n, m), then it is true for a pair $(n+1, m+1)$;

(c) if the theorem is true for the pair $(n, 1)$ then, also using the first step, one gets the theorem for the pair $(n+1, 0)$; this is a kind of Fubini theorem.

(3) In order to prove the (absolute) Hodge-Saito theorem with pair $(n, 0)$, one produces a relative situation with pair $(n-1, 1)$ by using a Lefschetz pencil. This is a morphism $\widetilde{X} \to \mathbb{P}^1$, where $\widetilde{X}$ is the blow-up space of the axis of the pencil in X. If the pencil is generic enough (with respect to the Hodge $\mathscr{D}$-module we start with), then it

is possible to pull-back the Hodge $\mathscr{D}$-module to $\widetilde{X}$ and obtain an object which is close to be a Hodge $\mathscr{D}$-module. Applying Step (2c) to it, one gets a Hodge structure. One then compares this Hodge structure to the data obtained on the hypercohomology of the original Hodge $\mathscr{D}$-module.

4.6. Polarizable Hodge $\mathscr{D}$-modules: some examples

The first main example is given by the following theorem, which is proven by induction on the dimension and, when proving the property ($\mathrm{MH}_{>0}$) with respect to a holomorphic function f, by using the resolution of singularities of $f=0$. In order to come back, one uses direct images by the resolution morphism; here, the Hodge-Saito theorem for a morphism proves useful.

***Theorem 4.6.1* (*cf.* [24]).** *A variation of polarized Hodge structure of weight w corresponds to a polarized Hodge module of weight $w+\dim X$.*

We thus see that the weight of a Hodge $\mathscr{D}$-module takes into account the dimension of its support. In order to understand the meaning of 'corresponds', let us consider the particular case of $\mathscr{O}_X$. The corresponding Hodge $\mathscr{D}$-module is the object $\big((\mathscr{O}_X, F_\bullet^{\mathrm{Saito}}\mathscr{O}_X), {}^p\mathbb{Q}_X, \alpha\big)$, with $F_\bullet^{\mathrm{Saito}}\mathscr{O}_X = F_\bullet[\dim X]\mathscr{O}_X$, ${}^p\mathbb{Q}_X := \mathbb{Q}_X[\dim X]$ and $\alpha : {}^p\mathrm{DR}\mathscr{O}_X \to {}^p\mathbb{Q}_X$ is the natural morphism. This is a pure Hodge module of weight $\dim X$. The polarization is that coming from the trivial pairing $\mathbb{Q}_X \otimes_{\mathbb{Q}_X} \mathbb{Q}_X \to \mathbb{Q}_X$, suitably shifted ant Tate-twisted.

4.6.a. The Hodge structure on vanishing cycles for an isolated singularity.

Let us explain how the results of Varchenko and Steenbrink can be recovered in this context. Let f be a germ of holomorphic function on $X=\mathbb{C}^n$ with an isolated singularity at the origin. Then ${}^p\Phi_f {}^p\mathbb{Q}_X$ is supported at the origin. If $\{0\} \xhookrightarrow{i_0} X$ is the inclusion, then ${}^p\Phi_f {}^p\mathbb{Q}_X = \boldsymbol{R}i_{0,*}H_{\mathbb{Q}}$, where $H_{\mathbb{Q}}$ is the reduced $\mathbb{Q}$-cohomology of the Milnor fibre of f at the origin.

The complex cohomology of the Milnor fibre is obtained by $\Phi_f {}^p\mathrm{DR}\mathscr{O}_X$, where we denote by Φ_f the functor $\bigoplus_{b\in[-1,0[}\psi_f^b$.

In such a situation, we find that $\phi_f\big((\mathscr{O}_X, F_\bullet^{\mathrm{Saito}}\mathscr{O}_X), {}^p\mathbb{Q}_X, \alpha\big)$ is a mixed Hodge structure, and more precisely that each $\mathrm{gr}_\ell^{\mathrm{M}}\phi_f\big((\mathscr{O}_X, F_\bullet^{\mathrm{Saito}}\mathscr{O}_X), {}^p\mathbb{Q}_X, \alpha\big)$ has weight $\ell+n=\ell+\dim X$.

Exercise 4.6.2. Prove that $i_{f,+}\mathscr{O}_X = \mathscr{O}_X[\partial_t]$ as a $\mathscr{O}_X$-module and determine the $\mathscr{D}_{X\times\mathbb{C}}$-action in this presentation. Show that the filtration $F_\bullet i_{f,+}\mathscr{O}_X$ is the filtration by the degree in ∂_t. Give the characterization of the V-filtration.

The result of Varchenko and Steenbrink are related to the local Gauss-Manin system M, which is the direct image by the projection $X\times\mathbb{C}\to\mathbb{C}$ of $i_{f,+}\mathscr{O}_X$. One has to restrict to a Milnor ball near the singularity in order to obtain something intrinsic.

We have (using the definition of direct images of $\mathscr{D}$-modules and the fact that the singularity is isolated)

$$M = \Omega^n_{X,0}[\partial_t]/(d - \partial_t df\wedge)\Omega^{n-1}_{X,0}[\partial_t].$$

This is known to be a germ of regular holonomic $\mathscr{D}$-module (here, we take the notation of § 2.0). The F-filtration induced by the filtration of $i_{f,+}\mathscr{O}_X$ is the filtration by the degree in ∂_t.

