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FOURIER TRANSFORMATION OF $\mathscr{D}$-MODULES AND APPLICATIONS

LECTURE NOTES (CIRM, LUMINY, SEPT. 2010)

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Abstract. These lectures are devoted to explaining some properties and some applications of the Fourier transformation of holonomic $\mathscr{D}$-modules. Firstly, a comparison is made with the usual Fourier transformation of tempered distribution. Soon, we restrict our attention to one-dimensional $\mathscr{D}$-modules, and explain the stationary phase formula in this context. This part relies on works of Katz, Bloch-Esnault and García López. As a first application, we explain the approach of Arinkin to the Katz algorithm, which reduces rigid irreducible bundles with connection to rank-one bundles with connection through various transformations, one of which is the Fourier (or Laplace) transformation. A second application, which relies on the work of Beilinson, Bloch, Deligne and Esnault, concerns the product formula for the period matrix of a bundle with connection. Both applications are strongly motivated by previous works of Katz and Laumon in the realm of ℓ-adic perverse sheaves, and are intended to pursue the analogy between the complex and the ℓ-adic theories. In this way, general holonomic $\mathscr{D}$-modules are analogous to wildly ramified ℓ-adic perverse sheaves.

CONTENTS

LECTURE 1

THE STATIONARY PHASE FORMULA

Summary. In this lecture, we introduce the notion of global and local Fourier (or Laplace) transform for a holonomic module over the Weyl algebra. After having explained the local structure of such a module in one variable, we give an explicit correspondence, called stationary phase formula, between the local structure of such a module at its singular points and that of its Fourier (or Laplace) transform.

1.a. Fourier transform of tempered distributions and Laplace transform of $\mathscr{D}$-modules. Let u be a tempered distribution on $\mathbb{C}^n$ (coordinates $z_j = x_j + iy_j$) and let ${}^F u$ be its Fourier transform. The Fourier kernel $e^{-i(x\cdot\xi + y\cdot\eta)}$ can also be written as $e^{\overline{z\zeta} - z\zeta}$ if we set $\zeta_j = \eta_j + i\xi_j$. If $P \in \mathbb{C}[z]\langle\partial_z\rangle$ is a holomorphic differential operator with polynomial coefficients, we have ${}^F(Pu) = {}^F P({}^F u)$, where ${}^F P \in \mathbb{C}[\zeta]\langle\partial_\zeta\rangle$ is the differential operator obtained from P by replacing z_j with $-\partial_{\zeta_j}$ and ∂_{z_j} with ζ_j.

Given a left $\mathbb{C}[z]\langle\partial_z\rangle$-module M of the form $\mathbb{C}[z]\langle\partial_z\rangle/(P_1, \dots, P_k)$, its Laplace transform ${}^F M$ (with kernel $e^{-z\zeta}$) is the left $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$-module $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle/({}^F P_1, \dots, {}^F P_k)$. More generally, note that the previous correspondence makes $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$ a left $\mathbb{C}[z]\langle\partial_z\rangle$-module and any $\mathbb{C}[z]\langle\partial_z\rangle$-module M defines a $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$-module ${}^F M := \mathbb{C}[\zeta]\langle\partial_\zeta\rangle \otimes_{\mathbb{C}[z]\langle\partial_z\rangle} M$, that is, ${}^F M$ is the $\mathbb{C}$-vector space M equipped with the structure of $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$-module where ζ_j acts by ∂_{z_j} and ∂_{ζ_j} by $-z_j$. The inverse Laplace transformation is the similar transformation with kernel $e^{z\zeta}$.

Example 1.1. Let $u \in \mathscr{S}'(\mathbb{C}^n_z)$ be a tempered distribution and let $M \subset \mathscr{S}'(\mathbb{C}^n_z)$ be the $\mathbb{C}[z]\langle\partial_z\rangle$-module generated by u. Then ${}^F M$ is the $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$-module generated by ${}^F u$ in $\mathscr{S}'(\mathbb{C}^n_\zeta)$.

1.b. Formal normal form of a holonomic $\mathscr{D}$-modules in dimension one

From now on, I assume that $n = 1$. In such a case, the notion of a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module is easy to define: a left $\mathbb{C}[z]\langle\partial_z\rangle$-module is holonomic if each element $m \in M$ satisfies a non-trivial equation $Pm = 0$ for some $P \in \mathbb{C}[z]\langle\partial_z\rangle \smallsetminus \{0\}$. It is said to be *irreducible* if it has no non-trivial submodule.

The main goal of these notes is to say something non-trivial about irreducible holonomic $\mathscr{D}$-modules, as a step to their classification. One should notice immediately that the Laplace transform ${}^F M$ of a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module is holonomic, and is irreducible as soon as ${}^F M$ is so.

Before considering this question, I will consider the simpler one for $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules. Given a $\mathbb{C}[z]\langle\partial_z\rangle$-module M, we get a $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-module (formalization at the origin)

$$\widehat{M}_0 := \mathbb{C}[\![z]\!]\langle\partial_z\rangle \otimes_{\mathbb{C}[z]\langle\partial_z\rangle} M \quad \text{i.e.,}\quad \widehat{M}_0 = \mathbb{C}[\![z]\!] \otimes_{\mathbb{C}[z]} M.$$

Changing the variable z to $z - z_o$ for some $z_o \in \mathbb{C}$ allows one to apply the same arguments at each z_o for $\widehat{M}_{z_o}$. One can also consider $\widehat{M}_\infty$: in such a case, denote by z' the coordinate $1/z$ at infinity; correspondingly, set $\partial_{z'} = -z^2\partial_z$; for a $\mathbb{C}[z]\langle\partial_z\rangle$-module M, consider $M_\infty := \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} M$ as a left $\mathbb{C}[z', z'^{-1}]\langle\partial_{z'}\rangle$-module and set

$$\widehat{M}_\infty := \mathbb{C}(\!(z')\!) \otimes_{\mathbb{C}[z',z'^{-1}]} M_\infty.$$

Note that $\widehat{M}_\infty$ is a priori a $\mathbb{C}(\!(z')\!)\langle\partial_{z'}\rangle$-module, not only a $\mathbb{C}[\![z']\!]\langle\partial_{z'}\rangle$-module, i.e., $z' : \widehat{M}_\infty \to \widehat{M}_\infty$ is invertible.

Singular points. Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module and let $m \in M$ be a non-zero element. Let P be a non-zero differential operator of minimal degree with respect to ∂_z annihilating m, written as $\sum_{k=0}^d p_k(z)\partial_z^k$, with $p_k \in \mathbb{C}[z]$ and $p_d \neq 0$. Near any $z_o \in \mathbb{C}$ such that $p_d(z_o) \neq 0$, Cauchy's theorem allows one to solve the equation $Pf = 0$, and algebraically this means that, in $\mathbb{C}[\![z - z_o]\!]\langle\partial_z\rangle$, the operator can be decomposed as a product of degree-one operators. Near each point of the zero set of p_d, such a statement is false, and a more precise analysis is needed.

Definition 1.2. Let $P \in \mathbb{C}[\![z]\!]\langle\partial_z\rangle$ written as $\sum_{k=0}^d p_k(z)\partial_z^k$, with $p_k \in \mathbb{C}[\![z]\!]$ and $p_d \neq 0$. The Newton polygon $N(P)$ of P is the compact boundary of the convex hull in $\mathbb{R}_+ \times \mathbb{R}$ of the sets $(k, v(p_k) - k) + ((-\mathbb{N}) \times \mathbb{N})$:

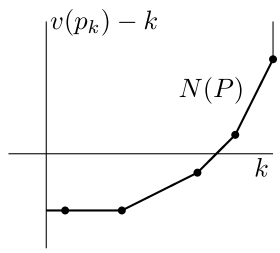


FIGURE 1

We say that $P \neq 0$ is *pure of slope* $\lambda \in \mathbb{Q}_+$ if $N(P)$ has only one edge, with slope λ.

