Claude Sabbah

VANISHING CYCLES OF POLYNOMIAL MAPS (TOPOLOGY, HODGE STRUCTURE, $\mathscr{D}$-MODULES)

LECTURE NOTES (NICE, NOVEMBER 2008)

PRELIMINARY VERSION

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Abstract. In these lectures, I will emphasize the global aspect (in the affine case) of the theory of singularities of holomorphic functions. I will focus on the case where the function is tame, a property analogous to the property for a germ of holomorphic function of having an isolated singularity. I will explain some recent results on the "vanishing cycles at infinity" of such a function as well as some applications to the supersymmetric index introduced by the physicists Cecotti and Vafa in 1991.

Résumé **(Les cycles évanescents des applications polynomiales (topologie, structure de Hodge, D-modules))**

Dans ce cours, je mettrai l'accent sur l'aspect global (dans le cadre affine) de la théorie des singularités de fonctions holomorphes. J'insisterai sur la propriété analogue à celle de singularité isolée d'une fonction holomorphe (fonction modérée) et j'expliquerai quelques résultats concernant les « cycles évanescents à l'infini » d'une telle fonction. J'indiquerai aussi quelques résultats récents sur l'indice supersymétrique, introduit par les physiciens Cecotti et Vafa en 1991.

CONTENTS

LECTURE 1

TOPOLOGY OF REGULAR FUNCTIONS ON AN AFFINE MANIFOLD

1.1. Introduction

The main guiding line of these notes (but not the only one) is summarized in the comparison table below. It consists in traducing properties known in the theory of germs of holomorphic functions (Milnor fibration, vanishing cycles, local Hodge theory, etc.) as similar properties of regular functions on an affine manifold. The goal is to input in this setting the new properties due to the fact that the function is algebraic. This setting is indeed more suitable to have comparison with (or to apply techniques of) arithmetics, for instance.

Comparison table

Germ of holomorphic function	Regular function
Germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$	Regular function $f : U \to \mathbb{A}^1$
Milnor ball B_ε (Stein)	U smooth affine over $\mathbb{C}$
Central fibre $f^{-1}(0)$	"fibre at infinity" $f^{-1}(\infty)$ not specified
$\mathrm{Crit}(f) \subset f^{-1}(0)$	$\mathrm{Crit}(f) \subset \bigcup_i f^{-1}(t_i)$
Milnor fibre $B_\varepsilon \cap f^{-1}(\eta)$, $0 \ll \eta \ll \varepsilon$	$f^{-1}(t)$, $\lvert t\rvert \gg 0$
Vanishing cocycles $H^*(B_\varepsilon, B_\varepsilon \cap f^{-1}(\eta))$	$H^*(U, f^{-1}(t))$
Vanishing cycles	Lefschetz thimbles
$\left\{\begin{array}{l}\text{Transversality with } \partial B_\varepsilon \\ \text{and isolated singularity}\end{array}\right\}$	Various notions of tameness
Gauss-Manin system	Algebraic Gauss-Manin system
Microlocalization	Fourier-Laplace transform
Regularity of the microlocal GM system	Irregularity of the Laplace GM system
(Microlocal) Brieskorn lattice	Brieskorn lattice
Microlocal duality	Duality for the Laplace transform
Order-two pole of the connection	Order-two pole of the connection
Varchenko-Steenbrink Hodge spectrum	Spectrum at infinity
———————————	Semi-simplicity of the Laplace GM syst.
Germ of variation of twistor structure	Variation of twistor structure

1.2. Setting and examples

Throughout these notes, the notation will be as follows:

• U is a smooth affine complex variety of dimension $n+1$, and $\mathscr{O}(U)$ is its ring of regular functions,

• $f \in \mathscr{O}(U)$ is also regarded as a map $f : U \to \mathbb{A}^1$ (I always assume that f is not constant),

• $F : X \to \mathbb{A}^1$ is some partial compactification of f, i.e., X is quasi-projective (not necessarily smooth) and contains U as a Zariski dense open subset, and $F_{|U} = f$; for example, if Y is any projective closure of U, one can choose for X the closure of the graph of f in $Y \times \mathbb{A}^1$ and for F the second projection.

Examples 1.2.1

(1) $\widetilde{X}$ is a compact Riemann surface (considered as smooth complex algebraic curve), $\widetilde{F} : \widetilde{X} \to \mathbb{P}^1$ is a meromorphic function on $\widetilde{X}$, $U = \widetilde{X} \smallsetminus \widetilde{F}^{-1}(\infty)$ is the complement of the poles of $\widetilde{F}$ and f is the restriction of $\widetilde{F}$ to U. Then $X = U$, $F = f$ and, in such a case, f is *proper* (more precisely, finite).

(2) $U = \mathbb{C}^{n+1}$ and f is a complex polynomial map.

(3) $U = (\mathbb{C}^*)^{n+1}$ and f is a Laurent polynomial,

(4) U is the complement of an affine hypersurface D in $\mathbb{C}^{n+1}$, e.g. an arrangement of hyperplanes, and f is the restriction of a polynomial to U.

1.3. Topology of the general fibre of f

The critical set $\mathrm{Crit}(f) = \{x \in U \mid df(x) = 0\}$ is a closed algebraic subset of U contained in a finite number of fibres $f^{-1}(t_j)$ of *critical values* t_j, whose set is denoted by C_f. If t is not a critical value, that is, $t \notin C_f$, then $f^{-1}(t)$ is a smooth affine variety of (complex) dimension n. Such a manifold is known (**[1]**) to have the homotopy type of a CW-complex of dimension $\leqslant n$. In particular, $H_k(f^{-1}(t), \mathbb{Z}) = 0$ for $k \geqslant n+1$, and the same holds for H^k. By Poincaré duality, we get $H^k_c(f^{-1}(t), \mathbb{Z}) = 0$ for $k \leqslant n-1$. The long exact sequence

$$\cdots \longrightarrow H^k_c(f^{-1}(t), \mathbb{Z}) \longrightarrow H^k(f^{-1}(t), \mathbb{Z}) \longrightarrow C^k_t \longrightarrow H^{k+1}_c(f^{-1}(t), \mathbb{Z}) \longrightarrow \cdots$$

reduces thus to the exact sequence

$$\begin{aligned} 0 \longrightarrow H^{n-1}(f^{-1}(t), \mathbb{Z}) \longrightarrow C^{n-1}_t \longrightarrow H^n_c(f^{-1}(t), \mathbb{Z}) \longrightarrow H^n(f^{-1}(t), \mathbb{Z}) \\ \longrightarrow C^n_t \longrightarrow H^{n+1}_c(f^{-1}(t), \mathbb{Z}) \longrightarrow 0 \end{aligned}$$

and isomorphisms

$$C^k_t \simeq \begin{cases} H^k(f^{-1}(t), \mathbb{Z}) & \text{if } k \leqslant n-2, \\ H^k_c(f^{-1}(t), \mathbb{Z}) & \text{if } k \geqslant n+1. \end{cases}$$

1.4. Topology of the map f

The first basic result, which is a consequence of Thom's stratification theory of maps to $F : X \to \mathbb{C}$ is that there exists a finite set $B_f \subset \mathbb{C}$ (called the *bifurcation set*, cf. [**30**]) such that $f : U \smallsetminus f^{-1}(B_f) \to \mathbb{C} \smallsetminus B_f$ is a C^∞ locally trivial fibration. As a consequence, as $\mathbb{C} \smallsetminus B_f$ is connected, the fibres $f^{-1}(t)$ ($t \in \mathbb{C} \smallsetminus B_f$) are all diffeomorphic to a given one. The set C_f of critical values is contained in B_f (this is not completely obvious), but B_f can be bigger than C_f. The difference[(1)] comes from the critical points of F (in a suitable sense) which are not contained in U.

The cohomology sheaves $\boldsymbol{R}^k f_* \mathbb{Z}_U$ ($k = 1, \dots$) are constructible sheaves of abelian groups on $\mathbb{A}^1$. When restricted to $U \smallsetminus B_f$, each $\boldsymbol{R}^k f_* \mathbb{Z}_U$ is a locally constant sheaf $\mathscr{H}_f^k$ with germ $(\boldsymbol{R}^k f_* \mathbb{Z}_U)_t = H^k(f^{-1}(t), \mathbb{Z})$ for any $t \in U \smallsetminus B_f$. In particular, $\boldsymbol{R}^k f_* \mathbb{Z}_U$ is supported on B_f for $k > n$ (but such a sheaf can be nonzero). Note also that, if $t \in B_f$, $(\boldsymbol{R}^k f_* \mathbb{Z}_U)_t = H^k(f^{-1}(D_t), \mathbb{Z})$, where D_t is a sufficiently small open neighbourhood of t, but as f is not proper, this can be distinct from $H^k(f^{-1}(t), \mathbb{Z})$.

On the other hand, the direct image with proper support $\boldsymbol{R}^k f_! \mathbb{Z}_U$ behaves well by base change. There is a natural morphism of complexes $\boldsymbol{R} f_! \mathbb{Z}_U \to \boldsymbol{R} f_* \mathbb{Z}_U$ whose cone $\mathscr{C}$ fits into a long exact sequence of cohomology sheaves

$$\cdots \longrightarrow \boldsymbol{R}^k f_! \mathbb{Z}_U \longrightarrow \boldsymbol{R}^k f_* \mathbb{Z}_U \longrightarrow \mathscr{C}^k \longrightarrow \boldsymbol{R}^{k+1} f_! \mathbb{Z}_U \longrightarrow \cdots$$

***Definition 1.4.1* (Weak tameness).** The map f is *weakly tame* if each cohomology sheaf $\mathscr{C}^k_{\mathbb{Q}}$ is a (locally) constant sheaf of $\mathbb{Q}$-vector spaces on $\mathbb{A}^1$.

This property has been considered by N. Katz [**21**, Th. 14.13.3].

***Example 1.4.2*.** Trivially, Example 1.2.1(1) is weakly tame, and the cone $\mathscr{C}$ is not only constant, but even zero.

***Remark 1.4.3*.** The cone $\mathscr{C}$ can be obtained in the following way. Let us choose an embedding

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & X \arrow[d, "F"] & X \smallsetminus U \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

where X is a locally compact topological space and F is proper. We have an exact triangle

$$\boldsymbol{R} j_! \mathbb{Q}_U \longrightarrow \boldsymbol{R} j_* \mathbb{Q}_U \longrightarrow \boldsymbol{R} i_* i^{-1}(\boldsymbol{R} j_* \mathbb{Q}_U) \xrightarrow{+1}$$

and $\mathscr{C}$ is identified with $\boldsymbol{R} g_* i^{-1}(\boldsymbol{R} j_* \mathbb{Q}_U)$. While $\mathscr{C}$ does not depend on the previous embedding, $\boldsymbol{R} i_* i^{-1}(\boldsymbol{R} j_* \mathbb{Q}_U)$ possibly depends on it.

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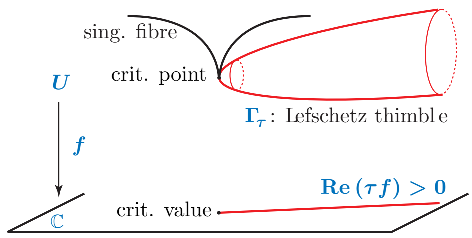
(1) I will not be much precise on this matter, as the main subject of these lectures concerns tame functions, for which such atypical values do not exist. See e.g., [**42**].

1.5. Why is Laplace transform useful?

Considering the oscillatory integrals attached to f is a classical subject. This means considering integrals

$$\int_{\Gamma_\tau} e^{-\tau f}\omega,$$

where ω is an algebraic differential form of degree $k+1$ on U and Γ_τ is a “Lefschetz thimble” of dimension $k+1$ attached to a critical point.



Such integrals satisfy linear differential equations with respect to the variable τ, and this defines the Laplace Gauss-Manin system of f. Classically, one relies the behaviour of such integrals when $\tau \to \infty$ with the critical values of f: this is called the stationary phase approximation. More geometrically, this makes a relation between the *vanishing cycles* of f at its critical values at finite distance and the *nearby cycles* of the Laplace Gauss-Manin system when $\tau \to \infty$.

On the other hand, one can consider the behaviour of such integrals when $\tau \to 0$. This is not trivial as large values of f have to be taken into account. We have the Laplace dual statement of the preceding one: *nearby cycles* of f when $f \to \infty$ are related with *vanishing cycles* of the Laplace Gauss-Manin system at $\tau = 0$.

Correspondingly, *vanishing cycles* of f at infinity (as defined in the comparison table of §1.1) are related to *nearby cycles* of the Laplace Gauss-Manin system at $\tau = 0$.

The Laplace Gauss-Manin system has singularities at $\tau = 0$ and $\tau = \infty$ only, and the interplay between both will be the main theme of these lectures. This is another way of encoding part of the geometric information contained in the map f, e.g., the monodromy representation on the cohomology of general fibres. It emphasizes the vanishing cycles of f, while the usual Gauss-Manin connection emphasizes the nearby cycles of f.

1.6. Topological Laplace transform

We look for a natural procedure to get rid of the constant sheaves which appear in the definition of weak tameness. The topological localized Laplace transform will do the job. In fact, the result of Proposition 1.6.1(1) below is similar to the well-known property that the Fourier transform of the constant function is a Dirac δ-function supported at the origin, hence vanishes identically away from the origin.

Let $\mathscr{F}$ be a constructible complex on $\mathbb{A}^1$. The (localized) Laplace transform (with kernel $e^{-t\tau}$) will be a complex $\widehat{\mathscr{F}}$ on the punctured affine line $\widehat{\mathbb{A}}^{1*}$ with coordinate τ. For $\tau_o \neq 0$, the germ $\widehat{\mathscr{F}}_{\tau_o}$ is the complex of global sections $\boldsymbol{R}\Gamma_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathscr{F})$ with support in the family Φ_{τ_o} defined as follows: a closed set in $\mathbb{A}^1$ belongs to Φ_{τ_o} if it is contained in an open angle of $\mathbb{A}^1$ of opening strictly less than π and with internal bisector $\mathbb{R}_+ \cdot (-\overline{\tau}_o)$. In other words, let $e : \widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ be the real oriented blowing-up of $\infty \in \mathbb{P}^1$ ($\widetilde{\mathbb{P}}^1$ is topologically a disc) and let $L^+_{\tau_o}$ be the closed subset $\{\operatorname{Re}(\tau_o e^{i \arg t}) \geqslant 0\}$ of $e^{-1}(\infty) \simeq S^1$. Then a closed set $F \subset \mathbb{A}^1$ belongs to Φ_{τ_o} iff its closure in $\widetilde{\mathbb{P}}^1$ does not meet $L^+_{\tau_o}$.

The definition of $\widehat{\mathscr{F}}$ can be given sheaf-theoretically (cf. [**24**]). We also denote by e the induced map $\widetilde{\mathbb{P}}^1 \times \widehat{\mathbb{A}}^{1*} \to \mathbb{P}^1 \times \widehat{\mathbb{A}}^{1*}$, by $L^+ \subset e^{-1}(\infty) \simeq S^1 \times \widehat{\mathbb{A}}^{1*}$ the closed subset $\operatorname{Re}(\tau e^{i \arg t}) \geqslant 0$ and by L^- its complement in $\widetilde{\mathbb{P}}^1 \times \widehat{\mathbb{A}}^{1*}$. We will consider the commutative diagram

$$\begin{tikzcd}
\mathbb{A}^1 \times \widehat{\mathbb{A}}^{1*} \arrow[r, hook, "\alpha"] \arrow[rr, bend left, "\widetilde{\ell}"] & L^- \arrow[r, hook, "\beta"] & \widetilde{\mathbb{P}}^1 \times \widehat{\mathbb{A}}^{1*} \arrow[d, "e"] \arrow[ddl, "\widetilde{p}"'] \arrow[ddr, "\widetilde{q}"] & \\
 & & \mathbb{P}^1 \times \widehat{\mathbb{A}}^{1*} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
 & \mathbb{P}^1 & & \widehat{\mathbb{A}}^{1*}
\end{tikzcd}$$

We then set

$$\widehat{\mathscr{F}} := \boldsymbol{R}\widetilde{q}_* \big[\beta_! \boldsymbol{R}\alpha_* p^{-1} \mathscr{F}\big][1],$$

Proposition 1.6.1

(1) *If $\mathscr{F}$ is a (locally) constant sheaf on $\mathbb{A}^1$, then $\widehat{\mathscr{F}} = 0$.*

(2) *If $\mathscr{F}$ is a perverse sheaf on $\mathbb{A}^1$, then $\widehat{\mathscr{F}}$ is a local system (up to a shift) whose germ at τ_o is $\mathbb{H}^0_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathscr{F})$, which has rank equal to $\dim \mathbb{H}^0(\mathbb{A}^1, t; \mathscr{F})$.*

Proof. For the first point, one has to show that $H^*_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathbb{Q}) = 0$. One uses the exact sequence

$$\cdots \longrightarrow H^k_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathbb{Q}) \longrightarrow H^k(\widetilde{\mathbb{P}}^1, \mathbb{Q}) \longrightarrow H^k(L^+, \mathbb{Q}) \longrightarrow H^{k+1}_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathbb{Q}) \cdots$$

and the fact that the restriction morphism $H^0(\widetilde{\mathbb{P}}^1, \mathbb{Q}) \to H^0(L^+, \mathbb{Q})$ is an isomorphism.