Scherk and Steenbrink [**28**] have proved that one can work on M instead of $i_{f,+}\mathscr{O}_X$ in order to obtain the Hodge filtration on the vanishing cycles. Namely, one identifies ${}^p\Phi_f{}^p\mathbb{C}_X$ with $\phi_t M$ (computed with the V-filtration of M) and one induces on it the F-filtration of M to get the Hodge filtration.

The local Gauss-Manin system has a specific property: the action of ∂_t is invertible. Moreover, each F_kM is stable by ∂_t^{-1}. It follows that, for any $k \geqslant 1$, $F_kM = \partial_t^k F_0M$.

It follows that the graded parts of the Hodge filtration on $\phi_t M$ can be recovered (applying a suitable power of ∂_t) of the graded part of the filtration induced by $V^\bullet M$ on the Jacobian module

$$F_0M/\partial_t^{-1}F_0M = \Omega^n_{X,0}/df \wedge \Omega^{n-1}_{X,0}.$$

4.6.b. The intersection complex of an isolated hypersurface singularity. Let us keep the setting of § 4.6.a, still assuming that the hypersurface $Y = \{f = 0\}$ has an isolated singularity at the origin. The constant sheaf ${}^p\mathbb{Q}_Y$ shifted by the dimension of Y is known to be a perverse sheaf on Y. Does it underlie a Hodge $\mathscr{D}_X$-module strictly supported on Y?

One remarks that, without strong assumptions on Y (for instance Y having a quotient singularity), it is not self-dual with respect to Poincaré-Verdier duality. This will be problematic for the polarization to exist.

The $\mathscr{D}_X$-module corresponding, via ${}^p\mathrm{DR}$, to ${}^p\mathbb{Q}_Y$ (or, more precisely, to the sheaf equal to ${}^p\mathbb{Q}_Y$ on Y and equal to 0 away from Y) can by obtained from $\psi_f^0\mathscr{O}_X$: it is equal to $\mathrm{Ker}[\mathrm{N} : \psi_f^0\mathscr{O}_X \to \psi_f^0\mathscr{O}_X]$. Indeed, let us denote by $i_Y : Y \hookrightarrow X$ the inclusion. At the level of the perverse complexes, we have a triangle

$$\boldsymbol{R}i_{Y,*}i_Y^{-1}{}^p\mathbb{Q}_Y \longrightarrow {}^p\Psi_f^1{}^p\mathbb{Q}_X \xrightarrow{\mathrm{Can}} {}^p\Phi_f^1{}^p\mathbb{Q}_X \xrightarrow{+1}$$

which is in fact an exact sequence in the abelian category of perverse sheaves

$$0 \longrightarrow \boldsymbol{R}i_{Y,*}i_Y^{-1}{}^p\mathbb{Q}_Y \longrightarrow {}^p\Psi_f^1{}^p\mathbb{Q}_X \xrightarrow{\mathrm{Can}} {}^p\Phi_f^1{}^p\mathbb{Q}_X \longrightarrow 0,$$

and, as Var is injective in the same category, we have $\boldsymbol{R}i_{Y,*}i_Y^{-1}{}^p\mathbb{Q}_Y = \mathrm{Ker}\,\mathrm{Can} = \mathrm{Ker}(T - \mathrm{Id})$. A similar reasoning can be made with the $\mathscr{D}_X$-modules.

Exercise 4.6.3. Let us consider an object of an abelian category which is acted on by a nilpotent endomorphism N. Let us denote by $\mathrm{M}_\bullet$ the corresponding monodromy filtration.

(1) Show that $\mathrm{Ker}\,\mathrm{N} \subset \mathrm{M}_0$.

(2) Show that the primitive part $P\mathrm{gr}_0^{\mathrm{M}}$ is equal to $\mathrm{Ker}\,\mathrm{N}/(\mathrm{Ker}\,\mathrm{N} \cap \mathrm{Im}\,\mathrm{N})$.

(*Hint:* one will first check these statements in the case of a nilpotent matrix having the Jordan normal form.

At the level of Hodge $\mathscr{D}_X$-modules, $\operatorname{Ker}\mathrm{N}$ is a *mixed Hodge module* (here, we use that $\psi_f^0\big((\mathscr{O}_X, F_\bullet^{\mathrm{Saito}}\mathscr{O}_X), {}^p\mathbb{Q}_X, \alpha\big)$ is a mixed Hodge $\mathscr{D}$-module and that the category of mixed Hodge $\mathscr{D}$-modules is abelian, although we did not define such a notion). By Exercise 4.6.3, $\operatorname{Ker}\mathrm{N}$ has weights less than or equal to $\dim Y$ (recall that the original Hodge module corresponding to $\mathscr{O}_X$ has weight $\dim X$). Moreover, the primitive object $\mathrm{Pgr}_0^{\mathrm{M}}\psi_f^0\big((\mathscr{O}_X, F_\bullet^{\mathrm{Saito}}\mathscr{O}_X), {}^p\mathbb{Q}_X, \alpha\big)$ is a polarized pure Hodge module of weight $\dim Y$. Of course, when we restrict away from the singularity of Y, we find the object $\big((\mathscr{O}_Y, F_\bullet^{\mathrm{Saito}}\mathscr{O}_Y), {}^p\mathbb{Q}_Y, \alpha\big)$.