Theorem 1.3 **(Structure of holonomic $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules).**

(1) *Each* $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$*-module* $\widehat{M}$ *has a unique finite decomposition* $\widehat{M} = \bigoplus_{\lambda\in\mathbb{Q}_+} \widehat{M}^{(\lambda)}$, *where* $m \in \widehat{M}^{(\lambda)}$ *iff there exists* $P \neq 0$ *pure of slope* λ *such that* $Pm = 0$.

(2) $\widehat{M}^{(0)}$ *is determined by the data of a quiver*

$$E \underset{v}{\overset{c}{\rightleftarrows}} F$$

where E, F *are finite dimensional* $\mathbb{C}$*-vector spaces and* c, v *are linear morphisms subject to the property that* $\mathrm{Id} + cv$ *and* $\mathrm{Id} + vc$ *are invertible.*

(3) *If* $\lambda > 0$, z *acts bijectively on* $\widehat{M}^{(\lambda)}$*, which is then a finite dimensional* $\mathbb{C}(\!(z)\!)$*-vector space.*

The space F is called the space of *vanishing cycles* of $\widehat{M}^{(0)}$. The structure of indecomposable $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules of pure slope 0 (also called regular) is a question of linear algebra. Let us consider the particular case of regular $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules on which z acts bijectively.

Proposition 1.4. *The action of* z *is invertible on* $\widehat{M}^{(0)}$ *if and only if* $v : F \to E$ *is an isomorphism. In such a case,*

(1) $\widehat{M}^{(0)} \simeq \mathbb{C}(\!(z)\!) \otimes_{\mathbb{C}} E$ *and the action of* $z\partial_z$ *is induced by that of* $\frac{1}{2\pi i}\log(vc)$,

(2) $\widehat{M}^{(0)}$ *is indecomposable if and only if* vc *has only one Jordan block,*

(3) $\widehat{M}^{(0)}$ *is irreducible if and only if* $\dim E = 1$.

Proposition 1.5 **(Indecomposable $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules of pure slope $\lambda > 0$)**

If $\widehat{M}^{(\lambda)}$ *is indecomposable, then* $\widehat{M}^{(\lambda)} \simeq \widehat{I}^{(\lambda)} \otimes_{\mathbb{C}(\!(z)\!)} \widehat{R}^{(0)}$, *where* $\widehat{I}^{(\lambda)}$ *is irreducible of pure slope* λ *and* $\widehat{R}^{(0)}$ *is indecomposable of pure slope* 0.

We now analyze the structure of irreducible $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules of positive slope. If $\widehat{I}^{(\lambda)}$ has rank one over $\mathbb{C}(\!(z)\!)$, then any generator of $\widehat{I}^{(\lambda)}$ is annihilated by an operator P of the form

$$P = z^{\lambda}(z\partial_z) + a(z), \quad \text{with } \lambda \in \mathbb{N}^*,\ a(z) \in \mathbb{C}[\![z]\!], \text{ and } a(0) \neq 0.$$

Up to changing the generator, one ca assume that $a(z)$ is a polynomial of degree $\leqslant \lambda$. We denote such a module by $\widehat{\mathscr{E}}^{\varphi}$, where $\varphi \in \mathbb{C}(\!(z)\!)/\mathbb{C}[\![z]\!]$ is such that $\varphi'(z) = -a(z)/z^{\lambda+1}$.

In order to get examples with non-integral slopes, it is necessary to introduce ramification and push-forward. In particular, if $\lambda = q/p$ with $(p, q) = 1$, then $\widehat{I}^{(\lambda)}$ has rank p. Let $\rho(u)$ be any element of $u\mathbb{C}[\![u]\!]$ with valuation $p \geqslant 1$ (e.g. $\rho(u) = u^p$). We regard ρ as a morphism of degree p from the formal disc with coordinate u to the formal disc with coordinate z through the correspondence $\rho : \mathbb{C}[\![z]\!] \to \mathbb{C}[\![u]\!]$, $z \mapsto \rho(u)$. Let N be a finite dimensional $\mathbb{C}(\!(u)\!)$-vector space equipped with a compatible action of $\mathbb{C}[\![u]\!]\langle\partial_u\rangle$. The *push-forward* $\rho_+ N$ is defined as follows:

(1) the $\mathbb{C}(\!(z)\!)$-vector space $\rho_* N$ is the $\mathbb{C}$-vector space N equipped with the structure of $\mathbb{C}(\!(z)\!)$-vector space given by $f(z) \cdot m := f(\rho(u))m$,

(2) the action of ∂_z is that of $\rho'(u)^{-1}\partial_u$.

Proposition 1.6 **(Irreducible $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules of pure slope $\lambda > 0$)**
If $\lambda = q/p > 0$ with $(p,q) = 1$, any irreducible $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-module $\widehat{I}^{(\lambda)}$ of pure slope λ takes the form $\mathrm{El}(\rho,\varphi,R) := \rho_+(\widehat{\mathscr{E}}^\varphi \otimes R)$ for some $\rho \in u\mathbb{C}[\![u]\!]$ of valuation p, $\varphi \in \mathbb{C}(\!(u)\!)/\mathbb{C}[\![u]\!]$ having a pole of order q and R of rank one and slope 0.

Up to modifying φ, one can assume that $\rho(u) = u^p$. Any indecomposable $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-module can also be written as $\mathrm{El}(\rho,\varphi,R) := \rho_+(\widehat{\mathscr{E}}^\varphi \otimes R)$, where R is indecomposable of pure slope 0 and $\rho_+\widehat{\mathscr{E}}^\varphi$ is irreducible.

Conclusion. Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module. For all $z_o \in \mathbb{C}$ except a finite set, the formalized module $\widehat{M}_{z_o}$ is isomorphic to $\mathbb{C}[\![z - z_o]\!]^n$ with its natural action of ∂_z. The complementary finite set, together with ∞, is called the set of singular points of M. At each singular point z_o, $\widehat{M}_{z_o}$ has a decomposition into pure-slope modules, and more precisely indecomposable objects, as described in the previous propositions for the positive-slope part, and as suggested by the quiver representation for the zero-slope part.

The formal stationary phase formula consists in describing the set of singular points and the corresponding decompositions for the Laplace transform ${}^F\!M$ of M, in terms of the similar data for M. Having a formal normal form for a differential equation allows one to solve it in terms of formal power series. A general theorem says that this amounts to finding asymptotic solutions of the differential equation. The consequence is that, knowing the asymptotic expansion of a basis of solutions of a differential equation in the neighbourhood of each of its singular points, we can recover the possible asymptotic expansions of the solutions of the Laplace transformed differential equation. The name "stationary phase" comes from the fact that the computation only involves the singular points of M.

1.c. The local Laplace transforms. Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module and let ${}^F\!M$ be its Laplace transform. For each $\zeta_o \in \mathbb{C}_\zeta \cup \{\infty_\zeta\}$, we can consider the formal object $\widehat{{}^F\!M}_{\zeta_o}$.