For the second point, one reduces to the case where $\mathscr{F} = j_* \mathscr{V}[1]$, with $\mathscr{V}$ a local system on $\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$ (if $\mathscr{F}$ is supported on points, the result is easy). There is clearly no $H^0_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathscr{V})$ hence no $\mathbb{H}^{-1}_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathscr{F})$. By duality (for perverse sheaves) one gets $\mathbb{H}^1_{\Phi_{\tau_o}}(\mathbb{A}^1, \mathscr{F})$. □

Corollary 1.6.2. *If f is weakly tame, then $\widehat{\boldsymbol{R}f_!\mathbb{Q}_U} \to \widehat{\boldsymbol{R}f_*\mathbb{Q}_U}$ is an isomorphism.* □

Remark 1.6.3. The Laplace transform does not behave well with respect to ordinary cohomology sheaves. That is, one should not expect $\widehat{\boldsymbol{R}^k f_*\mathbb{Q}_U} = [\widehat{\boldsymbol{R}f_*\mathbb{Q}_U}]^{k+1}$ in general. The right cohomology to be taken is the "perverse cohomology". This will be reflected in the next section by the fact that the Laplace transform of a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module is a holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module (not a complex).

LECTURE 2

TAME FUNCTIONS ON SMOOTH AFFINE VARIETIES

Tameness, or its various avatars, is a property which ensures week tameness of Definition 1.4.1. One looks for a computable criterion. M-tameness is a property analogous, in the affine case, to the existence of the Milnor fibration of hypersurface singularities. It emphasizes the behaviour of f on the affine manifold U. Cohomological tameness is more algebraic and emphasizes the behaviour of f at infinity for some compactification: there is only one "Milnor ball", namely U itself. The latter notion is only cohomological (with $\mathbb{Q}$ as coefficients) but the former is topological.

It is not clear whether one property is stronger than the other one. It is known that for polynomials on $\mathbb{C}^{n+1}$, cohomological tameness with respect to the standard projective compactification of the fibers is equivalent to the so called Malgrange condition ([**28**]), which in turn implies M-tameness. For instance, tame polynomials on $\mathbb{C}^{n+1}$ in the sense of Broughton [**3**] are both M-tame and cohomologically tame. Conversely, there exist polynomials which are both M-tame and cohomologically tame (with respect to some compactification) but do not satisfy the Malgrange condition (i.e., are not cohomologically tame with respect to the standard projective compactification): many examples of such polynomials have been constructed by L. Păunescu and A. Zaharia. Nevertheless, our interest in these notes will not be in finding such examples. We shall emphasize the common properties of various definitions of tameness.

2.1. A short reminder on perverse sheaves and nearby/vanishing cycles

This section only consists of directions for using perverse sheaves and I will not recall the main definitions here. A good reference for this section is [**9**, Chap. 4 & 5].

2.1.a. Perverse sheaves. In this section, 'sheaf' will mean 'sheaf of $\mathbb{Q}$-vector spaces' (or 'sheaf of $\mathbb{C}$-vector spaces'). On a complex algebraic variety or a complex analytic space X, one can define the notion of constructible sheaf (with respect to some algebraic or analytic stratification of X), as well as the notion of bounded complex with constructible cohomology sheaves (constructible complexes, for short).

Perverse sheaves on X belong to the category of constructible complexes on X. The following properties are useful:

(1) A perverse sheaf supported on a point is nothing other than a sky-scrapper sheaf corresponding to a finite dimensional vector space on this point.

(2) On a complex *manifold* of dimension n, a *smooth* perverse sheaf $\mathscr{F}$ takes the form $\mathscr{L}[n]$, where $\mathscr{L}$ is a locally constant sheaf. Hence $\mathscr{H}^j\mathscr{F} = 0$ if $j \neq -n$ and $\mathscr{H}^{-n}\mathscr{F} = \mathscr{L}$.

(3) Let X be a Riemann surface, S a discrete set of points in X and consider the inclusions

$$X \smallsetminus S \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} S.$$

Let $\mathscr{L}$ be a local system on $X \smallsetminus S$. Then

(a) $j_!\mathscr{L}[1]$ (extension by 0) is perverse and is a sheaf up to a shift by one,

(b) $j_*\mathscr{L}[1]$ (intermediate extension) is perverse and is a sheaf up to a shift by one,

(c) $\boldsymbol{R}j_*\mathscr{L}[1]$ (maximal extension) is perverse but is in general not a sheaf up to a shift by one: it has possibly two cohomology sheaves $\boldsymbol{R}^{-1}j_*\mathscr{L}[1] = j_*\mathscr{L}$ and $\boldsymbol{R}^0 j_*\mathscr{L}[1] = \mathscr{H}^1\boldsymbol{R}j_*\mathscr{L}$ (the latter is supported on S).

(4) If X is a smooth complex algebraic/analytic variety of dimension n and $S \subset X$ is a *divisor*, then $j_!\mathscr{L}[n]$ and $\boldsymbol{R}j_*\mathscr{L}[n]$ are perverse, but $j_*\mathscr{L}[n]$ not in general, and it has to be replace with the *intermediate extension* $\mathrm{IC}_X(\mathscr{L})$ of Deligne-Goresky-MacPherson.

(5) On the smooth affine manifold U considered in Lecture 1, the constant sheaf shifted by $\dim U$ is denoted by ${}^p\mathbb{Q}_U$.

(6) Morphisms of perverse sheaves (considered as constructible complexes) have kernels and cokernels in the category of perverse sheaves. Hence there is a notion of exact sequence of perverse sheaves (although such a notion does not exist for constructible complexes).

(7) Let $h : Y \to X$ be a morphism and let $\mathscr{F}$ be a perverse sheaf on Y. If h is finite on the support of $\mathscr{F}$, then $\boldsymbol{R}h_*\mathscr{F}$ is perverse on X.

2.1.b. Perverse cohomology. A constructible complex $\mathscr{F}$ has, by definition, constructible cohomology sheaves $\mathscr{H}^k\mathscr{F}$. One can also define its perverse cohomology sheaves ${}^p\mathscr{H}^k\mathscr{F}$: each ${}^p\mathscr{H}^k\mathscr{F}$ is a perverse sheaf, and if one has a triangle of constructible complexes, one deduces a long exact sequence of their perverse cohomology sheaves (as justified by the remark above).

Lemma 2.1.1. *Let $\mathscr{F}$ be a constructible complex. Assume that each ${}^p\mathscr{H}^k\mathscr{F}$ has only one nonzero cohomology sheaf. Then each cohomology sheaf of $\mathscr{F}$ is perverse, up to a shift, and equal to some ${}^p\mathscr{H}^k\mathscr{F}$.*

2.1.c. Nearby and vanishing cycles. Let $f : X \to \mathbb{C}$ be a holomorphic function and let $\mathscr{F}$ be a constructible complex on X. The complex of nearby cycles $\psi_f\mathscr{F}$ is a constructible complex supported on $f = 0$ and equipped with a monodromy T. For any $x \in f^{-1}(0)$, the cohomology $\mathscr{H}^k\psi_f\mathscr{F}_x$ of the germ $\psi_f\mathscr{F}_x$ is the cohomology of the Milnor fibre of f at x, with its monodromy. One can define a complex $\phi_f\mathscr{F}$, equipped with a monodromy T also, in such a way that we have a canonical morphism $\psi_f\mathscr{F} \to \phi_f\mathscr{F}$ giving rise to a triangle $i^{-1}\mathscr{F} \to \psi_f\mathscr{F} \to \phi_f\mathscr{F} \xrightarrow{+1}$, where $i : f^{-1}(0) \hookrightarrow X$ denotes the inclusion. It will be useful to shift these functors and set ${}^p\psi_f\mathscr{F} = \psi_f\mathscr{F}[-1]$, ${}^p\phi_f\mathscr{F} = \phi_f\mathscr{F}[-1]$.

If $\mathscr{F}$ is perverse (on X), then ${}^p\psi\mathscr{F}$ and ${}^p\phi\mathscr{F}$ are also perverse. It follows that the functors ${}^p\psi$ and ${}^p\phi$ commute with taking perverse cohomology.

If $h : Y \to X$ is proper, ${}^p\psi$ and ${}^p\phi$ commute with taking direct image by h, that is, if $\mathscr{F}$ is a constructible complex on Y, ${}^p\psi_{h\circ f}\mathscr{F} \simeq {}^p\psi_f\boldsymbol{R}h_*\mathscr{F}$, and similarly for ${}^p\phi$.

2.2. Various notions of tameness

A good reference for this section is [**9**, Chap. 6].

2.2.a. Tameness from inside. Let us choose a closed embedding of U in an affine space $\mathbb{A}^N$ and consider balls centered at some given point of $\mathbb{A}^N$ and of arbitrary large radius R (that is, consider the distance function $\delta(x) = \|x-a\|^2$, $a \in \mathbb{A}^N$). For any R large enough, the ball $B(a,R)$ intersects U transversally[(1)].

Definition 2.2.1 **(M-tameness).** We say that $f : U \to \mathbb{A}^1$ is M-tame if, for some choice of $U \hookrightarrow \mathbb{A}^N$ and δ, the following property is satisfied:

- for any $\eta > 0$, there exists R_η such that the ball $B(a,R)$ is transversal to $f^{-1}(t)$ for any $|t| < \eta$ and any $R > R_\eta$.

Note that the difference with the previous transversality statement lies in the fact that we are considering a family of affine submanifolds, not a single one.

Proposition 2.2.2. *If f is M-tame, it is weakly tame.*

[(1)] Assume there is a sequence $x_n \in U$ with $R_n = \|x_n - a\| \to \infty$ and $B(a,R_n) \not\pitchfork U$ at x_n, that is, $(x_n - a) \perp T_{x_n}U$; this is a real-analytic condition, and by the curve selection lemma, there exists a real-analytic path $x(t) \to +\infty$ on which the same property holds; on this path, $(x(t)-a) \perp x'(t)$ implies $\|x(t)-a\|$ constant, a contradiction.

Sketch of proof. Indeed, M-tameness implies cohomological M-tameness, that is, for any disc $D \subset \mathbb{A}^1$, there exists R_0 such that, for any $R \geqslant R_0$, the cone of the restriction morphism $\boldsymbol{R}f_*\mathbb{Q}_{f^{-1}(D)} \to \boldsymbol{R}f_*\mathbb{Q}_{f^{-1}(D)\cap B(R)}$ has constant cohomology sheaves (hence has no vanishing cycles). This follows from the property that the map $f : f^{-1}(D) \smallsetminus (f^{-1}(D) \cap B(R)) \to D$ is a locally trivial C^∞ fibration (cf. [**27**]). ☐

2.2.b. Tameness from outside. Instead of choosing a metric on U, let us choose an embedding

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & X \arrow[d, "F"] & X \smallsetminus U \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & \mathbb{A}^1 & \end{tikzcd} \tag{2.2.3}$$

as in Remark 1.4.3, insisting now that X is quasi-projective and F is projective.

***Definition 2.2.4* (Cohomological tameness).** We say that f is cohomologically tame if there exists an embedding (2.2.3) such that, for any $c \in \mathbb{A}^1$, the function $F - c$ has no vanishing cycles with respect to the complex $i^{-1}\boldsymbol{R}j_*\mathbb{Q}_U$, that is, $\phi_{F-c}\boldsymbol{R}j_*\mathbb{Q}_U = \phi_{f-c}\mathbb{Q}_U$.

***Proposition 2.2.5*.** *If f is cohomologically tame, then f is weakly tame.*

Proof. Since the functor ϕ commutes with direct images in a natural way, we deduce from cohomological tameness that $\phi_{t-c}\mathscr{C} = \boldsymbol{R}g_*\phi_{F-c}i^{-1}\boldsymbol{R}j_*\mathbb{Q}_U \simeq 0$. So the result follows from Lemma 2.1.1 and the following lemma. ☐

***Lemma 2.2.6*.** *Let $\mathscr{F}$ be a bounded complex (of $\mathbb{Q}$-vector spaces) on $\mathbb{A}^1$ with constructible cohomology. If $\phi_{t-c}\mathscr{F} \simeq 0$ for any $c \in \mathbb{A}^1$, then each perverse cohomology sheaf of $\mathscr{F}$ is a constant sheaf shifted by one.*

2.3. Examples

2.3.a. Polynomial functions. If $U = \mathbb{A}^{n+1}$, there is a natural projectivization of f, obtained by taking the closure of the graph of f in $\mathbb{P}^{n+1} \times \mathbb{A}^1$. Cohomological tameness with respect to this projectivization is equivalent to the *Malgrange condition* (cf. [**28**]) which implies also M-tameness.

A simple easy-to-check criterion has been given by Broughton [**3**]: If the total Milnor number of any small linear perturbation of f is constant and finite, then the previous condition holds (and even better). This shows in particular that polynomials which are convenient and non degenerate with respect to their Newton polyhedron (in the sense of Kouchnirenko, [**23**]) satisfy the previous property, as Broughton criterion is easily seen to be satisfied in such a case.

2.3.b. Laurent polynomials. The main example consists in convenient nondegenerate Laurent polynomials in the sense of Kouchnirenko [**23**]. They are shown (by Denef-Loeser, [**8**]) to be cohomologically tame with respect to the projectivization of f obtained by taking the closure of the graph of f in the toric projective variety defined by the Newton polyhedron of f.

2.3.c. Thom-Sebastiani sums. Let $f : U \to \mathbb{A}^1$ and $g : V \to \mathbb{A}^1$ be two tame functions and let $f \oplus g : U \times V \to \mathbb{A}^1$ be their Thom-Sebastiani sum defined by $f \oplus g(u, v) = f(u) + g(v)$. Then $f \oplus g$ is tame. (This is not completely obvious, see [**26**] for a proof.)

2.3.d. Alternate Thom-Sebastiani. This is an example where it is useful to consider tameness with respect to a perverse sheaf, not only the constant sheaf.

Let $f : U \to \mathbb{A}^1$ be a regular function on the affine manifold U. On the product U^r consider the Thom-Sebastiani sum $f^{\oplus r} = f \oplus \cdots \oplus f$ ($r \geqslant 2$ times). On the other hand, the quotient variety $U^{(r)} := U^r/\mathfrak{S}_r$ of U^r by the symmetric group is also affine (but not smooth if $\dim U \geqslant 2$). Let ρ be the quotient map. The function $f^{\oplus r}$ descends to a regular function $f^{(\oplus r)}$ on $U^{(r)}$, i.e., $f^{\oplus r} = f^{(\oplus r)} \circ \rho$.

The sheaf $\rho_* {}^p\mathbb{Q}_{U^r}$ is perverse on $U^{(r)}$ (cf. Property 2.1.a(7)) and it is equipped with an action of $\mathfrak{S}_r$. Denote by ${}^p\mathbb{Q}_U^{\wedge r}$ its anti-invariant part (in the perverse sense). This is a perverse subsheaf (in fact a direct summand) of $\rho_* {}^p\mathbb{Q}_{U^r}$.

Let $\delta : \Delta \hookrightarrow U^{(r)}$ be the image by ρ of the diagonals $\{u_i = u_j\} \hookrightarrow U^r$ ($i \neq j = 1, \dots, r$) and let V be the complementary subset (which is smooth but not affine, in general). We have the following properties:

- ${}^p\mathbb{Q}_{U|V}^{\wedge r}$ is a rank-one local system shifted by $r \dim U$,
- ${}^p\mathbb{Q}_U^{\wedge r} = \delta_!({}^p\mathbb{Q}_{U|V}^{\wedge r}) = \boldsymbol{R}\delta_*({}^p\mathbb{Q}_{U|V}^{\wedge r})$.