In fact, the pure component $\mathrm{Pgr}_0^{\mathrm{M}}\psi_f^0$ corresponds to the *intersection complex* IC_Y on Y. In such a way, we have constructed a canonical Hodge $\mathscr{D}$-module structure on IC_Y.

Exercise 4.6.4. Does the previous argument depend on the fact that Y has an isolated singularity?

LECTURE 5

HOW TO USE HODGE $\mathscr{D}$-MODULES

In this lecture, I will insist on the use of the Hodge filtration, from various points of view. I will allow an excursion to mixed Hodge modules to give more emphasis to some of its properties.

(1) *Strictness*: preservation of strictness under various operations implies, more or less directly, a strict control of the behaviour of the Hodge filtration by the topology.

(2) *Determining the Hodge filtration*: in many examples, some $\mathscr{D}$-module is proved to underlie a (mixed) Hodge module, but the Hodge filtration is not explicit. Making it explicit can be hard, but leads to interesting consequences.

5.1. Strictness

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $F_\bullet\mathscr{M}$ be a F-filtration (not necessarily good, but satisfying $F_k\mathscr{D}_X F_\ell\mathscr{M} \subset F_{k+\ell}\mathscr{M}$ for any k,ℓ). Then the Rees module $R_F\mathscr{M} := \bigoplus_k F_k\mathscr{M} z^k$ is a left module over the Rees ring $R_F\mathscr{D}_X$. Conversely, a left $R_F\mathscr{D}_X$-module takes the form $R_F\mathscr{M}$ for some filtered $\mathscr{D}_X$-module $\mathscr{M}$ if and only if it has no $\mathbb{C}[z]$-torsion, if we regard $\mathbb{C}[z]$ acting as a subring of $R_F\mathscr{D}_X$.

A bounded complex of filtered $\mathscr{D}_X$-modules is said to be *strict* if the cohomology of the associated Rees complex has no $\mathbb{C}[z]$-torsion.

5.1.a. Strictness of the dual module. If $\mathscr{M}$ is holonomic, then the dual complex $\mathbb{D}\mathscr{M}$ (which is the complex of left $\mathscr{D}_X$-modules associated to $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$) has cohomology in degree $\dim X$ only, and this is the dual holonomic $\mathscr{D}_X$-module $\mathscr{M}^\vee$.

Starting now from a well-filtered $\mathscr{D}_X$-module, we can similarly consider the dual complex $\mathbb{D}(R_F\mathscr{M})$. Two problems can occur at the same time:

– the cohomology possibly exists in degrees $\neq \dim X$; the corresponding cohomology classes are then annihilated by a power of z;

– the cohomology in degree $\dim X$ has possibly z-torsion.

In other words, the complex $\mathbb{D}(R_F\mathscr{M})$ is possibly not strict.

Corollary 5.1.1. *If* $(\mathscr{M}, F_\bullet\mathscr{M})$ *underlies a Hodge* $\mathscr{D}_X$*-module, then* $\mathbb{D}(R_F\mathscr{M})$ *is strict. In particular,* $\mathrm{gr}_F\mathscr{M}$ *is a Cohen-Macaulay* $\mathrm{gr}_F\mathscr{D}_X$*-module, i.e., the complex* $\boldsymbol{R}\mathscr{H}om_{\mathrm{gr}_F\mathscr{D}_X}(\mathrm{gr}_F\mathscr{M}, \mathrm{gr}_F\mathscr{D}_X)$ *has cohomology in degree* $\dim X$ *only, and*

$$\mathrm{Ext}^{\dim X}_{\mathrm{gr}_F\mathscr{D}_X}(\mathrm{gr}_F\mathscr{M}, \mathrm{gr}_F\mathscr{D}_X) = \mathrm{gr}_F\big(\mathrm{Ext}^{\dim X}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)\big),$$

where the filtration on the right-hand side is the natural one.

5.1.b. Strictness of the direct images. One of the statements of the Hodge-Saito theorem I did not insist on in § 4.5.b is the strictness property. It corresponds to Theorem 1.2.5 in classical Hodge theory. It is particularly useful to state it that way, instead of the decomposition $\bigoplus H^{p,q}$ as, in the relative case where the target space has dimension > 0, it can be stated in exactly the same way, although the decomposition needs introducing C^∞ coefficients, which is not desirable near singularities.

Let $(\mathscr{M}, F_\bullet\mathscr{M})$ be a holonomic $\mathscr{D}_X$-module equipped with a good filtration and let $R_F\mathscr{M}$ be the associated Rees module. It is $R_F\mathscr{D}_X$-coherent. If $f : X \to Y$ is a holomorphic map, the direct image functor f_+ extends as a filtered direct image functor, that is, as a direct image functor for $R_F\mathscr{D}_X$-modules. If f is proper, then $f_+R_F\mathscr{M}$ has $R_F\mathscr{D}_Y$-coherent cohomology.