Question 1.7. Is it possible to compute $\widehat{{}^F\!M}_{\zeta_o}$ only in terms of various $\widehat{M}_{z_o}$, where z_o varies in $\mathbb{C}_z \cup \{\infty_z\}$. More precisely, we look for an expression

$$\widehat{{}^F\!M}_{\zeta_o} = \bigoplus_{z_o \in \mathbb{C}_z \cup \{\infty_z\}} \mathscr{F}^{(z_o,\zeta_o)}(\widehat{M}_{z_o}).$$

If they exist, the transformations $\mathscr{F}^{(z_o,\zeta_o)}$ are called *local Fourier (Laplace) transform.*

Example 1.8. The local Laplace transform $\mathscr{F}^{(z_o,\infty)}$ ($z_o \in \mathbb{C}$) is nothing but microlocalization, namely, this is the tensor product with the ring $\widehat{\mathscr{E}}_{z_o}$ consisting of Laurent series $\sum_{k\geqslant k_o} a_k(z-z_o)\theta^k$ such that a_k is holomorphic in a fixed neighbourhood of z_o. It comes equipped with a suitable non-commutative product, such that in particular $\theta\cdot a(z-z_o)=\sum_{k\geqslant 0}(-1)^k a^{(k)}(z-z_o)\theta^{k+1}$. The ring $\widehat{\mathscr{E}}_{z_o}$ contains $\mathbb{C}[\![z-z_o]\!]\langle\partial_z\rangle$ as a subring by sending ∂_z to θ^{-1}. So we set

$$\mathscr{F}^{(z_o,\infty)}\widehat{M}_{z_o}=\widehat{\mathscr{E}}_{z_o}\otimes_{\mathbb{C}[\![z-z_o]\!]\langle\partial_z\rangle}\widehat{M}_{z_o}.$$

One can also define $\mathscr{F}^{(\infty,\infty)}$ with a suitable ring of microdifferential operators.

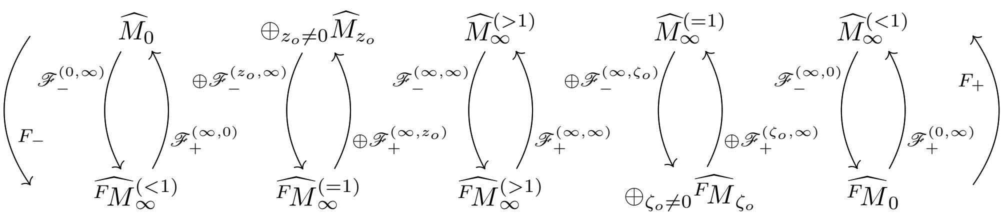
Theorem 1.9 **(Stationary phase theorem, cf. [?] and also [?]).** *Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module. Then*

$$(1.9)^{(0,\infty)}\qquad \widehat{{}^F\!M}{}_\infty^{(<1)}=\mathscr{F}^{(0,\infty)}\widehat{M}_0$$

$$(1.9)^{(\neq 0,\infty)}\qquad \widehat{{}^F\!M}{}_\infty^{(=1)}=\bigoplus_{z_o\in\mathbb{C}^*}\mathscr{F}^{(z_o,\infty)}\widehat{M}_{z_o}$$

$$(1.9)^{(\infty,\infty)}\qquad \widehat{{}^F\!M}{}_\infty^{(>1)}=\mathscr{F}^{(\infty,\infty)}\widehat{M}{}_\infty^{(>1)}$$

We can picture the theorem in the following diagram:



Remark 1.10

(1) There is no contribution of the kind (finite dist., finite dist.).

(2) There is a more precise correspondence for the slopes with respect to $\mathscr{F}^{(0,\infty)}$ or $\mathscr{F}^{(\infty,\infty)}$. For example, $\mathscr{F}^{(0,\infty)}\widehat{M}{}_0^{(\lambda)}=\widehat{{}^F\!M}{}_\infty^{(\mu)}$ with $1/\mu=1+1/\lambda$.

1.d. Explicit formulas for the local Laplace transforms. I will only consider the local Laplace transform $\mathscr{F}^{(0,\infty)}$. There are similar formulas for the other local Laplace transforms. According to the structure of holonomic $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules, it is enough to compute the local Laplace transform of a $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-module of pure slope 0 and of elementary $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-modules of pure slope $\lambda>0$.

Theorem 1.11 **(cf. [?] and [?])**

(1) *The local Laplace transform of a $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-module $\widehat{M}^{(0)}$ of pure slope 0, determined by a quiver 1.3(2), is isomorphic to the $\mathbb{C}(\!(\theta)\!)$-vector space $\mathbb{C}(\!(\theta)\!)\otimes_{\mathbb{C}}F$ equipped with the connection (of pure slope 0)such that $\theta\partial_\theta(1\otimes f)=1\otimes\frac{1}{2\pi i}\log(cv)(f)$.*

(2) *For any elementary* $\mathbb{C}[z]\langle\partial_z\rangle$*-vector space* $\mathrm{El}(\rho,\varphi,R)$ *with irregular connection (i.e., such that* $\varphi\notin\mathbb{C}[\![u]\!]$*), the local Laplace transform* $\mathscr{F}_{\pm}^{(0,\infty)}\mathrm{El}(\rho,\varphi,R)$ *is the elementary finite dimensional* $\mathbb{C}(\!(\theta)\!)$*-vector space with connection* $\mathrm{El}({}^F\!\rho_{\pm},{}^F\!\varphi,{}^F\!R)$ *with (setting* $L_q=(\mathbb{C}(\!(u)\!),d-\frac{q}{2}\frac{du}{u})$*)*

$$ {}^F\!\rho_{\pm}(u)=\mp\frac{\rho'(u)}{\varphi'(u)},\quad {}^F\!\varphi(u)=\varphi(u)-\frac{\rho(u)}{\rho'(u)}\,\varphi'(u),\quad {}^F\!R\simeq R\otimes L_q. $$

LECTURE 2

RIGID IRREDUCIBLE CONNECTIONS AND THE ALGORITHM OF KATZ

Summary. The purpose of this lecture is to consider the following "local-to-global" existence and uniqueness question: (a) given formal holonomic $\mathscr{D}$-modules at points $z_i \in \mathbb{A}^1_z$ ($i \in I$) and at ∞, does there exist a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module having $\{z_i \mid i \in I\} \cup \{\infty\}$ as singularities exactly and corresponding isomorphic local structure? (b) Is a $\mathbb{C}[z]\langle\partial_z\rangle$-module completely determined by the local formal structure at its singular points. When the answer to (b) is positive, the $\mathbb{C}[z]\langle\partial_z\rangle$-module is said to be rigid. The algorithm of Katz reduces any rigid holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module to a $\mathbb{C}[z]\langle\partial_z\rangle$-module having generic rank one through various transformations, one of which is the Fourier (Laplace) transformation.