Proposition 2.3.1. *If f is cohomologically tame, then $f^{(\oplus r)}$ is cohomologically tame on $U^{(r)}$ with respect to ${}^p\mathbb{Q}_U^{\wedge r}$.*

Clearly, cohomological tameness with respect to $\mathscr{F}$ means that there exists a projectivization F of f such that $\phi_{F-c}\boldsymbol{R}j_*\mathscr{F} = \phi_{f-c}\mathscr{F}$ for any $c \in \mathbb{A}^1$.

LECTURE 3

THE SPECTRUM AT INFINITY OF A TAME FUNCTION

3.1. The Laplace Gauss-Manin systems

Let $f : U \to \mathbb{A}^1$ be any regular function on a smooth affine variety U. The goal is to construct algebraically the holomorphic (flat) connection whose sheaf of horizontal sections is the topological Laplace transform $\widehat{\boldsymbol{R}f_*\mathbb{Q}_U}$. We look for various $\mathbb{C}[\tau,\tau^{-1}]$-modules $G^{(k)}$ equipped with an algebraic connection. The construction is a variant of the de Rham complex: $G^{(k)}$ is the $k+1$-th cohomology of the complex

$$(3.1.1)\quad 0 \to \mathscr{O}(U)[\tau,\tau^{-1}] \xrightarrow{d_f} \Omega^1(U)[\tau,\tau^{-1}] \xrightarrow{d_f} \cdots \xrightarrow{d_f} \Omega^{n+1}(U)[\tau,\tau^{-1}] \to 0,$$

where $d_f = e^{\tau f}\circ d_U \circ e^{-\tau f} = d_U - \tau df\wedge$. Clearly, $G^{(k)}$ is a $\mathbb{C}[\tau,\tau^{-1}]$-module, equipped with a connection $\widehat{\nabla}^{(k)}$, such that $\widehat{\nabla}^{(k)}_{\partial_\tau}$ is induced by $e^{\tau f}\circ\partial_\tau\circ e^{-\tau f} = \partial_\tau - f$.

Working in the coordinate $\theta = \tau^{-1}$, we can also write this complex as

$$(3.1.2)\quad 0 \to \mathscr{O}(U)[\theta,\theta^{-1}] \xrightarrow{\widetilde{d}_f} \Omega^1(U)[\theta,\theta^{-1}] \xrightarrow{\widetilde{d}_f} \cdots \xrightarrow{\widetilde{d}_f} \Omega^{n+1}(U)[\theta,\theta^{-1}] \to 0,$$

with $\widetilde{d}_f = \theta d_U - df\wedge$ and $\widehat{\nabla}^{(k)}_{\partial_\theta} = e^{f/\theta}\circ\partial_\theta\circ e^{-f/\theta} = \partial_\theta + f/\theta^2$.

Theorem 3.1.3. *Each $G^{(k)}$ is $\mathbb{C}[\tau,\tau^{-1}]$-free of finite rank. The connection $\widehat{\nabla}^{(k)}$ has a regular singularity at $\tau = 0$, but possibly an irregular one at $\tau = \infty$, and no other singularity at finite distance. The sheaf of horizontal sections of $\widehat{\nabla}^{(k)}$ is naturally identified with the local system $[\widehat{\boldsymbol{R}f_*\mathbb{Q}_U}]^{k+1}$.*

Sketch of proof.

(1) *Each $G^{(k)}$ is holonomic as a $\mathbb{C}[\tau,\tau^{-1}]\langle\tau\partial_\tau\rangle$-module.* On the affine variety $U\times\widehat{\mathbb{A}}^{1*}$, consider $\mathscr{O}_{U\times\widehat{\mathbb{A}}^{1*}}$ with connection $d-\tau df$. This connection is flat, and we get a holonomic $\mathscr{D}_{U\times\widehat{\mathbb{A}}^{1*}}$-module, that we denote by $\mathscr{O}_{U\times\widehat{\mathbb{A}}^{1*}}e^{-\tau f}$. By general results on holonomic $\mathscr{D}_{U\times\widehat{\mathbb{A}}^{1*}}$-modules (Bernstein, cf. **[15]**), the push-forward of $\mathscr{O}_{U\times\widehat{\mathbb{A}}^{1*}}e^{-\tau f}$ by the projection $p : U\times\widehat{\mathbb{A}}^{1*} \to \widehat{\mathbb{A}}^{1*}$ has holonomic cohomologies. As we work on

affine varieties, it is equivalent to consider global sections. The push-forward is then by definition the complex (3.1.1). Therefore, each $G^{(k)}$ is holonomic.

(2) To prove that $G^{(k)}$ is $\mathbb{C}[\tau,\tau^{-1}]$-free, it is now enough to prove that the connection on $G^{(k)}$ has no singularity. One could try to argue as follows: one can consider the parameter τ as a rescaling parameter, and as there is not much difference between f and τf for a fixed nonzero τ, one cannot create a singularity of $G^{(k)}$. The gap in this reasoning is that it does not take into account the behaviour of f when $f\to\infty$. So one should be more precise. Likewise, it seems that, at $\tau=0$, tensoring with $e^{-\tau f}$ does not introduce new singularity; but this argument forgets the behaviour when $f\to\infty$. It is therefore useful to consider a compactification $\widetilde{F}:\widetilde{X}\to\mathbb{P}^1$ of $f:U\to\mathbb{A}^1$. By a local analysis at $\widetilde{X}\smallsetminus U$, one checks that $\mathscr{O}_{U\times\widehat{\mathbb{A}}^{1*}}e^{-\tau f}$ has regular singularity *along* $\tau=0$ (but this connection does *not* have regular singularity in the usual sense), and this remains true after push-forward by p.

(3) In any case, the arguments are easier if we first apply the push-forward by f, in the sense of $\mathscr{D}$-modules, to $\mathscr{O}_U$, getting a complex $f_+\mathscr{O}_U$, and then apply the Laplace transform (twist by $e^{-t\tau}$ followed by integration along t) to $f_+\mathscr{O}_U$ for $\mathscr{D}$-modules on the affine line $\mathbb{A}^1$. The latter only amounts to changing names of variables, but is better understood.

The push-forward by f of $\mathscr{O}_U$ as a $\mathscr{D}_U$-module, denoted by $f_+\mathscr{O}_U$, is the complex of $\mathbb{C}[t]\langle\partial_t\rangle$-modules analogous to (3.1.1):

$$0\to\mathscr{O}(U)[\tau]\xrightarrow{d_f}\Omega^1(U)[\tau]\xrightarrow{d_f}\cdots\xrightarrow{d_f}\Omega^{n+1}(U)[\tau]\to 0, \tag{3.1.4}$$

with d_f as in (3.1.1). The left $\mathbb{C}[t]\langle\partial_t\rangle$-module structure is defined in the following way: ∂_t acts as the multiplication by τ, and t acts by multiplication by f on forms not depending on ∂_t and is extended on forms depending on ∂_t by using the rule $[\partial_t,t]=1$. This action commutes with d_f (check it), hence each cohomology module $M^{(k)}$ comes equipped with a left $\mathbb{C}[t]\langle\partial_t\rangle$-module structure i.e., is a $\mathbb{C}[t]$-module with a connection ∇, so that ∇_{∂_t} is the left action of ∂_t. Note however that, in general, $M^{(k)}$ does not have finite type over $\mathbb{C}[t]$.

Then, by general results on Gauss-Manin connection or systems, each $M^{(k)}$ is a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module with regular singularity. Its Laplace transform $\widehat{M}^{(k)}$ is a holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, with singularities at 0 and ∞ at most, the singularity at 0 being regular and the singularity at ∞ being in general irregular. Now, $G^{(k)}=\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]}\widehat{M}^{(k)}$.

(4) The last statement concerning the sheaf of horizontal sections of $\widehat{\nabla}^{(k)}$ decomposes in two steps: firstly, one shows that taking the de Rham complex commutes with taking the push-forward, so $\boldsymbol{R}f_*\mathbb{Q}_U$ is naturally interpreted as the de Rham complex of $f_+\mathscr{O}_U$; secondly, one shows that taking the de Rham complex commutes with taking the localized Laplace (or topological Laplace) transform. Only at the very end, one takes the cohomology of the various complexes that one considers. $\square$

3.2. The Brieskorn lattice

Let us keep the notation of §3.1. Set $\theta = \tau^{-1}$ and consider the sub-complex of (3.1.2)

$$0 \longrightarrow \mathscr{O}(U)[\theta] \xrightarrow{\widetilde{d}_f} \Omega^1(U)[\theta] \xrightarrow{\widetilde{d}_f} \cdots \xrightarrow{\widetilde{d}_f} \Omega^{n+1}(U)[\theta] \longrightarrow 0, \tag{3.2.1}$$

with $\widetilde{d}_f = \theta d_U - df\wedge$. Despite its apparent analogy with (3.1.4) (changing τ with τ^{-1}), this complex is not of the same kind. When f is tame, it has good finiteness properties.

Theorem 3.2.2. *If f is cohomologically tame (or M-tame), then this complex has cohomology in degree $n+1$ at most. Its cohomology module $G_0 := \Omega^{n+1}(U)[\theta]/\widetilde{d}_f\Omega^n(U)[\theta]$ is a free $\mathbb{C}[\theta]$-module whose rank is equal to the sum of the Milnor numbers of f at its critical points. G_0 is also equipped with a meromorphic connection having a pole of order two at $\theta = 0$ and a regular singularity at infinity. In particular, the fibre $G_0/\theta G_0$ of G_0 at $\theta = 0$ is equal to the Jacobian module $\Omega^{n+1}(U)/df \wedge \Omega^n(U)$.*

Sketch of proof. One starts with a geometric finiteness argument, i.e., working in the variable t. Let M_0 be the image of Ω_U^{n+1} in $M = \Omega_U^{n+1}[\tau]/d_f\Omega_U^n[\tau]$.

Proposition 3.2.3. *If f is cohomologically tame (or M-tame), then M_0 is a free $\mathbb{C}[t]$-module of finite rank generating M as a $\mathbb{C}[t]\langle\partial_t\rangle$-module.*

Indication for the proof (cf. [**33**, §10]*).* It is enough to show that M_0 is contained in some free $\mathbb{C}[t]$-module of finite rank (the generating property is by construction). The candidate is the Deligne lattice $V^{>-1}M$, which is the unique free $\mathbb{C}[t]$-submodule of M on which the connection ∇_{∂_t} has only logarithmic poles at C_f and the eigenvalues of the residue belong to $(-1, 0]$.

The proof is obtained by working locally analytically on X. In order to work with $\mathscr{D}$-modules, one embeds $F : X \to \mathbb{A}^1$ into $G : Y \to \mathbb{A}^1$ with Y smooth and G projective (the dimension of Y can be much bigger than that of X, this does not matter). Let κ denote the composed inclusion $U \hookrightarrow X \hookrightarrow Y$. The proof is first obtained for the $\mathscr{D}_Y$-module $\kappa_+\mathscr{O}_U$, with its natural good filtration. One then argue using the direct image by F (one knows that the V-filtration has a good behaviour by proper direct image). □

Remark 3.2.4. Dimca and Saito show in [**11**, Rem. 3.3] that this property characterizes the cohomologically tame functions (with respect to X) among those functions f which have only isolated singularities on X, that is, for which the support of $\phi_{F-c}\boldsymbol{R}j_*\mathbb{Q}_U$ is discrete (or empty) for any $c \in \mathbb{A}^1$.

We conclude with the general result:

***Lemma 3.2.5* (cf. e.g., [31, Cor. V.2.9]).** *Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module with regular singularities and let M_0 be a finitely generated $\mathbb{C}[t]$-submodule which generates M over $\mathbb{C}[t]\langle\partial_t\rangle$. Then $G_0 = \mathbb{C}[\theta]\cdot\widetilde{M}_0$, with $\widetilde{M}_0 := \mathrm{image}(M_0 \to G)$, is a free $\mathbb{C}[\theta]$-submodule of G of finite rank, and $G = \mathbb{C}[\theta,\theta^{-1}]\otimes_{\mathbb{C}[\theta]} G_0$.* □

3.3. The spectrum

We shall denote $U_0 = \operatorname{Spec}\mathbb{C}[\tau]$ and $U_\infty = \operatorname{Spec}\mathbb{C}[\theta]$ the two standard charts of $\mathbb{P}^1(\mathbb{C})$, where $\theta = 1/\tau$ on $U_0 \cap U_\infty$. We shall denote $0 = \{\tau = 0\}$ and $\infty = \{\theta = 0\}$.

Let G be a meromorphic bundle on $\mathbb{P}^1$, with a connection having singularities at 0 and ∞ only, the singularity at 0 being a regular singularity (but not necessarily the one at infinity). Then G is a free $\mathbb{C}[\tau,\tau^{-1}]$-module of finite rank μ, equipped with a derivation ∂_τ which makes it a left $\mathbb{C}[\tau,\tau^{-1}]\langle\partial_\tau\rangle$-module.

3.3.a. Lattices. A *lattice* L_0 (resp. L_∞) of G on U_0 (resp. L_∞) is a free $\mathbb{C}[\tau]$ (resp. $\mathbb{C}[\theta]$)-submodule of G such that $\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]} L_0 = G$ (resp. $\mathbb{C}[\theta,\theta^{-1}]\otimes_{\mathbb{C}[\theta]} L_\infty = G$). Given such lattices L_0 and L_∞, we produce a vector bundle on $\mathbb{P}^1$ by gluing them along the isomorphism $L_{0|\mathbb{C}^*} \simeq G \simeq L_{\infty|\mathbb{C}^*}$.

***Example 3.3.1*.** The Brieskorn lattice of a tame function is a U_∞-lattice of the Laplace Gauss-Manin system G of this function.

3.3.b. Deligne lattices on U_0. Since the connection on G has a regular singularity at $\tau = 0$, there exist canonical lattices, called *Deligne lattices*, on the chart U_0. For any real number α, there exists a unique U_0-lattice, denoted by $V^\alpha G$, satisfying the following properties:

(1) $V^\alpha G$ is logarithmic, that is, stable by $\tau\partial_\tau$,

(2) The residue of the connection on $V^\alpha G/\tau V^\alpha G$ has eigenvalues with real part in $[\alpha,\alpha+1)$.

In fact, we get a decreasing filtration ($V^\alpha G \supset V^{>\alpha}G$) by logarithmic lattices, and $V^{\alpha+1}G = \tau V^\alpha G$ for any α.

***Assumption 3.3.2*.** *From now on, we assume that the eigenvalues of the residue are real or, what amounts to the same, that the action induced by $\tau\partial_\tau - \alpha$ is nilpotent on $\mathrm{gr}_V^\alpha G := V^\alpha G/V^{>\alpha}G$, for any $\alpha \in \mathbb{R}$.*

3.3.c. The vector bundles $\mathcal{G}^\alpha$. Let us fix $\alpha \in \mathbb{R}$. The locally free sheaf $\mathcal{G}^\alpha$ is obtained by gluing $V^\alpha G$ on U_0 and G_0 on U_∞ using the isomorphisms on $U_0 \cap U_\infty$

$$\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]} V^\alpha G = G = \mathbb{C}[\theta,\theta^{-1}]\otimes_{\mathbb{C}[\theta]} G_0.$$

The following is easy to prove.

Lemma 3.3.3. *We have* $\mathcal{G}^\alpha \otimes \mathscr{O}(-\ell) \simeq \mathcal{G}^{\alpha+\ell}$ *and* $H^0(\mathbb{P}^1, \mathcal{G}^\alpha) = V^\alpha G \cap G_0$*, where the intersection is taken in* G. □

As a consequence,

- for any $\alpha \in \mathbb{R}$, $V^\alpha G \cap G_0$ is a finite dimensional vector space;
- for any fixed α and for $\ell \ll 0$ we have $V^{\alpha+\ell} G \cap G_0 = 0$, and for $\ell \gg 0$ we have $G_0 = (V^{\alpha-\ell} G \cap G_0) + \theta G_0$.