The Hodge-Saito theorem says, in the setting of § 4.5.c, that the direct image $f_+R_F\mathscr{M}$ of the filtered $\mathscr{D}_X$-module underlying a polarizable Hodge $\mathscr{D}_X$-module by a projective morphism is *strict.*

5.1.c. Application to vanishing cycles: analytic case. Let $f : X \to \mathbb{C}$ be a holomorphic function on a complex analytic manifold. Let $(\mathscr{M}, F_\bullet\mathscr{M})$ be a well-filtered coherent $\mathscr{D}_X$-module, $\mathrm{DR}(\mathscr{M}) = (\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla)$ its de Rham complex, which is naturally filtered by (4.1.2), and $(\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathrm{gr}^F\mathscr{M}, \mathrm{gr}^F\nabla)$ the associated graded complex.

Theorem 5.1.2. *Let* $(\mathscr{M}, F_\bullet\mathscr{M})$ *be (the filtered holonomic* $\mathscr{D}_X$*-module underlying) a mixed Hodge Module*[1] *on* X *such that the restriction of* f *to the support of* $\mathscr{M}$ *is a projective morphism in a neighbourhood of* $f^{-1}(0)$*. Then for all* $k \in \mathbb{Z}$ *one has*

$$\dim \mathbb{H}^{k-1}\big(f^{-1}(0), \Phi_f\,\mathrm{DR}(\mathscr{M})\big) = \dim \mathbb{H}^k\big(f^{-1}(0), (\Omega^\bullet_X \otimes_{\mathscr{O}_X} \mathrm{gr}^F\mathscr{M}, \mathrm{gr}^F\nabla - df\wedge)\big).$$

Letting $\mathscr{M} = \mathscr{O}_X$ in theorem 5.1.2, which is justified by Theorem 4.6.1 (the shift in the filtration does not affect the strictness property), we get:

Corollary 5.1.3. *If* $f : X \to \mathbb{C}$ *is projective in a neighbourhood of* $f^{-1}(0)$*, one has for all* $k \in \mathbb{Z}$

$$\dim \mathbb{H}^{k-1}\big(f^{-1}(0), \Phi_f\mathbb{C}_X\big) = \dim \mathbb{H}^k\big(f^{-1}(0), (\Omega^\bullet_X, df\wedge)\big).$$

---

1. The notion of a mixed Hodge module will not be defined here. One is referred to **[25]** for a precise definition. We will use that the strictness of direct images also applies to mixed Hodge $\mathscr{D}$-modules.

Notice that the corollary is well-known if one assumes that, in a neighbourhood of $f^{-1}(0)$, the function f has only a finite number of critical points; one then knows **[23]** that the complex $(\Omega_X^\bullet, df\wedge)$ has cohomology in degree $\dim X$ at most and has support in the set of critical points; so is then the perverse sheaf ${}^p\Phi_f{}^p\mathbb{C}_X$; in this case the corollary is consequence of the formula

$$\mu(f, x^o) = \dim \mathscr{O}_{X,x^o} \Big/ \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big)$$

where $\mu(f, x^o)$ is the number of vanishing cycles of f at x^o.

Let D be a divisor with normal crossings in X and $j: U = X - D \hookrightarrow X$ be the inclusion. Taking $\mathscr{M} = \mathscr{O}_X[*D]$ (*cf.* **[25**, § 2.d**]**), one gets:

Corollary 5.1.4. *Under the same assumptions as above one has, for all $k \in \mathbb{Z}$,*

$$\dim \mathbb{H}^{k-1}\big(f^{-1}(0), \Phi_f \boldsymbol{R}j_*\mathbb{C}_U\big) = \dim \mathbb{H}^k\big(f^{-1}(0), (\Omega_X^\bullet(\log D), df\wedge)\big).$$

Proof of theorem 5.1.2. Let $i: X \hookrightarrow X \times \mathbb{C}$ denote the inclusion defined by the graph of f. Identify $i_+\mathscr{M}$ with $\mathscr{M}[\partial_t]$. It is equipped with the good filtration

$$F_\ell\big(\mathscr{M}[\partial_t]\big) = \sum_{j+k=\ell,\ k\geqslant 0} F_j\mathscr{M}\partial_t^k$$

and $\mathrm{gr}_\ell^F\big(\mathscr{M}[\partial_t]\big)$ is identified with $\bigoplus_{j\leqslant\ell}\mathrm{gr}_j^F\mathscr{M}$. The relative de Rham complex

$$\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M}) = \big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}[\partial_t], \nabla - \partial_t\cdot df\wedge\big)$$

is filtered by

$$F_p\,\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M}) = \big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} F_{p+\bullet}\mathscr{M}[\partial_t], \nabla - \partial_t\cdot df\wedge\big)$$

so that

$$\mathrm{gr}_p^F\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M}) = \big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \big(\oplus_{j\leqslant p+\bullet}\mathrm{gr}_j^F\mathscr{M}\big), \mathrm{gr}^F\nabla - df\wedge\big).$$

The right-hand term is also the p-th term of a filtration

$$G_\bullet\big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathrm{gr}^F\mathscr{M}, \mathrm{gr}^F\nabla - df\wedge\big).$$

The graded complex G_p/G_{p-1} is the complex

$$\big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathrm{gr}_{p+\bullet}^F\mathscr{M}, \mathrm{gr}^F\nabla\big) = \mathrm{gr}_p^F\mathrm{DR}\,\mathscr{M}.$$