2.a. Irreducible $\mathbb{C}[z]\langle\partial_z\rangle$-modules and irreducible connections. Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module. For all $z_o \in \mathbb{A}^1$ except a finite number, the formalization $\widehat{M}_{z_o}$ is isomorphic to $\mathbb{C}[\![z - z_o]\!]^n$ equipped with its natural structure of $\mathbb{C}[\![z - z_o]\!]\langle\partial_z\rangle$-module. We say that z_o is a regular point of M. Otherwise, it is a singular point. The set of regular points of M is a Zariski open subset of $\mathbb{A}^1$. Let U be a Zariski open subset of $\mathbb{A}^1$ containing no singular point of M and let $\mathscr{O}(U) \subset \mathbb{C}(z)$ be the subring of rational fractions having no poles on U. Then (this is a theorem) $M_U := \mathscr{O}(U) \otimes_{\mathbb{C}[z]} M$ is a free $\mathscr{O}(U)$-module of rank n, equipped with a compatible action of ∂_z, that we call a *connection*. In a given $\mathscr{O}(U)$-basis of M_U, one can consider the matrix of ∂_z, although ∂_z is not $\mathscr{O}(U)$-linear and so the base change formula is not the usual one for linear operators. This matrix has entries in $\mathscr{O}(U)$. On the other hand, note that, in general, M is not a $\mathbb{C}[z]$-module of finite type, due to the presence of singular points.

Proposition 2.1. *Let M_U be a free $\mathscr{O}(U)$-module with a compatible action of ∂_z. Then there exists a unique holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module, denoted by $j_{!*}M_U$ if $j : U \hookrightarrow \mathbb{A}^1$ denotes the inclusion, such that*

- *$(j_{!*}M_U)_U = M_U$,*
- *$j_{!*}M_U$ has no sub or quotient module supported in $\mathbb{A}^1 \smallsetminus U$.*

The operation $j_{!*}$ is functorial and preserves inclusions. Moreover, if U' is a Zariski open subset of U and $M_{U'} := (M_U)_{U'}$, then $j'_{!*}M_{U'} = j_{!*}M_U$.

Corollary 2.2. *Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module and let U be a Zariski open subset of $\mathbb{A}^1$ containing no singular point of M. Then, if M is irreducible, M_U is irreducible and $M = j_{!*}M_U$. Conversely, if M_U is irreducible, then $j_{!*}M_U$ is irreducible.*

Proof of the corollary. If M is irreducible, then M satisfies both characteristic properties of $j_{!*}M_U$, hence it is equal to it. If M_U had a non-trivial submodule N_U, then $j_{!*}N_U$ would be a non-trivial submodule of $j_{!*}M_U$, a contradiction.

Conversely, if M_U is irreducible as a $\mathscr{O}(U)\langle\partial_z\rangle$-module, then the only possible submodules of $j_{!*}M_U$ are supported on $\mathbb{A}^1 \smallsetminus U$, hence are trivial. □

2.b. Katz transformations

The middle Laplace transformation. Let us start with a free $\mathscr{O}(U)$-module M_U with a connection. In order to define its Laplace transform, it is necessary to extend it first as a $\mathbb{C}[z]\langle\partial_z\rangle$-module. A minimal way to do this is to consider the middle extension $j_{!*}M_U$. The Laplace transform ${}^F(j_{!*}M_U)$ is a holonomic $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$-module, which defines a connection on its open set of regular points V. In general, the composed operation $M_U \mapsto {}^F(M_U) := \big[{}^F(j_{!*}M_U)\big]_V$, which associates a connection on V to a connection on U, is not invertible, since there is no reason why $({}^Fj)_{!*}(j_{!*}M_U)_V = {}^F(j_{!*}M_U)$, where ${}^Fj : V \hookrightarrow \mathbb{A}^1_\zeta$ is the inclusion.

However, if M_U is *irreducible*, then $j_{!*}M_U$ is also irreducible, as well as its Laplace transform ${}^F(j_{!*}M_U)$, which means that ${}^F(j_{!*}M_U) = ({}^Fj)_{!*}{}^F(j_{!*}M_U)_V$, according to the corollary. In conclusion, for *irreducible* free $\mathscr{O}(U)$-modules with connection, the minimal Laplace transformation is an *invertible* transformation.

Remark 2.3. If M is any holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module, the set of regular points of ${}^FM$ only depends on the behaviour of M near infinity, according to the (inverse) stationary phase formula. The non-zero ones are created from the part of the formal connection $\widehat{M}_\infty$ with slope one at infinity, while zero comes from the part $\widehat{M}_\infty^{(<1)}$. In particular, they only depend on M_U, whatever U is, provided M has no singular point in U.

Tensoring with a rank-one connection. Let $M_{U'}$ be a free $\mathscr{O}(U')$-module with connection and let $L_{U''}$ be a free $\mathscr{O}(U'')$-module of rank one with connection. Set $U = U' \cap U''$. The tensor product $L_U \otimes_{\mathscr{O}_U} M_U$ is naturally equipped with a connection by setting $\partial_z(\ell \otimes m) = (\partial_z\ell) \otimes m + \ell \otimes (\partial_z m)$. If $M_{U'}$ is irreducible, then M_U remains irreducible. Moreover, $L_U \otimes_{\mathscr{O}_U} M_U$ is irreducible: If it had a non-trivial submodule N_U, then $L_U^{-1} \otimes_{\mathscr{O}_U} N_U$ would be a non-trivial submodule of M_U, where L_U^{-1} is the dual $\mathscr{O}_U$-module equipped with the dual connection.

Möbius transformations. Let $M_{U'}$ be a free $\mathscr{O}(U')$-module with connection and let U be a Zariski open subset of U'. Let $\mu : \mathscr{O}(U) \xrightarrow{\sim} \mathscr{O}(U)$ be an automorphism of $\mathscr{O}(U)$

(a *Möbius transformation*). Let us set $M_U = (M_{U'})_U$. Then $\mu^* M_U$ is a free $\mathscr{O}(U)$-module, equipped with a connection (by using the chain rule). If $M_{U'}$ is irreducible, then M_U remains irreducible. Moreover, $\mu^* M_U$ is irreducible.

Remark 2.4. The interest of the Möbius transformations is that they allow to change the point at infinity. Followed by a Laplace transform, they allow to change the locus of singular points, according to Remark 2.3.

Middle convolution. This is the adjunction of tensoring with L by middle Laplace transformation. Let $M_{U'}$ be an irreducible free $\mathscr{O}(U')$-module with connection and let $L_{V''}$ be a free $\mathscr{O}(V'')$-module of rank one with connection. Let V' be the Zariski open set of regular points of ${}^F(j_{!*}M_{U'})$. Set $V = V' \cap V''$. The middle convolution of $M_{U'}$ by $L_{V''}$ is defined by

$$M_{U'} \star_{\mathrm{mid}} L_{V''} := {}^{F^{-1}}\big(({}^F M_{U'})_V \otimes_{\mathscr{O}(V)} L_V\big)_U,$$

where U is contained in the set of regular points of ${}^{F^{-1}}\big(({}^F M_U)_V \otimes_{\mathscr{O}(V)} L_V\big)$.

***Definition 2.5* (Algorithm of Katz).** An algorithm of Katz is a succession of transformations of the previous kind (Laplace transformation, tensoring with a rank-one connection, Möbius transformation) such that, starting from an irreducible free $\mathscr{O}(U)$-module with connection M_U, we reach a new one *having rank strictly smaller than* $\operatorname{rk} M_U$.

Question 2.6. Given an irreducible M_U, is it possible to find a Katz algorithm that terminates with a *rank-one connection* (equivalently, that terminates with $(\mathscr{O}(U'), \partial_z)$ for some $U' \subset U$)?

We will get a positive answer to Question 2.6 for *rigid* irreducible connections.

2.c. Rigidity and the Deligne-Simpson problem. Let us fix a finite set of points $Z \subset \mathbb{A}^1$ and a non-zero integer n. Set $U = \mathbb{A}^1 \smallsetminus Z$. The Deligne-Simpson problem is an existence question, while rigidity refers to a uniqueness question.