This can also be stated as

$$V^\alpha(G_0/\theta G_0) = \begin{cases} 0 & \text{if } \alpha \gg 0, \\ G_0/\theta G_0 & \text{if } \alpha \ll 0. \end{cases} \tag{3.3.4}$$

3.3.d. The spectrum at infinity. According to Birkhoff-Grothendieck's theorem, each $\mathcal{G}^\alpha$ decomposes as the sum of line bundles. Let v_α denote the number (with multiplicities) of such line bundles which are $\geqslant 0$, and by ν_α the difference $v_\alpha - v_{>\alpha}$.

Lemma 3.3.5. *For* $\alpha \in \mathbb{R}$*, we have*

$$\nu_\alpha = \dim (V^\alpha G \cap G_0)/(V^\alpha \cap \theta G_0 + V^{>\alpha} \cap G_0).$$

Proof. Consider the natural morphism

$$\mathscr{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} \Gamma(\mathbb{P}^1, \mathcal{G}^\alpha) \longrightarrow \mathcal{G}^\alpha, \tag{3.3.6}$$

whose image is denoted by $\mathcal{V}^\alpha$. This is a subbundle, in the sense that $\mathcal{V}^\alpha$ and $\mathcal{G}^\alpha/\mathcal{V}^\alpha$ are locally free. This property holds for any holomorphic bundle $\mathcal{G}$ on $\mathbb{P}^1$: indeed, the natural morphism is compatible with the Birkhoff-Grothendieck decomposition, so it suffices to prove it when $\mathcal{G}$ is a line bundle; in such a case, either $\mathcal{G}$ has global sections, and (3.3.6) is onto, or $\mathcal{G}$ has no global section, and (3.3.6) is zero.

The previous argument shows that the rank of $\mathcal{V}^\alpha$ is v_α, that the quotient sheaves $\mathcal{V}^\alpha/\mathcal{V}^{>\alpha}$ are locally free, and that $\nu_\alpha = \operatorname{rk} \mathcal{V}^\alpha/\mathcal{V}^{>\alpha}$.

Tensoring with $\mathscr{O}_{\mathbb{P}^1}(*0)$ and taking global section, (3.3.6) is the morphism

$$\mathbb{C}[\theta] \otimes_{\mathbb{C}} (G_0 \cap V^\alpha G) \longrightarrow G_0,$$

and $\Gamma(\mathbb{P}^1, \mathcal{V}^\alpha(*0))$ is the $\mathbb{C}[\theta]$-submodule of G_0 generated by the subspace $G_0 \cap V^\alpha G$, that is, $\mathbb{C}[\theta] \cdot (G_0 \cap V^\alpha G)$. Then,

$$\begin{aligned} \operatorname{rk} \mathcal{V}^\alpha = \dim \big[\mathbb{C}[\theta] \cdot (G_0 \cap V^\alpha G) + \theta G_0\big]/\theta G_0 &= \dim \big[(G_0 \cap V^\alpha G) + \theta G_0\big]/\theta G_0 \\ &= \dim (G_0 \cap V^\alpha G)\big/(\theta G_0 \cap V^\alpha G), \end{aligned}$$

and

$$\nu_\alpha = \dim (G_0 \cap V^\alpha G)/(\theta G_0 \cap V^\alpha G + G_0 \cap V^{>\alpha} G).$$ □

Definition 3.3.7. The spectral polynomial of (G, G_0) is

$$\mathrm{SP}_{G_0}(S) := \prod_{\alpha \in \mathbb{R}} (S - \alpha)^{\nu_\alpha}.$$

Theorem 3.3.8. *Let $f : U \to \mathbb{A}^1$ be a tame function and let $\mathrm{SP}_f^\infty(S) := \mathrm{SP}_{G_0}(S) = \prod_{\alpha\in\mathbb{R}}(S-\alpha)^{\nu_\alpha}$ be the spectral polynomial of its Brieskorn lattice. Then,*

(1) *the polynomial $\prod_{\alpha\in\mathbb{R}}(T-e^{2\pi i\alpha})^{\nu_\alpha}$ is the characteristic polynomial of the monodromy of $H^{n+1}(U, f^{-1}(t);\mathbb{Q})$ when t runs along a circle of big radius in the counterclockwise direction;*

(2) *we have $\nu_\alpha \neq 0 \Rightarrow (-\alpha) \in [0, n+1] \cap \mathbb{Q}$ (and in $(0, n+1) \cap \mathbb{Q}$ if $U = \mathbb{C}^{n+1}$), and $\nu_\alpha = \nu_{-(n+1)-\alpha}$;*

(3) *the spectral polynomial of the Brieskorn lattice is equal to the polynomial attached to the Steenbrink spectrum of the limit mixed Hodge structure of $\lim_{t\to\infty} H^{n+1}(U, f^{-1}(t);\mathbb{C})$.*

***Theorem 3.3.9* (cf. [26]).** *Given a family f_λ of regular functions on U parametrized by a disc. If f_λ is tame for any λ, then the spectrum at infinity of f_λ is semi-continuous in the sense of Varchenko. In particular, if the total Milnor number of f_λ is constant, the the spectral polynomial of the Brieskorn lattice of f_λ is constant.*

LECTURE 4

EXAMPLES

The following examples show two ways of approaching the spectrum at infinity of a tame function. The first one is geometric, and relies on the singularities of the section at infinity of the fibres of the function. The second one is more combinatorial.

4.1. García López-Némethi polynomials

Let $f = f_d + f_{d-1} + \cdots + f_0$ be a polynomial of $n+1$ variables $x_0, \dots, x_n$ decomposed according to its homogeneous components. We assume that f has degree d, so that $f_d \not\equiv 0$. The *natural* partial projectivization X of the graph of f is the closure of the graph of f in $\mathbb{P}^{n+1} \times \mathbb{A}^1$. If $X_0, \dots, X_n, Z$ are the projective coordinates on $\mathbb{P}^{n+1}$, $X \subset \mathbb{P}^{n+1} \times \mathbb{A}^1$ is defined by the equation

$$f_d(X_0, \dots, X_n) + Z f_{d-1}(X_0, \dots, X_n) + \cdots + Z^d f_0 = tZ^d.$$

A GL-N polynomial (also called $(*)$-polynomial by these authors, [**16, 17**]) is polynomial such that X *is non-singular*.

In general, the singular locus of X in $\mathbb{P}^{n+1} \times \mathbb{A}^1$ is given by the (X, Z)-homogeneous equations

$$\partial f_d = 0, \quad f_{d-1} = 0, \quad Z = 0.$$

In particular, it takes the form $\Sigma \times \mathbb{A}^1$. One also checks that the following properties are equivalent:

(1) X is non-singular,
(2) Each fibre X_t is nonsingular at $X_t \cap \{Z = 0\}$.

The latter property implies that, for any $c \in \mathbb{A}^1$, $\phi_{F-c}\mathbb{Q}_X$ is supported in U. Moreover, the restriction $i^{-1}\mathbb{Q}_X = \mathbb{Q}_{X_\infty}$ to $X_\infty = X \smallsetminus U$ satisfies the same property: indeed, X_∞ is defined by the equations $f_d = Z = 0$, hence takes the form $Y_\infty \times \mathbb{A}^1$, with $Y_\infty = \{f_d = 0\} \subset \mathbb{P}^n$. Consequently, $j_!\mathbb{Q}_U$ satisfies the same property, and by Verdier duality, so does $\boldsymbol{R}j_*\mathbb{Q}_U$. So:

Proposition 4.1.1. *A GL-N polynomial is cohomologically tame.* □

The projective variety Y_∞ defined by the homogenous part of f has singular locus S_∞ defined by the equations $\partial f_d = 0$. By assumption, this set is not met by the projective hypersurface $f_{d-1} = 0$, hence has dimension 0 at most. In other words, Y_∞ has at most isolated singularities.

Corollary 4.1.2. *Let f_d be a homogeneous polynomial of degree d defining a projective hypersurface $Y_\infty \subset \mathbb{P}^n$ having at most isolated singularities with respective Milnor numbers $\mu_1, \dots, \mu_r$. Then, for any choice of $f_{d-1}, \dots, f_0$, such that $f_{d-1} = 0$ does not contain any singularity of Y_∞, the polynomial $f = f_d + f_{d-1} + \dots + f_0$ is a GL-N polynomial with total Milnor number $(d-1)^{n+1} - \sum_i \mu_i$. In particular, the spectrum of f at infinity does not depend on such a choice of $f_{d-1}, \dots, f_0$.*

Proof. It remains to compute the total Milnor number μ. In order to avoid any computation, one can argue as follows. The Euler characteristic of a smooth fibre $f^{-1}(t)$ of f is $1 + (-1)^n \mu$. It is equal to $\chi(X_t) - \chi(Y_\infty)$. One has $\chi(X_t) = \chi(F_n^d)$, where F_n^d is any smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. Similarly, $\chi(Y_\infty) = \chi(F_{n-1}^d) + (-1)^{n-1} \sum_i \mu_i$. So, $1 + (-1)^n(\mu + \sum_i \mu_i) = \chi(F_n^d) - \chi(F_{n-1}^d)$. If we had done the computation starting with f'_d defining a smooth Y'_∞, we would have obtained $1 + (-1)^n \mu' = \chi(F_n^d) - \chi(F_{n-1}^d)$. For example, taking $f' = x_1^d + \dots + x_{n+1}^d$ we find $\mu' = (d-1)^{n+1}$. This gives the desired formula for μ. □

García López & Némethi compute the limit mixed Hodge structure on the cohomology space $H^n(f^{-1}(t), \mathbb{C}) = H^{n+1}(U, f^{-1}(t), \mathbb{C})$ when $t \to \infty$ in terms of the mixed Hodge structure of the (possibly) singular projective variety Y_∞.

Around $t = \infty$, one can choose a good compactification of X after taking a d-th root of $1/t$. Let D' be a disc with coordinate t' and let $\mathscr{X}' \subset \mathbb{P}^{n+1} \times D'$ defined by

$$f_d(X_0, \dots, X_n) + t' Z f_{d-1}(X_0, \dots, X_n) + \dots + t'^d Z^d f_0 = Z^d. \tag{4.1.3}$$

The map $\mathbb{P}^{n+1} \times D' \to \mathbb{P}^{n+1} \times D$ given by $(X, Z, t') \mapsto (X, t'^d Z, t'^d)$ sends $\mathscr{X}'_{|t' \neq 0}$ to the restriction $\mathscr{X}$ of X to $\mathbb{P}^{n+1} \times \{|t| \gg 0\}$. In other words, $\mathscr{X}'$ is a nice model (after ramification of t) of the behaviour of f near ∞. Moreover, the fibre of $\mathscr{X}'$ at $t' = 0$ is the subvariety $Y_\infty^{(d)}$ of $\mathbb{P}^{n+1}$ having equation $f_d - Z^d = 0$, that is, a d-fold cyclic cover of $\mathbb{P}^n$ along Y_∞.

One can roughly summarize the results of **[17]**:

Theorem 4.1.4.

(1) *The integral spectral numbers of f only depend on the Hodge numbers of Y_∞.*

(2) *The spectral numbers in $k/d + \mathbb{N}$, $k = 1, \dots, d-1$, only depend on the Hodge numbers of $Y_\infty^{(d)}$.*

(3) *The other spectral numbers are determined by the spectral numbers of each singular point of Y_∞.*

4.2. Nondegenerate Laurent polynomials

In this section, the ambient affine variety U will be a torus $(\mathbb{C}^*)^{n+1}$ and f will be a Laurent polynomial[(1)], if we fix global affine coordinates $u_0, \dots, u_n$ on U. One attaches to this Laurent polynomial a Newton polyhedron $\Gamma(f)$, obtained as the convex hull in $\mathbb{R}^{n+1}$ of the exponents of the monomial which effectively appear in f. We say, after Kouchnirenko ([**23**]) that

- f is nondegenerate if, for each facet Δ of the boundary of $\Gamma(f)$, denoting by f_Δ the corresponding part of f, then f_Δ has no critical point,
- f is convenient if 0 belongs to the interior of $\Gamma(f)$.

4.2.a. The Newton filtration. For any face σ of dimension $n-1$ of the boundary $\partial\Gamma(f)$, denote by L_σ the linear form with coefficients in $\mathbb{Q}$ such that $L_\sigma \equiv 1$ on σ. For $g \in \mathbb{C}[u, u^{-1}]$, put $\phi_\sigma(g) = \max_a L_\sigma(a)$, where the max is taken on the exponents of monomials appearing in g, and set $\phi(g) = \max_\sigma \phi_\sigma(g)$.

Remarks 4.2.1

(1) For $g, h \in \mathbb{C}[u, u^{-1}]$, we have

$$\phi(gh) \leqslant \phi(g) + \phi(h)$$

with equality if and only if there exists a face σ such that $\phi(g) = \phi_\sigma(g)$ and $\phi(h) = \phi_\sigma(h)$.

(2) As 0 belongs to the interior of $\Gamma(f)$, we have $\phi(g) \geqslant 0$ for any $g \in \mathbb{C}[u, u^{-1}]$ and $\phi(g) = 0$ if and only if $g \in \mathbb{C}$. This would not remain true without this convenient assumption.

(3) Set $\dfrac{du}{u} = \dfrac{du_0}{u_0} \wedge \cdots \wedge \dfrac{du_n}{u_n}$. If $\omega \in \Omega^{n+1}(U)$, write $\omega = g du/u$ and define $\phi(\omega) := \phi(g)$.

Consider the Newton increasing filtration $\mathscr{N}_\bullet\Omega^{n+1}(U)$ indexed by $\mathbb{Q}$, defined by

$$\mathscr{N}_\alpha\Omega^{n+1}(U) := \{g du/u \in \Omega^{n+1}(U) \mid \phi(g) \leqslant \alpha\}.$$

The previous remark shows that $\mathscr{N}_\alpha\Omega^{n+1}(U) = 0$ for $\alpha < 0$ and $\mathscr{N}_0\Omega^{n+1}(U) = \mathbb{C} \cdot du/u$.

Extend this filtration to $\Omega^{n+1}(U)[\theta]$ by setting

$$\mathscr{N}_\alpha\Omega^{n+1}(U)[\theta] := \mathscr{N}_\alpha\Omega^{n+1}(U) + \theta\mathscr{N}_{\alpha-1}\Omega^{n+1}(U) + \cdots + \theta^k\mathscr{N}_{\alpha-k}\Omega^{n+1}(U) + \cdots$$

and induce this filtration on G_0:

---

(1) One could also take $U = \mathbb{C}^{n+1}$ or $U = \mathbb{C}^k \times (\mathbb{C}^*)^{n+1-k}$ and adapt the constructions made in this section; in the polynomial case, for instance, the form ω to be considered is $du_0 \wedge \cdots \wedge du_n$.

Definition 4.2.2 **(Newton filtration of the Brieskorn lattice).** The Newton filtration of the Brieskorn lattice is defined by

$$\begin{aligned}\mathscr{N}_\alpha G_0 &:= \mathrm{image}\big[\mathscr{N}_\alpha \Omega^{n+1}(U)[\theta] \hookrightarrow \Omega^{n+1}(U)[\theta] \to G_0\big]\\ &= \mathscr{N}_\alpha \Omega^{n+1}(U)[\theta] \big/ \big(\widetilde{d}_f \Omega^n(U)[\theta] \cap \mathscr{N}_\alpha \Omega^{n+1}(U)[\theta]\big).\end{aligned}$$

***Lemma 4.2.3*.** *The Newton filtration on G_0 satisfies the following properties:*

(1) $\theta \mathscr{N}_\alpha G_0 \subset \mathscr{N}_{\alpha+1} G_0$,
(2) $\cup_\alpha \mathscr{N}_\alpha G_0 = G_0$,
(3) $\mathscr{N}_\alpha G_0 = 0$ *if* $\alpha < 0$ *and* $\dim \mathscr{N}_0 G_0 = 1$.