If p is large enough, this complex is acyclic in a neighbourhood of the compact fiber $f^{-1}(0)\cap\mathrm{Supp}\,\mathscr{M}$ (see *e.g.*, **[14, 17]**). We conclude, taking inductive limits, that, for p large enough and any k,

$$\begin{aligned}\dim \mathbb{H}^k\big(f^{-1}(0), \big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathrm{gr}^F\mathscr{M}, \mathrm{gr}^F\nabla - df\wedge\big)\big)\\ = \dim \mathbb{H}^k\big(X\times\{0\}, \mathrm{gr}_p^F\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M})\big).\end{aligned}$$

Let now $F_\bullet\mathscr{H}^k f_+\mathscr{M}$ be the good filtration defined as

$$\mathrm{image}\Big[\boldsymbol{R}^k f_*(F_p\,\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M})) \longrightarrow \boldsymbol{R}^k f_*(\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M}))\Big].$$

The strictness of the Hodge filtration on direct images (see [**25**, th. 2.14]) implies that

$$\dim \mathbb{H}^k\big(X\times\{0\}, \mathrm{gr}_p^F \mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(i_+\mathscr{M})\big) = \dim \mathrm{gr}_p^F \mathscr{H}^k f_+\mathscr{M}$$

for any p. Now, it follows from the local index theorem of Kashiwara (see *e.g.*, [**18**, p. 67]) that, for p large enough,

$$\dim \mathrm{gr}_p^F \mathscr{H}^k f_+\mathscr{M} = \dim {}^p\Phi_t {}^p\mathrm{DR}(\mathscr{H}^k f_+\mathscr{M}).$$

Then, on the one hand, using (4.5.3), we have ${}^p\mathrm{DR}(\mathscr{H}^k f_+\mathscr{M}) \simeq {}^p\boldsymbol{R}^k f_* {}^p\mathrm{DR}\mathscr{M}$ and, on the other hand, ${}^p\Phi_t {}^p\boldsymbol{R}^k f_* {}^p\mathrm{DR}\mathscr{M} = {}^p\boldsymbol{R}^k f_* {}^p\Phi_f {}^p\mathrm{DR}\mathscr{M}$, as Φ_f commutes with direct images and ${}^p\Phi_f$ preserves perversity, hence commute with perverse cohomology of the direct image. Now, up to a shift, the right-hand term is equal to the left-hand term in the theorem. □

5.1.d. Application to vanishing cycles: algebraic case. Let now $\widetilde{f}:\widetilde{X}\to\mathbb{P}^1$ be a morphism of algebraic varieties and let $f: X\to\mathbb{A}^1$ be its restriction over the affine line $\mathbb{A}^1$. Thus, X is quasi-projective and f is projective. In the following, we work in the *algebraic category*, so the differential forms are algebraic, etc.

Theorem 5.1.5. *Let $(\mathscr{M}, F_\bullet\mathscr{M})$ be (the filtered holonomic $\mathscr{D}_X$-module underlying) a mixed Hodge Module on X. The hypercohomology spaces on X of the complexes $(\Omega_X^\bullet\otimes_{\mathscr{O}_X} M, \nabla - df\wedge)$ and $(\Omega_X^\bullet\otimes_{\mathscr{O}_X}\mathrm{gr}^F\mathscr{M}, \mathrm{gr}^F\nabla - df\wedge)$ have the same (finite) dimension.*

Remark 5.1.6. It is well-known that one has

$$\dim \mathbb{H}^i\big(X, (\Omega_X^\bullet\otimes_{\mathscr{O}_X} M, \nabla - df\wedge)\big) = \sum_{c\in\mathbb{C}} \dim \mathbb{H}^{i-1}\big(f^{-1}(c), \Phi_{f-c}\,\mathrm{DR}(M)\big).$$

One can apply Theorem 5.1.5 to the pure Hodge $\mathscr{D}$-module $\big((\mathscr{O}_X, F_\bullet^{\mathrm{Saito}}\mathscr{O}_X), \mathbb{Q}_X, \alpha\big)$ (and we can use $F_\bullet\mathscr{O}_X$ instead of $F_\bullet^{\mathrm{Saito}}\mathscr{O}_X$, as the shift of the filtration does not affect strictness. One then has $\mathrm{gr}^F\mathscr{M} = \mathscr{O}_X$ and $\mathrm{gr}^F\nabla = 0$. We thus get:

***Corollary 5.1.7* (Kontsevitch and Barannikov).** *The hypercohomology spaces on X of the complexes $(\Omega_X^\bullet, d - df\wedge)$ and $(\Omega_X^\bullet, df\wedge)$ have the same (finite) dimension.*

One also can apply this result to $\mathscr{M} = \mathscr{O}_X[*D]$, if D is a divisor of X. Then we need here the theory of mixed Hodge $\mathscr{D}$-modules. Assume that D is a divisor with normal crossings. We get

Corollary 5.1.8. *The hypercohomology spaces on X of the complexes*

$$(\Omega_X^\bullet(\log D), d - df\wedge) \quad \textit{and} \quad (\Omega_X^\bullet(\log D), df\wedge)$$

have the same (finite) dimension.