(a) Given a n-dimensional $\mathbb{C}(\!(z - z_o)\!)$ (resp. $\mathbb{C}(\!(z')\!)$) vector space with connection $\widehat{M}_{z_o}$ for each $z_o \in \mathbb{A}^1$ (resp. $\widehat{M}_\infty$), *does there exist an irreducible free $\mathscr{O}(U)$-module with connection M_U having the given formalization at Z and at ∞?*

This is called the *Deligne-Simpson problem* for the formal data $(\widehat{M}_{z_o})_{z_o \in Z \cup \{\infty\}}$. There is a trivial necessary condition, which follows from the residue theorem: the sum over $Z \cup \{\infty\}$ of the traces of the residues of the connection should be an integer.

The case where each formal data has only slope zero (regular singularity) is equivalent to the following. Given conjugacy classes $C_{z_o} \subset \mathrm{GL}(n, \mathbb{C})$ for $z_o \in Z \cup \{\infty\}$ such that the product of the determinants of these classes is one, does there exist representatives $c_{z_o} \in C_{z_o}$ such that $\prod_{z_o \in Z \cup \{\infty\}} c_{z_o} = \mathrm{Id}$ and the family $(c_{z_o})_{z_o \in Z \cup \{\infty\}}$ is irreducible?

(b) On the other hand, we will say that a free $\mathscr{O}(U)$-module with connection M_U is *rigid* if any other M'_U having isomorphic formal data $\widehat{M}'_{z_o} \simeq \widehat{M}_{z_o}$ ($z_o \in Z \cup \{\infty\}$) is isomorphic to M_U.

***Example 2.7* (Rank one).** Assume M_U has rank one. We have thus $M_U \simeq \mathscr{O}(U)$ and the action of ∂_z on the generator 1 is given by some $f \in \mathscr{O}(U)$: $\partial_z(1) = f \cdot 1$, $\partial_z(g(z)) = (g'(z) + f(z)g(z)) \cdot 1$. Changing the generator consists in choosing $p(z) \in \mathscr{O}(U)^*$. Then the "matrix" f of the connection is changed to $f(z) + p'(z)/p(z)$. For $z_o \in \mathbb{A}^1$, we denote by $[f_{z_o}]$ the polar part of f at z_o, and we denote $[\widetilde{f}_\infty]$ the polynomial part (in the variable z) of $-z^2 f(z)$ (that we regard as the polar part in the variable $z' = 1/z$). The residue formula is written as

$$\sum_{z_o \in \mathbb{A}^1} \operatorname{Res}_{z_o}[f_{z_o}] + \operatorname{Res}_\infty[\widetilde{f}_\infty] = 0.$$

A similar property holds for the localizations $\widehat{M_U}_{z_o}$ and $\widehat{M_U}_\infty$. For $z_o \in \mathbb{A}^1$, the isomorphism class of $\widehat{M_U}_{z_o}$ is determined by the data of a polar part in $\mathbb{C}(\!(z - z_o)\!)/\mathbb{C}[\![z - z_o]\!]$ up to adding a integer to its residue. At infinity, a similar result hods in the variable z'?

The Deligne-Simpson problem consists in the following: given polar parts $(f_{z_o})_{z_o \in Z}$ and $\widetilde{f}_\infty$, is it possible to find $f \in \mathscr{O}(U)$ such that $[f_{z_o}] - f_{z_o}$ ($z_o \in Z$) and $[\widetilde{f}_\infty] - \widetilde{f}_\infty$ have integral residue?

A necessary condition is that the sum of the residues of f_{z_o} and $\widetilde{f}_\infty$ is an integer. Conversely, if such a condition on the given polar parts is satisfied, one can naturally modify one of these polar parts in order that the sum of the residues is zero, and one can consider them as elements of $\mathscr{O}(U)$ and define f as the sum of these elements at finite distance, and $-\widetilde{f}_\infty/z^2$. Then f is a solution to the problem. In other words, the Deligne-Simpson problem has a solution in rank one.

The rigidity question asks for uniqueness: assume that $f \in \mathscr{O}(U)$ has simple poles with integral residues n_{z_o} at finite distance. Then rigidity means that there should exists $p(z) \in \mathscr{O}(U)^*$ such that $f(z) = p'(z)/p(z)$. Such a p exists: take $p(z) = \prod_{z_o \in Z}(z - z_o)^{n_o}$. In other words, any rank one $\mathscr{O}(U)$-module with ∂_z is rigid.

We will be mainly concerned with rigidity and we will not consider the existence problem. We will give a criterion, due to Katz [**?**] and Bloch-Esnault [**?**], for rigidity. Let M_U be a free $\mathscr{O}(U)$-module with connection. Then the free $\mathscr{O}(U)$-module $\operatorname{End}(M_U) := \operatorname{Hom}_{\mathscr{O}(U)}(M_U, M_U)$ is equipped with a natural action of ∂_z, defined by the formula

$$(\partial_z \varphi)(m) = \partial_z(\varphi(m)) - \varphi(\partial_z m).$$

If N_U is a free $\mathscr{O}(U)$-module with connection, we attach to it the following numbers:

$$\begin{aligned}
\chi(N_U) &:= \chi(\partial_z : N_U \longrightarrow N_U) = \dim \operatorname{Ker} - \dim \operatorname{Coker}, \\
h_{z_o}(N_U) &:= \dim \operatorname{Ker} \partial_z : \widehat{N}_{U,z_o} \longrightarrow \widehat{N}_{U,z_o}, \quad (z_o \in \mathbb{A}^1 \smallsetminus U), \\
h_\infty(N_U) &:= \dim \operatorname{Ker} \partial_{z'} : \widehat{N}_{U,\infty} \longrightarrow \widehat{N}_{U,\infty}.
\end{aligned}$$

***Definition 2.8* (Index of rigidity).** Let M_U be a free $\mathscr{O}(U)$-module with connection. Its *rigidity index* is

$$\operatorname{rig} M_U := \chi(\operatorname{End}(M_U)) - \sum_{z_o \in Z \cup \{\infty\}} h_{z_o}(\operatorname{End}(M_U)).$$

***Theorem 2.9* (Katz, Bloch-Esnault).** *Let M_U be an irreducible free $\mathscr{O}(U)$-module with connection. Then M_U is rigid if and only if* $\operatorname{rig} M_U = 2$.

***Theorem 2.10* (Katz, Bloch-Esnault).** *Let M_U be an irreducible free $\mathscr{O}(U)$-module with connection. Then* $\operatorname{rig} M_U = \operatorname{rig} {}^F M_U$. *In particular, M_U is rigid if and only if ${}^F M_U$ is rigid.*

2.d. The algorithm of Katz for rigid connections, after Arinkin. The purpose of this section is to prove:

***Theorem 2.11* (Arinkin[1] [?]).** *Let M_U be a rigid irreducible free $\mathscr{O}(U)$-module with connection. Then there exists a choice of the Katz algorithm which stops at rank one.*

***Lemma 2.12*.** *The Katz transformations preserve the index of rigidity.*

Proof. For the Laplace transform, this is Theorem 2.10. For the tensor with a rank-one $\mathscr{O}(U)$-module with connection, note that $\operatorname{End}(M_U \otimes L_U) = \operatorname{End}(M_U)$. Lastly, for a Möbius transformation, one notices that $\operatorname{End}(\mu^* M_U) = \mu^* \operatorname{End}(M_U)$ and that, for any N_U, $\chi(\mu^* N_U) = \chi(N_U)$ and

$$\sum_{z_o \in \mu^{-1}(Z \cup \{\infty\})} h_{z_o}(\mu^* N_U) = \sum_{z_o \in \mu^{-1}(Z \cup \{\infty\})} h_{\mu(z_o)}(N_U) = \sum_{z_o \in Z \cup \{\infty\}} h_{z_o}(N_U). \qquad \square$$

Therefore, the theorem is a consequence of the following proposition.