Proof. (1) is clear, (2) follows from $\cup_\alpha \mathscr{N}_\alpha \Omega^{n+1}(U) = \Omega^{n+1}(U)$, and (3) follows from the similar statement for $\Omega^{n+1}(U)$. ☐

Definition 4.2.4 **(The Newton filtration on the Gauss-Manin system)**
For any $\alpha \in \mathbb{Q}$, we set

$$\mathscr{N}_\alpha G := \mathscr{N}_\alpha G_0 + \tau \mathscr{N}_{\alpha+1} G_0 + \cdots + \tau^k \mathscr{N}_{\alpha+k} G_0 + \cdots .$$

For instance, we have

$$\mathscr{N}_0 G = \mathrm{image}\big[(\mathscr{N}_0 \Omega^{n+1}(U) + \tau \mathscr{N}_1 \Omega^{n+1}(U) + \cdots + \tau^k \mathscr{N}_k \Omega^{n+1}(U) + \cdots) \to G\big].$$

From the definition, we clearly get $\tau \mathscr{N}_\alpha G \subset \mathscr{N}_{\alpha-1} G$, which implies that $\mathscr{N}_\alpha G$ is a $\mathbb{C}[\tau]$-module, and the Newton filtration on G is exhaustive. It will be useful to consider the decreasing Newton filtration, defined by $\mathscr{N}^\alpha := \mathscr{N}_{-\alpha}$.

4.2.b. Newton filtration and Deligne filtration

Theorem 4.2.5 **(cf. [13]).** *Assume that f is convenient and nondegenerate with respect to its Newton polyhedron. Then the (decreasing) Newton filtration $\mathscr{N}^\bullet G_0$ on the Brieskorn lattice coincides with the filtration $G_0 \cap V^\bullet G$ induced on G_0 by the filtration by Deligne lattices $V^\bullet G$.*

Corollary 4.2.6 **(The spectrum, cf. [13]).** *The spectrum (or the spectral polynomial) of the Newton filtration on G_0 is equal to the spectrum (or the spectral polynomial) of the Brieskorn lattice (which is also the spectrum at infinity in the sense of Steenbrink of the Laurent polynomial f).* ☐

***Remark 4.2.7*.** This does not mean that the spectrum is combinatorially easy to compute. A simple example has been computed in **[14]**. In mirror symmetry, the spectrum corresponds to the "age" indexing the orbifold cohomology of the projective toric variety having $\Gamma(f)$ as Newton polyhedron.

Sketch of the proof of Theorem 4.2.5. The main argument is the strictness of the morphism

$$\mathbb{C}[u, u^{-1}]^{n+1} \longrightarrow (u_0 \partial f/\partial u_0, \ldots, u_n \partial f/\partial u_n)$$

with respect to the Newton filtration (cf. [**23**, Th. 4.1]). One deduces the strictness result $\mathscr{N}^\alpha G_0 \cap \theta G_0 = \theta \mathscr{N}^{\alpha+1} G_0$ (cf. the decreasing version of Definition 4.2.2). One then proves that $\mathscr{N}^\bullet G$, as defined in Definition 4.2.4, coincides with the filtration by Deligne lattices. The strictness result above is then used to identify $\mathscr{N}^\alpha G_0$ with $\mathscr{N}^\alpha G \cap G_0$, hence with $V^\alpha G \cap G_0$. $\square$

LECTURE 5

FOURIER-LAPLACE TRANSFORM OF A VARIATION OF HODGE STRUCTURE

5.1. Introduction

Given a regular function $f : U \to \mathbb{A}^1$, each cohomology space of fibres $f^{-1}(t)$ underlies a *mixed Hodge structure*, after Deligne [**7**]. This mixed Hodge structure varies with the same numerical data as long as t remains away from the bifurcation set B_f. When t tends to $t_o \in B_f$ or to ∞, this mixed Hodge structure has a limit, called the limit mixed Hodge structure (a good general reference is [**29**]).

Such mixed Hodge structures may be quite involved and difficult to compute. Even for a tame function, the general mixed Hodge structure is possibly not pure. Going to the Laplace transform simplifies things and kills the constant Hodge structures which occur as subquotients of the variation of mixed Hodge structure.

Here is an example of such a simplification produced by the Laplace transform. Recall that, given a polarized variation of Hodge structure on $\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$, the corresponding monodromy representation is semisimple (if we were dealing with unitary representations, the result would easy). Although the variation attached to a tame function is usually not a polarized variation of pure Hodge structures, we have:

Theorem 5.1.1. *Let $f : U \to \mathbb{A}^1$ be a cohomologically tame function. Then the Laplace Gauss-Manin system G is semisimple as a $\mathbb{C}[\tau, \tau^{-1}]$-module with connection.*

Be careful that this does not mean that the monodromy of the corresponding connection on $\widehat{\mathbb{A}}^{1*}$ is semisimple, because the connection usually has an irregular singularity at infinity. A similar result had been proved by N. Katz in an arithmetic setting [**21**]. However, the price to pay is the need to extend the notion of variation of Hodge structure in order to take account the structure on G coming, by Fourier-Laplace transform, of a variation of Hodge structure.

Proof. Let me first give the sketch of the proof. Consider a good compactification of f (i.e., that coming in the definition of tameness):

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & X \arrow[d, "F"] \\ & \mathbb{A}^1 \end{tikzcd}$$

The fibres $F^{-1}(c)$ (for $c \in \mathbb{A}^1 \smallsetminus C_f$) are smooth on U, but are possibly singular on $X \smallsetminus U$, and X itself can be singular on some subset of $X \smallsetminus U$. If we look for purity or semisimplicity, it is therefore natural to consider the intersection cohomology $\mathrm{IH}^n(F^{-1}(c), \mathbb{Q})$ instead of the cohomology $H^n(F^{-1}(c), \mathbb{Q})$.(1)

The family of $\mathrm{IH}^n(F^{-1}(c), \mathbb{Q})$, when $c \in \mathbb{A}^1 \smallsetminus C_f$, forms a locally constant sheaf $\mathscr{V}_{!*}$. By a general result (the Decomposition Theorem, **[2]** and **[37]**, cf. also the survey article **[4]**), this locally constant sheaf is semisimple, that is, the corresponding monodromy representation decomposes into direct sum of simple representations. I will consider the (perverse) sheaf $j_*\mathscr{V}_{!*}$ on $\mathbb{A}^1$. It is then semisimple as a perverse sheaf(2). Via the Riemann-Hilbert correspondence, it corresponds to a $\mathbb{C}[t]\langle\partial_t\rangle$-module $M_{!*}$ with regular singularities at C_f and at ∞. This $\mathbb{C}[t]\langle\partial_t\rangle$-module is then semisimple.

The natural composed morphism

$$\mathrm{IH}^n(F^{-1}(c), \mathbb{Q}) \longrightarrow H^n(F^{-1}(c), \mathbb{Q}) \longrightarrow H^n(f^{-1}(c), \mathbb{Q})$$

induces a morphism $M_{!*} \to M$, where M is the Gauss-Manin system of f (denoted $M^{(n+1)}$ after (3.1.4)). The kernel and cokernel of this morphism are smooth, that is, isomorphic to powers of $(\mathbb{C}[t], d)$. As a consequence, their localized Laplace transforms $G_{!*}$ and G are isomorphic as $\mathbb{C}[\tau, \tau^{-1}]$-modules with connection. But $M_{!*}$ semisimple as a $\mathbb{C}[t]\langle\partial_t\rangle$-module trivially implies that its Laplace transform $\widehat{M_{!*}}$ is semisimple as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module. The following lemma shows that this implies that the localized module $G_{!*}$ is so as a module with connection. Therefore, $G = G_{!*}$ is so as a module with connection.

Lemma 5.1.2. *Let N be a simple $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module with singularity at 0 only. Then the localized module $H = \mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}[\tau]} N$ is a simple $\mathbb{C}[\tau, \tau^{-1}]$-module with connection.*

Proof. Indeed, note first that H is also a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module and that the map $N \to H$ is injective. If $H' \subset H$ is a strict $\mathbb{C}[\tau, \tau^{-1}]$-submodule with connection, it is also a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-submodule. Therefore, $H' \cap N = 0$. As τ acts bijectively on H', we also have $H' \cap \tau^{-k}N = 0$ for any k, hence $0 = H' \cap \bigcup_{k \geqslant 0}(\tau^{-k}N) = H' \cap H = H'$. □

I will now justify the previous statements sketched above. The main point is to remark that the intersection complex $\mathrm{IC}_X(\mathbb{Q})$ sits between $j_!\mathbb{Q}_U$ and $\boldsymbol{R}j_*\mathbb{Q}_U$:

$$j_!\mathbb{Q}_U \longrightarrow \mathrm{IC}_X(\mathbb{Q}) \longrightarrow \boldsymbol{R}j_*\mathbb{Q}_U,$$

and that the cones of both morphisms do not have vanishing cycles either, so that their direct images have constant *perverse* cohomology sheaves. Note that, in the case of GL-N polynomials considered in §4.1, $\mathrm{IC}_X(\mathbb{Q})$ reduces to $\mathbb{Q}_X$, as X is smooth. I will now use the perverse shift convention, as in §2.1. So the perverse cohomology of

(1) Of course, in the case of GL-N polynomials considered in §4.1, considering the cohomology is enough.

(2) The proof goes the other way, by proving first that $j_*\mathscr{V}_{!*}$ is a simple perverse sheaf.

the cone of $\boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q}) \to \boldsymbol{R}f_*{}^p\mathbb{Q}_U$ consists of constant sheaves shifted by one. On the other hand, the absence of vanishing cycles above implies that ${}^p\mathscr{H}^\ell \boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q})$ and ${}^p\mathscr{H}^\ell \boldsymbol{R}f_*{}^p\mathbb{Q}_U$ are constant sheaves shifted by one if $\ell \neq 0$. In the long exact sequence of perverse cohomology

$$\begin{aligned} \cdots \longrightarrow {}^p\mathscr{H}^{-1}\boldsymbol{R}f_*{}^p\mathbb{Q}_U \longrightarrow {}^p\mathscr{H}^0\mathscr{C} \\ \longrightarrow {}^p\mathscr{H}^0\boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q}) \longrightarrow {}^p\mathscr{H}^0\boldsymbol{R}f_*{}^p\mathbb{Q}_U \\ \longrightarrow {}^p\mathscr{H}^1\mathscr{C} \longrightarrow {}^p\mathscr{H}^1\boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q}) \longrightarrow \cdots \end{aligned}$$

all the terms but those in the middle are constant sheaves (shifted by one), hence we get a short exact sequence

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow {}^p\mathscr{H}^0\boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q}) \longrightarrow {}^p\mathscr{H}^0\boldsymbol{R}f_*{}^p\mathbb{Q}_U \longrightarrow \mathscr{F}_2 \longrightarrow 0$$

in the perverse category, where $\mathscr{F}_1, \mathscr{F}_2$ are constant sheaves shifted by 1. By the Riemann-Hilbert correspondence, this gives the exact sequence

$$0 \longrightarrow N_1 \longrightarrow M_{!*} \longrightarrow M \longrightarrow N_2 \longrightarrow 0,$$

with N_1, N_2 isomorphic to powers of $(\mathbb{C}[t], d)$. □

Example 5.1.3. Consider the function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ given by $f(x_0, \dots, x_n) = \sum x_i^2$. We have $r = 1$ and $p_1 = 0$.

- If $n = 1$, then ${}^p\mathscr{H}^0\boldsymbol{R}f_*{}^p\mathbb{Q}_U = j_!{}^p\mathbb{Q}_{\mathbb{A}^1 \smallsetminus \{p_1,\dots,p_r\}}$, ${}^p\mathscr{H}^0\boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q}) = i_{0,*}\mathbb{Q}_0$, so $\mathscr{V}_{!*} = 0$, $\mathscr{F}_1 = 0$ and $\mathscr{F}_2 = {}^p\mathbb{Q}_{\mathbb{A}^1}$.
- If $n = 2$, then ${}^p\mathscr{H}^0\boldsymbol{R}f_*{}^p\mathbb{Q}_U = j_*\mathscr{V}$, where $\mathscr{V}$ has rank one, ${}^p\mathscr{H}^0\boldsymbol{R}F_* \operatorname{IC}_X({}^p\mathbb{Q}) = j_*\mathscr{V}_{!*}$, where $\mathscr{V}_{!*}$ has rank two, $\mathscr{F}_1$ has rank one and $\mathscr{F}_2 = 0$.

5.2. Variation of Hodge structure

5.2.a. Definition. Let us fix a finite set of points $\{p_1, \dots, p_r\}$ in $\mathbb{A}^1$, and let us set $X = \mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$. Let $\mathscr{V}$ be a locally constant sheaf on X of rank d (i.e., a representation of $\pi_1(X) \to \operatorname{GL}_d(\mathbb{C})$). It defines a holomorphic vector bundle V on X, equipped with a holomorphic connection ∇, such that $V^\nabla = \mathscr{V}$.

Let $H = \mathscr{C}_X^\infty \otimes_{\mathscr{O}_X} V$ be the associated C^∞ bundle, equipped with the C^∞ connection $D = \nabla + \overline{\partial}$. A *polarized variation of Hodge structure* of weight $w \in \mathbb{Z}$ on V consists of the following data:

(1) a *decomposition* of the C^∞ bundle H, usually denoted by $H = \bigoplus_p H^{p,w-p}$, by C^∞ subbundles,

(2) a *Hermitian metric* h on H,

subject to the following conditions:

(a) the decomposition is orthogonal with respect to h and the nondegenerate Hermitian form $S = \oplus_p i^w(-1)^p h_{|H^{p,w-p}}$ is D-flat,

(b) (Griffiths' transversality)

$$\begin{aligned} \nabla(H^{p,w-p}) &\subset (H^{p,w-p} \oplus H^{p-1,w-p+1}) \otimes_{\mathscr{O}_X} \Omega^1_X \\ \overline{\partial}(H^{p,w-p}) &\subset (H^{p,w-p} \oplus H^{p+1,w-p-1}) \otimes_{\mathscr{O}_{\overline{X}}} \Omega^1_{\overline{X}}. \end{aligned} \tag{5.2.1}$$

Given such a variation, we set $F^pH = \bigoplus_{p'\geqslant p} H^{p',w-p'}$. This defines a holomorphic subbundle of H (i.e., $\overline{\partial}F^pH \subset F^pH \otimes \Omega^1_{\overline{X}}$, and $F^pV := \operatorname{Ker} \overline{\partial}_{|F^pH}$ is a holomorphic subbundle of V. Moreover, $\nabla F^pV \subset F^{p-1}V \otimes \Omega^1_X$.

5.2.b. Extension to $\mathbb{P}^1$. The previous data were holomorphic on $\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$. We wish to extend them to $\mathbb{P}^1$ and apply to them a GAGA theorem in order to get algebraic objects.

Extension of (V, ∇) as a meromorphic bundle. According to Deligne **[6]**, there exists, up to isomorphism, a unique meromorphic bundle on $\mathbb{P}^1$ with poles at $D = \{p_1, \dots, p_r, \infty\}$, that I denote by $\widetilde{\mathscr{M}}$, that is a locally free $\mathscr{O}_{\mathbb{P}^1}(*D)$-module) equipped with a meromorphic connection having regular singularities at D, which restricts to (V, ∇) on $X = \mathbb{P}^1 \smallsetminus D$.

According to a theorem of Schmid (**[40]**), $\widetilde{\mathscr{M}}$ is characterized as the subsheaf of j_*V ($j : X \hookrightarrow \mathbb{P}^1$) consisting of local sections whose h-norm has moderate growth along D.

Extension of (V, ∇) as a $\mathbb{C}[t]\langle\partial_t\rangle$-module. One can regard $\widetilde{\mathscr{M}}$ as a $\mathscr{D}_{\mathbb{P}^1}$-module. As such, it contains a submodule $\mathscr{M}$ such that $\mathrm{DR}\,\mathscr{M} = j_*\mathscr{V}$. We will consider its localization at ∞, denoted $\mathscr{M}(*\infty)$ (which coincides with $\widetilde{\mathscr{M}}$ near ∞). By an analogue of GAGA, $\mathscr{M}$ is algebraic, and $M := \Gamma(\mathbb{P}^1, \mathscr{M}(*\infty))$ is a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.

Extension of the filtration. Schmid results (**[40]**) applied to the filtration $F^\bullet V$, together with the previous algebraization procedure, produce a *good filtration* (written in a decreasing way) $F^\bullet M$ of M as a $\mathbb{C}[t]\langle\partial_t\rangle$-module.

Extension of the flat sesquilinear pairing. Lastly, the flat sesquilinear pairing S extends as a sesquilinear pairing

$$M \otimes_{\mathbb{C}} \overline{M} \longrightarrow \mathscr{S}'(\mathbb{A}^1)$$

which takes values in the space of *temperate distributions* (Schwartz space) and is $\mathbb{C}[t]\langle\partial_t\rangle \otimes_{\mathbb{C}} \overline{\mathbb{C}[t]\langle\partial_t\rangle}$-linear.