Proof of theorem 5.1.5. By Theorem 5.1.2 and remark 5.1.6, it is enough to prove that the natural map

$$(*)\quad \mathbb{H}^k\big(X, (\Omega_X^\bullet \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{M}, \operatorname{gr}^F\nabla - df\wedge)\big) \longrightarrow \mathbb{H}^k\big(X^{\mathrm{an}}, (\Omega_X^\bullet \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{M}, \operatorname{gr}^F\nabla - df\wedge)^{\mathrm{an}}\big)$$

is an isomorphism for all k. The cohomology of the complex

$$\big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{M}, \operatorname{gr}^F\nabla - df\wedge\big)^{\mathrm{an}}$$

is supported by a finite number of fibers $f^{-1}(c)$ and, by faithful flatness of $\mathscr{O}_{X^{\mathrm{an}}}$ over $\mathscr{O}_X$, the same holds for the complex $\big(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{M}, \operatorname{gr}^F\nabla - df\wedge\big)$. These complexes can be regarded as having $\mathscr{O}_{\widetilde{X}^{\mathrm{an}}}$- or $\mathscr{O}_{\widetilde{X}}$-coherent cohomology, and GAGA implies that

$$(5.1.9)\quad H^k\big(X, \mathscr{H}^j(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{M}, \operatorname{gr}^F\nabla - df\wedge)\big) \longrightarrow H^k\big(X^{\mathrm{an}}, \mathscr{H}^j(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{M}, \operatorname{gr}^F\nabla - df\wedge)^{\mathrm{an}}\big)$$

is an isomorphism. The analytization morphism is compatible with the natural spectral sequences with E_2^{jk}-term respectively given by the left-hand and right-hand term in (5.1.9). As E_2^{jk} is finite dimensional, the spectral sequence degenerates at a finite rank and consequently $(*)$ is an isomorphism too. □

5.1.e. Strictness and families. Another way to regard the strictness property is the following consequence of the definition of a Hodge $\mathscr{D}$-module:

Corollary 5.1.10. *If $(\mathscr{M}, F_\bullet\mathscr{M})$ is the filtered $\mathscr{D}_X$-module underlying a Hodge $\mathscr{D}$-module and if ${}^p\mathrm{DR}\mathscr{M}$ is a local system up to a shift, then each $\operatorname{gr}_p^F\mathscr{M}$ (and each $F_p\mathscr{M}$) is a holomorphic vector bundle on X.*

In other words, *no topological jump implies no analytic jump*. For instance, let $f : X \to \mathbb{C}$ be a holomorphic function on the smooth manifold X having a smooth critical locus C. Locally, C is included in one fibre of f, say $f^{-1}(0)$. Let us moreover assume that $\Phi_f\mathbb{Q}_X$ is a local system on C. As it underlies a mixed Hodge module, it follows that, for any $b \in [-1, 0[$ and any p, $\operatorname{gr}_F^p\psi_f^b\mathscr{O}_X$ is locally free of finite rank on C (the shift between $F_\bullet\mathscr{O}_X$ and $F_\bullet^{\mathrm{Saito}}\mathscr{O}_X$ has no effect here). Assume for instance that $\dim C = 1$. By choosing a local transverse section to C, we can regard f as a μ-constant family $(f_t)_{t\in C}$ of isolated hypersurface singularities defined by $f_t : X_t \to \mathbb{C}$. We conclude that, in such a situation, the spectrum of f_t is locally constant when t varies on C.

However, it is not known if any μ-constant family $(f_t)_{t\in C}$ gives rise to a local system $\Phi_f\mathbb{Q}_X$, and such a proof cannot be applied to get a general proof that the spectrum is constant in such a situation. On the one hand, there is a (quite tricky) proof that the spectrum is constant in such a situation (and even a more general semi-continuity result) by Varchenko **[32]** and Steenbrink **[30]**.

The condition that $\Phi_f\mathbb{Q}_X$ is a local system on C is a particular case of the general condition (without assuming C smooth) that the transversal slice is non-characteristic. Recent results of Dimca, Maisonobe, M. Saito and Torrelli **[9]** analyze the behaviour of the spectrum by a *non-characteristic restriction.* The topological statement is as follows:

***Theorem 5.1.11* (*cf.* [9]).** *Let $Y=\{f=0\}$ be a smooth divisor in X and let $i_T: T\to X$ be the inclusion of a closed submanifold which is transversal to Y. Assume that $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module such that T is non-characteristic for $\mathscr{M}$ and for $\psi_f\mathscr{M}$. Let $V^\bullet\mathscr{M}$ denote the Kashiwara-Malgrange filtration of $\mathscr{M}$ along the hypersurface Y and also the Kashiwara-Malgrange filtration of the restriction $i_T^+\mathscr{M}$ along the hypersurface $T\cap Y$. Then, for any $b\in\mathbb{R}$ and any $\beta\in\mathbb{C}$, the following hold:*

(a) $i_T^+(V^b\mathscr{M})=V^bi_T^+\mathscr{M}$;

(b) $i_T^+\psi_f^\beta\mathscr{M}=\psi_{f_{|T}}^\beta i_T^+\mathscr{M}$.