***Proposition 2.13* (Arinkin [?]).** *Let M_U be a rigid irreducible free $\mathscr{O}(U)$-module with connection. If* $\operatorname{rk} M_U > 1$, *there exists a choice of the Katz algorithm which strictly decreases the rank.*

Sketch of proof. I will not try to give a detailed proof, that one can find in [?]. I will only indicate the role of the rigidity assumption in the proof.

(1) There is a Euler-Poincaré formula (analogue of the Grothendieck-Ogg-Shafarevich formula) expressing the rigidity index. One introduces, for any $x \in \mathbb{P}^1$, the following number, also called the number of vanishing cycles of $j_{!*} M_U$ at x, where now j denotes the inclusion $U \hookrightarrow \mathbb{P}^1$: we set

$$\delta_x(M_U) = \operatorname{rk} M_U - h_x(M_U) + \operatorname{irr}_x(M_U),$$

1. A similar result has been previously (2006) proved by Deligne in an unpublished letter to N. Katz.

where $\mathrm{irr}_x(M_U)$ (the irregularity number of M_U at x) is defined in terms of the decomposition of $\widehat{M}_{U,x}$:

$$\mathrm{irr}_x(M_U) = \sum_{\lambda>0} \lambda \,\mathrm{rk}\, \widehat{M}_{U,x}^{(\lambda)}.$$

Then

$$\mathrm{rig}\, M_U = 2(\mathrm{rk}\, M_U)^2 - \sum_{x\in\mathbb{P}^1\smallsetminus U} \delta_x(\mathrm{End}(M_U)),$$

(where 2 is the topological Euler characteristic of $\mathbb{P}^1$). If M_U is rigid, we thus have

$$(*) \qquad \sum_{x\in\mathbb{P}^1\smallsetminus U} \delta_x(\mathrm{End}(M_U)) < 2(\mathrm{rk}\, M_U)^2.$$

(2) For each $x \in \mathbb{P}^1 \smallsetminus U$, choose a local coordinate t (if $x = z_o \in \mathbb{A}^1$, choose $t = z - z_o$, and if $x = \infty$, choose $t = 1/z$). One shows that there exists an *irreducible* $\mathbb{C}(\!(t)\!)$-vector space with connection $\widehat{L}_x$ such that

$$\delta_x(\mathrm{End}(M_U)) \geqslant \frac{\mathrm{rk}\, M_U}{\mathrm{rk}\, \widehat{L}_x} \cdot \delta_x(\widehat{L}_x \otimes \widehat{M}_{U,x}),$$

and that $\mathrm{rk}\, \widehat{L}_x \geqslant 2$ implies that the RHS is $\geqslant (\mathrm{rk}\, M_U)^2 = \mathrm{rk}\,\mathrm{End}(\widehat{M}_U)$. Then $(*)$ implies that there is *at most one* $x \in \mathbb{P}^1 \smallsetminus U$ such that $\mathrm{rk}\, \widehat{L}_x \geqslant 2$. One can assume that $\infty \notin U$ and, if such a x exists, one can assume that it is equal to ∞ after applying a suitable Möbius transformation.

(3) If $\mathrm{rk}\, \widehat{L}_x = 1$ for every $x \in \mathbb{P}^1 \smallsetminus U$, there exists L_U such that $\widehat{L}_{U,x} = \widehat{L}_x$ for $x \in \mathbb{A}^1 \smallsetminus U$ and $\widehat{L}_{U,\infty}$ is obtained from $\widehat{L}_\infty$ by adding $\alpha\, dt/t$ to the connection (This is the Cousin problem: given polar parts of meromorphic 1-forms at $x \in \mathbb{P}^1 \smallsetminus U$, find a global meromorphic 1-form giving rise to these polar parts; the sum of residues should be zero in order to have a solution, which explains the introduction of $\alpha\, dt/t$). Then an explicit formula for the rank of the middle convolution implies that $(L_U \otimes M_U) \star K_\alpha$ has rank strictly smaller than that of M_U, as a consequence of $(*)$, where $K_\alpha = (\mathbb{C}[\zeta, \zeta^{-1}], d + \alpha\, d\zeta/\zeta)$.

(4) If $\mathrm{rk}\, \widehat{L}_x = 1$ for every $x \in \mathbb{A}^1 \smallsetminus U$ and $\mathrm{rk}\, \widehat{L}_\infty \geqslant 2$, one can find a suitable rank-one L_U such that $\widehat{L}_{U,x} = \widehat{L}_x$ for $x \in \mathbb{A}^1 \smallsetminus U$, such that ${}^F(L_U \otimes M_U)$ has rank smaller than $\mathrm{rk}\, M_U$. □

Remark 2.14. From the point of view of the physicist or of the analyst, the Fourier or Laplace transformation relates two different worlds, the space of positions and the space of (spacial) frequencies. The variables z and ζ reflect two very different ways of encoding a function. This justifies the different names for the variables.

The point of view of the algebraist is different, since the Laplace transformation is regarded as an automorphism of the Weyl algebra $\mathbb{C}[z]\langle\partial_z\rangle$, inducing an equivalence from the category of holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-modules to itself. This point of view is present in the approach of Arinkin to Katz transformations, since the Laplace transformation is one of them, together with Möbius transformations for instance.

Previous works on the Katz algorithm did not take this point of view however. Indeed, these works only considered free $\mathscr{O}(U)$-modules with connection having slope 0 at each point of $\mathbb{P}^1 \smallsetminus U$, and more specifically the rigid irreducible ones. This property is not preserved by the Laplace transformation, and the main transformation was then middle convolution, which can be expressed without using the Laplace transformation (in the same way as the usual convolution of L^1 functions can be defined without using Fourier transform).

Considering the whole category of holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-modules with no restriction on slopes makes the Laplace transformation completely symmetric, and it can be used as a fundamental transformation.

LECTURE 3

THE DETERMINANT OF THE DE RHAM COHOMOLOGY

Summary. This lecture explains the role of the Laplace transformation in proving a product formula for the determinant of a period matrix. This can be regarded as an analogue of the product formula proved by Laumon [**?**] in the ℓ-adic setting, and the method is similar. We will restrict to showing the mechanism providing the product formula, and we will not prove (not even express) such a formula, for which we refer to [**?, ?**].

3.a. Statement of the problem. Let M be a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module with singularity set $Z \cup \{\infty\}$. The de Rham complex is the complex

$$M \xrightarrow{\partial_z} M$$

where the second term M is in degree 0. The de Rham cohomology of M is the cohomology of this complex, that is,

$$H^{-1}_{\mathrm{DR}}(M) = \operatorname{Ker}\partial_z, \quad H^0_{\mathrm{DR}}(M) = \operatorname{Coker}\partial_z.$$

These spaces are both finite dimensional $\mathbb{C}$-vector spaces (this is a theorem). If M is defined over a subfield K of $\mathbb{C}$ (for example if $M = \mathbb{C}[z]\langle\partial_z\rangle/(P)$ where P is a differential operator having coefficients in $K[z]$), then the de Rham cohomology is also defined over K.