5.3. Fourier-Laplace transform of the algebraic data

Given $(M, F^\bullet M, S)$, where M is any regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module, $F^\bullet M$ is a good filtration of M (written in a decreasing way) and $S : M \otimes_{\mathbb{C}} \overline{M} \to \mathscr{S}'(\mathbb{A}^1)$ is a $\mathbb{C}[t]\langle\partial_t\rangle \otimes_{\mathbb{C}} \overline{\mathbb{C}[t]\langle\partial_t\rangle}$-linear morphism, we will define the Fourier-Laplace transform of this set of data.

5.3.a. Laplace transform $\widehat{M}$ of M. This is very simple: using the identification $\mathbb{C}[t]\langle\partial_t\rangle \simeq \mathbb{C}[\tau]\langle\partial_\tau\rangle$ given by $t \mapsto -\partial_\tau$ and $\partial_t \to \tau$, we regard M as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module and call it $\widehat{M}$. Later, we will consider $G := \mathbb{C}[\tau,\tau^{-1}] \otimes_{\mathbb{C}[\tau]} \widehat{M}$, equipped with its natural connection.

5.3.b. Laplace transform of the good filtration $F^\bullet M$. The result of this Laplace transform will be the *Brieskorn lattice* of the good filtration. This is a *free* $\mathbb{C}[\theta]$-module $G_0^{(F^\bullet)}$ of G such that $G = \mathbb{C}[\theta,\theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0^{(F^\bullet)}$. Let us explain its construction.

For any $\mathbb{C}[t]$-submodule L of M of finite type, we regard it as a $\mathbb{C}[\partial_\tau]$-submodule of $\widehat{M}$, and we denote by L' its image in G via the natural morphism $\widehat{M} \to G$. We then set

$$G_0^{(L)} = \sum_{j\geqslant 0} \theta^j L' \subset G.$$

This is the $\mathbb{C}[\theta]$-submodule of G generated by L'. The action of ∂_τ on L' naturally extends as an action of $\theta^2\partial_\theta$ on $G_0^{(L)}$. Using that M has a regular singularity at infinity, one proves that $G_0^{(L)}$ is *free of finite rank* over $\mathbb{C}[\theta]$.

We apply this to any F^pM, and because F^pM is good, we get $\theta G_0^{(F^{p-1})} \supset G_0^{(F^p)}$ for any p, with equality for $p \ll 0$. it is therefore natural to apply a normalization:

$$G_0^{(F^\bullet,p)} := \theta^{-p} G_0^{(F^p)}$$

so that $G_0^{(F^\bullet,p-1)} \supset G_0^{(F^\bullet,p)}$ with equality for $p \ll 0$. We denote by $G_0^{(F^\bullet)}$ the $\mathbb{C}[\theta]$-module $G_0^{(F^\bullet,p)}$ for $p \ll 0$. It is equipped with an action of $\theta^2\partial_\theta$ and satisfies $\mathbb{C}[\theta,\theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0^{(F^\bullet)} = G$.

***Example 5.3.1* (Relation with §3.2).** Assume that $f : U \to \mathbb{A}^1$ is tame. Let $M = M^{(n+1)}$ be the Gauss-Manin system of f, as defined by (3.1.4). By the general theory of mixed Hodge modules **[38]**, M has a distinguished good filtration $F^{\mathrm{H}}_\bullet M$, called its Hodge filtration, which restricts, for each $t \notin C_f$, to the Hodge filtration (taken in an increasing way) of the mixed Hodge structure on $H^n(f^{-1}(t),\mathbb{C})$. Let us set $F_pM = F^{\mathrm{H}}_{p-(n+1)}M$ for any p. Then,

$$G_0^{(F^\bullet)} = G_0.$$

This is not obvious, as the construction of F_pM takes into account the behaviour near $X \smallsetminus U$ (e.g., the order of the pole along $X \smallsetminus U$ of algebraic differential forms on U), while that of G_0 does not rely on any restriction along $X \smallsetminus U$.

5.3.c. Fourier transform of the flat sesquilinear pairing S. Note that $\overline{t\tau} - t\tau = -2i\,\mathrm{Im}(t\tau) = -2i(\mathrm{Re}\,t\,\mathrm{Im}\,\tau + \mathrm{Im}\,t\,\mathrm{Re}\,\tau)$. Then the Fourier transform $\mathcal{F}_t$ with kernel $\exp(\overline{t\tau} - t\tau)\frac{i}{2\pi} dt \wedge d\bar{t}$ is an isomorphism between $\mathscr{S}'(\mathbb{A}^1)$ (t-plane) and $\mathscr{S}'(\widehat{\mathbb{A}}^1)$ (τ-plane). Given a 2-form ψ in the Schwartz space $\mathscr{S}(\widehat{\mathbb{A}}^1)$ (i.e., $\psi = \chi(\tau) d\tau \wedge d\overline{\tau}$

with χ C^∞, rapidly decaying as well as all its derivatives when $\tau \to \infty$), we set, for $u \in \mathscr{S}'(\mathbb{A}^1)$,

$$\langle \mathcal{F}_t u, \psi \rangle := \langle u, \mathcal{F}_\tau \psi \tfrac{i}{2\pi} dt \wedge d\bar{t} \rangle, \quad \text{with } \mathcal{F}_\tau \psi = \int e^{\overline{t\tau} - t\tau} \psi.$$

(Recall that $\mathcal{F}_\tau \psi$ belongs to $\mathscr{S}(\mathbb{A}^1)$.) If $S : M \otimes_{\mathbb{C}} \overline{M} \to \mathscr{S}'(\mathbb{A}^1)$ is a sesquilinear pairing, we denote by $\widehat{S}$ the composition $\mathcal{F}_t \circ S$ of S with the Fourier transform of temperate distributions. Then $\widehat{S}$ becomes a $\mathbb{C}[\tau]\langle \partial_\tau \rangle \otimes_{\mathbb{C}} \overline{\mathbb{C}[\tau]\langle \partial_\tau \rangle}$-linear morphism (that we call a *sesquilinear pairing*)

$$\widehat{S} : \widehat{M} \otimes_{\mathbb{C}} \overline{\iota^+ \widehat{M}} \longrightarrow \mathscr{S}'(\widehat{\mathbb{A}}^1).$$

where ι is the map $\tau \mapsto -\tau$ (the ι^+ is needed as we use the kernel $e^{\overline{t\tau}}$ for the Laplace transform of $\overline{M}$, not $e^{-\overline{t\tau}}$). If we localize we respect to τ, we get a sesquilinear pairing

$$\widehat{S} : G \otimes_{\mathbb{C}} \overline{\iota^+ G} \longrightarrow \mathscr{S}'(\widehat{\mathbb{A}}^{1*}),$$

where $\mathscr{S}'(\widehat{\mathbb{A}}^{1*})$ denotes the set of distributions on $\widehat{\mathbb{A}}^{1*}$ having moderate growth at $\tau = 0$ and $\tau = \infty$. In fact, the distributions which are in the image of $\widehat{S}$ do not have singularities on $\widehat{\mathbb{A}}^{1*}$, hence they are C^∞ there.

LECTURE 6

THE 'NEW SUPERSYMMETRIC INDEX' OF A TAME FUNCTION

We continue to investigate the Fourier-Laplace transform of a polarized variation of Hodge structure on $X = \mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$. We keep the notation introduced in §5.2. Starting from a polarized variation of Hodge structure $(V, F^\bullet V, S)$ on X, we have obtained in §5.3 a free $\mathbb{C}[\theta, \theta^{-1}]$-module G with connection, a $\mathbb{C}[\theta]$-lattice $G_0^{(F^\bullet)}$ of it on which the connection has a pole of order 2 at most, and a sesquilinear pairing $\widehat{S}$.

The ideas developed here come from various sources. The idea of the twistor construction with metric goes back to C. Simpson [**41**]. Applying this to singularities or tame functions goes back to the work of Cecotti and Vafa [**5**], and has been emphasized by C. Hertling [**19**], followed by his work with C. Sevenheck ([**20**] and recent preprints). On the other hand, the application to tame functions has been developed in [**32, 34, 35**]. However, this is the visible part of an iceberg (cf. [**25**], see also [**36**]).

6.1. Positivity property of the Fourier-Laplace transform

6.1.a. The twistor construction. In order to express the positivity property in the Fourier-Laplace setting, which should be analogous to the positivity property of the Hermitian metric h in the definition of §5.2.a, let us look more closely at $\widehat{S}$. Working in the analytic topology on $\widehat{\mathbb{A}}^{1*}$, G becomes a holomorphic vector bundle $\mathscr{G}$ with holomorphic connection, $\widehat{S}$ becomes a sesquilinear pairing

$$\widehat{S} : \mathscr{G} \otimes_{\mathbb{C}} \overline{\iota^* \mathscr{G}} \longrightarrow \mathscr{C}^\infty_{\widehat{\mathbb{A}}^{1*}}.$$

Let $\mathcal{G}$ denote the sheaf of horizontal sections of $\mathscr{G}$. This is a local system on $\widehat{\mathbb{A}}^{1*}$, and $\widehat{S}$, being compatible with the connection and its conjugate, induces a sesquilinear pairing (of sheaves of $\mathbb{C}$-vector spaces)

$$\widehat{S} : \mathcal{G} \otimes_{\mathbb{C}} \overline{\iota^{-1} \mathcal{G}} \longrightarrow \mathbb{C}_{\widehat{\mathbb{A}}^{1*}}.$$

Let $\mathbf{S}$ denote the circle $|\theta| = 1$. Clearly, giving $\widehat{S}$ is equivalent to giving $(\widehat{S})_{|\mathbf{S}}$. Now, on $\mathbf{S}$, the map ι is also equal to the restriction to $\mathbf{S}$ of the map $\sigma : \theta \mapsto -1/\overline{\theta}$. Taking the same way in the opposite direction, we find that $\widehat{S}$ induces a pairing

$$\widehat{C} : \mathscr{G} \otimes_{\mathscr{O}_{\widehat{\mathbb{A}}^{1*}}} \overline{\sigma^*\mathscr{G}} \longrightarrow \mathscr{O}_{\widehat{\mathbb{A}}^{1*}}, \tag{6.1.1}$$

where now every term is holomorphic, and which is compatible with the holomorphic connection.

Theorem 6.1.2. *Assume, for simplicity, that the weight w of the given polarized variation of Hodge structure on $\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$ is zero. Then,*

(1) *$\widehat{C}$ is nondegenerate, hence defines a gluing of $\mathscr{G}_0^\vee$ with $\sigma^*\overline{\mathscr{G}_0}$ giving rise to a holomorphic bundle $\mathscr{H}^\vee$ on $\mathbb{P}^1$; let $\mathscr{H}$ be the dual bundle;*

(2) *this holomorphic bundle $\mathscr{H}$ is trivializable, and we set $H = \Gamma(\mathbb{P}^1, \mathscr{H})$,*

(3) *the canonical identifications*

$$\mathscr{G}_0/\theta\mathscr{G}_0 \xleftarrow{\sim} \Gamma(\mathbb{P}^1, \mathscr{H}) = H \xrightarrow{\sim} \sigma^*\overline{\mathscr{G}_0}^\vee/\tau\sigma^*\overline{\mathscr{G}_0^\vee} = \overline{\mathscr{G}_0/\theta\mathscr{G}_0}^\vee$$

define a Hermitian form on the vector space $\mathscr{G}_0/\theta\mathscr{G}_0 = G_0/\theta G_0$, hence on H and on any fibre $\mathscr{G}_0/(\theta - \theta_o)\mathscr{G}_0$, $\theta_o \in \mathbb{C}$,

(4) *this Hermitian form is positive definite.*

Exercise 6.1.3. This exercise shows that, in the case of a Hodge structure (cf. §5.2.a), going from S to h can be regarded as a particular case of the construction used in Theorem 6.1.2.

Let H be a complex vector space with a decomposition $H = \bigoplus_p H^{p,-p}$ (we assume here that the weight is zero) and a sesquilinear pairing $S : H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ such that the decomposition is S-orthogonal and the sesquilinear form h defined by $h_{|H^{p,-p}} = (-1)^p S_{|H^{p,-p}}$ and $h(H^{p,-p}, H^{q,-q}) = 0$ if $q \neq p$, is Hermitian positive definite. Set $F^pH = \bigoplus_{p' \geqslant p} H^{p,-p}$. In other words, $(H, F^\bullet H, S)$ is a polarized complex Hodge structure of weight 0.

(1) Set $G_0 = \bigoplus_k (F^pH)\theta^{-p}$. Show that G_0 is a free $\mathbb{C}[\theta]$-module isomorphic to $\bigoplus_r \big(\theta^{-r}\mathbb{C}[\theta] \otimes_{\mathbb{C}} H^{r,-r}\big)$.

(2) Show that $G := \mathbb{C}[\theta, \theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0$ is isomorphic to $\mathbb{C}[\theta, \theta^{-1}] \otimes_{\mathbb{C}} H$.

(3) Extend S in a $\mathbb{C}[\theta, \theta^{-1}]$-linear way to get

$$\mathscr{S} : G \otimes_{\mathbb{C}[\theta,\theta^{-1}]} \sigma^*\overline{G} \longrightarrow \mathbb{C}[\theta, \theta^{-1}].$$

(4) Show that the bundle $\mathscr{H}$ defined (after duality) by the gluing $\mathscr{S}$ is trivializable.

(5) Identify H with $\Gamma(\mathbb{P}^1, \mathscr{H})$ and h with the Hermitian form of Theorem 6.1.2(3).

(6) Identify $G_0/\theta G_0$ with $\mathrm{gr}^F H$.

6.1.b. Indication for the proof of Theorem 6.1.2. Let me clarify the meaning of the theorem. Recall the construction of G: starting from the $\mathbb{C}[t]\langle\partial_t\rangle$-module M constructed in §5.2.b, obtained from the $\mathscr{D}_{\mathbb{P}^1}$-module $\mathscr{M}(*\infty)$ whose de Rham complex is $j_*\mathscr{V}$ near $p_1,\dots,p_r$ and $\boldsymbol{R}j_*V$ near ∞, we can regard G as the cokernel of the morphism

$$\mathbb{C}[\theta,\theta^{-1}]\otimes_{\mathbb{C}} M \xrightarrow{\partial_t-\theta^{-1}} \mathbb{C}[\theta,\theta^{-1}]\otimes_{\mathbb{C}} M,$$

that we can also write as $e^{t/\theta}\partial_t e^{-t/\theta}$, and so, for any $\theta_o\neq 0$, the fibre $G/(\theta-\theta_o)G$ is the cokernel of

$$M \xrightarrow{e^{t/\theta_o}\partial_t e^{-t/\theta_o}} M.$$

In the theorem, we remark that $\mathscr{G}$ coincides with $\mathscr{G}_0$ away from $\theta=0$, so for any $\theta_o\neq 0$, $\mathscr{G}_0/(\theta-\theta_o)\mathscr{G}_0=\operatorname{Coker}(e^{t/\theta_o}\partial_t e^{-t/\theta_o})$. In particular, for any such θ_o, we should obtain a Hermitian metric on this cokernel. Moreover, we should have a canonical identification of these cokernels with metric, when θ_o varies in $\mathbb{C}^*$. Lastly, this identification should extend to $\theta_o=0$, but now on $\mathscr{G}_0/\theta\mathscr{G}_0$ (because $\mathscr{G}$ is not defined at $\theta=0$).

I will only explain how to get a Hermitian metric on the cokernel when $\theta_o=1$, by comparing with the proof by Zucker **[44]** of the construction of a polarized Hodge structure on the cohomology $H^1(\mathbb{P}^1,j_*\mathscr{V})$ (cf. notation of §5.2.b). I will use the abbreviation *regular case* for the case considered by Zucker, and *exponentially twisted case* for the case considered in Theorem 6.1.2.

Step zero: the metric and the connection

• *Regular case.* On the bundle V we consider the metric h and the connection ∇. Note that ∇ is not a metric connection. This is related to the fact that ∇ does not preserve the Hodge filtration F^pV.

• *Exponentially twisted case.* We now twist the connection ∇ and consider ${}^F\nabla:=e^t\nabla e^{-t}=\nabla-dt$. Correspondingly, we change the metric near ∞ and consider ${}^Fh:=e^{2\operatorname{Re}t}h$.

