# Vanishing cycles and their algebraic computation (III)

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- $X$: smooth quasi-projective $/\mathbb{C}$ (Zariski top.)
- $f : X \longrightarrow \mathbb{A}^1_t$ ***not necess. projective***
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- Recall: $\phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}$ is a constructible complex supported on $\mathrm{Sing}(f^{-1}(c))$, equipped with

$$\mathrm{T}_c : \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}} \longrightarrow \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}$$

  It induces

$$\mathrm{T}_c : H^k(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}) \longrightarrow H^k(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}})$$

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- $\nabla : F \to F \otimes \mathrm{d}z$ a ***connection***, *i.e.*, $\nabla_{\partial_z} : F \to F$ is $\mathbb{C}$-linear $+$ Leibniz rule.

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- $v = (v_1, \dots, v_\mu)$: $\mathbb{C}((z))$-basis of $F$,
- Matrix of $\nabla$: $\boxed{\nabla_{\partial_z} v = v \cdot A_v(z)}$
- Change of basis: $w = v \cdot P(z) \Rightarrow$

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- DEFINITION: $(F, \nabla)$ has ***reg. sing.*** if $\exists$ a basis $v$ s.t. $A_v(z) = \mathrm{M}/z$, $\quad \mathrm{M}$ cst. matrix. Set $E = \mathbb{C} \cdot v$.

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- $\mathrm{T} : \exp(-2\pi i \mathrm{M}) : E \to E$ indept. of such choice: ***Monodromy*** of $(F, \nabla)$ (***reg. sing.***)
- $\boxed{\widehat{\mathrm{RH}}(F, \nabla) = (E, \mathrm{T})}$

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- $\widehat{\mathrm{RH}}^{-1}(E, \mathrm{T})$: $\mathbb{C}((z))$-vect. space with ***reg. sing.*** connection.
- $\Rightarrow \widehat{\mathrm{RH}}(\mathbb{C}((z)) \otimes_{\mathbb{C}} E, \nabla) = (E, \mathrm{T})$

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$$(F, \nabla) \simeq \bigoplus_{\varphi \in \mathbb{C}((z))} (F_\varphi, \nabla^{\mathrm{reg}} + \mathrm{d}\varphi \, \mathrm{Id})$$

Moreover, $[\varphi] \in \mathbb{C}((z))/\mathbb{C}[\![z]\!]$ and $(F_{[\varphi]}, \nabla^{\mathrm{reg}})$ ***uniquely determined***.

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  Moreover, $[\varphi] \in \mathbb{C}((z))/\mathbb{C}[\![z]\!]$ and $(F_{[\varphi]}, \nabla^{\mathrm{reg}})$ ***uniquely determined***.
- Write $(F_\varphi, \nabla^{\mathrm{reg}} + \mathrm{d}\varphi\,\mathrm{Id}) = \widehat{\mathscr{E}}^{\varphi} \otimes (F_\varphi, \nabla^{\mathrm{reg}})$.
- $(F,\nabla)$ of ***exponential type*** if any $[\varphi]$ has simple pole, *i.e.*, $[\varphi] = c/z$ for some $c \in \mathbb{C}$.

# $\mathbb{C}((z))$-vect. spaces with connection

- EXAMPLE: Given $(E, \mathrm{T}, c \in \mathbb{C})$, consider

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- Choose $(E_c^k, \mathrm{T}) = \big(H^k(f^{-1}(c), \phi_{f-c}\mathbb{C}_X), \mathrm{T}\big)$ ($c \in \mathbb{C}$.)

PROBLEM: To give an algebraic formula (in terms of $f$) for $\bigoplus_{c \in \mathbb{C}} \big(\widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}(E_c^k, \mathrm{T})\big)$.

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FORMAL BRIESKORN COMPLEX ATTACHED TO $f$:

$$0 \to \mathscr{O}_X(\!(z)\!) \xrightarrow{\mathrm{d}-\mathrm{d}f/z} \Omega^1_X(\!(z)\!) \xrightarrow{\mathrm{d}-\mathrm{d}f/z} \cdots \Omega^{n+1}_X(\!(z)\!) \to 0$$

Equipped with $\boxed{\nabla_{\partial_z} := \partial/\partial z + f/z^2}$

# Goal for today

THEOREM 3.3.2. For each $k$,

$$\Big(H^k\big(X, (\Omega^\bullet_X((z)), \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)$$

$$\simeq \bigoplus_{c\in\mathbb{C}} \widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}\Big(H^{k-1}\big(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big), \mathrm{T}\Big)$$

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REMARKS.

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REMARKS.