On the other hand, in many interesting cases, the de Rham cohomology is also equipped with a natural $\mathbb{Q}$-structure.

Example 3.1. Let $T_1, \dots, T_r$ be r elements of $\mathrm{GL}_n(\mathbb{Q})$ ($r \geqslant 1$, $n \geqslant 1$). Let us fix r points $z_1, \dots, z_r$ in $\mathbb{A}^1$ and let us set $U = \mathbb{A}^1 \smallsetminus \{z_1, \dots, z_r\}$. The matrices $T_1, \dots, T_r$ determine a representation of the fundamental group $\pi_1(U)$ when we identify it with the free group on r elements, that we can choose as loops γ_i encircling once z_i and not having index zero with respect to z_j, $j \neq i = 1, \dots, r$. This representation gives rise to a local system $\mathscr{L}_{\mathbb{Q}}$ of n-dimensional k-vector spaces on U. The cohomology

$H^k(U, \mathscr{L}_{\mathbb{Q}})$ consists of finite-dimensional $\mathbb{Q}$-vector spaces. It is possibly non-zero for $k = 0, 1$ at most. A Euler-Poincaré formula gives

$$\chi(U, \mathscr{L}_{\mathbb{Q}}) = \dim H^0 - \dim H^1 = \chi(U) \operatorname{rk} \mathscr{L}_{\mathbb{Q}} = (1-r)n.$$

The Riemann-Hilbert correspondence produces a free $\mathscr{O}(U)$-module M_U of rank n with a connection ∂_z having slope 0 at $z_1, \dots, z_r$ and at ∞. It determines a holonomic $\mathbb{C}[z]\langle\partial_z\rangle$-module $M = j_* M_U$ ($j : U \hookrightarrow \mathbb{A}^1$), such that $H^{k-1}_{\mathrm{DR}}(M) \simeq H^k(U, \mathscr{L}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$.

Given a $\mathbb{Q}$-basis and a K-basis of $H^k_{\mathrm{DR}}(M)$ ($k = -1, 0$), the determinant of the base change $\mathcal{P}_j$ between these bases is a complex number, independent of the choice of the bases when considered modulo $\mathbb{Q}^\times$ and $K^\times$. The *period determinant* is

$$\frac{\det \mathcal{P}_0}{\det \mathcal{P}_{-1}} \in \mathbb{C}^\times / \mathbb{Q}^\times K^\times.$$

Example 3.2. Let α be a complex number. We define $M = \mathbb{C}[z, z^{-1}]$ with the action of ∂_z defined by

$$\partial_z(1) = -1 + \alpha/z, \quad \partial_z(p(z)) = p'(z) + p(z)\partial_z(1) \text{ if } p(z) \in \mathbb{C}[z, z^{-1}].$$

Then M is defined over $K = \mathbb{Q}(\alpha)$. The singularity set of M reduces to $\{0, \infty\}$ (one can check that M has slope zero at $z = 0$ and slope one at $z = \infty$). The de Rham cohomology $H^{-1}_{\mathrm{DR}}(M)$ is zero since the equation

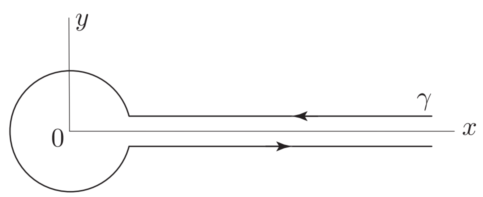
$$\frac{p'(z)}{p(z)} = 1 - \alpha/z$$

has no solution in $\mathbb{C}[z, z^{-1}]$ (easily checked). The $\mathbb{C}$-linear map $\partial_z : M \to M$ is expressed on the $\mathbb{C}$-basis $(z^k)_{k\in\mathbb{Z}}$ by the formula $z^k \mapsto -z^k + (\alpha + k)z^{k-1}$. Assume for simplicity that $\alpha \notin \mathbb{Z}$. Then $\dim \operatorname{Coker} \partial_z = 1$, since each z^k is equivalent, modulo $\operatorname{Im} \partial_z$, to a constant (e.g. if $k \geqslant 1$, $z^k \sim \Gamma(\alpha + k + 1)/\Gamma(\alpha + 1)$). This computation can be done over K, and we find $H^0_{\mathrm{DR}}(M_K) \simeq K$.

We will now define an isomorphism $H^0_{\mathrm{DR}}(M) \xrightarrow{\sim} \mathbb{C}$ (the period isomorphism) and the $\mathbb{Q}$-structure on $H^0_{\mathrm{DR}}(M)$ will be, by definition, the $\mathbb{Q}$-subspace corresponding to $\mathbb{Q} \subset \mathbb{C}$. We first define a $\mathbb{C}$-linear map $M \to \mathbb{C}$:

$$p(z) \longmapsto \lim_{\gamma} \int_\gamma e^{-z} z^\alpha p(z)\, dz,$$

where γ is a path as in the picture below, and the limit is taken with respect to the radius of the circle, which tends to 0. Moreover, we fix a determination of the logarithm along this path to define z^α.



The first point to check is that any element of the form $\partial_z(p(z))$ is mapped to 0. Notice that, working analytically, $\partial_z(p(z))$ can be written as

$$e^z z^{-\alpha} \circ \frac{\partial}{\partial z}\big(e^{-z} z^\alpha p(z)\big),$$

so that the integral is written as

$$\int_\gamma \frac{\partial}{\partial z}(e^{-z} z^\alpha p(z))\, dz,$$

and is zero by integration by parts, since e^{-z} is exponentially decreasing at infinity on γ. The image of 1 is equal to

$$\lim_\gamma \int_\gamma e^{-z} z^\alpha dz = (e^{2\pi i\alpha} - 1)\Gamma(\alpha + 1),$$

if we assume for simplicity that $\operatorname{Re}\alpha > 0$. In other words,

$$1_K = (e^{2\pi i\alpha} - 1)\Gamma(\alpha + 1) \cdot 1_{\mathbb{Q}}.$$

In this product formula, we can regard $(e^{2\pi i\alpha} - 1)$ as produced by the singularity of M at 0, while $\Gamma(\alpha + 1)$ is produced by the singularity at infinity.

Question 3.3. Is it possible to express the period determinant in terms of local data of M at its singularity set $Z \cup \{\infty\}$?

One possible proof, developed in [**?**], consists in expressing the K-vector space $\det H^0_{\mathrm{DR}}(M) \otimes (\det H^{-1}_{\mathrm{DR}}(M))^{-1}$ (the *de Rham determinant*) in terms of local data, and similarly for the corresponding $\mathbb{Q}$-vector space, also called the *Betti determinant*, and then compute the local multiplicative factors. In this lecture, I will give a hint for the computation of the de Rham determinant and explain the use of the Laplace transform.

3.b. The local de Rham determinant and the local de Rham data. The purpose of this section is to introduce the local terms in the desired product formula. For M as above, we have defined the de Rham (global) determinant as the one-dimensional vector space $\det H^0_{\mathrm{DR}}(M) \otimes (\det H^{-1}_{\mathrm{DR}}(M))^{-1}$.