Step one: the meromorphic L^2 complex and Poincaré lemma

• *Regular case.* Considering the meromorphic de Rham complex $(\Omega^\bullet_{\mathbb{P}^1}(\widetilde{\mathscr{M}}),\nabla)$, we get a resolution of the complex $\boldsymbol{R}j_*\mathscr{V}$ (by construction). Among the meromorphic sections, let us consider those which are locally L^2 (on $\mathbb{P}^1$): in order to define such a notion, we need a metric on V (take h) and a metric on forms (take a complete metric on X locally equivalent to the Poincaré metric at $p_1,\dots,p_r,\infty$). We then denote by $\Omega^1_{\mathbb{P}^1}(\widetilde{\mathscr{M}})_{(2)}$ these L^2-forms, and by $\widetilde{\mathscr{M}}_{(2)}$ the sections of $\widetilde{\mathscr{M}}$ whose restrictions to $\mathbb{A}^1\smallsetminus\{p_1,\dots,p_r\}$ are L^2 near the punctures, and whose ∇-derivative is L^2, i.e., lies in $\Omega^1_{\mathbb{P}^1}(\widetilde{\mathscr{M}})_{(2)}$. Therefore we get the *meromorphic L^2 de Rham complex*

$$\mathrm{DR}(\widetilde{\mathscr{M}})_{(2)}=\{0\longrightarrow\widetilde{\mathscr{M}}_{(2)}\xrightarrow{\ \nabla\ }\Omega^1_{\mathbb{P}^1}(\widetilde{\mathscr{M}})_{(2)}\longrightarrow 0\}.$$

• *Exponentially twisted case.* Same construction with ${}^Fh$ instead of h and ${}^F\nabla$ instead of ∇. Note that the only difference is at infinity, as e^t is holomorphic and invertible at finite distance, and the twist is inessential there. We get $\mathrm{DR}(\widetilde{{}^F\mathscr{M}})_{(2)}$.

***Lemma 6.1.4* (Meromorphic L^2 Poincaré lemma)**

• *Regular case. The complex* $\mathrm{DR}(\widetilde{\mathscr{M}})_{(2)}$ *is a resolution of* $j_*\mathscr{V}$*, i.e., is quasi-isomorphic to* $\mathrm{DR}\,\mathscr{M}$

• *Exponentially twisted case. The complex* $\mathrm{DR}(\widetilde{{}^F\!\mathscr{M}})_{(2)}$ *is quasi-isomorphic to* $\mathrm{DR}(\mathscr{M}(*\infty), \nabla - dt)$.

Taking global hypercohomology we get:

• *Regular case.* $\mathbb{H}^*(\mathbb{P}^1, \mathrm{DR}(\widetilde{\mathscr{M}})_{(2)}) = H^*(\mathbb{P}^1, j_*\mathscr{V})$,

• $\mathbb{H}^1(\mathbb{P}^1, \mathrm{DR}(\widetilde{{}^F\!\mathscr{M}})_{(2)}) = G/(\theta - 1)G$.

Step two: the L^2 complex and Poincaré lemma

• *Regular case.* We now work in the C^∞, and even distributional, context. We consider the bundle $H = \mathscr{C}_X^\infty \otimes V$ ($X = \mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$) with its flat connection $D := \nabla + \overline{\partial}$ and its metric h. The L^2 complex $\mathscr{L}_{(2)}(H, D, h)$ on $\mathbb{P}^1$ is made of sections of H (or $H \otimes dt$, $H \otimes d\bar{t}$, $H \otimes (dt \wedge d\bar{t})$) which are L^2 as well as their D-derivative, with respect to the metric h and the Poincaré metric. It is quite clear that we have an inclusion of complexes $\mathrm{DR}(\widetilde{\mathscr{M}})_{(2)} \hookrightarrow \mathscr{L}_{(2)}(H, D, h)$.

• Same construction with ${}^F h$ and ${}^F\nabla$, to get $\mathscr{L}_{(2)}(H, {}^F D, {}^F h)$ and an inclusion $\mathrm{DR}(\widetilde{{}^F\!\mathscr{M}})_{(2)} \hookrightarrow \mathscr{L}_{(2)}(H, {}^F D, {}^F h)$.

***Lemma 6.1.5* (L^2 Poincaré lemma).** *Both inclusions are quasi-isomorphisms.*

Step three: conclusion. By the two Poincaré lemmas, we conclude that the L^2 cohomology is finite dimensional. It is then possible to apply Hodge Theory to the Laplace operator attached to (H, D, h) (resp. $(H, {}^F D, {}^F h)$) and the chosen complete metric on X. This gives representatives of cohomology classes in $H^*(\mathbb{P}^1, j_*\mathscr{V})$ (resp. elements of $G/(\theta - 1)G$) which are *harmonic*. The L^2 metric on the L^2 spaces (the terms of the L^2 complexes) induces a metric on the harmonic sections: this is the desired metric.

Idea for the end of the proof. Such an argument can be extended to any θ_o, provided $\theta_o \neq 0$. The identification of the spaces and metrics for various θ_o comes from the fact that the L^2 complexes have the same terms (but not the same differentials) and the corresponding Laplacians are positively proportional, hence give rise to the *same* harmonic sections.

At $\theta_o = 0$, this argument breaks down and one has to invoke another argument, called the L^2 Dolbeault lemma.

6.1.c. Application to tame functions. Recall the philosophy developed up to now: although the cohomology $H^*(f^{-1}(t), \mathbb{C})$ of a generic fibre of a tame function $f : U \to \mathbb{A}^1$ is equipped with a *mixed Hodge structure*, and more precisely although the Gauss-Manin system M underlies a *mixed Hodge module*, its Laplace transforms behaves as if M were a pure Hodge module. This also holds for the polarization.

Therefore, by the general result above we get a positive definite Hermitian form on $G_0/\theta G_0 = \Omega^{n+1}(U)/df\wedge\Omega^n(U)$. This form should have a geometric expression.

For any $t\in\mathbb{A}^1\smallsetminus C_f$, the fibre $f^{-1}(t)$ is smooth, hence there exists a canonical duality pairing

$$H^n_c(f^{-1}(t),\mathbb{C})\otimes_{\mathbb{C}} H^n(f^{-1}(t),\mathbb{C})\longrightarrow\mathbb{C},$$

that we can make sesquilinear:

$$P_t: H^n_c(f^{-1}(t),\mathbb{C})\otimes_{\mathbb{C}}\overline{H^n(f^{-1}(t),\mathbb{C})}\longrightarrow\mathbb{C}.$$

This pairing is realized at the level of differential forms: if α,β are C^∞ forms of degree n on $f^{-1}(t)$, and if α has compact support, then

$$P_t([\alpha],[\overline{\beta}])=\int_{f^{-1}(t)}\alpha\wedge\overline{\beta}.$$

This is the restriction of a duality at the level of complexes (sesquilinear Poincaré-Verdier duality)

$$P:\boldsymbol{R}f_!{}^p\mathbb{C}_U\otimes_{\mathbb{C}}\overline{\boldsymbol{R}f_*{}^p\mathbb{C}_U}\longrightarrow\mathbb{C}_{\mathbb{A}^1}[2].$$

Let us apply the topological Laplace transform of §1.6. Both complexes $\boldsymbol{R}f_!{}^p\mathbb{C}_U$ and $\boldsymbol{R}f_*{}^p\mathbb{C}_U$ have the same transform, which is identified with $\mathcal{G}[1]$ (where $\mathcal{G}$ is defined in §6.1.a). Because of the conjugation, the Laplace transform of $\overline{\boldsymbol{R}f_*{}^p\mathbb{C}_U}$ is $\overline{\iota^{-1}\mathcal{G}}$, and we get a nondegenerate sesquilinear pairing

$$\widehat{P}:\mathcal{G}\otimes\overline{\iota^{-1}\mathcal{G}}\longrightarrow\mathbb{C}_{\widehat{\mathbb{A}}^{1*}}.$$

Notice that $\widehat{P}$ can be computed from the restriction

$$P_!:\boldsymbol{R}f_!{}^p\mathbb{C}_U\otimes_{\mathbb{C}}\overline{\boldsymbol{R}f_!{}^p\mathbb{C}_U}\longrightarrow\mathbb{C}_{\mathbb{A}^1}[2].$$

One can also give the following more concrete interpretation of $\widehat{P}$. Let us fix $\theta_o\in\widehat{\mathbb{A}}^{1*}$. Then the stalk

$$\widehat{P}_{\theta_o}:\mathcal{G}_{\theta_o}\otimes_{\mathbb{C}}\overline{\mathcal{G}_{-\theta_o}}\longrightarrow\mathbb{C}$$

is identified with the natural duality pairing

$$H^{n+1}_{\Phi_{f_*,\theta_o}}(U,\mathbb{C})\otimes_{\mathbb{C}}\overline{H^{n+1}_{\Phi_{f_!,-\theta_o}}(U,\mathbb{C})}\longrightarrow H^{2n+2}_c(U,\mathbb{C})\xrightarrow{\int_{[U]}}\mathbb{C},$$

where Φ_{f_*,θ_o} denotes the family of closed sets of U which are contained in $f^{-1}(F)$, $F\in\Phi_{\theta_o}$, and $\Phi_{f_!,-\theta_o}$ is the subfamily of $\Phi_{f_*,-\theta_o}$ consisting of those sets on which f is proper. In particular, $\Phi_{f_*,\theta_o}\cap\Phi_{f_!,-\theta_o}$ consists of closed sets on which f is proper and whose image by f is compact, hence is the family of compact sets.

Set $\widehat{S}=[(-1)^{n(n+1)/2}/(2\pi i)^{n+1}]\widehat{P}$, let $\widehat{C}:\mathscr{G}\otimes_{\mathscr{O}_{\widehat{\mathbb{A}}^{1*}}}\sigma^*\overline{\mathscr{G}}\to\mathscr{O}_{\widehat{\mathbb{A}}^{1*}}$ be the sesquilinear pairing given by (6.1.1). Using the Brieskorn lattice G_0 of f, we construct, as in Theorem 6.1.2, a holomorphic bundle $\mathscr{H}$.

***Theorem 6.1.6* (Positivity theorem for tame functions).** *The conclusion of Theorem* 6.1.2 *applies to* $\mathscr{H}$.

6.2. The 'new supersymmetric index'

6.2.a. The self-adjoint operator $\mathscr{Q}$. Recall that the pairing (6.1.1) is compatible with the connections. This is equivalent to saying that the meromorphic connections on $\mathscr{G}_0^\vee$ and that on $\sigma^*\overline{\mathscr{G}_0}$ glue together as a meromorphic connection on $\mathscr{H}$, with a pole of order two at $\theta = 0$ and a pole of order two at $\theta = \infty$, and no other pole.

As $\mathscr{H}$ is trivial, we can write $\mathscr{H}^\vee = H \otimes \mathscr{O}_{\mathbb{P}^1}$, and using this trivialization, the connection has to be written in the following form:

$$\nabla_{\partial_\theta} = \partial_\theta + \mathscr{U}/\theta^2 - \mathscr{Q}/\theta - \mathscr{U}^\dagger,$$

where $\mathscr{U}, \mathscr{Q}, \mathscr{U}^\dagger$ are endomorphisms of H. Moreover, $\mathscr{U}^\dagger$ is the h-adjoint of $\mathscr{U}$ and $\mathscr{Q}$ is self-adjoint. This operator, which was first introduced by Cecotti and Vafa, is called the 'new supersymmetric index' of the variation of Hodge structure. In the case of the Brieskorn lattice of a tame function, as in §6.1.c, it is called the 'new supersymmetric index' of f.

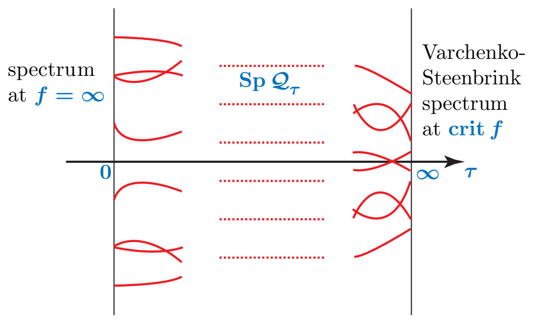
The spectrum of $\mathscr{Q}$ consists of real numbers. It is in general difficult to compute explicitly. I denote its characteristic polynomial by Susy.

6.2.b. Limits of the characteristic polynomial of $\mathscr{Q}$ by rescaling f. Instead of computing the polynomial Susy_f for a given tame function, one may try to compute the polynomial $\mathrm{Susy}_{\tau f}$ for any rescaling parameter $\tau \in \mathbb{C}^*$. One can show that the coefficients of this polynomial are real-analytic with respect to τ. The following was conjectured by C. Hertling:

Theorem 6.2.1. *Assume $f : U \to \mathbb{C}$ is cohomologically tame. Then,*

$$\lim_{\tau\to 0} \mathrm{Susy}_{\tau f}(T) = \mathrm{SP}_f^\infty(T - (n+1)/2),$$
$$\lim_{\tau\to \infty} \mathrm{Susy}_{\tau f}(T) = \mathrm{SP}_f^0(T - (n+1)/2),$$

where SP_f^∞ is the spectral polynomial of the Brieskorn lattice of f as in Definition 3.3.7, and SP_f^0 is the product of all spectral polynomials of f at its critical points.



LECTURE 7

MONODROMY AT INFINITY

In this lecture, I will consider the properties of the monodromy at infinity of a function $f : U \to \mathbb{A}^1$ which is not supposed to be tame. More precisely, I will consider the monodromy around a circle with $|t|$ large of $H^k(U, f^{-1}(t), \mathbb{Q})$. It is denoted by $T_\infty^{(k)}$.

7.1. The monodromy at infinity is important

Let us start with a general result. Let $\mathscr{F}$ be a $\mathbb{Q}$-constructible sheaf (*not* a complex) on $\mathbb{A}^1$, with singular points $p_1, \dots, p_r$, so that $\mathscr{F}_{|\mathbb{A}^1 \smallsetminus \{p_1,\dots,p_r\}}$ is a locally constant sheaf of $\mathbb{Q}$-vector spaces.

Proposition 7.1.1. *For such a sheaf, the property that $H^k(\mathbb{A}^1, \mathscr{F}) = 0$ for all k (that is, for $k = 0, 1, 2$) is equivalent to the property that, for $|p_o| \gg 0$ and a suitable system of paths $[p_o, p_i]$ (e.g., a star centered at p_o), the natural morphism $\mathscr{F}_{p_i} \to \mathscr{F}_{p_o}$ is injective and the "diagonal" map $\mathscr{F}_{p_o} \to \bigoplus_{i=1}^r \mathscr{F}_{p_o}/\mathscr{F}_{p_i}$ is an isomorphism.*

The proof is done by using a Mayer-Vietoris presentation of the cohomology of $\mathscr{F}$ (cf. [**22**, Th. 2.29]). Notice that the proof, being purely topological, applies to constructible sheaves of $\mathbb{Z}$-modules. Notice also that a sheaf satisfying the property of the proposition is a perverse complex, when shifted by one (cf. *loc. cit.*).

Corollary 7.1.2. *Let $\mathscr{F}$ be a constructible sheaf with no cohomology on $\mathbb{A}^1$ and let $\rho : \pi_1(\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}) \to \mathrm{Aut}(\mathscr{F}_{p_o})$ be the monodromy representation attached to the local system $\mathscr{F}_{|\mathbb{A}^1 \smallsetminus \{p_1,\dots,p_r\}}$. Then the following are equivalent:*

(1) *ρ is trivial, i.e.,* $\mathrm{image}(\rho) = \{\mathrm{Id}\}$,

(2) $T_\infty = \mathrm{Id}$.