In particular,

$$i_T^+(\psi_f\mathscr{M})=\psi_{f_{|T}}(i_T^+\mathscr{M})\quad\text{and}\quad i_T^+(\phi_f\mathscr{M})=\phi_{f_{|T}}(i_T^+\mathscr{M}). \tag{5.1.11 $*$}$$

If T is a smooth hypersurface defined by the equation $t=0$ in a local coordinate system, then the non-characteristic assumption in the theorem implies that T is also non-characteristic with respect to $\psi_f\mathscr{M}$ and to $\phi_f\mathscr{M}$. Therefore, the functor ϕ_t vanishes on $\mathscr{M}$, $\psi_f\mathscr{M}$ and $\phi_f\mathscr{M}$ and, in the last line of the theorem, one can replace i_T^+ with ψ_t^0 or with ψ_t.

A consequence of the definition of the Hodge filtration is

Corollary 5.1.12. *With the assumptions of Theorem* 5.1.11*, assume moreover that* $(\mathscr{M},F_\bullet\mathscr{M})$ *underlies a Hodge $\mathscr{D}$-module. Then* (5.1.11 $*$) *holds after taking* gr_p^F *for the naturally induced filtration on all terms.*

This corollary implies the constancy of the spectrum by a non-characteristic (in the sense of Theorem 5.1.11) hyperplane section.

5.2. Determining the Hodge filtration

5.2.a. What does a polarizable Hodge $\mathscr{D}$-module look like? One can give the following description of a polarized Hodge $\mathscr{D}$-module on a complex manifold X (*cf.* **[24, 25]**. We know, by definition, that a Hodge $\mathscr{D}$-module is S-decomposable. On other words, it is decomposed as the direct sum of Hodge $\mathscr{D}$-modules with strict support. We are reduced to consider these Hodge $\mathscr{D}$-modules. Let Z denote the support of the Hodge $\mathscr{D}$-module. By assumption, Z is then a closed irreducible analytic subset of X. There exists a closed analytic subset Σ of Z, containing the singular locus of Z, and a variation of polarized Hodge structure on $Z\smallsetminus\Sigma$, with underlying local system $\mathscr{H}_\mathbb{Q}$, such that the underlying constructible complex of the Hodge $\mathscr{D}$-module is the intermediate extension of $\mathscr{H}_\mathbb{Q}$ on Z, denoted by $\mathrm{IC}_Z\,\mathscr{H}_\mathbb{Q}$.

This result is by no means trivial, even when Z is smooth and Σ is a normal crossing divisor. In such a case, this is a generalization of Schmid's theorem 3.3.7 to dimension $\geqslant 2$ (see [**12, 13, 2, 3**]).

Going back from the previous simple geometric situation to the general case of (Z,Σ) uses resolution of singularities and the Hodge-Saito theorem for a projective morphism.

Nevertheless, such a structure theorem does not give precise indication on the Hodge filtration. We would like to have a formula generalizing (2.2.4) in order to extend the filtration from $Z \smallsetminus \Sigma$ to Z. This kind of formula exists in order to extend the filtration from $Z \smallsetminus Y$ to Z, where Y is a principal divisor of Z containing Σ (*cf.* [**24**, Prop. 3.2.2]), but the terms of the V-filtration involved are not $\mathscr{O}$-coherent, hence let too much freedom to the Hodge filtration. In this direction, more precise results are contained in [**27**].

5.2.b. A mixed example: the polar filtration. Let X be a complex manifold and let Y be a reduced divisor in X. The sheaf $\mathscr{O}_X(*Y)$ of meromorphic functions on X with poles along Y at most is a $\mathscr{D}_X$-module. It does not underlie a pure Hodge $\mathscr{D}$-module, but a mixed one. Nevertheless, it is interesting to try to determine its Hodge filtration. When $X=\mathbb{P}^n$, it is also interesting to determine the Hodge filtration on the hypercohomology of its de Rham complex, that is, on the cohomology of $X \smallsetminus Y$, after the comparison theorem. The latter question has been considered by Deligne and Dimca [**5**].

The former question is treated in [**26**]. Let us denote by $F_\bullet\mathscr{O}_X(*Y)$ the Hodge filtration and by $P_\bullet\mathscr{O}_X(*Y)$ the filtration by the order of the pole, defined as $P_k\mathscr{O}_X(*Y) = \mathscr{O}_X((k+1)Y)$ (*i.e.*, a pole of order $\leqslant k+1$ along Y). Then $P_k\mathscr{O}_X(*Y) \subset F_k\mathscr{O}_X(*Y)$ for any k and there is equality at $x\in Y$ if $k \leqslant \alpha_{f,x}-1$, where $\alpha_{f,x}$ is the minimal exponent of f at x.

BIBLIOGRAPHY

[1] J. Briançon & Ph. Maisonobe – "Idéaux de germes d'opérateurs différentiels à une variable", *Enseign. Math.* **30** (1984), p. 7–38.

[2] E. Cattani, A. Kaplan & W. Schmid – "Degeneration of Hodge structures", *Ann. of Math.* **123** (1986), p. 457–535.

[3] ———, "L^2 and intersection cohomologies for a polarizable variation of Hodge structure", *Invent. Math.* **87** (1987), p. 217–252.

[4] P. Deligne – "Théorie de Hodge II", *Publ. Math. Inst. Hautes Études Sci.* **40** (1971), p. 5–57.

[5] P. Deligne & A. Dimca – "Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières", *Ann. Sci. École Norm. Sup. (4)* **23** (1990), no. 4, p. 645–656.

[6] P. Deligne & L. Illusie – "Relèvements modulo p^2 et décomposition du complexe de de Rham", *Invent. Math.* **89** (1987), p. 247–270.