For each singular point z_i of M, the formalized module $\widehat{M}_{z_i}$ has a slope decomposition $\widehat{M}_{z_i} = \bigoplus_{\lambda\in\mathbb{Q}} \widehat{M}^{(\lambda)}_{z_i}$ (cf. Lecture 1). The term of slope 0 is determined by a quiver

$$E_i \underset{v_i}{\overset{c_i}{\rightleftarrows}} F_i$$

where $\mathrm{Id} + v_i c_i$ and $\mathrm{Id} + c_i v_i$ are invertible. On the other hand, any term of positive slope is decomposed as the direct sum of elementary indecomposable modules $\mathrm{El}(\rho, \varphi, R_{i,\rho,\varphi})$, and each $R_{i,\rho,\varphi}$ is written as $E_{i,\rho,\varphi} \otimes_{\mathbb{C}} \mathbb{C}(\!(u)\!)$, where u is the ramified coordinate such that $\rho(u) = z - z_i$, with a suitable connection of slope 0.

We have a similar decomposition for M at infinity, where the term F_∞ is identified with E_∞. In order to simplify the argument, I will assume that M_∞ has slope 0.

Question 3.3 **(de Rham aspect made precise).** Express the global de Rham determinant in terms of the various $\det F_i$, $\det E_{i,\rho,\varphi}$, $\det E_\infty$.

3.c. The role of the Laplace transform. The main problem is to make explicit a relation between the global and the local determinants. Another related problem is to compute the de Rham determinant in terms of a morphism between finite dimensional vector spaces.

Firstly, the de Rham complex $M \xrightarrow{\partial_z} M$ can be written as

$$ {}^F\!M \xrightarrow{\ \zeta\ } {}^F\!M. $$

Now, it is classical that this complex is quasi-isomorphic to

$$ \widehat{{}^F\!M}_0 \xrightarrow{\ \zeta\ } \widehat{{}^F\!M}_0 $$

because of the flatness of $\mathbb{C}[\![\zeta]\!]$ over $\mathbb{C}[\zeta]$. Even without the simplifying assumption on M_∞, one can use the decomposition with respect to slopes and remark that ζ is bijective on the part with positive slope. The complex is then computed only in terms of the quiver attached to $\widehat{{}^F\!M}{}_0^{(0)}$, that is,

$$ {}^F\!F_0 \xrightarrow{\ {}^F\!v\ } {}^F\!E_0. $$

Both ${}^F\!E_0$ and ${}^F\!F_0$ are finite dimensional, so we finally find

$$ \det H^0_{\mathrm{DR}}(M) \otimes (\det H^{-1}_{\mathrm{DR}}(M))^{-1} = \det {}^F\!E_0 \otimes (\det {}^F\!F_0)^{-1}. $$

We now use the simplifying assumption. The inverse local Laplace transform $(\mathscr{F}^{(\infty,0)})^{-1}$ identifies ${}^F\!F_0$ with E_∞ (cf. Theorem 1.11(1)), and therefore identifies $\det {}^F\!F_0$ with $\det E_\infty$.

It remains now to express $\det {}^F\!E_0$ in terms of the local data $\det F_i$, $\det E_{i,\rho,\varphi}$. From the stationary phase formula, the local data can be recovered at $\zeta = \infty$, while $\det {}^F\!E_0$ is an object defined at $\zeta = 0$. The simplifying assumption above implies that the singularities of ${}^F\!M$ are $\zeta = 0$ and $\zeta = \infty$, because, according to the (inverse) stationary phase formula, other singularities would be caused by the part of slope 1 at $z = \infty$, which is assumed to be zero. In particular, $\mathbb{C}[\zeta,\zeta^{-1}] \otimes_{\mathbb{C}[\zeta]} {}^F\!M$ is a free $\mathbb{C}[\zeta,\zeta^{-1}]$-module of finite rank.

Let us notice that $\mathbb{C}(\!(\zeta)\!) \otimes_{\mathbb{C}[\zeta]} {}^F\!M$ is identified with $\mathbb{C}(\!(\zeta)\!) \otimes_{\mathbb{C}} {}^F\!E_0$. It follows that there is a unique $\mathbb{C}[\zeta]$-submodule E of M such that

$$ \mathbb{C}[\![\zeta]\!] \otimes_{\mathbb{C}[\zeta]} {}^F\!E = \mathbb{C}[\![\zeta]\!] \otimes_{\mathbb{C}[\zeta]} \otimes_{\mathbb{C}} {}^F\!E_0, $$

and we have $\mathbb{C}[\zeta,\zeta^{-1}] \otimes_{\mathbb{C}[\zeta]} {}^F\!E = \mathbb{C}[\zeta,\zeta^{-1}] \otimes_{\mathbb{C}[\zeta]} {}^F\!M = {}^F\!M_\infty$. Moreover, ${}^F\!E_0$ is the fibre of E at $\zeta = 0$.

Similarly, setting $\zeta' = 1/\zeta$, $\widehat{{}^F\!M}_\infty$ is decomposed as

$$\bigoplus_i \Big[\big(\mathbb{C}(\!(\zeta')\!) \otimes_{\mathbb{C}} F_i \big) \oplus \bigoplus_{\rho,\varphi} \mathrm{El}({}^F\!\rho, {}^F\!\varphi, R_{i,\rho,\varphi} \otimes L_q) \Big],$$

according to the stationary phase formula, where $R_{i,\rho,\varphi} = \mathbb{C}(\!(u)\!) \otimes_{\mathbb{C}} E_{i,\rho,\varphi}$, and the natural differential on each term indexed by i is twisted by $e^{z_i/\zeta'}$. As a consequence, there exists a unique free $\mathbb{C}[\zeta']$-module ${}^F\!E'$ such that

$$\mathbb{C}[\![\zeta']\!] \otimes_{\mathbb{C}[\zeta']} {}^F\!E' = \bigoplus_i \Big[\big(\mathbb{C}[\![\zeta']\!] \otimes_{\mathbb{C}} F_i \big) \oplus \bigoplus_{\rho,\varphi} {}^F\!\rho_* (\mathbb{C}[\![u]\!] \otimes_{\mathbb{C}} E_{i,\rho,\varphi}) \Big].$$

The fibre of ${}^F\!E'$ at ∞ (that is, at $\zeta' = 0$), is then identified with

$$\bigoplus_i \Big[F_i \oplus \bigoplus_{\rho,\varphi} (E_{i,\rho,\varphi})^{p+q} \Big],$$

where p is the valuation of ρ and q is the order of the pole of φ. We also notice that $\mathbb{C}[\zeta', \zeta'^{-1}] \otimes_{\mathbb{C}[\zeta']} {}^F\!E' = {}^F\!M_\infty$.

As a consequence, both modules ${}^F\!E$ and ${}^F\!E'$ glue as a vector bundle on $\mathbb{P}^1$, that we denote by $\widetilde{{}^F\!E}$. This bundle is not trivial in general. However, its determinant is a rank-one bundle on $\mathbb{P}^1$, and up to a simple twist, is trivial. As a consequence, omitting the twist in the notation, the fibre of $\det \widetilde{{}^F\!E}$ at $\zeta = 0$, which is nothing but $\det {}^F\!E_0$, is identified canonically with the fiber of $\det \widetilde{{}^F\!E}$ at $\zeta = \infty$, which is

$$\bigotimes_i \Big[\det F_i \otimes \bigotimes_{\rho,\varphi} (\det E_{i,\rho,\varphi})^{\otimes p+q} \Big].$$

Putting everything together, we find

$$\det H^0_{\mathrm{DR}}(M) \otimes (\det H^{-1}_{\mathrm{DR}}(M))^{-1} = \bigotimes_i \Big[\det F_i \otimes \bigotimes_{\rho,\varphi} (\det E_{i,\rho,\varphi})^{\otimes p+q} \Big] \otimes (\det E_\infty)^{-1},$$

which is the expected product formula.