- Conj. by Kontsevich. ($\overset{?}{\Longrightarrow} \phi_f$ on formal schemes)
- For $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ isol. sing., Brieskorn ('70) $\Rightarrow$

$$\Big(\mathscr{H}^k\big(\Omega^\bullet_{\mathbb{C}^{n+1},0}((z)), \mathrm{d}{-}\mathrm{d}f/z\big), \nabla_{\partial_z}\Big) = \begin{cases} 0 & \text{if } k \neq n+1 \\ \widehat{\mathrm{RH}}^{-1}(\mathscr{H}^n\phi_f\mathbb{C}, \mathrm{T}) \end{cases}$$

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- No Hodge theory needed, today, only resol. of sing.

# Assume $f$ proper

THEOREM 3.2.1.(1) For each $k$,

$$\Big(\mathbb{C}(\!(z)\!) \otimes_{\mathbb{C}[z,z^{-1}]} H^k\big(X, (\Omega_X^\bullet[z,z^{-1}], \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)$$

$$\simeq \bigoplus_{c\in\mathbb{C}} \widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}\Big(H^{k-1}\big(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big), \mathrm{T}\Big)$$

Can prove: $f$ is proper $\Rightarrow$ this is equiv. to Th. 3.3.2.

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  $$G := \mathrm{coker}\Big[\mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}} M \xrightarrow{z\partial_t - 1} \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}} M\Big]$$

  equipped with $\nabla_{\partial_z} = \partial/\partial z + t/z^2$.

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  equipped with $\nabla_{\partial_z} = \partial/\partial z + t/z^2$.
- ***Formal stationary phase formula*** gives

$$(\mathbb{C}((z))\otimes_{\mathbb{C}[z]} G, \nabla_{\partial_z}) \simeq \bigoplus_{c\in\mathbb{C}} \widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}({}^p\phi_{t-c}{}^p\mathrm{DR} M^{\mathrm{an}}, \mathrm{T}).$$

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- Apply to $M = \mathscr{H}^k f_+ \mathscr{O}_X$ ($\Rightarrow {}^p\phi_{t-c}{}^p\mathrm{DR} M^{\mathrm{an}} = E_c^k$)

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- $G$ is $\mathbb{C}[z, z^{-1}]$-free of rank $\deg f \cdot \#\mathrm{Crit}\, f$

$\Rightarrow \mathbb{C}(\!(z)\!) \otimes_{\mathbb{C}[z,z^{-1}]} G$ does not give the right answer ($= 0$)

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- $\widehat{G} = 0$ with
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Proof:

- $\ker = 0$ easy
- Given $\psi_{k_o}, \psi_{k_o+1}, \cdots \in \mathbb{C}[t, 1/f']$, can find $\varphi_{k_o}, \varphi_{k_o+1}, \cdots \in \mathbb{C}[t, 1/f']$ s.t.

$$\psi_{k_o} = -f'\varphi_{k_o}, \ldots, \quad \psi_{k+1} = \partial_t\varphi_k - f'\varphi_{k+1}, \ldots,$$

(solve induct. with $f'$ invertible in $\mathbb{C}[t, 1/f']$).

# Strategy of the proof

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$$\simeq \bigoplus_{c\in\mathbb{C}} \widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}\Big(H^{k-1}\big(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big), \mathrm{T}\Big)$$

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- Introduce an intermediate term

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- Choose a comm. diag.

$$(3.3.4)\qquad \begin{array}{ccc} X & \overset{j}{\hookrightarrow} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1 \end{array}$$

$Y$: smooth proj.

$D := Y \smallsetminus X$: ncd

# Strategy of the proof

- GAGA type thm by standard methods:

$$\Big(H^k\big(X, (\Omega^\bullet_X((z)), \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)$$
$$\simeq \Big(H^k\big(Y, (\Omega^\bullet_Y(*D)((z)), \mathrm{d} - \mathrm{d}F/z)\big), \nabla_{\partial_z}\Big)$$
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- ***Local*** statement on $Y^{\mathrm{an}}$:

$$\boxed{(\Omega^\bullet_{Y^{\mathrm{an}}}(*D)((z)), \mathrm{d} - \mathrm{d}F/z)\big) \simeq Rj_*(\Omega^\bullet_{X^{\mathrm{an}}}((z)), \mathrm{d} - \mathrm{d}f/z))}$$

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  - Assume that $(D\cup$ ***crit. fibres***$)$ is ***ncd***. Then local computations $\Rightarrow$ OK.
  - Need a ***decomposition thm for proper modifications*** to reduce to the ncd case.