[7] J.-P. Demailly – "Théorie de Hodge L^2 et théorèmes d'annulation", in *Introduction à la théorie de Hodge*, Panoramas & Synthèses, vol. 3, Société Mathématique de France, 1996, p. 3–111.

[8] A. Dimca, F. Maaref, C. Sabbah & M. Saito – "Dwork cohomology and algebraic $\mathscr{D}$-modules", *Math. Ann.* **318** (2000), no. 1, p. 107–125.

[9] A. Dimca, Ph. Maisonobe, M. Saito & T. Torrelli – "Multiplier ideals, V-filtrations and transversal sections", *Math. Ann.* **336** (2006), no. 4, p. 901–924.

[10] P.A. Griffiths & J. Harris – *Principles of Algebraic Geometry*, A. Wiley-Interscience, New York, 1978.

[11] F. Guillén & V. Navarro Aznar – "Sur le théorème local des cycles invariants", *Duke Math. J.* **61** (1990), p. 133–155.

[12] M. Kashiwara – "The asymptotic behaviour of a variation of polarized Hodge structure", *Publ. RIMS, Kyoto Univ.* **21** (1985), p. 853–875.

[13] ———, "A study of variation of mixed Hodge structure", *Publ. RIMS, Kyoto Univ.* **22** (1986), p. 991–1024.

[14] G. Laumon – "Sur la catégorie dérivée des $\mathscr{D}$-modules filtrés", in *Algebraic geometry (Tokyo/Kyoto, 1982)*, Lect. Notes in Math., vol. 1016, Springer-Verlag, 1983, p. 151–237.

[15] Ph. Maisonobe & Z. Mebkhout – "Le théorème de comparaison pour les cycles évanescents", in *Éléments de la théorie des systèmes différentiels géométriques*, Séminaires & Congrès, vol. 8, Société Mathématique de France, Paris, 2004, p. 311–389.

[16] Ph. Maisonobe & C. Sabbah – "Aspects of the theory of $\mathscr{D}$-modules", Lecture Notes (Kaiserslautern 2002), version de juillet 2011, disponible à `http://www.math.polytechnique.fr/~sabbah/livres.html`, 2011.

[17] B. Malgrange – "Sur les images directes de $\mathscr{D}$-modules", *Manuscripta Math.* **50** (1985), p. 49–71.

[18] ———, *Équations différentielles à coefficients polynomiaux*, Progress in Math., vol. 96, Birkhäuser, Basel, Boston, 1991.

[19] Z. Mebkhout & L. Narváez Macarro – "Le théorème de constructibilité de Kashiwara", in *Éléments de la théorie des systèmes différentiels* (Ph. Maisonobe & C. Sabbah, eds.), Les cours du CIMPA, Travaux en cours, vol. 46, Hermann, Paris, 1993, p. 47–98.

[20] Z. Mebkhout & C. Sabbah – "§ III.4 $\mathscr{D}$-modules et cycles évanescents", in *Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}$-modules cohérents*, Travaux en cours, vol. 35, Hermann, Paris, 1989, p. 201–239.

[21] C. Peters & J.H.M. Steenbrink – *Mixed Hodge Structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 52, Springer-Verlag, Berlin, 2008.

[22] E. Reyssat – *Quelques aspects des surfaces de Riemann*, Progress in Math., vol. 77, Birkhäuser, Basel, Boston, 1989.

[23] K. Saito – "On a generalisation of de Rham lemma", *Ann. Inst. Fourier (Grenoble)* **26** (1976), p. 165–170.

[24] M. Saito – "Modules de Hodge polarisables", *Publ. RIMS, Kyoto Univ.* **24** (1988), p. 849–995.

[25] ———, "Mixed Hodge Modules", *Publ. RIMS, Kyoto Univ.* **26** (1990), p. 221–333.

[26] ———, "On b-function, spectrum and rational singularity", *Math. Ann.* **295** (1993), p. 51–74.

[27] ———, "On the Hodge filtration of Hodge modules", *Mosc. Math. J.* **9** (2009), no. 1, p. 161–191.

[28] J. Scherk & J.H.M. Steenbrink – "On the mixed Hodge structure on the cohomology of the Milnor fiber", *Math. Ann.* **271** (1985), p. 641–655.

[29] W. Schmid – "Variation of Hodge structure: the singularities of the period mapping", *Invent. Math.* **22** (1973), p. 211–319.

[30] J.H.M. Steenbrink – "Semi-continuity of the singularity spectrum", *Invent. Math.* **79** (1985), p. 557–565.

[31] J.H.M. Steenbrink & S. Zucker – "Variation of mixed Hodge structure I", *Invent. Math.* **80** (1985), p. 489–542.

[32] A.N. Varchenko – "On semicontinuity of the spectrum and an upper bound for the number of singular points of projective hypersurfaces", *Soviet Math. Dokl.* **27** (1983), p. 735–739.

[33] C. Voisin – *Théorie de de Hodge et géométrie algébrique complexe*, Cours spécialisés, vol. 10, Société Mathématique de France, Paris, 2002.

[34] S. Zucker – "Hodge theory with degenerating coefficients: L_2-cohomology in the Poincaré metric", *Ann. of Math.* **109** (1979), p. 415–476.