Proof. For the given system of paths, consider the monodromy T_i along a loop following the path $[p_o, p_i - \varepsilon]$, winding once around p_i and going back along the same path to p_o. Then $\mathscr{F}_{p_i} \subset \operatorname{Ker}(T_i - \mathrm{Id})$. On the other hand, using the decomposition above, $\mathscr{F}_{p_i} \simeq \bigoplus_{j\neq i} \mathscr{F}_{p_o}/\mathscr{F}_{p_j}$. Moreover, T_∞ decomposes as $T_\infty = T_1 \cdots T_r$ (cf. [**22**]). From this presentation one deduces that the monodromy representation ρ is determined from the T_i. Then the proof follows from Lemma 7.1.3 below. □

Lemma 7.1.3. *Let F be a vector space having a decomposition $F = \bigoplus_{i=1}^r F_i$ and let $T_1, \dots, T_r$ be automorphisms such that $T_i = \mathrm{Id} + V_i$ and $\forall\, i$, $\operatorname{Ker} V_i \supset \bigoplus_{j\neq i} F_j$. Set $T_\infty = T_1 \cdots T_r$. Then $T_\infty = \mathrm{Id} \Leftrightarrow \forall\, i,\ T_i = \mathrm{Id}$.*

Proof. For any $v \in F_1$, $T_\infty(v) = T_1(v)$. So the assumption $T_\infty = \mathrm{Id}$ implies that $T_1 = \mathrm{Id}$ on F_1. As $T_1 = \mathrm{Id}$ on $\bigoplus_{j\neq 1} F_j$, it follows that $T_1 = \mathrm{Id}$ and the lemma follows by induction. □

Example 7.1.4. Let $f : U \to \mathbb{A}^1$ be a regular function on a smooth affine variety. Let $p : \mathbb{A}^1 \to \operatorname{Spec} \mathbb{C}$ be the constant map to a point. There is a natural morphism $p^{-1}\boldsymbol{R}\Gamma(U, \mathbb{Q}) \to \boldsymbol{R}f_*\mathbb{Q}_U$ whose cone $\mathscr{C}$ has cohomology sheaves $\mathscr{H}^k(\mathscr{C})$ with stalk at $t \notin B_f$ given by $H^k(U, f^{-1}(t))$. Taking global sections over $\mathbb{A}^1$ gives an isomorphism (in the derived category of vector spaces) $\boldsymbol{R}\Gamma(U, \mathbb{Q}) \xrightarrow{\sim} \boldsymbol{R}p_*\boldsymbol{R}f_*\mathbb{Q}_U$ (because $p \circ f$ is the constant map $U \to \operatorname{Spec}\mathbb{C}$). This implies $\boldsymbol{R}\Gamma(\mathbb{A}^1, \mathscr{C}) = 0$, i.e., the complex $\mathscr{C}$ has no global hypercohomology on $\mathbb{A}^1$.

If moreover f is *tame* then, as $\boldsymbol{R}f_*\mathbb{Q}_U$ is compatible with base change (cf. e.g., [**9**, Th. 6.2.15]), the stalk of $\mathscr{H}^k(\mathscr{C})$ at any $t \in \mathbb{A}^1$ is equal to $H^k(U, f^{-1}(t))$. But the tameness condition implies that, for $k \neq n+1$, $H^k(U, f^{-1}(t)) = 0$ (cf. [**33**, Th. 8.1] or [**9**, Cor. 6.2.16]). Therefore the sheaf $\mathscr{H}^{n+1}(\mathscr{C})$ satisfies the property of Proposition 7.1.1, and therefore the monodromy at infinity is equal to identity if and only if the representation $\rho : \pi_1(\mathbb{A}^1 \smallsetminus C_f) \to \operatorname{Aut}\big(H^{n+1}(U, f^{-1}(t_o))\big)$ is trivial.

***Example 7.1.5* (Dimca-Némethi [10]).** Assume now that $U = \mathbb{C}^{n+1}$, but f is any polynomial (possibly not tame). Consider the various representations $\rho^{(k)} : \pi_1(\mathbb{A}^1 \smallsetminus B_f, p_o) \to \operatorname{Aut}\big(H^k(U, f^{-1}(p_o), \mathbb{Q})\big)$. Notice that, if $k \geqslant 1$, $H^k(U, f^{-1}(p_o), \mathbb{Q}) = \widetilde{H}^{k-1}(f^{-1}(p_o), \mathbb{Q})$ (reduced cohomology).

***Theorem* (Dimca-Némethi [10]).** *Fix $k \geqslant 1$. The following are equivalent:*

(1) $T_\infty^{(k)} = \mathrm{Id}$,
(2) $\mathrm{image}(\rho^{(k)}) = \{\mathrm{Id}\}$.

Proof. The proof is first done in homology. One shows that a decomposition exists on the relative homology (i.e., the reduced homology), by using a star with center p_o and with ends the points of B_f. Taking benefit of the representation of homology classes by cycles in the fibres of f, one shows that the homological monodromy T_i satisfies a property dual of that of Lemma 7.1.3, namely, $\mathrm{image}(T_i - \mathrm{Id}) \subset F_i$. Corollary 7.1.2 applies then in the same way and the result follows by duality on cohomology. □

7.2. Size of the Jordan blocks of T_∞, examples in the tame case

In this section, I will assume that f is cohomologically tame.

7.2.a. Multiplication by f and monodromy at infinity. Recall that multiplication by f on its Brieskorn lattice G_0 is interpreted as the action of the differential operator $\theta^2\partial_\theta$. When induced on the finite dimensional vector space $G_0/\theta G_0 = \Omega^{n+1}(U)/df \wedge \Omega^n(U)$, it defines an endomorphism having C_f as its eigenvalues.

On the other hand, f acts on the Laplace Gauss-Manin system as $-\partial_\tau$, and thus shifts by -1 the Deligne filtration $V^\bullet G$, that is,

$$f \cdot V^\alpha G = (-\partial_\tau)V^\alpha G \subset V^{\alpha-1}G.$$

We can then consider the graded action of f:

$$[f] : \mathrm{gr}_V^\alpha(G_0/\theta G_0) \longrightarrow \mathrm{gr}_V^{\alpha-1}(G_0/\theta G_0)$$

which is a *nilpotent endomorphism* on the graded space

$$\mathrm{gr}_V^\bullet(G_0/\theta G_0) := \bigoplus_\alpha \mathrm{gr}_V^\alpha(G_0/\theta G_0).$$

Example 7.2.1 **(Convenient and nondegenerate Laurent polynomials)**
If $U = (\mathbb{C}^*)^{n+1}$ with coordinates $u_0, \dots, u_n$, let ω be the differential form $\dfrac{du_0}{u_0} \wedge \cdots \wedge \dfrac{du_n}{u_n}$. Then the identification $\Omega^{n+1}(U) = \mathscr{O}(U) \cdot \omega$ gives an algebra structure on $\Omega^{n+1}(U)/df \wedge \Omega^n(U) \simeq \mathscr{O}(U)/(\partial f)$ (the Jacobian algebra). The Newton filtration is compatible with this structure, and we get a graded algebra $\mathrm{gr}^{\mathscr{N}}[\mathscr{O}(U)/(\partial f)]$. By definition, f has degree one with respect to the Newton filtration (considered in an increasing way), and induces by multiplication a nilpotent endomorphism on this graded algebra.

Such an algebra occurs in a completely different framework. Let $X(\Gamma)$ be the toric variety defined by the polyhedron Γ, together with its embedding in a projective space with associated canonical bundle $\mathscr{O}(1)$. Then, in many examples, this algebra is interpreted as the *orbifold cohomology* of $X(\Gamma)$ and $[f]$ is the cup product by $c_1(\mathscr{O}(1))$.

Theorem 7.2.2 **(Varchenko [43]).** *Let T_∞ be the monodromy of $H^{n+1}(U, f^{-1}(t))$ along a circle of big radius positively oriented. Then the Jordan structure of the nilpotent part of T_∞ relative to an eigenvalue λ is the same as the Jordan structure of $[f]$ acting on $\bigoplus_{\alpha | e^{2\pi i\alpha}=\lambda} \mathrm{gr}_V^\alpha(G_0/\theta G_0)$.*

The proof of the theorem in the global setting follows the same lines as in the local setting, cf. [**39**, Th. 7.1].

***Remark 7.2.3*.** Recall that the spectrum is contained in $[0, n+1]$ (or $(0, n+1)$ if $U = \mathbb{C}^{n+1}$). This description makes then clear that the maximal possible size of a Jordan block of T_∞ is $n+1$ if $\lambda \neq 1$. However, if $\lambda = 1$, we expect in general a possible maximal size equal to $n+2$, while, when $U=\mathbb{C}^{n+1}$, this can be at most n. This difference is produced by the cohomology of U.

One could expect that, nevertheless, the size of the Jordan blocks of the monodromy at infinity on $H^{n+1}(f^{-1}(t))$ is n. The example below shows that this is not to be expected (see also Remark 7.3.3).

Example 7.2.4. Assume $n=1$ and $f:(\mathbb{C}^*)^2\to\mathbb{C}$ is defined by $f(u_0,u_1)=u_0+u_1+1/u_0u_1$. Then f has three nondegenerate critical points, with critical values 3ζ ($\zeta^3=1$), and if t is a regular value, $\dim H^2(U,f^{-1}(t))=3$. We have the exact sequence

$$0\longrightarrow H^1(U)\longrightarrow H^1(f^{-1}(t))\xrightarrow{\ \delta\ } H^2(U,f^{-1}(t))\longrightarrow H^2(U)\longrightarrow 0,$$

showing that $\dim H^1(f^{-1}(t))=4$. The monodromy at infinity on $H^2(U,f^{-1}(t))$ has a Jordan block of size $3=n+2$ (cf. the computation in [**14**]).

On the other hand, one can check that the monodromy at infinity on $H^1(f^{-1}(t))$ has a Jordan block of size $2=n+1$. This is done by showing that there exists a variation map $v:H^2(U,f^{-1}(t))\to H^1(f^{-1}(t))$ such that $v\delta$ and δv are $T_\infty-\mathrm{Id}$ on the corresponding spaces. Therefore, if $\omega\in H^2(U,f^{-1}(t))_1$ is such that $(T_\infty-\mathrm{Id})^3\omega=0$ and $(T_\infty-\mathrm{Id})^2\omega\neq 0$, then $(T_\infty-\mathrm{Id})^2v(\omega)=0$ and $(T_\infty-\mathrm{Id})v(\omega)\neq 0$ (apply δ on the left).

7.2.b. Monodromy at infinity of GL-N polynomials. I keep the notation of §4.1. García López & Némethi [**18**] compute the number of Jordan blocks of a given size for any eigenvalue of the monodromy at infinity of a GL-N polynomial. As one may expect from the computation of the spectrum, whether an eigenvalue is a dth root of unity or not will have influence on the answer, in the sense that, for dth roots of unity, the result depends on global invariants of Y_∞, while for other eigenvalues, it only depends on the local monodromies of the singularities of Y_∞.

Let D be a disc centered at ∞ in $\mathbb{P}^1$ and let $D'\to D$ be the ramified covering $t'\mapsto t'^d$. Recall that we have implicitly used in §4.1 the diagram

$$\begin{array}{ccc}\mathscr{X}' & \longrightarrow & \mathscr{X}\\ {\scriptstyle F'}\big\downarrow & & \big\downarrow{\scriptstyle F}\\ D' & \longrightarrow & D\end{array}$$

We denote with a $*$ the effect of restricting away from ∞ (e.g., $t'\neq 0$). On the one hand, the map $X_{|D^*}\to D^*$ is a smooth proper fibration (provided C_f does not intersect D) and we are interested in the monodromy of the restriction $U_{|D^*}\to D^*$. We also know that $X_{\infty|D^*}\to D^*$ is a trivial fibration. Therefore, for what concerns eigenvalues not equal to 1, there is no difference in considering the proper fibration or the open fibration. For the eigenvalue 1, the main argument uses Hodge theory to show that $H^n_{Y_\infty}(X_t)$ is a direct summand in $H^n(X_t)$, in a way compatible to the monodromy. Setting $\#_\ell T$ for the number of Jordan blocks of size ℓ of the operator T with respect to the eigenvalue 1, this leads to

***Theorem 7.2.5* (García López & Némethi [16])**

- $\#_2(T_\infty(f)) = \#_1(T_\infty(F)) - b_n(Y_\infty)$,
- $\#_\ell(T_\infty(f)) = \#_{\ell-1}(T_\infty(F))$ *($\ell \geqslant 3$),*

and $\dim \operatorname{Ker}(T_\infty(f) - \mathrm{Id}) = p_{n-1}(Y_\infty) := b_{n-1}(Y_\infty) - b_{n-1}(\mathbb{P}^n)$.

The next step consists in analyzing $T_{F'}$ and then go from $T_{F'}$ to T_F. Firstly, notice that $T_{F'}$ is identified with T_F^d.

The singularities of $\mathscr{X}'$ are easily analyzed from Equation (4.1.3): they are located on $\{t' = 0\}$, hence on $Y_\infty^{(d)} := \{f_d - Z^d = 0\}$, and are defined by the supplementary equations $\partial f_d/\partial X_i = 0$, $Z = 0$. Hence these are the isolated singular points of $Y_\infty = Y_\infty^{(d)} \cap \{Z = 0\}$. With a similar computation, one shows that, away from these points, the map F' is smooth. It follows that, when considering the nearby cycle complex $\psi_{F'}\mathbb{Q}_{\mathscr{X}'}$, the subcomplex $\psi_{F'}\mathbb{Q}_{\mathscr{X}',\lambda}$ corresponding to an eigenvalue $\lambda \neq 1$ is supported at the singular points. On the other hand, $\psi_{F'}\mathbb{Q}_{\mathscr{X}',1}$ has support equal to $Y_\infty^{(d)}$. This leads to the precise expression of $\#_\ell(T_\infty(f))$ in terms of the corresponding $\#\ell$ of the monodromies of the local singularities of Y_∞ and global invariants of Y_∞ (cf. [**18**]).

7.3. Weights and the monodromy at infinity

The function $f : U \to \mathbb{A}^1$ is not supposed to be tame anymore. A. Dimca and M. Saito have emphasized in [**12**] the relationship between the weights of the general fibre of f, that of U, and the size of the Jordan block of the monodromy with respect to the eigenvalue 1. This relation applies in fact to a very general framwork.

Given any complex algebraic variety Y, the cohomology $H^j(Y, \mathbb{C})$ is equipped, according to Deligne (cf. [**29**]), with a canonical mixed Hodge structure, from which we only retain the weight filtration, which already exists on $H^j(Y, \mathbb{Q})$ and which we denote by $W_\bullet H^j(Y, \mathbb{Q})$. It has the following properties:

- If Y is smooth and compact, $H^j(Y, \mathbb{Q})$ is pure of weight j, that is, $\mathrm{gr}_k^W H^j(Y, \mathbb{Q}) = 0$ if $k \neq j$.
- If Y is smooth (but not compact), let $\overline{Y}$ be a smooth projective compactification of Y such that $D := \overline{Y} \smallsetminus Y$ is a divisor with normal crossings. Denote by $m(D)$ the maximal number of local smooth components of D which have a nonempty intersection (hence $1 \leqslant m(D) \leqslant \dim Y$ and by $m(Y)$ the smallest possible $m(D)$ for various choices of $\overline{Y}$. Then the weights of $H^j(Y, \mathbb{Q})$ belong to $[j, j + m(Y)]$.
- If Y is smooth and affine, then the weights are leq than $2j$, so they belong to $[j, j + \min(m(Y), j)]$.

***Theorem 7.3.1* (Dimca & Saito, [12]).** *Let $f : X \to S$ be a morphism of complex algebraic varieties such that $\dim X = n+1$ and S is a smooth curve. Let $\overline{S}$ be the smooth compactification of S. Let j be a positive integer, and r, r' be integers such that $r' < r$. Assume that, for a general $s \in S$, $H^j(X_s, \mathbb{Q})$ has weights in $[j+r', j+r]$ (i.e., $\mathrm{gr}_k^W H^j(X_s, \mathbb{Q}) = 0$ if $k \notin [j+r', j+r]$), and $H^{j+1}(X, \mathbb{Q})$ has weights $\leqslant j+r$. Then the Jordan blocks of the monodromies of $H^j(X_s, \mathbb{Q})$ around $s_\infty \in \overline{S} \smallsetminus S$ for the eigenvalue 1 have size $\leqslant r - r'$.*

***Corollary 7.3.2* (Dimca & Saito, [12]).** *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function. Then the Jordan blocks of the monodromies of $H^j(f^{-1}(t), \mathbb{Q})$ around ∞ for the eigenvalue 1 have size $\leqslant \min(m(f^{-1}(t)), j)$.*

***Remark 7.3.3*.** If $U = (\mathbb{C}^*)^{n+1}$, then $H^{j+1}(U)$ has weight $2(j+1) = j + (j+2)$, and the bound for the size of the Jordan blocks is $j+2$.

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