# Strategy of the proof

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$$(\Omega^\bullet_{Y^{\rm an}}(*D)((z)), \mathrm{d} - \mathrm{d}F/z)\big) \simeq Rj_*(\Omega^\bullet_{X^{\rm an}}((z)), \mathrm{d} - \mathrm{d}f/z)\big),$$

we get

$$\boxed{\begin{aligned}&\Big(H^k\big(X, (\Omega^\bullet_X((z)), \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)\\ &\qquad\qquad \simeq \Big(H^k\big(X^{\rm an}, (\Omega^\bullet_{X^{\rm an}}((z)), \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)\end{aligned}}$$

# Strategy of the proof

- From

$$\big(\Omega^\bullet_{Y^{\mathrm{an}}}(*D)((z)), \mathrm{d}-\mathrm{d}F/z\big) \simeq Rj_*\big(\Omega^\bullet_{X^{\mathrm{an}}}((z)), \mathrm{d}-\mathrm{d}f/z\big),$$

  we get

$$\boxed{\begin{aligned}&\Big(H^k\big(X, (\Omega^\bullet_X((z)), \mathrm{d}-\mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)\\ &\qquad\simeq \Big(H^k\big(X^{\mathrm{an}}, (\Omega^\bullet_{X^{\mathrm{an}}}((z)), \mathrm{d}-\mathrm{d}f/z)\big), \nabla_{\partial_z}\Big)\end{aligned}}$$

- We now focus on $X^{\mathrm{an}}$. Try to relate

$$(\Omega^\bullet_{X^{\mathrm{an}}}((z)), \mathrm{d}-\mathrm{d}f/z)_{|f^{-1}(c)} \quad \text{with} \quad \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}$$

  $\Rightarrow$ ***local statement*** on $X^{\mathrm{an}}$.

# Strategy of the proof

- Would like to compare

$$(\Omega^\bullet_{X^{\mathrm{an}}}((z)), \mathrm{d} - \mathrm{d}f/z)$$

with

$$\mathbb{C}((z)) \otimes_{\mathbb{C}[z^{-1}]} (\Omega^\bullet_{X^{\mathrm{an}}}[z^{-1}], \mathrm{d} - \mathrm{d}f/z)$$

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- More useful: $\partial_t \leftrightarrow z^{-1}$, compare with

$$\widehat{\mathscr{E}_{\mathbb{C}}} \otimes_{\mathscr{D}_{\mathbb{C}}} (\Omega^{\bullet}_{X^{\mathrm{an}}}[z^{-1}], \mathrm{d} - \mathrm{d}f/z)$$

# Strategy of the proof

- Comparison OK (loc. comput. on resol. of sing. $+$ come back)

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- Comparison OK (loc. comput. on resol. of sing. $+$ come back)
- Use previous results of M. Saito and M. Kapranov to conclude

$$
\begin{aligned}
&\Big(H^k\big(X^{\mathrm{an}}, (\Omega^\bullet_{X^{\mathrm{an}}}(\!(z)\!), \mathrm{d} - \mathrm{d}f/z)\big), \nabla_{\partial_z}\Big) \\
&\quad\simeq \Big(H^k\big(X^{\mathrm{an}}, \widehat{\mathscr{E}}_{\mathbb{C}} \otimes_{\mathscr{D}_{\mathbb{C}}} (\Omega^\bullet_{X^{\mathrm{an}}}[z^{-1}], \mathrm{d} - \mathrm{d}f/z), \nabla_{\partial_z}\Big) \\
&\qquad \bigoplus_{c\in\mathbb{C}} \widehat{\mathscr{E}}^{-c/z} \otimes \widehat{\mathrm{RH}}^{-1}\Big(H^{k-1}\big(f^{-1}(c), \phi_{f-c}\mathbb{C}_{X^{\mathrm{an}}}\big), \mathrm{T}\Big)
\end{aligned}
$$

# Conclusion for the 3 lectures

# Conclusion for the 3 lectures

- The case $f$ proper is more comfortable and brings more info.
- If $f$ not proper, consider a diagram

$$\begin{array}{ccc} X & \overset{\kappa}{\hookrightarrow} & X' \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\ & & \mathbb{A}^1 \end{array}$$

  $\Rightarrow (\Omega_X\bullet[z, z^{-1}], \mathrm{d} - \mathrm{d}f/z)$ captures info for $\phi_{g-c}R\kappa_*\mathbb{C}_X$.
- ***But*** in general $\phi_{g-c}R\kappa_*\mathbb{C}_X \neq R\kappa_{c*}\phi_{f-c}\mathbb{C}_X$.
  ($\kappa_c : f^{-1}(c) \hookrightarrow g^{-1}(c)$)

# Tame case

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- However, if ***tameness assumption***:

$$\forall c \in \mathbb{C}, \quad \operatorname{Supp} \phi_{g-c} R\kappa_* \mathbb{C}_X \cap (X' \setminus X) = \varnothing$$

  $\Rightarrow$ can work with $f$ "***as if***" it were proper.
- Useful in Mirror Symmetry for Fano manifolds (or Fano orbifolds).